이학석사 학위논문

Compactification of Character Varieties via Triangulation on Surfaces

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서울대학교 대학원
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이 논문을 이학석사 학위논문으로 제출함

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Abstract

In this survey paper, first we introduce the result of [1]. It treats (relative) $\mathrm{SL}_2(\mathbb{C})$-character varieties of surfaces and their compactification via word length, giving a partial answer to the forklore conjecture that relative character varieties are log Calabi-Yau. We interpret this in the context of [2], which introduces compactification using triangulation. As in [2], result of Mondal [3] is used as a critical tool to prove properties of algebras associated to the compactification of character varieties. Such algebraic properties will lead to desirable interpretation of combinatoric results related to counting of multicurves on surfaces.

Keywords: Character Variety, Degree-like function, Multicurve, Triangulation.
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Chapter 1

Introduction

Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with $n \geq 1$ punctures. For a reductive algebraic Lie group G, let $X_{g,n}$ be the coarse moduli space of G-local systems on $\Sigma_{g,n}$. Let $X_{g,n,k}$ be the subvariety of $X_{g,n}$ obtained by prescribing traces $k \in \mathbb{A}^n(\mathbb{C})$ of peripheral loops around the punctures. There is a forklore conjecture regarding its compactification:

Conjecture 1.0.1 *There exists a normal compactification $\overline{X_{g,n,k}}$ of $X_{g,n,k}$ with boundary $D_{g,n,k}$ such that the pair $(\overline{X_{g,n,k}}, D_{g,n,k})$ is log Calabi-Yau.*

This is especially interesting when $G = \mathrm{SL}_2(\mathbb{C})$. [1] gives an example by word compactification.

Theorem 1.0.2 *The conjecture is true for $G = \mathrm{SL}_2(\mathbb{C})$, where $\overline{X_{g,n,k}} = X^{word}_{g,n,k}$ is defined by word compactification of $X_{g,n,k}$.*

In [2], another way of compactification is given.

Theorem 1.0.3 *The conjecture is true for $G = \mathrm{SL}_2(\mathbb{C})$, where $\overline{X_{g,n,k}} = X^{\tilde{\Delta}}_{g,n,k}$ is defined by compactification of $X_{g,n,k}$ via triangulation Δ.*

All these compactification process boils down to introducing a filtration on the function ring of our character variety, and especially both are examples of filtration via counting curves, counted with weights. In such settings, the associated graded ring structure can be revealed by inspecting the generators and their relations carefully. The desired result is that such graded rings are actually isomorphic to certain monomial rings, and from that we know they are graded Gorenstein algebras. As the combinatorial aspects of Gorenstein algebras suggest, it will result in a combinatorial reciprocity formula that forebodes the log Calabi-Yau property of relative character varieties.

Chapter 2

Background

2.1 Filtration and Degree-like Function

We introduce the basic algebraic notions required to define compactification of character varieties. The varieties of our concern would be spectra of rings. We consider compactification of those induced by giving filtration on those rings. By filtration, we mean multiplicative, increasing, and exhaustive ones as defined below. Let R be an integral finitely generated algebra over $\mathbb{C}$.

Definition 2.1.1 (Filtration) *A filtration $\mathcal{F}$ on R is a collection $\{\mathcal{F}_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of $\mathbb{C}$-subspaces of R such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, $\mathcal{F}_i\mathcal{F}_j \subset \mathcal{F}_{i+j}$, $\mathbb{C} \subset \mathcal{F}_0$, $R = \bigcup_i \mathcal{F}_i$.*

When R is filtered(=given a filtration), there are two graded algebras induced by the filtration. One is the Rees algebra $R^{\mathcal{F}} = \bigoplus_{i\geq 0} \mathcal{F}_i X^i$ and the other is what we call the associated graded algebra, $\mathrm{Gr}^{\mathcal{F}} R = \bigoplus_{i\geq 0} \mathcal{F}_i/\mathcal{F}_{i-1}$. Then we have $R \cong R^{\mathcal{F}}/(X-1) \cong R^{\mathcal{F}}_{(X)}$, $\mathrm{Gr}^{\mathcal{F}} R \cong R^{\mathcal{F}}/(X)$. For $Z = \operatorname{Spec} R$, $Z^{\mathcal{F}} = \operatorname{Proj} R^{\mathcal{F}}$ would be our desired compactification. Note that in this case, the complement of Z is exactly $\operatorname{Proj} Gr^{\mathcal{F}} R$.

We consider exhaustive filtrations(i.e. $R = \bigcup_i \mathcal{F}_i$) because it guarantees the filtered ring conveys certain properties from its associated graded algebra.

Lemma 2.1.2 *If* $\operatorname{Gr}^{\mathcal{F}} R$ *is an integral domain, so is* R*. If* $\operatorname{Gr}^{\mathcal{F}} R$ *is a normal domain, so is* R*.*

Another equivalent way of describing filtrations is degree-like functions.

Definition 2.1.3 (Degree-like Function) $\delta : R \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ *is a degree-like function if*

$\delta(\mathbb{C}^*) = 0,\ \delta(0) = -\infty,$

$\delta(f_1 + f_2) \leq \max\{\delta(f_1), \delta(f_2)\}$ *with equality when* $\delta(f_1) \neq \delta(f_2)$,

$\delta(f_1 f_2) \leq \delta(f_1) + \delta(f_2)$.

If the last equality holds, δ *is a semi-degree(=valuation).*

Indeed, a filtration $\mathcal{F}$ defines a degree-like function

$$\delta : x \mapsto \deg(\in x).$$

Conversely, given a degree-like function δ we can define filtration $\mathcal{F}$ by

$$\mathcal{F}_i = \{x | \delta(x) \leq i\}.$$

Via this equivalence, we can consider Rees algebra R^δ and $\operatorname{Gr}^\delta R$ induced by degree-like function δ defined on R.

2.2 Character Variety

Let π be a finitely generated group, and G a reductive algebraic Lie group over $\mathbb{C}$. Then we define the G–representation variety Rep_π as the affine scheme $\operatorname{Hom}(\pi, G) = \operatorname{Spec} \mathbb{C}[\operatorname{Rep} \pi]$. Indeed, it is the closed subscheme of G^m cut by the relations of m generators of π. When G is a matrix group, we define $\operatorname{tr}_a \in \mathbb{C}[\operatorname{Rep} \pi]$ by $\rho \mapsto \operatorname{tr} \rho(a)$ for $a \in \pi$.

Then we define the G–character variety X_π by $\mathrm{Hom}(\pi, G) /\!\!/ G = \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}_\pi]^G$, which is GIT quotient by the conjugate action of G. We can regard $\mathrm{tr}_a \in \mathbb{C}[\mathrm{Rep}_\pi]^G$ since trace is invariant under conjugate actions. Our main interest in this paper is the case when $G = \mathrm{SL}_2(\mathbb{C})$, in which case a lot is known. For example, first we have $\mathrm{tr}_1 = 2$ and for $a, b \in \pi$ and $\rho \in \mathrm{Rep}_\pi$, we have $\rho(a) - \mathrm{tr}_a(\rho)1 + \rho(a)^{-1} = 0$ by Cayley-Hamilton theorem, hence $\mathrm{tr}_a \mathrm{tr}_b = \mathrm{tr}_{ab} + \mathrm{tr}_{ab^{-1}}$. We finish this chapter with a theorem that these are all the relations we have in X_π.

Theorem 2.2.1 *For* $G = \mathrm{SL}_2(\mathbb{C})$,

$$\mathbb{C}[X_\pi] = \mathbb{C}\left[\mathrm{tr}_a \,|\, a \in \pi\right] / \left(\mathrm{tr}_1 - 2, \mathrm{tr}_a \mathrm{tr}_b = \mathrm{tr}_{ab} + \mathrm{tr}_{ab^{-1}}\right).$$

Chapter 3

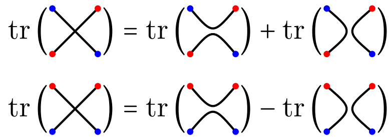
Compactification via Filtration

3.1 Relations on Generators

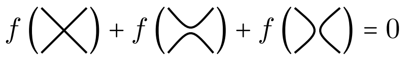
Let $\Sigma_{g,n}$ be an n-punctured Riemann surface with genus g and n We fix $\pi = \pi_1(\Sigma_{g,n}), G = \mathrm{SL}_2(\mathbb{C})$ for the rest of this paper. Our initial aim is to describe the structure of $R_{g,n}$ by inspecting its generators and their relations, following [2]. A theorem in [4] introduces a natural basis that we can work with, namely multicurves. We call a union of finite disjoint curves on $\Sigma_{g,n}$ a multicurve, and denote by $\mathfrak{Mult}$ the collection of isotopy classes of multicurves. We may define their traces as follows: first, since traces of conjugate elements are equivalent, we may define traces of baseless loops. By 2.2.1, we have $\mathrm{tr}_1 - 2, \mathrm{tr}_a \mathrm{tr}_b = \mathrm{tr}_{ab} + \mathrm{tr}_{ab^{-1}}$ as relations of traces of loops which induces $\mathrm{tr}_c = \mathrm{tr}_{c^{-1}}$ for any loop c, so we may ignore the orientation of loops and consider their traces as traces of curves. Then we define the trace of a union of curves as the product of traces of its components. The set of traces of mulcicurves (up to isotopy) is exactly our desired basis.

Theorem 3.1.1 (Charles-Marché) $R_{g,n} = \bigoplus_{c \in \mathfrak{Mult}} \mathbb{C}\,\mathrm{tr}_c$.

By 2.2.1, the relations between traces of multicurves is described by $\mathrm{tr}_1 - 2, \mathrm{tr}_a \mathrm{tr}_b = \mathrm{tr}_{ab} + \mathrm{tr}_{ab^{-1}}$, where 1 is the empty multicurve. The latter relation can be visualized as follows.

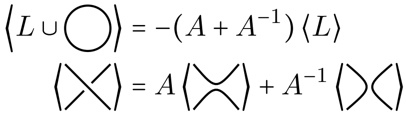


where the omitted part of curves connects between ends with same colors. We might want to have a better description of relations where we do not have to care how the curves are connected, so we define $f(c) = (-1)^{|c|} \mathrm{tr}_c$ where $|c|$ is the number of components of c. Then we have



We shall see this means $R_{g,n}$ is isomorphic to a specialized skein algebra.

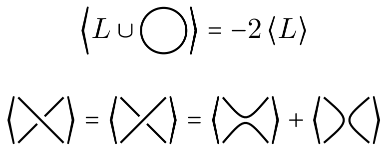
The skein algebra $\mathrm{Sk}_A(\Sigma_{g,n})$ is the $\mathbb{C}\left[A, A^{-1}\right]$-algebra generated by framed links in $\Sigma_{g,n} \times (-1,1)$ up to isotopy defined by relations

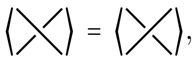


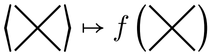
disjoint union with trivial loop

skein relation

Thus $\mathrm{Sk}_A(\Sigma_{g,n})/(A+1) \cong \mathrm{Sk}_A(\Sigma_{g,n})/(A^{-1}+1)$ is the $\mathbb{C}$-algebra defined by



Under isotopy, we can always choose a framed link on which the projection $\Sigma_{g,n} \times (-1,1) \to \Sigma_{g,n}$ is immersion to a multicurve. So generators of $\mathrm{Sk}_A(\Sigma_{g,n})/(A+1)$ can be represented by multicurves; Since , upper-crossings and under-crossings of framed links coincide, so that the correspondence between generators representing framed links and multicurves is one-to-one. That is, by

 we obtain the ring isomorphism $\mathrm{Sk}_A(\Sigma_{g,n})/(A-1) \cong R_{g,n}$.

The following automorphism from [5] gives another formulation of defining relations of $R_{g,n}$.

Theorem 3.1.2 *Each spin structure for $\Sigma_{g,n} \times (-1,1)$ defines a non-canonical linear map*

$$\phi : \mathrm{Sk}_A(\Sigma_{g,n}) \to \mathrm{Sk}_{-A}(\Sigma_{g,n})$$

defined by changing signs of each linked frame.

The above theorem, along with the isomorphism $\mathrm{Sk}_A(\Sigma_{g,n})/(A-1) \cong R_{g,n}$, shows that there is some $\tilde{f} = \pm f$ where the signs may differ for each multicurve, and satisfies the following relation:

3.2 Triangulation of $\Sigma_{g,n}$

As mentioned, giving a compactification of $R_{g,n}$ is accomplished by defining a filtration on $R_{g,n}$. First, we introduce a filtration defined by a set of 'counting curves' on $\Sigma_{g,n}$. Let $S = \{s_1, \ldots, s_r\}$ be a collection of closed curves and arcs(curves that connect between punctures), $d \in \mathbb{Z}^r_{\geq 0}$. Then define $\iota_{(S,d)}(c) = \sum_{k=1}^{r} d_k \iota(c, s_k)$ for $c \in \mathfrak{Mult}$, where $\iota(x,y)$ is the least number of intersections between curves in the isotopy classes x, y. So $\iota_{(S,d)}$ is basically the intersection number with respect to S counted with weight d.

Proposition 3.2.1 $\delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{c_i}\right) = \max_i \iota_{(S,d)}(c_i)$ *(c_i is a connected curve, $p_i \neq 0 \quad \forall i$) is a degree-like function.*

Proof. $\delta_{(S,d)}(\mathbb{C}^*), \delta_{(S,d)}(0) = -\infty$ is direct from the definition.

$\delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{a_i} + \sum_j q_j \operatorname{tr}_{b_j}\right) = \max\left\{\iota_{(S,d)}(c) | c = a_i = b_j, a_i + b_j \neq 0\right\}$

$\leq \max\left\{\max_i \iota_{(S,d)}(a_i), \max_j \iota_{(S,d)}(b_j)\right\} = \max\left\{\delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{a_i}\right), \delta_{(S,d)}\left(\sum_j q_j \operatorname{tr}_{b_j}\right)\right\}$

with equality when $\delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{a_i}\right) \neq \delta_{(S,d)}\left(\sum_j q_j \operatorname{tr}_{b_j}\right)$,

$\delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{a_i} \cdot \sum_j q_j \operatorname{tr}_{b_j}\right) = \delta_{(S,d)}\left(\sum_{i,j} p_i q_j \left(\operatorname{tr}_{a_i b_j} + \operatorname{tr}_{a_i b_j{}^{-1}}\right)\right)$

$\leq \max_i \iota_{(S,d)}(a_i) + \max_j \iota_{(S,d)}(b_j) = \delta_{(S,d)}\left(\sum_i p_i \operatorname{tr}_{a_i}\right) + \delta_{(S,d)}\left(\sum_j q_j \operatorname{tr}_{b_j}\right)$

the last inequality is due to $\delta_{(S,d)}(\operatorname{tr}_{ab}) \leq \iota_{(S,d)}(a) + \iota_{(S,d)}(b)$.

□

One important case is that $S = \Delta$ is the set of arcs where these arcs divide $\Sigma_{g,n}$ into areas homeomorphic to disk and each of those areas is bounded by three arcs, counted with multiplicity. Then we call each of those areas *a triangle*, and S a triangulation of $\Sigma_{g,n}$. Next, we argue how a triangulation parametrizes multicurves on $\Sigma_{g,n}$.

Since an arc bounds two triangles and a triangle is bounded by three arcs, counted with multiplicity, we have $2N$ triangles and $r = 3N$ arcs for some N. Then $2N$ triangles, $3N$ arcs, n points filling in the punctures form a cell structure on a surface of genus g and we have $2N - 3N + n = 2 - 2g$, hence $N \equiv n \ (mod\ 2)$.

Next we inspect how a multicurve appears in each triangles. When we consider a multicurve intersecting Δ at minimum in its isotopy class, their should be no bigons bounded by part of an arc and part of the multicurve. So when restricted to a triangle, multicurve is a disjoint union of segments connecting between arcs in Δ. Its isotopy type is characterized by a triple of nonnegative integers representing the number of segments connecting each pair of arcs. Since we have $2N$ triangles the configuration on the whole surface is represented by a $6N$-tuple of nonnegative integers. There must be the same number of segments connected from each side of an arc so the tuple must satisfy linear relations corresponding to the arcs. There are $3N$ arcs and the tuple must be a nonnegative integer solution of some $3N \times 6N$ integer matrix Φ. Conversely, if we have a nonnegative linear solution of such Φ, we can glue those curves in triangles to obtain a multicurve. In summary:

Proposition 3.2.2 *A triangulation* $\Delta = \{s_1, \ldots, s_{3N}\}$ *defines a bijection*

$$\begin{aligned} \phi : \mathfrak{Mult} &\longleftrightarrow \left\{\alpha \in \mathbb{Z}^{6N}_{\geq 0} | \Phi(\alpha) = 0\right\} \\ c &\longmapsto \alpha(c) \end{aligned}$$

for some $\Phi \in \mathfrak{M}_{3N \times 6N}(\mathbb{Z})$, *where* $\alpha(c)_i$ = *number of times* c *wraps around* i*th angle.*

Example 3.2.3 *In Figure 3.1,* $\Delta = a \cup b \cup c \cup d \cup e \cup f$ *is a triangulation on* $\Sigma_{1,2}$. *In the view of Proposition 3.2.2, multicurves on* $S_{1,2}$ *corresponds(up to isotopy) to nonnegative integer solutions of the following* Φ.

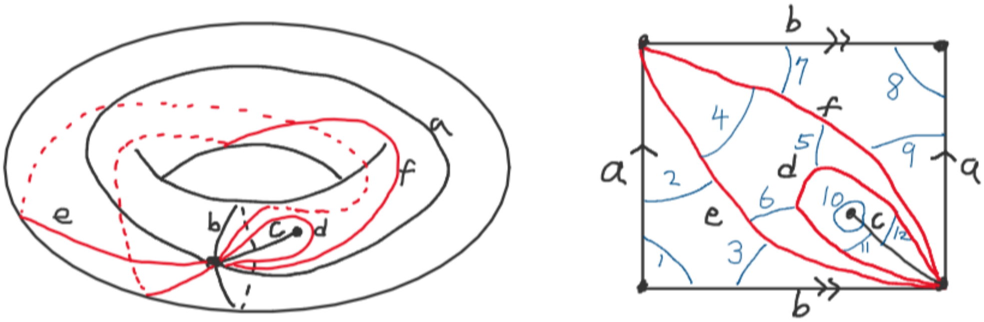


Figure 3.1: Edges and triangles on $S_{1,2}$

$$\Phi = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Combined with 3.1.1, 3.2.2 induces

$$\begin{aligned} \phi : R_{g,n} &\underset{\mathbb{C}-\mathfrak{vect}}{\cong} \mathbb{C}\left[x^{\alpha} | \Phi(\alpha) = 0, \alpha \in \mathbb{Z}^{6N}_{\geq 0}\right] \\ \tilde{f}(c) &\longmapsto x^{\alpha(c)} \end{aligned}$$

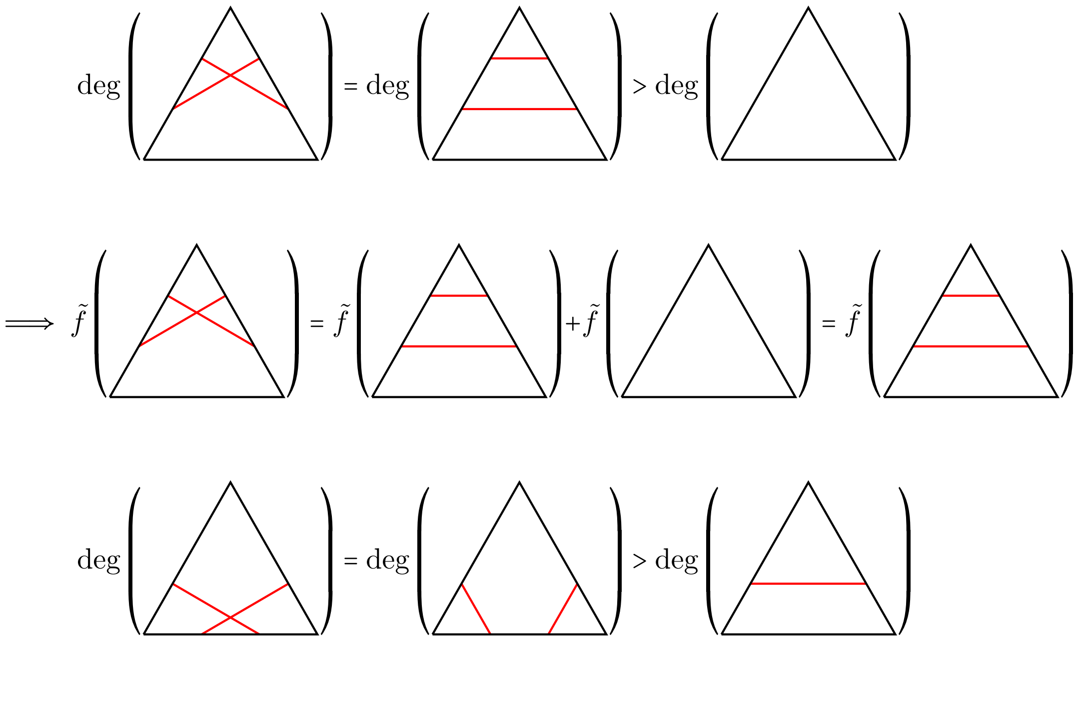
Note that this $\mathbb{C}$-linear isomorphism preserves filtration where the filtration on the right-hand side is given by the weighted degree of monomials.

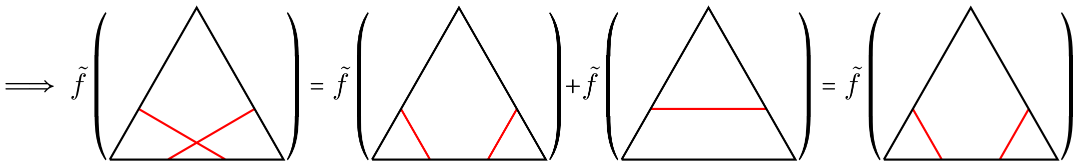
Next is our main theorem, characterizing the structure of the compactification of $X_{g,n}$.

Theorem 3.2.4 *ϕ induces $\mathbb{C}$-algebra isomorphism*

$$\mathrm{Gr}^{\Delta}\,\phi : \mathrm{Gr}^{\Delta}\,R_{g,n} \cong \mathbb{C}\left[x^{\alpha} | \Phi(\alpha) = 0, \alpha \in \mathbb{Z}_{\geq 0}^{6N}\right] \subset \mathbb{C}[x_1, \ldots, x_{6N}]$$

Proof. Now that we know the generators of $\mathrm{Gr}^{\Delta}\,R_{g,n}$ and how they are multiplied, we only need to check if it coincides with how monomials in monomial subring are multiplied, i.e. has the structure of a free abelian group. While multiplying two generators(multicurves) in $\mathrm{Gr}^{\Delta}\,R_{g,n}$, there might occur an intersection of those two multicurves. Then we have





in $\mathrm{Gr}^{\Delta} R_{g,n}$. Hence we may resolved any intersection occuring while multiplying multicurves and it is enough to count only the numbers of each segment in each triangle. In other words:

$$\mathrm{Gr}^{\Delta} R_{g,n} \cong \mathbb{C}\left[x^{\alpha} | \Phi(\alpha) = 0, \alpha \in \mathbb{Z}_{\geq 0}^{6N}\right].$$

□

Note that $\mathrm{Gr}^{\Delta} \phi$ in the theorem preserves the weighted degree of both rings. i.e. δ_{Δ} is a semi-degree.

Much is known about the monomial ring $\mathbb{C}\left[x^{\alpha} | \Phi(\alpha) = 0, \alpha \in \mathbb{Z}_{\geq 0}^{6N}\right]$. For example, we invoke the following theorem in [6].

Theorem 3.2.5 *For any $\Phi \in \mathfrak{M}_{M \times N}(\mathbb{Z})$ with $\operatorname{rk} \Phi = M \leq N$ and $\Phi(1, \ldots, 1) = 0$, $\mathbb{C}\left[x^{\alpha} | \Phi(\alpha) = 0, \alpha \in \mathbb{Z}_{\geq 0}^{N}\right] \subset \mathbb{C}[x_1, \ldots, x_N]$ is a graded Gorenstein domain.*

Corollary 3.2.6 *For a triangulation Δ, $\mathrm{Gr}^{\Delta} R_{g,n}$ is a normal graded Gorenstein domain.*

Proof. That $\mathrm{Gr}^{\Delta} R_{g,n}$ is a normal domain is immediate from 3.2.4. For the rest, it is enough to show that $\operatorname{rk} \Phi = 3N$ and $\Phi(1, \ldots, 1) = 0$. Let $v_1, v_2 \ldots, v_{3N}$ be row vectors of Φ. Suppose $\sum_{i=1}^{3N} a_i v_i = 0$. Choose a triangle. Pick columns corresponding to segments connecting between arcs enclosing each triangle. If the triangle has three distinct arcs, say v_1, v_2, v_3. Then Φ looks as follows after reordering columns so that the picked columns come first, up to sign of rows.

$$\begin{pmatrix} 1 & 1 & 0 & \\ 1 & 0 & 1 & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & * \\ & \vdots & & \\ 0 & 0 & 0 & \end{pmatrix}$$

From this, we have $a_1 = a_2 = a_3 = 0$. Meanwhile, if two arcs coincide so that the chosen triangle is enclosed by two distinct edges, say v_1, v_2, then Φ looks as follows up to reordering of rows and columns and the sign of rows.

$$\begin{pmatrix} 1 & 1 & 0 & \\ 1 & -1 & 0 & \\ 0 & 0 & 0 & * \\ & \vdots & & \\ 0 & 0 & 0 & \end{pmatrix}$$

Thus $a_1 = a_2 = 0$. Since every arc bounds some triangle, we have $a_i = 0$ for all i. We conclude that $v_1, v_2, \cdots, v_{3N}$ are independent and $\operatorname{rk}\Phi = 3N$.

Next, consider the multicurve that consists of n peripheral curves on $\Sigma_{g,n}$. Since it has to wrap around each angle of the triangles for exactly once, its corresponding monomial is $x^{(1,\dots,1)}$. i.e. $\Phi(1,\dots,1) = 0$. □

The following theorem originates from [3].

Theorem 3.2.7 *δ is a subdegree (that is, $\delta(f^n) = n\delta(f) \quad \forall n \geq 0, f \in R$) implies Z^δ is relatively normal at infinity, i.e. $\Gamma(U, \mathcal{O}_{Z^\delta})$ is integrally closed in $\Gamma(U \cap Z, \mathcal{O}_{Z^\delta})$ for any U open in Z^δ.*

Corollary 3.2.8 *If R is a normal domain and δ is a subdegree, then R^δ is a normal domain.*

Corollary 3.2.9 *$X_{g,n}, X^\Delta_{g,n}$ are normal varieties.*

Proof. The isomorphism $\operatorname{Gr}^{\Delta}\phi$ in 3.2.4 preserves grades. i.e. δ^{Δ} is a semi-degree, hence a subdegree. By 2.1.2 and 3.2.6, $R_{g,n}$ is a normal domain. By 3.2.8, $R_{g,n}^{\Delta}$ is a normal domain. □

3.3 Combinatorial Results

Now we shall bring out a reciprocity formula, which is a fragmentary clue about the log Calabi-Yau property of the relative character variety of $\Sigma_{g,n}$. The result of this section depends on the following formula by [6].

Theorem 3.3.1 *For any $\Phi \in \mathfrak{M}_{M\times N}(\mathbb{Z})$ with $\operatorname{rk}\Phi = M \leq N$, K be the set of its nonnegative integer solutions and K° its positive integer solutions. Then $k(z) = \sum_{\alpha\in K} z^{\alpha}$, $k^{\circ}(z) = \sum_{\alpha\in K^{\circ}} z^{\alpha}$ are rational functions. If $k^{\circ} \neq 0$, $k(z^{-1}) = (-1)^{N-M}k^{\circ}(z)$.*

Of course, the theorem is to be applied to Φ in 3.2.4. The main ingredient in the proof of 3.3.1 is the Grothendieck local duality, one of whose immediate consequence is as follows:

Theorem 3.3.2 *For a graded Cohen-Macaulay d-dimensional algebra R with graded canonical module ω_R, $H_R(t^{-1}) = (-1)^d H_{\omega_R}(t)$.*

Here, H denotes the Hilbert series. By applying this directly to $\operatorname{Gr}_{g,n}^{\Delta}$, which we have seen to be a graded Gorenstein ring so that its graded canonical module is a grade shift of itself, we obtain

Corollary 3.3.3 $H_{D_{g,n}^{\Delta}}(t^{-1}) = (-1)^r t^{\sum_i d_i} H_{D_{g,n}^{\Delta}}(t)$

where $D_{g,n}^{\Delta} = \operatorname{Proj}\operatorname{Gr}^{\Delta} R_{g,n}$ is the boundary of $X_{g,n}$ in $X_{g,n}^{\Delta}$.

To apply 3.3.1, it is required to inspect the space of positive integer solutions, which is simple in the case of 3.2.4 since $(1,\ldots,1)$ is the unique positive solution of Φ. Here we have

$$k\left(z^{-1}\right) = (-1)^{6N-\operatorname{rk}(\Phi)}k^{\circ}(z) = (-1)^{3N}t^{(1,1,\ldots,1)}k(z)$$

Now define

$$g_\Delta(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{3N}} G_\alpha x^\alpha$$

$$f_\Delta(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{3N}} F_\alpha x^\alpha$$

where

$$G_\alpha = \left\{ c \in \mathfrak{Mult} | \iota_{(\Delta,d)}(c) = \alpha \right\}$$

$$F_\alpha = \left\{ c \in \mathfrak{Mult} | \text{c has no peripheral components,} \iota_{(\Delta,d)}(c) = \alpha \right\}$$

Then the substitution $z_{(i,j)} = x_i^{\frac{1}{2}} x_j^{\frac{1}{2}}$, where x_i, x_j are variables corresponding to arcs C_i, C_j and $z_{(i,j)}$ is the variable corresponding to the segment connecting those arcs, gives $g_\Delta(x) = k(z)$. Since we have already seen that the multicurve comprised of the peripheral curves corresponds to the monomial $z^{(1,\ldots,1)}$, we have

$$\begin{aligned}
f_\Delta\left(x^{-1}\right) &= g_\Delta\left(x^{-1}\right) \prod_{i=1}^{n} \left(1 - x^{-\alpha_i}\right) \\
&= (-1)^{6N - \text{rank}(\Phi)} z^{(1,1,\ldots,1)} g_\Delta(x) \prod_{i=1}^{n} \left(1 - x^{-\alpha_i}\right) \\
&= (-1)^N \left(\prod_{i=1}^{n} x^{\alpha_i} \right) g_\Delta(x) \prod_{i=1}^{n} \left(1 - x^{-\alpha_i}\right) \\
&= (-1)^n g_\Delta(x) \prod_{i=1}^{n} \left(x^{\alpha_i} - 1\right) \\
&= g_\Delta(x) \prod_{i=1}^{n} \left(1 - x^{\alpha_i}\right) \\
&= f_\Delta(x).
\end{aligned}$$

which results in a reciprocity formula that suggests a log Calabi-Yau property of the relative character variety.

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국문초록

본 조사 논문에서는 $SL_2(\mathbb{C})$-특성 다양체의 옹골화를 제시한 Whang과 Farajzadeh-Tehrani, Frohman의 결과를 살펴본다. 이것은 널리 알려진 추측처럼 log Calabi-Yau 성질을 만족하는데, 우리는 이것을 Farajzadeh-Tehrani, Frohman의 논문을 따라서 세는 곡선을 통한 옹골화로써 이해할 것이다. 특히 삼각화를 통한 옹골화를 살펴볼 것인데, 여기에는 Mondal의 유사 정도 함수에 대한 이론이 중요하게 쓰일 것이다. 이러한 대수적 성질들을 살펴본 뒤에는 이것들에 의해 나타나는 조합론적 현상이 어떻게 대수적으로 설명될 수 있을지 직접 보일 것이다.

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