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Viscosity solution for geometric Asian barrier option

(기하학적 아시안 베리어 옵션에 대한 점성해)

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Viscosity solution for geometric Asian barrier option

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Abstract

Viscosity solution for geometric Asian barrier option

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In this thesis, we consider Barrier option and Geometric Asian option based on Black-Scholes model and derive partial differential equation which these two options satisfy. Also, we calculate its closed form solution as the option value at time t. Moreover, by combining Barrier option and Geometric Asian option, we consider Geometric Asian Barrier option and its modeling partial differential problem. However, It is not known that this problem has classical solution. Instead, we show that the value of geometric Asian barrier option becomes a viscosity solution of the modeling problem.

Key words: Black-Scholes equation, Barrier option, Geometric Asian option, Geometric Asian Barrier option,Viscosity solution
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Contents

Chapter 1

Introduction

Option pricing is an interesting subject in the quantitive finance. The basic model in option pricing is the Black-Scholes model(equation)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rV = 0, \tag{1.0.1}$$

which Fisher Black and Myron Scholes [6] discovered in 1973. It is a parabolic equation and also can be transformed by the heat equation by change of variable $x = e^{\tilde{x}}$. So by using the fundamental solution of heat equation, we calculate the solution of Black-Scholes equation([9]). The regular call and put options satisfy (1.0.1). In the Black-Scholes model, we assume some following conditions ([11]) :

1. The risk-free rate r is known.
2. The stock price process X_t follows a geometric Brownian motion process

   $$dX_t = \mu X_t\, dt + \sigma X_t dW_t$$

   where μ and σ are constant and W_t is a standard Brownian motion.
3. The stock pays no dividends.
4. There are no transaction costs and taxes.

5. There are no penalties for short sales.

6. The market is arbitrage free.

7. Option trading operates continuously.

Since (1.0.1) can be transformed by the heat equation. the value $v(t, x)$ of the option which satisfies (1.0.1) is represented as the closed form solution. In addition to (1.0.1), there are different equation with specific conditions which the exotic options satisfy. In particular, we focus on pricing Barrier option, Geometric Asian option and Geometric Asian Barrier option. In the case of them, the closed form solution sometime is not guaranteed.

In chapter 2, we consider the basic notions which are needed for option pricing including Itô calculus with Brownian motion. Also, we study the risk-neutral measure. We first prove the Girsanov theorem which represents how to transform the real measure $\mathbb{P}$ into the risk-neutral measure $\widetilde{\mathbb{P}}$. Since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, we price the option under the risk-neutral measure. By the Feynman-Kac formula, we also get the risk-neutral pricing formula and then we derive the Black-Scholes formula.

In chapter 3, we study Barrier option. Barrier option is a derivative that the payoff depends on whether the stock price hits a predetermined barrier during the option period. In addition to regular call and put options, the barrier option has additional barrier condition, B which is constant. First, there are Up option and Down option. If the barrier is set above the initial stock price, it is a up-option. Also, if the barrier is set below the initial stock price, it is a down-option. In addition, there are Knock-in option and Knock-out option. A Knock-in option becomes valid when the stock price hits the barrier. For example, we consider knock-in call option. When the stock price hit the barrier, a knock-in call option acts as a regular call option. However, a knock-out option becomes invalid if the barrier is touched. Merton [13] firstly priced the value of the down-out call option. M. Rubinstein introduced the various types of payoff of barrier options and priced the value of barrier options([14]). We calculate the value of down-in and up-out call option([15])

and confirm the equation with conditions for the barrier option.

In chapter 4, we study Geometric Asian option. Geometric Asian option is a derivative whose payoff contains the geometric average of stock price instead of the stock price. If the stock price has log-normal distribution, so does the geometric average of the stock price. Therefore, only for the geometric Asian option, there exists the closed solution formula in Black-Scholes model and the solution has uniqueness([5]). The continuous geometric average of the stock price is given by

$$\exp\left(\frac{1}{t}\int_0^t \log X_u du\right).$$

Kemna and Vorst [12] derived the closed solution form for the geometric Asian option specially at time $t = 0$. More generally, the closed form solution for the geometric Asian option price at time t is given in [4]. So we derive the value and the equation which the geometric Asian options satisfy using the property of martingale and Markov.

In chapter 5, we consider the geometric Asian barrier option which is the combination of the barrier option and the geometric Asian option. It has being currently researched by Aimi and Guardasoni [3] and Aimi et al [1], [2]. Unlike the geometric Asian option, the geometric Asian barrier option has no closed priced formula ([1]) in Black-Scholes model. Therefore, we adopt the viscosity solution. We introduce the definition of viscosity solution of pde for the geometric Asian barrier option([8]). Moreover, we confirm the equation with conditions for the geometric Asian barrier option and we show that its value become a viscosity solution.

Chapter 2

Black-Scholes equation

2.1 Itô Calculus

We consider stocks as the underlying asset for the option. We know that the stock price is unexpected. That is, a stock price process must have randomness. For this randomness, we use the Brownian motion.

Definition 2.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_t$ be an associated filtration. A continuous stochastic process $W = (W_t)_{t \geq 0}$ is a standard Brownian motion if it satisfies the followings

1. $W_0 = 0$

2. For all $0 \leq s \leq t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and follows normal distribution $\mathcal{N}(0, t - s)$.

For option pricing, the most important properties of Brownian motion is a martingale and it accumulates quadratic variation at rate one per unit time.

Theorem 2.1.2. *Let $(W_t)_{t \geq 0}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ be an associated filtration. Then the Brownian motion W_t is a martingale.*

Proof. For $s < t$,

$$\begin{aligned}\mathbb{E}[W_t \mid \mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s \mid \mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s \mid \mathcal{F}_s] + W_s \\ &= \mathbb{E}[W_t - W_s] + W_s \\ &= W_s\end{aligned}$$

By Hölder inequality, we can get $\mathbb{E}[|W_t|] \leq \sqrt{\mathbb{E}[W_t^2]} = \sqrt{t} < \infty$. In all, Brownian motion is a martingale.

□

Definition 2.1.3. Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time t is

$$[f, f]_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} [f(t_{i+1}) - f(t_i)]^2$$

where $t_i = \frac{it}{n}$ and $0 = t_0 < t_1 < \cdots < t_n = t$.

Theorem 2.1.4. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the quadratic variation of Brownian motion is*

$$[W, W]_t = t.$$

Proof. The quadratic variation of Brownian motion is given by

$$[W, W]_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2$$

We know $W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, \frac{t}{n})$. So we have

$$Var[W_{t_{i+1}} - W_{t_i}] = \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = \frac{t}{n}.$$

Therefore, we can derive

$$\mathbb{E}\left[\lim_{n\to\infty}\sum_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\right]=\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}}-W_{t_i})^2]=t$$

We need to show that the variance is zero. Then we write

$$\begin{aligned} Var\left[\lim_{n\to\infty}\sum_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\right] &= \mathbb{E}\left[\left(\lim_{n\to\infty}\sum_{i=0}^{n-1}\Delta W_{t_i}^2-t\right)^2\right] \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[\left(\Delta W_{t_i}^2-\frac{t}{n}\right)^2\right] \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\frac{3t^2}{n^2}-\frac{2t^2}{n^2}+\frac{t^2}{n^2}\right) \\ &= \lim_{n\to\infty}\frac{2t^2}{n} \\ &= 0. \end{aligned}$$

In all, we get the quadratic variation $[W,W]_t = t$. □

Since we employ the Brownian motion process for the randomness, the stochastic differential equation of the stock price process have the Brownian motion term. However, we know that the Brownian motion is nowhere differentiable. So for considering the Brownian motion term, we need new calculus, called Itô calculus.

Definition 2.1.5. Let $(W_t)_{t\geq 0}$ be the Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and $(X_t)_{t\geq 0}$ be a stochastic process. If $(X_t)_{t\geq 0}$ satisfies the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

where σ and μ are locally bounded in t and progressively measurable, it is called the Itô process. We can also write in integrated form

$$X_t = X_o + \int_0^t \mu(s, X_s)\, ds \; + \int_0^t \sigma(s, X_s)\, dW_s$$

We call $\mu(t, X_t)$ the drift and $\sigma(t, X_t)$ the volatility of X_t.

Definition 2.1.6. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\geq 0}$ be a associated filtration. Assume

$$\mathbb{E}\left[\int_0^t f(s, W_s)^2\, dW_s\right] < \infty.$$

Then Itô integral is defined by

$$I_t = \int_0^t f(s, W_s)\, dW_s := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i})$$

where $f(t_i, W_{t_i})$ is a simple process and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$.

Lemma 2.1.7. *Let $(X_t)_{t\geq 0}$ be an Itô process and $f(x, t)$ be C^2-function. Then, for every $t \geq 0$, we have almost surely*

$$df(t, X_t) = f_t(t, X_t)\, dt + f_x(t, X_t)\, dX_t + \frac{1}{2} f_{xx}(t, X_t)\, dX_t dX_t.$$

2.2 Risk-neutral measure

Theorem 2.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\widetilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. If $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, there exists almost surely positive random variable Z such that $\mathbb{E}[Z] = 1$ and*

$$\widetilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}$$

for $A \in \mathcal{F}$.
Such random variable Z *is called Radon-Nikodym derivative and denoted by*

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

Proposition 2.2.2. *Let* $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$ *be probability measures on* $(\Omega, \mathcal{F})$. *Suppose for a random variable* Z *and* $A \in \mathcal{F}$, *we define*

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$$

such that

$$\widetilde{\mathbb{P}}(A) = \int_A Z \, d\mathbb{P}$$

and $\mathbb{P}(Z > 0) = 1$. *Then the probability measures* $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$ *are equivalent.*

Proof. We say that two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if they have same null-set. For $A \in \mathcal{F}$, suppose that $\mathbb{P}(A) = 0$. Then we have

$$\widetilde{\mathbb{P}}(A) = \int_\Omega \mathbb{1}_A Z \, d\mathbb{P} = 0.$$

On the contrary, for $\widetilde{\mathbb{P}}(B) = 0$, suppose $B \in \mathcal{F}$. Then we have

$$\mathbb{P}(B) = \int_\Omega \mathbb{1}_B \frac{1}{Z} \, d\widetilde{\mathbb{P}} = 0.$$

□

Proposition 2.2.3. *Let* $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ *be a filtered probability space and* $\widetilde{\mathbb{P}}$ *be another probability measure on* $(\Omega, \mathcal{F})$. *For the Radon-Nikodym derivative* $Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$, *define the Radon-Nikodym derivative process*

$$Z_t = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$$

Then Z_t *is a martingale under* $\mathbb{P}$.

Proof. By Theorem 2.2.1, we get

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z : \Omega \to (0, \infty).$$

Moreover, we derive

$$\mathbb{E}\left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right] = \int_\Omega \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} d\mathbb{P} = \int_\Omega d\widetilde{\mathbb{P}} = 1.$$

Therefore, we can deduce $Z_t > 0$ for $o \leq t \leq T$. Now, we want to check that Z_t is non-negative martingale. Clearly Z_t is $\mathcal{F}_t$-measurable. By the tower property of the conditional expectation, we get the equality

$$\begin{aligned}\mathbb{E}\left[Z_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[Z \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\ &= \mathbb{E}\left[Z \mid \mathcal{F}_s\right] \\ &= Z_s.\end{aligned}$$

In addition, we obtain

$$\begin{aligned}\mathbb{E}\left[|Z_t|\right] &= \mathbb{E}\left[\mathbb{E}\left[Z \mid \mathcal{F}_t\right]\right] \\ &= \mathbb{E}\left[Z\right] \\ &= 1 < \infty.\end{aligned}$$

In all, Z_t is non-negative $\mathbb{P}$-martingale. □

Lemma 2.2.4. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $\widetilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ such that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. Let $(X_t)_{0\leq t\leq T}$ be a $\mathcal{F}_t$-measurable and consider the positive $\mathbb{P}$-martingale $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t$. Then the expectation value of X_t under $\widetilde{\mathbb{P}}$ is given by*

$$\widetilde{\mathbb{E}}\left[X_t\right] = \mathbb{E}\left[X_t Z_t\right].$$

Proof. By the definition of the Radon-Nikodym derivative and the tower property of the conditional expectation, we derive the equality

$$\begin{aligned}\widetilde{\mathbb{E}}\left[X_t\right] &= \mathbb{E}\left[X_t Z\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X_t Z \mid \mathcal{F}_t\right]\right] \\ &= \mathbb{E}\left[X_t \mathbb{E}\left[Z \mid \mathcal{F}_t\right]\right] \\ &= \mathbb{E}\left[X_t Z_t\right].\end{aligned}$$

□

Lemma 2.2.5. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $\widetilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ such that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent. Let $(X_t)_{0\leq t\leq T}$ be a $\mathcal{F}_t$-measurable and consider the positive $\mathbb{P}$-martingale $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}= Z_t$. For $0 \leq s \leq t \leq T$,*

$$\widetilde{\mathbb{E}}\left[X_t \mid \mathcal{F}_s\right] = \frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right].$$

Proof. We know clearly $\frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right]$ is $\mathcal{F}_s$-measurable. Then we need to check the partial averaging property such that for any $A \in \mathcal{F}_s$,

$$\int_A \frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right]\, d\widetilde{\mathbb{P}} = \int_A X_t\, d\widetilde{\mathbb{P}}.$$

Since we have

$$\begin{aligned}\int_A \frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right]\, d\widetilde{\mathbb{P}} &= \int_A \frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right] Z_s\, d\mathbb{P} \\ &= \int_A X_t Z_t\, d\mathbb{P} \\ &= \int_A X_t\, d\widetilde{\mathbb{P}}.\end{aligned}$$

In all, we get the result

$$\tilde{\mathbb{E}}\left[X_t \mid \mathcal{F}_s\right] = \frac{1}{Z_s}\mathbb{E}\left[X_t Z_t \mid \mathcal{F}_s\right].$$

□

Theorem 2.2.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(X_t)_{t\geq 0}$ be a martingale with respect to a filtration $\mathcal{F}_t$ with a continuous paths and $X_0 = 0$. If for all $t \geq 0$, the quadratic variation of X_t is*

$$[X, X]_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 = t,$$

X_t is a Brownian motion.

Proof. We want to show $X_t \sim \mathcal{N}(0, t)$ by using the moment generating function. For fixed u, define $f(t, X_t) = e^{\theta X_t - \frac{1}{2}\theta^2 t}$. By Itô formula,

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} \right) dt + \frac{\partial f}{\partial X_t} dX_t \\ &= \theta f(t, X_t) dX_t. \end{aligned}$$

By integrating and taking expectation both sides, we get

$$\mathbb{E}[f(t, X_t)] = 1 + \theta\, \mathbb{E}\left[\int_0^t f(s, X_s) dX_s \right].$$

Since Itô integral $\int_0^t f(s, X_s) dX_s$ is a martingale and its value at time 0 is zero, its expectation value is also zero. In all. the moment generating function of X_t is

$$\mathbb{E}[e^{\theta X_t}] = e^{\frac{1}{2}\theta^2 t}.$$

which is the moment generating function for the normal distribution with mean zero and variance t. That is, $X_t \sim \mathcal{N}(0, t)$. Since X_t is a martingale, we can write for $s < t$

$$\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0$$

This means that the increment $X_t - X_s$ is independent to $\mathcal{F}_s$. Consider the variance of $X_{t+s} - X_t$

$$\begin{aligned} Var[X_{t+s} - X_t] &= Var[X_{t+s}] + Var[X_t] - 2Cov[X_{t+s}, X_t] \\ &= t + s + t - 2\left(\mathbb{E}\left[X_{t+s}X_t - \mathbb{E}[X_{t+s}]\mathbb{E}[X_t]\right]\right) \\ &= 2t + s - 2\mathbb{E}[X_{t+s}X_t] \\ &= 2t + s - 2\mathbb{E}[X_t(X_{t+s} - X_t) - X_t^2] \\ &= 2t + s - 2\mathbb{E}[X_t^2] \\ &= s. \end{aligned}$$

Since clearly $\mathbb{E}[X_{t+s} - X_t] = 0$, we get $X_{t+s} - X_t \sim \mathcal{N}(0, s)$. We know that X_0 and X_t has continuous paths. In all, X_t is a standard Brownian motion. □

Theorem 2.2.7. ***(Girsanov Theorem)***
Let $(W_t)_{0 \leq t \leqslant T}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration for Brownian motion. For an adapted process $(\theta_t)_{0 \leq t \leq T}$, define

$$\begin{aligned} Z_t &= exp\left(-\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right), \\ \widetilde{W}_t &= W_t + \int_0^t \theta_s \, ds. \end{aligned}$$

and assume

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \theta_t^2 \, dt}\right] < \infty.$$

Then Z_t is a martingale under $\mathbb{P}$ and $(\widetilde{W}_t)_{0 \leq t \leq T}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$ defined by

$$Z_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}.$$

Proof. We will use Theorem 2.2.6. First, we want to show Z_t is a martingale under $\mathbb{P}$. Put $f(x) = e^x$. Then

$$Z_t = f(A_t)$$

where

$$A_t = -\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds.$$

By Itô formula, we get the equation

$$\begin{aligned} dZ_t &= f'(A_t)\, dA_t + \frac{1}{2} f''(A_t)\, dA_t dA_t \\ &= -\theta_t Z_t \, dW_t. \end{aligned}$$

By integrating both sides, we get

$$Z_t = Z_s - \int_s^t \theta_u Z_u \, dW_u.$$

Since it has no drift term, Z_t is also a martingale. Conditioned on $Z = Z_T$, by the martingale property, we can derive

$$Z_t = \mathbb{E}\left[Z_T \mid \mathcal{F}_t\right] = \mathbb{E}\left[Z \mid \mathcal{F}_t\right], \quad 0 \le t \le T.$$

So Z_t is a Radon-Nikodym derivative process.
Now, we are going to consider $\widetilde{W}_t Z_t$. By Itô formula, we get

$$\begin{aligned} d(\widetilde{W}_t Z_t) &= \widetilde{W}_t \, dZ_t + Z_t \, d\widetilde{W}_t + d\widetilde{W}_t \, dZ_t \\ &= (1 - \theta_t \widetilde{W}_t) Z_t \, dW_t. \end{aligned}$$

By integrating both sides, we get

$$\widetilde{W}_t Z_t = \widetilde{W}_s Z_s + \int_s^t (1 - \theta_u \widetilde{W}_u) Z_u \, dW_u.$$

Since it has no drift term and $\int_s^t (1 - \theta_u \widetilde{W}_u) Z_u \, dW_u$ is independent to $\mathcal{F}_s$, we can get the equation

$$\mathbb{E}[\widetilde{W}_t Z_t \mid \mathcal{F}_s] = \widetilde{W}_s Z_s.$$

Therefore, we consider $\widetilde{W}_t$ and then

$$\widetilde{\mathbb{E}}[\widetilde{W}_t \mid \mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[\widetilde{W}_t Z_t \mid \mathcal{F}_s] = \frac{1}{Z_s}\widetilde{W}_s Z_s = \widetilde{W}_s.$$

Also, by the condition, $\widetilde{\mathbb{E}}[|\widetilde{W}_t|] < \infty$. Clearly $\widetilde{W}_0 = 0$ and has continuous paths for $t > 0$. We want to derive the quadratic variation of $\widetilde{W}_t$. For $t_i = \frac{i}{n}t$, set $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$. Then the quadratic variation of $\widetilde{W}_t$ is

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n-1}\left(\widetilde{W}_{t_{i+1}} - \widetilde{W}_{t_i}\right)^2 &= \lim_{n\to\infty}\sum_{i=1}^{n-1}\left(W_{t_{i+1}} - W_{t_i}\right)^2 \\ &\quad + \lim_{n\to\infty}\sum_{i=1}^{n-1} 2\left(W_{t_{i+1}} - W_{t_i}\right)\left(\int_{t_i}^{t_{i+1}} \theta_s ds\right) \\ &\quad + \lim_{n\to\infty}\sum_{i=1}^{n-1}\left(\int_{t_i}^{t_{i+1}} \theta_s ds\right)^2 \\ &= t.\end{aligned}$$

Since the integration is continuous, $\lim_{n\to\infty}\int_{t_i}^{t_{i+1}} \theta_s ds = 0$, and then we can get above result. By the Lévy theorem, $\widetilde{W}_t$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$. □

Remark 2.2.8. $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ in Theorem 2.2.7 are equivalent.

Definition 2.2.9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space and let $\widetilde{\mathbb{P}}$ be another probability measure such that they are equaivalent. Then the probability measure $\widetilde{\mathbb{P}}$ is said to be the risk-neutral measure if it satisfies the following conditions

1. $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent.
2. Under $\widetilde{\mathbb{P}}$, the discounted underlying asset price is a martingale.

In the Black-Scholes model, we assumed that the stock price follows a geometric Brownian motion. Now, we consider the general geometric Brownian motion form

$$dX_t = \mu(t)X_t dt + \sigma(t)X_t dW_t, \quad 0 \le t \le T \tag{2.2.1}$$

where $\mu(t)$ and $\sigma(t)$ are adapted process. In the integral form, we have

$$X_t = X_0 \exp\left[\int_0^t \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^t \sigma(u) dW_u\right]. \tag{2.2.2}$$

Define the discounted factor

$$D_t = e^{-\int_0^t r(u)\, du} \tag{2.2.3}$$

where $r(t)$ is the interest rate. Then the discounted stock price process is given by

$$D_t X_t = X_0 \exp\left[\int_0^t \left(\mu(u) - r(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^t \sigma(u)\, dW_u\right]. \tag{2.2.4}$$

In the differential form, (2.2.4) is expressed by

$$\begin{aligned} d(D_t X_t) &= (\mu(t) - r(t)) D_t X_t\, dt + \sigma(t) D_t X_t\, dW_t \\ &= \sigma(t) D_t X_t\, (\theta(t)\, dt + \, dW_t) \end{aligned} \tag{2.2.5}$$

where $\theta(t) := \frac{\mu(t)-r(t)}{\sigma(t)}$ is an adapted process. With $\theta(t)$, consider Z_t in Theorem 2.2.7 and

$$\widetilde{W}_t = \theta(t) + \int_0^t \theta(u)\, du. \tag{2.2.6}$$

Define the probability measure $\widetilde{\mathbb{P}}$ with Radon-Nikodym derivative process Z_t with respect to $\widetilde{\mathbb{P}}$. By Theorem 2.2.7, (2.2.6) is a standard Brownian motion under $\widetilde{\mathbb{P}}$. Therefore, (2.2.5) becomes

$$d(D_t X_t) = \sigma(t) D_t X_t\, \widetilde{W}_t. \tag{2.2.7}$$

and so it is a martingale under $\widetilde{\mathbb{P}}$. Finally, we propose that $\widetilde{\mathbb{P}}$ is a risk-neutral measure. Moreover, under $\widetilde{\mathbb{P}}$, we have

$$dX_t = r(t)X_t\,dt + \sigma(t)X_t\,d\widetilde{W}_t. \tag{2.2.8}$$

Then (2.2.8) means that the return rate from the stock under the risk-neutral measure is a risk-free rate.

Proposition 2.2.10. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Define the process Z_t such that*

$$Z_t = exp\left(-\int_0^t \theta_s\,dW_s - \frac{1}{2}\int_0^t \theta_s^2\,ds\right)$$

Then the probability measure $\widetilde{\mathbb{P}}$ defined by

$$Z_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$$

is a risk-neutral measure.

Proof. By Proposition 2.2.2, we know that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. From (2.2.7), the discounted stock price process is a martingale. By Definition 2.2.9, $\widetilde{\mathbb{P}}$ is the risk-neutral measure. □

2.3 Black-Scholes formula

Theorem 2.3.1. ***(Feynman-Kac formula)***
Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ be a related filtration. Suppose that the stochastic process X_t satisfies the generalized stochastic differential equation such that

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

For $X_t = x$, we consider the following partial differential equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 v}{\partial x^2} + \mu(t,x)\frac{\partial v}{\partial x} - r(t)v(t,x) = 0 \tag{2.3.1}$$

with the boundary condition $v(T,x) = V(X_T)$. Then the solution is given by

$$v(t,x) = \mathbb{E}\left[e^{-\int_t^T r(u)du} V(X_T) \mid \mathcal{F}_t\right]. \tag{2.3.2}$$

Proof. Define the process Z_u such that

$$Z_u = e^{-\int_t^u r(z)dz} v(u, X_u).$$

By Itô formula, we get

$$\begin{aligned}
dZ_u &= \frac{\partial Z_u}{\partial u}du + \frac{\partial Z_u}{\partial X_u}dX_u + \frac{1}{2}\frac{\partial^2 Z_u}{\partial X_u^2}(dX_u)^2 \\
&= e^{-\int_t^u r(z)dz}\left(\frac{\partial v}{\partial u} + \frac{1}{2}\sigma^2(u,X_u)\frac{\partial^2 v}{\partial X_u^2} + \mu(u,X_u)\frac{\partial v}{\partial X_u} - r(u)v(u,X_u)\right)du \\
&\quad + e^{-\int_t^u r(z)dz}\sigma(u,X_u)\frac{\partial v}{\partial X_u}dW_u \\
&= e^{-\int_t^u r(z)dz}\sigma(u,X_u)\frac{\partial v}{\partial X_u}dW_u.
\end{aligned}$$

By taking integral from t to T, we derive

$$Z_T - Z_t = \int_t^T e^{-\int_t^u r(z)dz}\sigma(u,X_u)\frac{\partial v}{\partial X_u}dW_u. \tag{2.3.3}$$

Since the right side of (2.3.3) is Itô integral, we obtain

$$\mathbb{E}[Z_t] = \mathbb{E}[Z_T].$$

Thus the conditional expectation can be given by

$$\mathbb{E}[Z_t \mid \mathbb{F}_t] = \mathbb{E}[Z_T \mid \mathbb{F}_t].$$

Finally, we can get the result

$$v(t,x) = \mathbb{E}\left[e^{-\int_t^T r(z)dz} V(X_T) \mid \mathcal{F}_t\right]. \tag{2.3.4}$$

□

Consider Theorem 2.3.1 under the risk-neutral measure $\widetilde{\mathbb{P}}$. By Theorem 2.3.1, under the Black-Scholes model, the value of option satisfies the Black-Scholes equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv = 0.$$

for some constant $r, \sigma > 0$. In addition, (2.3.4) becomes

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(X_T) \mid \mathcal{F}_t\right]. \tag{2.3.5}$$

We refer (2.3.5) as the risk-neutral pricing formula. We are going to use the risk-neutral pricing formula for pricing the option. Therefore, the value in this paper means the risk-neutral value.

Now, we consider a European call and a put option whose expiration time is T and strike price is K under Black-Scholes model. The payoff of a call option is given by

$$(X_T - K)^+$$

and a put option is

$$(K - X_T)^+.$$

By the risk-neutral pricing formula, the value of call option is given by

$$c(t,x) = \widetilde{\mathbb{E}}[e^{-r(T-t)}(X_T - K)^+ \mid \mathcal{F}_t]. \tag{2.3.6}$$

We have the stock price process

$$dX_t = rX_t dt + \sigma X_t d\widetilde{W}_t$$

which is the geometric Brownian motion form under $\widetilde{\mathbb{P}}$. Then we have

$$X_T = X_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(\widetilde{W}_T - \tilde{W}_t)\right] \tag{2.3.7}$$

Let $Z = -\frac{\widetilde{W}_T - \widetilde{W}_t}{\sqrt{T-t}}$. Then Z follows the standard normal distribution and is independent to $\mathcal{F}_t$. Therefore, conditioned on $X_t = x$ and $Z_t = z$, (2.3.6) holds with

$$\begin{aligned} c(t,x) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(x\exp\left[\left(r-\frac{1}{2}\sigma^2\right)(T-t) - \sigma\sqrt{T-t}Z\right] - K\right)^+\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r(T-t)}\left(x\exp\left[\left(r-\frac{1}{2}\sigma^2\right)(T-t) - \sigma\sqrt{(T-t)}\,z\right] - K\right)^+ e^{-\frac{1}{2}z^2}dz \end{aligned} \tag{2.3.8}$$

Moreover, we can only define (2.3.8) if

$$z < \frac{\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} := d_2.$$

Thus we get

$$\begin{aligned} c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} x\exp\left(-\frac{1}{2}z^2 - \sigma\sqrt{(T-t)}\,z - \frac{\sigma^2}{2}(T-t)\right)dz \\ &\quad - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-r(T-t)}Ke^{-\frac{1}{2}z^2}dz \\ &= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\exp\left[-\frac{1}{2}(z+\sigma\sqrt{(T-t)})^2\right]dz - Ke^{-r(T-t)}N(d_2) \\ &= xN(d_1) - Ke^{-r(T-t)}N(d_2) \end{aligned}$$

where N is a standard normal cumulative distribution function. Finally, the value of European call option is

$$c(t,x) = xN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log \frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Theorem 2.3.2. *Let $c(t)$ be a value of the call option and $p(t)$ be a value of the put option at time t. Suppose that call and put options have the same maturity T and the strike price K. Then $c(t)$ and $p(t)$ have the relation*

$$c(t) - p(t) = X_t - Ke^{-r(T-t)}$$

By Theorem 2.3.2, we can get the value of European put option at time t as follows:

$$p(t, x) = e^{-r(T-t)}KN(-d_2) - xN(-d_1).$$

Moreover, the value of a call option satisfies the equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + rx\dfrac{\partial v}{\partial x} - rv = 0. & \text{in } \mathcal{H}_T \\ u(T, x, y) = f(y) := (X_T - K)^+ & x \in \mathbb{R}^+ \end{cases} \tag{2.3.9}$$

where $\mathcal{H}_T := [0, T) \times \mathbb{R}^+$.

Chapter 3

Barrier Option

3.1 Pricing Barrier option

For the barrier option, whether the stock price hits the barrier determines the validity of option. Thus the payoff must contain barrier restriction. Consider the up-out call option. If the stock price hit the barrier, the up-out option become worthless. For the validity of option, the maximum of the stock price must be lower than the barrier. Therefor, the payoff is given by

$$(X_T - K)^+ \mathbb{1}_{\left\{\max\limits_{0 \le t \le T} X_t \le B\right\}}.$$

where K is the strike price, B is the barrier and T is the maturity. On the other hand, the down-in option must hit the barrier for the validity. So the payoff of down-in put option is given by

$$(K - X_T)^+ \mathbb{1}_{\left\{\min\limits_{0 \le t \le T} X_t \le B\right\}}.$$

Between the values of these barrier options, there are relations.

Theorem 3.1.1. *The value of the regular European call option is the sum of the value of down-out call and a down-in call.*

Proof. The value of down-in call is given by

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+\mathbb{1}_{\left\{\min\limits_{t\le u\le T}X_u\le B\right\}}\middle|\mathcal{F}_t\right]. \tag{3.1.1}$$

and the value of down-out call is given by

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+\mathbb{1}_{\left\{\min\limits_{t\le u\le T}X_u\ge B\right\}}\middle|\mathcal{F}_t\right]. \tag{3.1.2}$$

Then the sum of (3.1.1) and (3.1.2) is expressed by

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+\left(\mathbb{1}_{\left\{\min\limits_{t\le u\le T}X_u\le B\right\}}+\mathbb{1}_{\left\{\min\limits_{t\le u\le T}X_u\ge B\right\}}\right)\middle|\mathcal{F}_t\right]\\
&=\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+\left(\mathbb{1}_{\left\{\left(\min\limits_{t\le u\le T}X_u\le B\right)\cup\left(\min\limits_{t\le u\le T}X_u\ge B\right)\right\}}\right)\middle|\mathcal{F}_t\right]\\
&=\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+\mid\mathcal{F}_t\right]
\end{aligned} \tag{3.1.3}$$

The last expression of (3.1.3) is the value of regular European call option. □

Lemma 3.1.2. *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and let* $(W_t)_{t\ge0}$ *be a standard Brownian motion. For a stopping time* T*, we define*

$$\bar{W}_t=\begin{cases}W_t & \text{if } t\le T\\ 2W_T-W_t & \text{if } t>T.\end{cases}$$

Then $(\bar{W}_t)_{t\le0}$ *is also a standard Brownian motion.*

Also, we can derive the reflection equality. Given $m>0$, define the stopping time $T_m=\inf\{t\ge0:W_t=m\}$. Since $W_{T_m}=m$, we can derive the equation

$$\begin{aligned}
\mathbb{P}(T_m\le t,W_t\le w)&=\mathbb{P}(T_m\le t,2W_{T_m}-W_t\le w)\\
&=\mathbb{P}(W_t\ge 2m-w).
\end{aligned} \tag{3.1.4}$$

Consider the maximum of the Brownian motion

$$M_t = \max_{0 \le s \le t} W_s. \tag{3.1.5}$$

Then (3.1.4) can be expressed by

$$\begin{aligned} \mathbb{P}(T_m \le t, W_t \le w) &= \mathbb{P}(M_t \ge m, W_t \le w) \\ &= \mathbb{P}(W_t \ge 2m - w). \end{aligned} \tag{3.1.6}$$

For the minimum of the Brownian motion

$$m_t = \min_{0 \le s \le t} W_s, \tag{3.1.7}$$

the reflection equality is given by

$$\begin{aligned} \mathbb{P}(T_m \le t, W_t \ge w) &= \mathbb{P}(m_t \le m, W_t < w) \\ &= \mathbb{P}(W_t \le 2m - w). \end{aligned} \tag{3.1.8}$$

Theorem 3.1.3. *Let $(W_t)_{t \le 0}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $(M_t)_{t \ge 0}$ as (3.1.5). Conditioned on $M_t = m$ and $W_t = w$, the joint density function of the pair of (M_t, W_t) is given by*

$$f_{M_t, W_t}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \ge m, m < 0.$$

Proof. We have

$$\mathbb{P}(M_t \ge m, W_t \le w) = \int_m^\infty \int_{-\infty}^w f_{M_t, W_t}(x, y)\, dy dx. \tag{3.1.9}$$

Since the Brownian motion has a normal distribution, $W_t \sim \mathcal{N}(0, t)$, we obtain

$$\mathbb{P}(W_t \ge 2m - w) = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}}\, dz. \tag{3.1.10}$$

By the equality (3.1.8), (3.1.9) and (3.1.10) are equivalent as

$$\int_m^\infty \int_{-\infty}^w f_{M_t,W_t}(x,y)\,dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}}\,dz.$$

Therefore, we have

$$f_{M_t,W_t}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \leq m, m > 0.$$

□

Theorem 3.1.4. *Let $(W_t)_{t\leq 0}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $(m_t)_{t\geq 0}$ as (3.1.7). Conditioned on $W_t = w$ and $m_t = m$, the joint density function of the pair (m_t, W_t) is given by*

$$f_{m_t,W_t}(m,w) = -\frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \leq m, m > 0.$$

Proof. We have

$$\mathbb{P}(m_t < m, W_t \geq w) = \int_{-\infty}^m \int_w^\infty f_{m_t,W_t}(x,y)\,dydx. \tag{3.1.11}$$

From the normal distribution of Brownian motion, we obtain

$$\mathbb{P}(W_t \leq 2m-w) = \int_{-\infty}^{2m-w} \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\,dz. \tag{3.1.12}$$

Since (3.1.11) and (3.1.12) are equivalent, we can get

$$f_{m_t,W_t}(m,w) = -\frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \geq m, m < 0.$$

□

We dealt with the standard Brownian motion with zero drift. Now, we consider the Brownian motion with a drift. Let $(\widetilde{W}_t)_{0\leqslant t\leqslant T}$ be the standard

Brownian motion on a probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. Define

$$\widehat{W}_t = \alpha t + \widetilde{W}_t. \tag{3.1.13}$$

Then $\widehat{W}_t$ is the Brownian motion with drift α on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$.

Theorem 3.1.5. *Consider the Brownian motion defined by* (3.1.3)*. Define the maximum of* $\widehat{W}_t$ *such that*

$$\widehat{M}_T = \max_{0 \le t \le T} \widehat{W}_t.$$

Then the joint density function of $(\widehat{M}_T, \widehat{W}_T)$ *under* $\widetilde{\mathbb{P}}$ *is given by*

$$\widetilde{f}_{\hat{M}_T, \hat{W}_T}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, m \le w, m > 0.$$

Proof. With (3.1.13), we can define the Radon-Nikodym derivative

$$\widehat{Z}_t = \exp\left(-\alpha \widetilde{W}_t - \frac{1}{2}\alpha^2 t\right) = \exp\left(-\alpha \widehat{W}_t + \frac{1}{2}\alpha^2 t\right). \tag{3.1.14}$$

By Theorem 3.1.8, a probability measure $\widehat{\mathbb{P}}$ is defined by

$$\widehat{Z}_t = \frac{d\widehat{\mathbb{P}}}{d\widetilde{\mathbb{P}}} |_{\mathcal{F}_t} \tag{3.1.15}$$

and $\widehat{W}_t$ is the standard Brownian motion under $\widehat{\mathbb{P}}$. Therefore, by Theorem 4.1.3, the joint density function of $(\widehat{M_T}, \widehat{W_T})$ under $\widehat{\mathbb{P}}$ is given by

$$\widehat{f}_{\widehat{M_T}, \widehat{W_T}}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}$$

We want to derive the joint density function of $(\hat{M}_T, \hat{W}_T)$ under $\widetilde{\mathbb{P}}$. By Lemma

3.1.6, we can derive

$$\begin{aligned}
\widetilde{\mathbb{P}}(\hat{M}_T \leqslant m, \hat{W}_T \leqslant w) &= \widetilde{\mathbb{E}}[\mathbb{1}_{\{\hat{M}_T \leqslant m, \hat{W}_T \leqslant w\}}] \\
&= \hat{\mathbb{E}}\left[\frac{1}{\hat{Z}_T}\mathbb{1}_{\{\hat{M}_T \leqslant m, \hat{W}_T \leq w\}}\right] \\
&= \int_{-\infty}^{w}\int_{\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T}\hat{f}_{\hat{M}_T, \hat{W}_T}(x,y)\,dxdy.
\end{aligned}$$

Thus, we arrive at

$$\int_{-\infty}^{m}\int_{-\infty}^{w}\widetilde{f}_{\widehat{M}_T, \widehat{W}_T}(x,y)\,dxdy = \int_{-\infty}^{w}\int_{\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T}\widehat{f}_{\widehat{M}_T, \widehat{W}_T}(x,y)\,dxdy.$$

Therefore, we get the joint probability density function of $(\widehat{M_T}, \widehat{W_T})$ under $\widetilde{\mathbb{P}}$ such that

$$\widetilde{f}_{\hat{M}_T, \hat{W}_T}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}}e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, m \geq w, m > 0.$$

□

Theorem 3.1.6. *Consider the Brownian motion defined by* (3.1.13)*. Define the minimum of* $\widehat{W_t}$ *such that*

$$\widehat{m}_T = \min_{0 \leq t \leq T}\widehat{W}_t.$$

Then the joint density function of $(\widehat{m}_T, \widehat{W_T})$ *under* $\widetilde{\mathbb{P}}$ *is given by*

$$\widetilde{f}_{\widehat{M}_T, \widehat{W}_T}(m,w) = -\frac{2(2m-w)}{T\sqrt{2\pi T}}e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, m \leq w, m < 0.$$

Proof. It is the same as the proof of Theorem 3.1.5. □

Proposition 3.1.7. *Consider the down-in call option which expires at* T *with the strike price* K *and the barrier* B*. Then for* $B < K$*, the risk-neutral* *value of down-in call option at time t is given by*

$$
\begin{aligned}
v(t,x) = {} & x\left(\frac{x}{B}\right)^{\left(-1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_+\left(T-t,\frac{B^2}{Kx}\right)\right) \\
& - Ke^{-r(T-t)}\left(\frac{x}{B}\right)^{\left(1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_-\left(T-t,\frac{B^2}{Kx}\right)\right).
\end{aligned}
$$

where we define

$$
\lambda_\pm\left(T-t,\frac{B^2}{Kx}\right) = \frac{\log\frac{B^2}{Kx} + \left(r\pm\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}
$$

Proof. Assume the strike price is larger than the barrier, $B < K$. Suppose that the stock price satisfies

$$
dX_t = rX_t dt + \sigma X_t d\widetilde{W}_t.
$$

where $\widetilde{W}_t$ is the standard Brownian motion under $\widetilde{\mathbb{P}}$. The risk-neutral value of down-in call is given by

$$
\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T-K\right)^+ \mathbb{1}_{\left\{\min\limits_{t\le u\le T} X_u<B\right\}}\middle|\mathcal{F}_t\right] \tag{3.1.16}
$$

For $u \ge t$, we have

$$
\begin{aligned}
X_u &= X_t\exp\left[\left(r-\frac{\sigma^2}{2}\right)(u-t)+\sigma(\widetilde{W}_u-\widetilde{W}_t)\right] \\
&= X_t\exp\left[\sigma\widehat{W}_{u-t}\right]
\end{aligned} \tag{3.1.17}
$$

where we define

$$
\widehat{W}_{u-t} = \alpha(u-t)+\widetilde{W}_u-\widetilde{W}_t \tag{3.1.18}
$$

with $\alpha = \frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)$. Define the minimum of (3.1.18) such that

$$
\widehat{m}_{T-t} = \min_{t\le u\le T}\widehat{W}_{u-t}. \tag{3.1.19}
$$

Therefore, by using (3.1.17) and (3.1.19), we have

$$\begin{aligned}
&(X_T - K)^+ \mathbb{1}_{\left\{\min\limits_{t\le u\le T} X_u \le B\right\}} \\
&= \left(X_t e^{\sigma \widehat{W}_{T-t}} - K\right) \mathbb{1}_{\left\{X_t e^{\sigma \widehat{W}_{T-t}} > K,\ X_t e^{\sigma \widehat{m}_{T-t}} \le B\right\}} \\
&= \left(X_t e^{\sigma \widehat{W}_{T-t}} - K\right) \mathbb{1}_{\left\{\widehat{W}_{T-t} > k,\ \widehat{m}_{T-t} < b\right\}}
\end{aligned} \tag{3.1.20}$$

where we define

$$k = \frac{1}{\sigma} \log \frac{K}{X_t} \quad \text{and} \quad b = \frac{1}{\sigma} \log \frac{B}{X_t}.$$

We have that (3.1.20) is valid only on $\{(m, w) : -\infty < m \le b, k < w < \infty\}$. Thus, since $\widehat{W}_{T-t}$ and $\widehat{m}_{T-t}$ are independent of $\mathcal{F}_t$, conditioned on $\widehat{W}_{T-t} = w$, $\widehat{m}_{T-t} = m$ and $X_t = x$, (3.1.16) is expressed by

$$\begin{aligned}
v(t, x) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \left(X_t e^{\sigma \widehat{W}_{T-t}} - K\right) \mathbb{1}_{\left\{\widehat{W}_{T-t} > k,\ \widehat{m}_{T-t} < b\right\}}\right] \\
&= \int_k^\infty \int_{-\infty}^b e^{-r(T-t)} \left(x e^{\sigma w} - K\right) \frac{-2(2m - w)}{(T - t)\sqrt{2\pi(T - t)}} e^{\alpha w - \frac{1}{2}\alpha^2 (T-t) - \frac{(2m-w)^2}{2(T-t)}}\, dm dw
\end{aligned} \tag{3.1.21}$$

Put

$$y = \frac{(2m - w)^2}{2(T - t)}, \quad dy = \frac{2(2m - w)}{(T - t)}\, dm. \tag{3.1.22}$$

If we apply (3.1.22) to (3.1.21), we obtain

$$\begin{aligned}
v(t, x) &= \int_k^\infty \int_\infty^{\frac{(2b-w)^2}{2(T-t)}} -e^{-r(T-t)} \left(x e^{\sigma w} - K\right) \frac{1}{\sqrt{2\pi(T - t)}} e^{\alpha w - \frac{1}{2}\alpha^2 (T-t) - y}\, dy dw \\
&= \frac{1}{\sqrt{2\pi(T - t)}} \int_k^\infty e^{-r(T-t)} \left(x e^{\sigma w} - K\right) e^{\alpha w - \frac{1}{2}\alpha^2 (T-t) - \frac{(2b-w)^2}{2(T-t)}}\, dw \\
&= x I_1 - K I_2
\end{aligned}$$

where we define

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi(T-t)}}\int_k^\infty e^{-r(T-t)+\sigma w+\alpha w-\frac{1}{2}\alpha^2(T-t)-\frac{(2b-w)^2}{2(T-t)}}\,dw \\
I_2 &= \frac{1}{\sqrt{2\pi(T-t)}}\int_k^\infty e^{-r(T-t)+\alpha w-\frac{1}{2}\alpha^2(T-t)-\frac{(2b-w)^2}{2(T-t)}}\,dw
\end{aligned}
\tag{3.1.23}
$$

The integral form of (3.1.23) is generalized by

$$
\frac{1}{\sqrt{2\pi(T-t)}}\int_k^\infty e^{\beta+\gamma w-\frac{1}{2(T-t)}w^2}\,dw \tag{3.1.24}
$$

Put

$$
z = \frac{w-\gamma(T-t)}{\sqrt{T-t}}. \tag{3.1.25}
$$

If we put (3.1.25) into (3.1.24), (3.1.24) is transformed to cumulative density function of standard normal distribution such that

$$
e^{\beta+\frac{\gamma^2(T-t)}{2}}\frac{1}{\sqrt{2\pi}}\int_{\frac{k-\gamma(T-t)}{\sqrt{T-t}}}^\infty e^{-\frac{1}{z^2}}\,dz \;=\; e^{\beta+\frac{\gamma^2(T-t)}{2}}N\left(\frac{-k+\gamma(T-t)}{\sqrt{T-t}}\right) \tag{3.1.26}
$$

where N is the cumulative density function of the standard normal distribution. Therefore, (3.1.24) and (3.1.25) are expressed by

$$
\begin{aligned}
I_1 &= \left(\frac{x}{B}\right)^{\left(-1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_+\left(T-t,\frac{B^2}{Kx}\right)\right) \\
I_2 &= e^{-r(T-t)}\left(\frac{x}{B}\right)^{\left(1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_-\left(T-t,\frac{B^2}{Kx}\right)\right).
\end{aligned}
$$

In all, we have the risk-neutral value of the up-in call option such that

$$
\begin{aligned}
v(t,x) = {} & x\left(\frac{x}{B}\right)^{\left(-1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_+\left(T-t,\frac{B^2}{Kx}\right)\right) \\
& - Ke^{-r(T-t)}\left(\frac{x}{B}\right)^{\left(1-\frac{2r}{\sigma^2}\right)} N\left(\lambda_-\left(T-t,\frac{B^2}{Kx}\right)\right).
\end{aligned}
$$

□

Remark 3.1.8. By Theorem 3.1.1, we can get the value of down-in call option

Proposition 3.1.9. *Consider the up-out call option which expires at T with the strike price K and the barrier B. Then for $B > K$, the risk-neutral value of up-out call option at time t is given by*

$$\begin{aligned} v(t,x) = x &\left[N\left(\lambda_+\left(T-t,\frac{x}{K}\right)\right) - N\left(\lambda_+\left(T-t,\frac{x}{B}\right)\right)\right] \\ &- Ke^{-r(T-t)}\left[N\left(\lambda_-\left(T-t,\frac{x}{K}\right)\right) - N\left(\lambda_-\left(T-t,\frac{x}{B}\right)\right)\right] \\ &- x\left(\frac{x}{B}\right)^{\left(-1-\frac{2r}{\sigma^2}\right)}\left[N\left(\lambda_+\left(T-t,\frac{B^2}{Kx}\right)\right) - N\left(\lambda_+\left(T-t,\frac{B}{x}\right)\right)\right] \\ &+ Ke^{-r(T-t)}\left(\frac{x}{B}\right)^{\left(1-\frac{2r}{\sigma^2}\right)}\left[N\left(\lambda_-\left(T-t,\frac{B^2}{Kx}\right)\right) - N\left(\lambda_-\left(T-t,\frac{B}{x}\right)\right)\right] \end{aligned}$$

where we define

$$\begin{aligned} \lambda_\pm\left(T-t,\frac{x}{K}\right) &= \frac{\log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ \lambda_\pm\left(T-t,\frac{x}{B}\right) &= \frac{\log\frac{x}{B} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ \lambda_\pm\left(T-t,\frac{B}{x}\right) &= \frac{\log\frac{B}{x} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Proof. It is the same as the proof of Proposition 3.1.7. □

3.2 PDE for Barrier option

Now, we consider the partial differential equation which the barrier options satisfy. Consider the up-out call option. Since the option becomes invalid when the stock price hits the barrier, the payoff can be described by

$$V(T) = (X_T - K)^+ \mathbb{1}_{\{\max_{0\le t\le T} X_t \le B\}}.$$

Then the risk-neutral value of up-out call option is given by

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T - K\right)^+ \mathbb{1}_{\left\{\max\limits_{t\leq u\leq T} X_u \leq B\right\}} \middle| \mathcal{F}_t\right]. \tag{3.2.1}$$

Since we assume that the stock price X_t follows the Markov process and the payoff $V(T)$ only depends on the stock prices, there is a function $v(t, X_t)$

$$V(t) = v(t, X_t). \tag{3.2.2}$$

Now, we assume that the up-out call option has not knocked out prior to time t, conditioned on $X_t = x$. Then we have

$$v(t, x) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(X_T - K\right)^+ \mathbb{1}_{\left\{\max\limits_{0\leq t\leq T} X_u \leq B\right\}} \middle| \mathcal{F}_t\right]. \tag{3.2.3}$$

Moreover, (3.2.1) can be transformed into

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T) \mid \mathcal{F}_t\right].$$

Therefore, we can derive the equality

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{-rt}V(t) \mid \mathcal{F}_s\right] &= \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[e^{-rT}V(T) \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT}V(T) \mid \mathcal{F}_s\right] \\
&= e^{-rs}V(s).
\end{aligned} \tag{3.2.4}$$

The equality (3.2.4) means that $e^{-rt}v(t, x)$ is a martingale. Thus we want to derive the partial differential equation which the barrier option value $v(t, x)$ follows.

Define the stopping time T_B which is the first time that the stock price hits the barrier B and then $X_{T_B} = B$. Since the stock price oscillates, T_B can be regarded as the knock-out time. By Optional sampling theorem, the stopped process of a martingale is also a martingale. Thus the process

$$e^{-r(t\wedge T_B)}V(t\wedge T_B) = \begin{cases} e^{-rt}V(t) & 0 \le t \le T_B \\ e^{-rt}V(T_B) & T_B < t \le T \end{cases}$$

is a martingale. Since we have (3.2.2), we also get $e^{-rt}v(t, X_t)$ is a martingale by the stopping time T_B. Therefore, we can derive the equation

$$\begin{aligned} d(e^{-rt}v(t,X_t)) &= -re^{-rt}v dt + e^{-rt}\frac{\partial v}{\partial t}dt + e^{-rt}\frac{\partial v}{\partial x}dX_t + \frac{1}{2}e^{-rt}\frac{\partial^2 v}{\partial x^2}dX_t dX_t \\ &= e^{-rt}\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv\right)dt + e^{-rt}\sigma x\frac{\partial v}{\partial x}v\, d\widetilde{W}_t. \end{aligned} \tag{3.2.5}$$

Since the martingale has no dt term, dt term must be zero for $0 \le t \le T_B$ in (3.2.5). Thus we obtain the equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv = 0. \tag{3.2.6}$$

Moreover, the pair (t, x) can have any value in $\mathcal{D}_T := [0, T) \times (0, B)$ only before the option knocks out. That is, $v(t, x)$ holds the Black-Scholes equation in $\mathcal{D}_T$. In addition, we need a condition which express the barrier restriction such that

$$v(t, B) = 0, \quad t \in [0, T).$$

In all, the value of the up-out call option follows the problem

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv = 0 & \text{in } \mathcal{D}_T \\ v(T, x) = g(x) := (x-K)^+ & x \in (0, B) \\ v(t, B) = 0 & t \in [0, T). \end{cases} \tag{3.2.7}$$

If we consider a down-out put option, clearly the value of the down-out put option satisfies the Black-Scholes equation. But, It has different domain. Since the option have an effect before the barrier is reached at any time

$t \in [0, T)$, the value $v(t, x)$ satisfies the Black-Scholes equation in $\mathcal{D}_T := [0, T) \times (B, \infty)$. In addition, the final condition is given by the payoff at the maturity

$$v(T, x) = (K - x)^+, \quad x > B.$$

and the barrier restriction is represented by

$$v(t, B) = 0, \quad t \in [0, T).$$

Chapter 4

Geometric Asian Option

4.1 Pricing Geometric Asian option

A geometric Asian option contains a geometric average instead of the stock price. A geometric average can replace the stock price or the strike price. In this paper, we only consider the case of replacing the stock price. Under the price fluctuation, the average is less affected than just stock price. So the probability that the option is out of money abruptly at the maturity can be decreased. This is why we use the geometric Asian option. Moreover, the decline of volatility derive lower value than a regular call or put option.

Now, we define a process

$$Y_t = \int_0^t \log X_u du. \tag{4.1.1}$$

where the stock price X_t satisfies

$$dX_t = \mu X_t \, dt + \sigma X_t \, d\widetilde{W}_t.$$

Then the continuous geometric average of the stock price at time t is expressed by

$$\exp\left(\frac{Y_t}{t}\right).$$

Therefore, the payoff of the geometric Asian call option is given by

$$\left(\exp\left(\frac{Y_T}{T}\right) - K\right)^+. \tag{4.1.2}$$

and its put option is given by

$$\left(K - \exp\left(\frac{Y_T}{T}\right)\right)^+. \tag{4.1.3}$$

Compared with the payoff of a regular call option, we can find that the geometric average replaces the stock price.

Now, we consider the value at the general time t. The risk-neutral value of geometric Asian option at time t is given by

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\exp\frac{Y_T}{T} - K\right)^+ \middle| \mathcal{F}_t\right]$$

Since for $u \geq t$, we have

$$X_u = X_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma\left(\widetilde{W}_u - \widetilde{W}_t\right)\right],$$

we have

$$\begin{aligned} Y_T &= \int_0^t \log X_u \, du + \int_t^T \log X_u \, du \\ &= Y_t + \int_t^T \left[\log X_t + \left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma\left(\widetilde{W}_u - \widetilde{W}_t\right)\right] du \\ &= Y_t + (T-t)\log X_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma\int_t^T \left(\widetilde{W}_u - \widetilde{W}_t\right) du. \end{aligned} \tag{4.1.4}$$

The stochastic integral in (4.1.4) has the distribution

$$\int_t^T \left(\widetilde{W}_u - \widetilde{W}_t\right) du \sim \mathcal{N}\left(0, \frac{(T-t)^3}{3}\right). \tag{4.1.5}$$

Let

$$Z = -\frac{\int_t^T \left(\widetilde{W}_u - \widetilde{W}_t\right) du}{\sqrt{\frac{(T-t)^3}{3}}}.$$

Then Z has a standard normal distribution and is independent of $\mathcal{F}_t$. By the form (4.1.4), we can write

$$\exp\left(\frac{Y_T}{T}\right) = \exp\left(\frac{Y_t}{T}\right) X_t^{\frac{T-t}{T}} \exp\left[\frac{\left(r - \frac{\sigma^2}{2}\right)(T-t)^2}{2T} - \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}} Z\right].$$

For conciseness, we denote

$$\begin{aligned} A &= \exp\left(\frac{Y_t}{T}\right) X_t^{\frac{T-t}{T}}, \\ \overline{\mu} &= \frac{\left(r - \frac{\sigma^2}{2}\right)(T-t)^2}{2T}, \\ \overline{\sigma} &= \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}. \end{aligned}$$

Therefore, conditioned on $X_t = x$ and $Y_t = y$, the risk-neutral value of the geometric Asian option at time t can be given by

$$v(t,x,y) = e^{-r(T-t)} \widetilde{\mathbb{E}}\left[(A \exp(-\overline{\sigma} Z + \overline{\mu}) - K)^+\right] \tag{4.1.6}$$

Then the payoff is nonzero where

$$Z < \frac{\log \frac{K}{A} + \overline{\mu}}{\overline{\sigma}} := d_2.$$

Thus (4.1.6) is expressed by

$$
\begin{aligned}
v(t,x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-r(T-t)} \left(A \exp\left(-\overline{\sigma} z + \overline{\mu}\right) - K\right) e^{-\frac{z^2}{2}}\, dz \\
&= e^{-r(T-t)} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} A \exp\left(\overline{\mu}\right) \exp\left(-\frac{z^2}{2} - \overline{\sigma} z\right) dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} K e^{-\frac{z^2}{2}}\, dz \right] \\
&= e^{-r(T-t)} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} A \exp\left(\overline{\mu} + \frac{\overline{\sigma}^2}{2}\right) \exp\left(-\frac{1}{2}(z+\overline{\sigma})^2\right) dz - K N(d_2) \right] \\
&= e^{-r(T-t)} \left[A \exp\left(\overline{\mu} + \frac{\overline{\sigma}^2}{2}\right) N(d_1) - K N(d_2) \right]
\end{aligned}
$$

where $d_1 = d_2 + \overline{\sigma}$. In all, the value of the geometric Asian option at general time t is given by

$$
v(t,x,y) = e^{-r(T-t)} \left[X_t^{\frac{T-t}{T}} \exp\left(\frac{Y_t}{T}\right) \exp\left(\frac{\left(r - \frac{\sigma^2}{2}\right)}{2T} (T-t)^2 + \frac{\sigma^2 (T-t)^3}{6T^2} \right) N(d_1) - K N(d_2) \right] \tag{4.1.7}
$$

where

$$
d_1 = \frac{T \log \frac{X_t^{\frac{T-t}{T}}}{K} + Y_t + \frac{\left(r - \frac{\sigma^2}{2}\right)(T-t)^2}{2} + \frac{\sigma^2 (T-t)^3}{3T}}{\sigma \sqrt{\frac{(T-t)^3}{3}}}, \tag{4.1.8}
$$

$$
d_2 = d_1 - \frac{\sigma}{T} \sqrt{\frac{(T-t)^3}{3}}. \tag{4.1.9}
$$

4.2 PDE for Geometric Asian option

Now, we derive the partial differential equation which the geometric Asian options satisfy. From (4.1.1) and (4.1.2), we can know that the payoff $V(T)$ depends on X_t and Y_t. Since Y_t is an integral of the stock price from 0 to t, it depends on all past values between 0 and t. Therefore, unlike the stock price X_t, Y_t itself does not follow the Markov process. However, Y_t is related with X_t such that (X_t, Y_t) is the Markov pair. Therefore, there is a function

$v(t, X_t, Y_t)$ such that

$$V(t) = v(t, X_t, Y_t). \tag{4.2.1}$$

We know that (4.2.1) is a martingale by the same method of (3.2.4). Conditioned on $X_t = x$ and $Y_t = y$, we want to derive the general partial differential equation of which $v(t, x, y)$ is a solution. By taking the differential and two-dimensional Itô formula, we obtain the equation

$$d(e^{-rt}v(t, X_t, Y_t)) = e^{-rt}\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} + \log x \frac{\partial v}{\partial y} - rv\right) dt + e^{-rt}\sigma x \frac{\partial v}{\partial x} d\tilde{W}_t.$$

By the property of martingale, dt term must be zero and then we have

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} + \log x \frac{\partial v}{\partial y} - rv = 0. \tag{4.2.2}$$

In all, the geometric Asian call option generally satisfies the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + rx\dfrac{\partial v}{\partial x} + \log x \dfrac{\partial v}{\partial y} - rv = 0 & \text{in } \mathcal{O}_T \\ u(T, x, y) = h(y) := \left(\exp\left(\dfrac{y}{T}\right) - K\right)^+ & \text{in } \mathcal{O} \end{cases} \tag{4.2.3}$$

where $\mathcal{O} := \mathbb{R}^+ \times \mathbb{R}$ and $\mathcal{O}_T := [0, T) \times \mathcal{O}$.

Chapter 5

Viscosity Solution as Value of Geometric Asian Barrier Option

5.1 Geometric Asian Barrier Option

We considered the barrier option and the geometric Asian option. Now, we combine these two options, called Geometric Asian Barrier option. It is the geometric Asian option with an additional barrier condition.
We have two process, X_t such that

$$dX_t = rX_t\,dt + \sigma X_t\,d\widetilde{W}_t,$$

and Y_t such that

$$dY_t = \log X_t\,dt.$$

For the geometric Asian barrier option, the path of the stock price is also contained in the payoff. That is, whether the stock price reaches the barrier is important.

Suppose that the geometric Asian barrier options have the barrier B, the strike price K and the maturity T. Then the payoff (see [10]) is given by

$$\text{up-out call} : \left(\exp\left(\frac{Y_T}{T}\right) - K\right)^+ \mathbb{1}_{\left\{\max\limits_{0\le t\le T} X_t < B\right\}} \tag{5.1.1}$$

$$\text{down-in call} : \left(\exp\left(\frac{Y_T}{T}\right) - K\right)^+ \mathbb{1}_{\left\{\min\limits_{0\le t\le T} X_t \le B\right\}} \tag{5.1.2}$$

$$\text{up-in put} : \left(K - \exp\left(\frac{Y_T}{T}\right)\right)^+ \mathbb{1}_{\left\{\max\limits_{0\le t\le T} X_t \ge B\right\}} \tag{5.1.3}$$

$$\text{down-out put} : \left(K - \exp\left(\frac{Y_T}{T}\right)\right)^+ \mathbb{1}_{\left\{\min\limits_{0\le t\le T} X_t > B\right\}}. \tag{5.1.4}$$

Now, we study the geometric Asian up-out call option. This option is only valid before the stock price hits the barrier B. Therefore, the maximum value of the stock price must be restricted below B and so the characteristic functions on (5.1.1) express this barrier restriction. Finally, the payoff of the geometric Asian up-out call option is given by (5.1.1). On the other hand, the down-in option takes an effect after the stock price reaches the barrier B. So the stock price must be lower than the barrier. That is, the minimum of the stock price is lower that the barrier. Therefore, we have (5.1.2). Other things can be obtained similarly.

In the previous section, we derived (4.2.3) which the value of the geometric Asian call option depending on t, X_t and Y_t generally satisfies. The risk-neutral value of geometric Asian up-out call option is given by

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\exp\left(\frac{Y_T}{T}\right) - K\right)^+ \mathbb{1}_{\left\{\max\limits_{t\le u\le T} X_u \le B\right\}} \middle| \mathcal{F}_t\right]. \tag{5.1.5}$$

Since the geometric Asian barrier option also have same property with the geometric Asian options, (5.1.5) basically satisfies the equation (4.2.2). However, we additionally need to consider the existence of the barrier B. For the up-out call option, the interval of the stock price is restricted by

$x \in (0, B)$ and so we have additional boundary condition

$$v(t, B, y) = 0, \quad t \in [0, T), \quad y \in \mathbb{R}.$$

In all, we can consider the modeling differential problem for the value of Geometric Asian up-out call option

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + rx\dfrac{\partial v}{\partial x} + \log x \dfrac{\partial v}{\partial y} - rv = 0 & \text{in } \mathcal{S}_T \\ v(T, x, y) = g(y) := \left(\exp\left(\dfrac{y}{T}\right) - K\right)^+ & \text{in } \mathcal{S} \\ v(t, B, y) = 0 & \text{in } [0, T) \times \mathbb{R}. \end{cases} \tag{5.1.6}$$

where $\mathcal{S} := (0, B) \times \mathbb{R}$ and $\mathcal{S}_T := [0, T) \times \mathcal{S}$.

5.2 Viscosity solution

Now, we study the value of geometric Asian up-out call option

$$v(t, x, y) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\exp\left(\frac{Y_T}{T}\right) - K\right)^+ \mathbb{1}_{\left\{\max\limits_{t \le u \le T} X_u \le B\right\}} \middle| \mathcal{F}_t\right] \tag{5.2.1}$$

with the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + rx\dfrac{\partial v}{\partial x} + \log x \dfrac{\partial v}{\partial y} - rv = 0 & \text{in } \mathcal{S}_T \\ v(T, x, y) = g(y) := \left(\exp\left(\dfrac{y}{T}\right) - K\right)^+ & \text{in } \mathcal{S} \\ v(t, B, y) = 0 & \text{in } [0, T) \times \mathbb{R}. \end{cases} \tag{5.2.2}$$

It is not known that the problem (5.2.2) has classical closed form solution. Therefore, we are going to define the solution in the viscosity sense. If we mention the function $v(t, x, y)$, it means (5.2.1).

Definition 5.2.1. Let $v \in C(\overline{\mathcal{S}_T})$ be a locally bounded function.

(i) v is a viscosity subsolution of (5.2.2) in $\mathcal{S}_T$ if v satisfies

$$\begin{aligned} v(T,x,y) &\leq g(x,y) \qquad \text{in } \mathcal{S}, \\ v(t,B,y) &\leq 0 \qquad \text{in } [0,T)\times\mathbb{R}, \end{aligned}$$

and for any $\varphi \in C^2(S_T)$ such that

$$\begin{aligned} v(t_0,x_0,y_0) &= \varphi(t_0,x_0,y_0) \\ v(t,x,y) &\leq \varphi(t,x,y) \quad \text{for } (t,x,y)\in\mathcal{S}_T, \end{aligned}$$

we have

$$\varphi_t + L\varphi \geq 0 \quad \text{at } (t_0,x_0,y_0)\in\mathcal{S}_T.$$

(ii) v is a viscosity supersolution of (5.2.2) in $\mathcal{S}_T$ if v satisfies

$$\begin{aligned} v(T,x,y) &\geq g(x,y) \qquad \text{in } \mathcal{S}, \\ v(t,B,y) &\geq 0 \qquad \text{in } [0,T)\times\mathbb{R}, \end{aligned}$$

and for any $\varphi \in C^2(S_T)$ such that

$$\begin{aligned} v(t_0,x_0,y_0) &= \varphi(t_0,x_0,y_0) \\ v(t,x,y) &\geq \varphi(t,x,y) \quad \text{for } (t,x,y)\in\mathcal{S}_T, \end{aligned}$$

we have

$$\varphi_t + L\varphi \leq 0 \quad \text{at } (t_0,x_0,y_0)\in\mathcal{S}_T.$$

(iii) v is a viscosity solution of (5.2.2) if it is both a viscosity subsolution and supersolution of (5.2.2).

Lemma 5.2.2. *The function v(t, x, y) is continuous on* $\overline{\mathcal{S}}_T$.

Proof. By Markov property, we can represent v as

$$\begin{aligned} v(t,x,y) &= e^{-r(T-t)}\widetilde{E}\left[g(Y_T)\mathbb{1}_{\left\{\max\limits_{t\le u\le T} X_u \le B\right\}} \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)}\tilde{E}\left[g(Y_{T-t})\mathbb{1}_{\left\{\max\limits_{0\le u\le T-t} X_s \le B\right\}} \right]. \end{aligned} \tag{5.2.3}$$

So (5.2.3) starts at time $t=0$. Conditioned on $X_0 = x$ and $Y_0 = y$, we can write

$$X_t^{0,x} = x\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\widetilde{W}_t\right]$$

and

$$Y_t^{0,x,y} = y + \int_0^t \log X_u^{0,x}\,du$$

For other $(t',x',y') \in \mathcal{S}_T$, we have

$$\frac{X_t^{0,x}}{X_{t'}^{0,x'}} = \frac{x}{x'}\exp\left[\left(r - \frac{1}{2}\sigma^2\right)(t'-t) + \sigma\left(\widetilde{W}_t - \widetilde{W}_{t'}\right)\right]$$

and

$$\begin{aligned} \left|Y_t^{0,x,y} - Y_{t'}^{0,x',y'}\right| &\le |y-y'| + \int_0^t \left|\log\frac{X_u^{0,x}}{X_u^{0,x'}}\right| du + \left|\int_t^{t'} \log X_u^{0,x'}\,du\right| \\ &= |y-y'| + t\left|\log x - \log x'\right| + \left|\int_t^{t'} \log X_u^{0,x'} du\right|. \end{aligned}$$

Since $\widetilde{W}_t$ is continuous a.s., we observe

$$\left(X_{t'}^{0,x'}, Y_{t'}^{0,x',y'}\right) \to \left(X_t^{0,x}, Y_t^{0,x,y}\right) \quad \text{a.s.}$$

as $(t', x', y') \to (t, x, y)$ and

$$\max_{0 \le u \le T-t'} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)u + \sigma \widetilde{W}_u\right] \to \max_{0 \le u \le T-t} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)u + \sigma \widetilde{W}_u\right] \quad \text{a.s.}$$

as $t \to t'$. Moreover, the payoff in (5.2.3) is given by

$$g(Y_{T-t'}^{0,x',y'})\mathbb{1}_{\left\{\max\limits_{0 \le u \le T-t'} X_u^{0,x'} \le B\right\}} \to g(Y_{T-t}^{0,x,y})\mathbb{1}_{\left\{\max\limits_{0 \le u \le T-t} X_u^{0,x} \le B\right\}} \quad \text{a.s.} \tag{5.2.4}$$

as $(t', x', y') \to (t, x, y)$. We know that (5.2.4) is bounded such like

$$0 \le g(Y_{T-t}^{0,x,y})\mathbb{1}_{\left\{\max\limits_{0 \le u \le T-t} X_u^{0,x} \le B\right\}} \le e^{\frac{B}{T}},$$

Therefore, by the Lebesgue's dominated convergence theorem, we have

$$v(t', x', y') \to v(t, x, y)$$

as $(t', x', y') \to (t, x, y)$. □

Theorem 5.2.3. *The function $v(t, x, y)$ is a viscosity subsolution of* (5.2.2).

Proof. Take any point $(t_0, x_0, y_0) \in \mathcal{S}_T$ and any function $\varphi \in C^2(\mathcal{S}_T)$ such that

$$v(t_0, x_0, y_0) = \varphi(t_0, x_0, y_0)$$

and

$$v \le \varphi \quad \text{in} \quad \mathcal{S}_T.$$

We assume that X_t and Y_t start at time t_0 with $X_{t_0} = x_0$ and $Y_{t_0} = y_0$. That is, we now denote $X_t := X_t^{t_0,x_0}$ and $Y_t := Y_t^{t_0,x_0,y_0}$. Define the stopping time

τ and τ_h such that

$$\tau = \inf\left\{t_0 < t : X_t \geq B \text{ or } |X_t - x_0| + |Y_t - y_0| \geq \frac{x_0}{2}\right\} \tag{5.2.5}$$

and

$$\tau_h = \tau \wedge (t_0 + h) \tag{5.2.6}$$

for small $h > 0$. By the strong Markov property, we have

$$v(\tau_h, X_{\tau_h}, Y_{\tau_h}) = e^{-r(T-\tau_h)}\widetilde{\mathbb{E}}\left[g(Y_T)\mathbb{1}_{\left\{\max\limits_{\tau_h \leq u \leq T} X_u \leq B\right\}}\middle|\mathcal{F}_{\tau_h}\right]. \tag{5.2.7}$$

Therefore, we can derive the equality

$$\begin{aligned}
&v(t_0, x_0, y_0)\\
&= e^{-r(T-t_0)}\widetilde{\mathbb{E}}\left[g(Y_T)\mathbb{1}_{\left\{\max\limits_{t_0 \leq u \leq T} X_u \leq B\right\}}\middle|\mathcal{F}_{t_0}\right]\\
&= e^{-r(T-t_0)}\widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[g(Y_T)\mathbb{1}_{\left\{\max\limits_{\tau_h \leq u \leq T} X_u \leq B\right\}}\mathbb{1}_{\left\{\max\limits_{t_0 \leq u \leq \tau_h} X_u \leq B\right\}}\middle|\mathcal{F}_{\tau_h}\right]\middle|\mathcal{F}_{t_0}\right]\\
&= e^{r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[e^{-r(T-\tau_h)}\widetilde{\mathbb{E}}\left[g(Y_T)\mathbb{1}_{\left\{\max\limits_{\tau_h \leq u \leq T} X_u \leq B\right\}}\middle|\mathcal{F}_{\tau_h}\right]\mathbb{1}_{\left\{\max\limits_{t_0 \leq u \leq \tau_h} X_u \leq B\right\}}\middle|\mathcal{F}_{t_0}\right]\\
&= e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[v(\tau_h, X_{\tau_h}, Y_{\tau_h})\mathbb{1}_{\left\{\max\limits_{t_0 \leq u \leq \tau_h} X_u \leq B\right\}}\middle|\mathcal{F}_{t_0}\right].
\end{aligned} \tag{5.2.8}$$

Moreover, by definition of τ_h, we know

$$\mathbb{1}_{\left\{\max\limits_{t_0 \leq u \leq \tau_h} X_u \leq B\right\}} = 1.$$

Thus, we obtain

$$\begin{aligned}\varphi(t_0,x_0,y_0) &= v(t_0,x_0,y_0)\\ &= e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[v(\tau_h,X_{\tau_h},Y_{\tau_h})\middle|\mathcal{F}_{t_0}\right]\\ &\le e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[\varphi(\tau_h,X_{\tau_h},Y_{\tau_h})\middle|\mathcal{F}_{t_0}\right].\end{aligned} \tag{5.2.9}$$

Using Ito formula, we have

$$\begin{aligned}&\varphi(\tau_h,X_{\tau_h},Y_{\tau_h})-\varphi(t_0,x_0,y_0)\\ &=\int_{t_0}^{\tau_h}\left(\varphi_t+\frac{1}{2}\sigma^2(X_u)^2\varphi_{xx}+rX_u\varphi_x+\log X_u\varphi_y\right)(u,X_u,Y_u)\,du\\ &\quad+\int_{t_0}^{\tau_h}\sigma X_u\varphi(u,X_u,Y_u)\,d\widetilde{W}_u.\end{aligned} \tag{5.2.10}$$

Putting (5.2.10) into (5.2.9), we obtain

$$\begin{aligned}\varphi(t_0,x_0,y_0) &\le e^{-r(\tau_h-t_0)}\varphi(t_0,x_0,y_0)\\ &\quad+e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[\int_{t_0}^{\tau_h}\left(\varphi_t+\frac{1}{2}\sigma^2(X_u)^2\varphi_{xx}+rX_u\varphi_x+\log X_u\varphi_y\right)(u,X_u,Y_u)\,du\middle|\mathcal{F}_{t_0}\right]\\ &\quad+e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[\int_{t_0}^{\tau_h}\sigma X_u\varphi(u,X_u,Y_u)\,d\widetilde{W}_u\middle|\mathcal{F}_{t_0}\right]\\ &= e^{-r(\tau_h-t_0)}\varphi(t_0,x_0,y_0)\\ &\quad+e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[\int_{t_0}^{\tau_h}\left(\varphi_t+\frac{1}{2}\sigma^2(X_u)^2\varphi_{xx}+rX_u\varphi_x+\log X_u\varphi_y\right)(u,X_u,Y_u)\,du\middle|\mathcal{F}_{t_0}\right]\end{aligned} \tag{5.2.11}$$

For each event $\omega\in\Omega$ and sufficiently small $h>0$, we can write

$$\tau_h(\omega)=t_0+h.$$

By the mean value theorem, we derive

$$\begin{aligned}&\frac{1}{h}\int_{t_0}^{\tau_h}\left(\varphi_t+\frac{1}{2}\sigma^2(X_u)^2\varphi_{xx}+rX_u\varphi_x+\log X_u\varphi_y\right)(u,X_u,Y_u)\,du\\&\quad\to\left(\varphi_t+\frac{1}{2}\sigma^2x_0^2\varphi_{xx}+rx_0\varphi_x+\log x_0\varphi_y\right)(t_0,x_0,y_0)\quad\text{a.s.}\end{aligned}$$

as $h\to 0$. We know that

$$\left(\varphi_t+\frac{1}{2}\sigma^2(X_t)^2\varphi_{xx}+rX_t\varphi_x+\log X_t\varphi_y\right)(t,X_t,Y_t)$$

is uniformly bounded. By the Lebesgue's dominated convergence theorem and (5.2.11), we have

$$\begin{aligned}0\le{}&\frac{1}{h}\left(e^{-r(\tau_h-t_0)}-1\right)\varphi(t_0,x_0,y_0)\\&+e^{-r(\tau_h-t_0)}\widetilde{\mathbb{E}}\left[\frac{1}{h}\int_{t_0}^{\tau_h}\left(\varphi_t+\frac{1}{2}\sigma^2(X_u)^2\varphi_{xx}+rX_u\varphi_x+\log X_u\varphi_y\right)(u,X_u,Y_u)\,du\middle|\mathcal{F}_{t_0}\right]\\\to{}&\left(\varphi_t+\frac{1}{2}\sigma^2x_0^2\varphi_{xx}+rx_0\varphi_x+\log x_0\varphi_y-r\varphi\right)(t_0,x_0,y_0)\\&=(\varphi_t+L\varphi)(t_0,x_0,y_0)\end{aligned}\tag{5.2.12}$$

as $h\to 0$. □

Theorem 5.2.4. *The function $v(t,x,y)$ is a viscosity supersolution of* (5.2.2)*.*

Proof. It is the same as the proof of Theorem 5.2.3 except the direction of inequality. □

From Theorem 5.2.3 and Theorem 5.2.4, we know that $v(t,x,y)$ are viscosity solution of (5.2.2).

Theorem 5.2.5. *Let u and v be a viscosity subsolution and a supersolution of* (5.2.2)*, respectively. Assume that there exists $A>0$ such that*

$$u(t,x,y)\le A\exp\left(A(\log x)^2+Ay^2\right)\quad\textit{and}\quad v(t,x,y)\ge-\exp A\left(A(\log x)^2+Ay^2\right)$$

in $\mathcal{S}_T$. Then we obtain

$$u \le v \quad in \quad \mathcal{S}_T.$$

For the proof, we consider a barrier function ϕ as

$$\phi(t,x,y) = \exp\left(\frac{1}{6T_0} + \frac{(\sigma^2+2r)^2}{8\sigma^2}\right)(T-t) \\ \exp\left(\frac{(\log x)^2}{6\sigma^2(t-T+2T_0)} + \frac{y^2}{6\sigma^2(t-T+2T_0)^3}\right) \quad (5.2.13)$$

for $(t,x,y) \in [T-T_0,T] \times \mathcal{S}$, where $0 < T_0 < \min\left\{\frac{1}{24A\sigma^2}, \frac{1}{2}\right\}$. We know that the barrier function ϕ goes to ∞ as $x \to 0$ or $y \to \pm\infty$. Then, we have

$$\phi_t + L\phi < 0$$

and

$$\frac{1}{6\sigma^2(t-T+2T_0)} > 2A \quad \text{and} \quad \frac{1}{6\sigma^2(t-T+2T_0)^3} > 2A \qquad (5.2.14)$$

for $T - T_0 \le t \le T$.

Proof of Theorem 5.2.5. For each $\varepsilon > 0$ and the function (5.2.13), we consider

$$u^\varepsilon = u - \varepsilon\phi \quad \text{and} \quad v^\varepsilon = v + \varepsilon\phi, \qquad (5.2.15)$$

Then u^ε and v^ε are also a viscosity subsolution and a supersolution of (5.2.2), respectively. Moreover, there exists $N > 0$ such that

$$A\exp\left(A(\log x)^2 + Ay^2\right) \le \varepsilon\phi(t,x,y) \qquad (5.2.16)$$

if $T - T_0 \le t \le T$ and $x \le \frac{1}{N}$ or $|y| > N$.
Let

$$\mathcal{R}_T = [T-T_0,T) \times (\frac{1}{N}, B) \times (-N,N).$$

Then clearly u^ε and v^ε are a viscosity subsolution and a supersolution of (5.2.2), respectively in $\mathcal{R}_T$ with

$$u^\varepsilon \le v^\varepsilon \quad \text{on} \quad \partial_p \mathcal{R}_T. \tag{5.2.17}$$

By the comparison principle, we obtain

$$u^\varepsilon \le v^\varepsilon \quad \text{in} \quad \mathcal{R}_T. \tag{5.2.18}$$

In addition, we have

$$u^\varepsilon \le v^\varepsilon \quad \text{in} \quad ([T - T_0, T] \times \mathcal{S}) \setminus \mathcal{R}_T. \tag{5.2.19}$$

Therefore, from (5.2.18) and (5.2.19), we have

$$u^\varepsilon \le v^\varepsilon \quad \text{in} \quad [T - T_0, T] \times \mathcal{S}. \tag{5.2.20}$$

By sending ε to 0, we obtain

$$u \le v \quad \text{in} \quad [T - T_0, T] \times \mathcal{S}. \tag{5.2.21}$$

By iterating this procedure to $[T - (k+1)T_0, T - kT_0]$ for $k = 1, 2, \cdots$, we conclude that

$$u \le v \quad \text{in} \quad \mathcal{S}_T. \tag{5.2.22}$$

□

Proposition 5.2.6. *The problem* (5.2.2) *has a unique viscosity solution.*

Proof. Suppose that u and v are viscosity solution of (5.2.2). Then, by Theorem 5.2.5, we have

$$u \le v \quad \text{and} \quad u \ge v \quad \text{in} \quad \mathcal{S}_T. \tag{5.2.23}$$

□

Therefore, the value of geometric Asian up-out call option

$$v(t,x,y) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \left(\exp\left(\frac{Y_T}{T}\right) - K \right)^+ \mathbb{1}_{\left\{ \max\limits_{t \le u \le T} X_u \le B \right\}} \middle| \mathcal{F}_t \right] \tag{5.2.24}$$

is unique solution of (5.2.2).

Appendix A

Itô integral

A.1 Properties of Itô integral

Theorem A.1.1. *Let* $(W_t)_{t\geq 0}$ *be a standard Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\mathcal{F}_t)_{t\geq 0}$ *be the associated filtration. Then Itô integral is defined by*

$$I_t = \int_0^t f(u, W_u) dW_u = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i}).$$

Then I_t *is a martingale.*

Proof. For $s < t$, we write

$$\begin{aligned} I_t &= \int_0^s f(u, W_u)\, dW_u + \int_s^t f(u, W_u)\, dW_u \\ &= I_s + \lim_{n\to\infty} \sum_{i=m}^{n-1} f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i}) \end{aligned}$$

where $m < n-1$. Since W_t is a martingale, we have

$$\begin{aligned}
\mathbb{E}[I_t \mid \mathcal{F}_s] &= I_s + \mathbb{E}\left[\lim_{n\to\infty}\sum_{i=m}^{n-1} f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_s\right] \\
&= I_s + \lim_{n\to\infty}\sum_{i=m}^{n-1} \mathbb{E}\left[\mathbb{E}\left[f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \mid \mathcal{F}_s\right] \\
&= I_s + \lim_{n\to\infty}\sum_{i=m}^{n-1} \mathbb{E}\left[f(t_i, W_{t_i})(W_{t_i} - W_{t_i})\right] \\
&= I_s.
\end{aligned}$$

Since W_t is continuous and then $\lim_{n\to\infty}\max_{0\le k\le n-1}|W_{t_{k+1}} - W_{t_k}| = 0$, we get $\mathbb{E}[|I_t|] < \infty$. So Itô integral I_t is a martingale. □

Theorem A.1.2. ***(Itô isometry)***
Let $(W_t)_{t\ge 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\ge 0}$ be the associated filtration. Then the Itô integral satisfies

$$\mathbb{E}\left[\left(\int_0^t f(s, W_s)dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t f(s, W_s)^2 ds\right].$$

Proof. First, we want to show $W_t^2 - t$ is also a martingale.
Clearly, $W_t^2 - t$ is $\mathcal{F}_t$ - adapted. Since $W_t - W_s$ is independent to $\mathcal{F}_s$ for $s < t$, we get

$$\begin{aligned}
\mathbb{E}[W_t^2 - t \mid \mathcal{F}_s] &= \mathbb{E}[(W_t - W_s + W_s)^2 \mid \mathcal{F}_s] - t \\
&= \mathbb{E}[(W_t - W_s)^2 \mid \mathcal{F}_f] + 2\mathbb{E}[W_s(W_t - W_s) \mid \mathcal{F}_s] + \mathbb{E}[W_s^2 \mid \mathcal{F}_s] - t \\
&= W_s^2 - s.
\end{aligned}$$

Moreover, $\mathbb{E}[|W_t^2 - t|] \le \mathbb{E}[W_t^2 + t] = 2t < \infty$. It means that $W_t^2 - t$ is a martingale.

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t f(s,W_s)dW_s\right)^2\right] - \mathbb{E}\left[\int_0^t f(s,W_s)^2 ds\right] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}[f(t_i,W_{t_i})^2(W_{t_{i+1}}-W_{t_i})^2] - \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}[f(t_i,W_{t_i})^2(t_{i+1}-t_i)] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[f(t_i,W_{t_i})^2\left((W_{t_{i+1}}-W_{t_i})^2-(t_{i+1}-t_i)\right) \mid \mathcal{F}_{t_i}\right]\right] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[f(t_i,W_{t_i})^2(W_{t_{i+1}}^2-t_{i+1}) \mid \mathcal{F}_{t_i}\right]\right] \\
&\quad - 2\lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[f(t_i,W_{t_i})^2 W_{t_{i+1}}W_{t_i} \mid \mathcal{F}_{t_i}\right]\right] \\
&\quad + \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[f(t_i,W_{t_i})^2(W_{t_i}^2+t_i) \mid \mathcal{F}_{t_i}\right]\right] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\mathbb{E}\left[f(t_i,W_{t_i})^2(W_{t_i}^2-t_i-2W_{t_i}^2+W_{t_i}^2+t_i)\right] \\
&= 0.
\end{aligned}
$$

Therefore, we get the result

$$
\mathbb{E}\left[\left(\int_0^t f(s,W_s)\,dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t f(s,W_s)^2\,ds\right].
$$

□

Theorem A.1.3. ***(Quadratic variation of Itô integral)***
The Itô integral, $I_t = \int_0^t f(s,W_s)dW_s$, has quadratic variation process such that

$$
[I,I]_t = \int_0^t f(s,W_s)^2 ds
$$

Proof. By definition of the quadratic variation,

$$[I,I]_t = \lim_{n\to\infty} \sum_{k=0}^{m-1} (I_{t_{k+1}} - I_{t_k})^2.$$

where $t_k = \frac{kt}{m}$, $0 = t_0 < t_1 < \cdots < t_m = t$. Since f is a simple process , $f(t_k, W_{t_k})$ is a constant value on $[t_k, t_{k+1})$. We partition the subinterval $[t_k, t_{k+1})$ such that

$$t_k = s_0 < s_1 < \cdots < s_n = t_{k+1}$$

Then we can write

$$I_{s_{i+1}} - I_{s_i} = \int_{s_i}^{s_{i+1}} f(t_k, W_{t_k}) dW_u = f(t_k, W_{t_k})(W_{s_{i+1}} - W_{s_i}).$$

Hence we have

$$\begin{aligned}
(I_{t_{k+1}} - I_{t_k})^2 &= \lim_{n\to\infty} \sum_{i=0}^{n-1} (I_{s_{i+1}} - I_{s_i})^2 \\
&= f(t_k, W_{t_k})^2 \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{s_{i+1}} - W_{s_i})^2 \\
&= f(t_k, W_{t_k})^2 (t_{k+1} - t_k)
\end{aligned}$$

Finally, the quadratic variation of Itô integral can be described by

$$\begin{aligned}
[I,I]_t &= \lim_{m\to\infty} \sum_{k=0}^{m-1} \left(I_{t_{k+1}} - I_{t_k}\right)^2 \\
&= \lim_{m\to\infty} \sum_{k=0}^{m-1} f(t_k, W_{t_k})^2 (t_{k+1} - t_k) \\
&= \int_0^t f(s, W_s)^2 \, ds
\end{aligned}$$

□

A.2 Proof of (4.1.5)

First, we want to prove that the expectation value of Itô integral is zero.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Then the Itô integral is defined by

$$I_t = \int_0^t f(s, W_s)dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i})$$

Since W_t is a martingale, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f(s, W_s)dW_s\right] &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}[f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i})] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{E}[f(t_i, W_{t_i})(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}]\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}[f(t_i, W_{t_i})(W_{t_i} - W_{t_i})] \\
&= 0.
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
\int_0^t W_s\, ds &= sW_s|_0^t - \int_0^t s\frac{dW_s}{ds}\, ds \\
&= tW_t - \int_0^t s\, dW_s \\
&= \int_0^t (t-s)\, dW_s
\end{aligned}$$

This means that $I_t = \int_0^t (t-s)\, dW_s$ is an Itô integral and so a martingale. Moreover, according to above fact, we have

$$\mathbb{E}[I_t] = 0.$$

Thus the variance of I_t is

$$Var[I_t] = \mathbb{E}\left[I_t^2\right] = \mathbb{E}\left[\left(\int_0^t (t-s)\,dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t (t-s)^2\,ds\right] = \frac{t^3}{3}.$$

We want to show that I_t follows a normal distribution by using the moment generating function. Consider the quadratic variation of I_t. By above theorem, we obtain

$$[I,I]_t = \int_0^t (t-s)^2 ds = \frac{t^3}{3}$$

In differential form, we can write $dI_t dI_t = t^2 dt$. Define the function $f(t, I_t)$ such that

$$f(t, I_t) = e^{\theta I_t - \frac{1}{2}\theta^2\left(\frac{t^3}{3}\right)}$$

By Itô formula, we develop

$$\begin{aligned} df(t, I_t) &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial I_t}\,dI_t + \frac{1}{2}\frac{\partial^2 f}{\partial I_t^2}\,dI_t dI_t \\ &= \theta f(t, I_t)\,dI_t \end{aligned}$$

By taking integration, we have

$$f(t, I_t) = f(0, I_0) + \theta \int_0^t f(t, I_t)\,dI_t$$

By taking expectation, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{\theta I_t - \frac{1}{2}\theta^2\left(\frac{t^3}{3}\right)}\right] &= 1 + \mathbb{E}\left[\theta \int_0^t f(t, I_t)\,dI_t\right] \\ &= 1 \end{aligned}$$

Finally, we get $\mathbb{E}[e^{\theta I_t}] = e^{\frac{1}{2}\theta^2\frac{t^3}{3}}$ which is the moment generating function of a normal distribution with mean zero and variance $\frac{t^3}{3}$. That is, I_t follows a normal distribution with mean zero and variance $\frac{t^3}{3}$

$$I_t \sim \mathcal{N}\left(0, \frac{t^3}{3}\right).$$

Appendix A

Martingale with zero drift term

Theorem A.0.1. ***(Martingale Representation, [7])***
Let $(W_t)_{o\leq t\leq T}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the filtration generated by this Brownian motion. If $(M_t)_{0\leq t\leq T}$ is a martingale with respect to this filtration, there exists an adapted process $(\phi_t)_{0\leq t\leq T}$ such that

$$M_t = M_0 + \int_0^t \phi_u \, dW_u, \quad 0 \leq t \leq T.$$

Theorem A.0.2. *Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Define the process M_t such that*

$$dM_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

with $\mathbb{E}\left[\left(\int_0^T \sigma_s \, ds\right)^{\frac{1}{2}}\right] < \infty$. Then M_t is a martingale if and only if M_t has no drift term.

Proof. $(\Rightarrow)$ By the Martingale Representation theorem, we have an $\mathcal{F}$-measurable process ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_u \, dW_u$$
$$\Rightarrow \quad dM_t = \phi_t \, dW_t.$$

Therefore, a martingale process M_t has no drift term.
($\Leftarrow$) Consider Itô process

$$dM_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

By taking integration, we obtain

$$M_t - M_s = \int_s^t \mu(u, M_u)\, du + \int_s^t \sigma(u, M_u)\, dW_u$$

If we take conditional expectation with respect to the filtration $\mathcal{F}_s$ for $s < t$, we get the formula

$$\begin{aligned} \mathbb{E}[M_t \mid \mathcal{F}_s] &= \mathbb{E}[M_s] + \mathbb{E}\left[\int_s^t \mu(u, M_u)\, du \mid \mathcal{F}_s\right] \\ &= M_s + \mathbb{E}\left[\int_s^t \mu(u, M_u)\, du \mid \mathcal{F}_s\right] \end{aligned}$$

Thus, if $\mu(t, M_t)$ is zero, M_t is a $\mathbb{P}$ - martingale.

$\square$

Corollary A.0.3. *Let* $(W_t)_{0\le t\le T}$ *be a standard Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and let* $(\mathcal{F}_t)_{0\le t\le T}$ *be the filtration generated by this Brownian motion. For an adapted process* $(\theta_t)_{0\le t\le T}$*, define*

$$\begin{aligned} Z_t &= \exp\left[-\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du\right] \\ \widetilde{W}_t &= W_t + \int_0^t \theta_u\, du \end{aligned}$$

where $\widetilde{W}_t$ *is a standard Brownian motion and* $\widetilde{\mathbb{E}}\left[\int_0^T \theta_u^2 Z_u^2\, du\right] < \infty$*. Let* $(\tilde{M}_t)_{0\le t\le T}$ *be a martingale under* $\widetilde{\mathbb{P}}$*. Then there exists an adapted process* $(\widetilde{\phi}_t)_{0\le t\le T}$ *such that*

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\phi}_u\, d\widetilde{W}_u, \quad 0 \le t \le T.$$

Proof. Let $f(x) = e^x$, $g(x) = \frac{1}{x}$ and $\gamma_t = -\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du$. Then we have

$$
\begin{aligned}
dZ_t &= df(\gamma_t) \\
&= f'(\gamma_t)\, d\gamma_t + \frac{1}{2} f''(\gamma_t)\, d\gamma_t d\gamma_t \\
&= -\theta_t Z_t \, dW_t
\end{aligned}
$$

and

$$
\begin{aligned}
d\left(\frac{1}{Z_t}\right) &= f'(Z_t)\, dZ_t + \frac{1}{2} f''(Z_t)\, dZ_t dZ_t \\
&= \frac{1}{Z_t}\theta_t \, dW_t + \frac{\theta_t^2}{Z_t}\, dt.
\end{aligned}
$$

By Lemma 2.6, for $s < t$, we can derive

$$
\widetilde{M}_s = \widetilde{\mathbb{E}}\left[\widetilde{M}_t \mid \mathcal{F}_s\right] = \mathbb{E}\left[\frac{Z_t \tilde{M}_t}{Z_s} \mid \mathcal{F}_s\right].
$$

Therefore, we have

$$
Z_s \widetilde{M}_s = \mathbb{E}\left[Z_t \widetilde{M}_t \mid \mathcal{F}_s\right].
$$

It means that $M_t = Z_s \widetilde{M}_s$ is a martingale under $\mathbb{P}$. By Theorem 4.1, there is an adapted process $(\phi)_{0 \le t \le T}$ such that

$$
M_t = M_0 + \int_0^t \phi_u dW_u, \quad 0 \le t \le T.
$$

Thus the differential of $\widetilde{M}_t$ has the form

$$
\begin{aligned}
d\widetilde{M}_t &= d\left(M_t \frac{1}{Z_t}\right) \\
&= \frac{1}{Z_t}\, dM_t + M_t\, d\frac{1}{Z_t} + dM_t d\frac{1}{Z_t} \\
&= \frac{\phi_t}{Z_t}\, dW_t + \frac{M_t\theta_t}{Z_t}\, dW_t + \frac{M_t\theta_t^2}{Z_t}\, dt + \frac{\phi_t\theta_t}{Z_t} dt \\
&= \frac{\phi_t}{Z_t}\, (dW_t + \theta_t\, dt) + \frac{M_t\theta_t}{Z_t}\, (dW_t + \theta_t\, dt) \\
&= \widetilde{\phi}_t d\widetilde{W}_t
\end{aligned}
$$

where

$$
\widetilde{\phi}_t = \frac{\phi_t + M_t\theta_t}{Z_t}.
$$

Finally, it can be written by

$$
\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\phi}_u\, d\widetilde{W}_u, \quad 0 \le t \le T.
$$

□

Remark A.0.4. Corollary A.0.3 is the Martingale Representation theorem on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. Therefore, we have the results that Theorem A.0.1 also works for $\widetilde{\mathbb{P}}$ - martingale.

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국문초록

본 학위 논문에서는 블랙-숄즈 모델에 기초한 장벽옵션과 기하학적 아시안 옵션을 다루고 이 두 옵션이 만족하는 편미분 방정식을 유도한다. 옵션시간 동안, 두 옵션가격을 이 편미분방정식의 고전해로써 계산한다. 더욱이, 장벽옵션과 기하학적 아시안 옵션을 결합한 기하학적 아시안 장벽 옵션을 다룬다. 덧붙여, 기하학적 아시안 장벽 옵션의 모델링 문제를 다루고, 기하학적 아시안 장벽 옵션의 가격이 점도해로써 이 모델링 문제를 만족하는 것을 보인다.

주요어휘: 블랙-숄즈 방정식, 베리어 옵션, 기하평균 아시안 옵션, 기하평균 아시안 베리어 옵션, 점성해

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