이학박사 학위논문

Extensions of Bourgain's circular maximal theorem

(부르갱 원 극대 함수 정리의 확장)

2023년 8월

서울대학교 대학원

수리과학부

이 주 영

Extensions of Bourgain's circular maximal theorem

(부르갱 원 극대 함수 정리의 확장)

지도교수 이 상 혁

이 논문을 이학박사 학위논문으로 제출함

2023년 4월

서울대학교 대학원

수리과학부

이 주 영

이 주 영의 이학박사 학위논문을 인준함

2023년 6월

위 원 장 이 훈 희 (인)

부위원장 이 상 혁 (인)

위 원 김 준 일 (인)

위 원 서 인 석 (인)

위 원 Bez, Neal (인)

Extensions of Bourgain's circular maximal theorem

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Lee, Juyoung

Dissertation Director : Professor Sanghyuk Lee

Department of Mathematical Sciences
Seoul National University

August 2023

Abstract

Extensions of Bourgain's circular maximal theorem

Lee, Juyoung

Department of Mathematical Sciences
The Graduate School
Seoul National University

The estimates for maximal functions play important roles in various problems in mathematical analysis such as those in partial differential equations, geometric measure theory, and harmonic analysis. Since the 1950s, the maximal functions defined by averages have been extensively studied in the field of classical harmonic analysis and a huge literature has been devoted to the subject. In 1976, Stein proved his seminal result: L^p bound on the spherical maximal operator on the optimal range for every dimension bigger than 2. Its two-dimensional counterpart, the bound on the circular maximal function, turned out to be more difficult since the traditional L^2 based argument did not work. In 1986, however, Bourgain settled the problem by proving his celebrated theorem: the circular maximal operator is bounded on L^p for $p > 2$. In this thesis, we prove three results which strengthen Bourgain's circular maximal theorem. First, we establish on the sharp range of p, q the L^p–L^q boundedness of the circular maximal operator on the Heisenberg group for Heisenberg radial functions. Secondly, we obtain the sharp L^p–L^q boundedness of the two-parametric maximal operator defined by averages over tori. Lastly, we prove L^p estimates on the elliptic maximal operators which are multiparametric maximal operators given by averages over ellipses.

Key words: Averaging operator, Maximal bound, Sobolev regularity, Local smoothing
Student Number: 2016-29031

Contents

Abstract **i**

1 Introduction **1**
1.1 Maximal averages over rectangles . . . 1
1.2 Maximal averages over submanifolds . . . 3
1.3 The circular maximal function and L^p improving property . . 5
1.4 Maximal averages on the Heisenberg group . . . 6
1.5 Two parameter maximal averages over tori . . . 7
1.6 Multiparameter maximal averages over ellipses . . . 9
1.7 Notations . . . 10

2 Preliminaries **12**
2.1 Decoupling inequalities . . . 12
2.2 Local smoothing estimates of the wave operator . . . 16

3 The Heisenberg circular maximal operator **19**
3.1 Heisenberg radial functions and main estimates . . . 21
3.2 Local maximal estimates . . . 23
3.3 Global maximal estimates . . . 27
3.4 Proof of main estimates . . . 30
3.5 Proof of Proposition 3.1.1 . . . 31
3.6 Proof of Proposition 3.1.2 . . . 34
3.7 Sharpness of the range of p, q . . . 41

4 Two parameter averages over tori **42**
4.1 Comparison with one parameter maximal average . . . 43
4.2 Local smoothing estimates of averages over tori . . . 44
4.3 Two parameter propagator . . . 45

4.4 Estimates for the averaging operator $\mathcal{A}_t^s$ 50
4.5 Global maximal estimates 58
4.6 Local maximal estimates 64
4.7 Proof of smoothing estimates 66
4.8 Optimality of the estimates 75

5 Multiparameter averages over ellipses **79**
5.1 Local smoothing estimates for averaging operators over ellipses. 80
5.2 Proof of maximal bounds 82
5.3 Variable coefficient decoupling inequalities 87
5.4 Proof of local smoothing estimates 92
5.5 Proof of Theorem 5.3.2 99
5.6 Optimality of the estimates 106

Abstract (in Korean) **i**

Acknowledgement (in Korean) **ii**

Chapter 1

Introduction

Average is one of the most important concepts in mathematics. It helps to understand overall behaviour of a family of objects in many contexts. Usually, averaging over a class of objects gives rise to better properties which we can not claim for each object. Among many different forms of average depending on particular purposes, what we are interested in is the arithmetic mean. In particular, we focus on its beauty in mathematical analysis using the language of harmonic analysis. A main objectivity in analysis is to understand functions defined on a space G. Average of a function on G is given by integration which generalizes the arithmetic mean. Under some structures of measure and integration on the space G, the particular value of a function f at each point of G is generally not important. Instead, a family of averages of f completely determines f almost everywhere. This is the main idea of generalized function, distribution, and a power of average. Over the last half century, boundedness of the maximal averages, which allows us to say continuity of averages, has been extensively studied. In this thesis, we study generalizations of the monumental results, Bourgain's circular maximal theorem. We start with briefly reviewing the history of the study of maximal averages.

1.1 Maximal averages over rectangles

One advantage of average is that it makes a function regular. To be concrete, let $\mathbb{R}^n$ be the n dimensional Euclidean space with the Lebesgue measure dx, and $B_r(c)$ be the ball of radius $r > 0$ with center $c \in \mathbb{R}^n$. Then, for a locally integrable function f on $\mathbb{R}^n$,

$$\frac{1}{|B_r(0)|}\int_{B_r(0)} f(x-y)dy$$

is an average of f over a ball of radius r centered at x where $|A|$ is a volume of a set $A \subset \mathbb{R}^n$. To see differentiability of averages, one may ask if

$$\lim_{r\to 0}\frac{1}{|B_r(0)|}\int_{B_r(0)} f(x-y)dy = f(x) \tag{1.1.1}$$

holds. This obviously holds when f is a continuous function. Thus, continuity gives differentiability of averages.

This gives rise to two questions. The first question is that instead of continuous functions, what happens when we consider merely (locally) integrable functions, L^p functions. This is closely related to the L^p boundedness of the Hardy-Littlewood maximal operator

$$M_{HL}f(x) = \sup_{r>0}\frac{1}{|B_r(0)|}\int_{B_r(0)} |f(x-y)|dy.$$

It is well known that M_{HL} is bounded on L^p if and only if $p > 1$ and weakly bounded on L^1. This implies that (1.1.1) holds almost everywhere if f is merely locally integrable. The second question is that for which family of sets where we are taking averages, (1.1.1) holds almost everywhere for a suitable family of functions (it is usually a family of locally L^p functions for a suitable p). Let $\mathfrak{O}$ be a family of sets with nonzero bounded measure, $\{O\}$. The second question asks whether

$$\lim_{\mathrm{diam}(O)\to 0}\frac{1}{|O|}\int_O f(x-y)dy = f(x)$$

holds. Similarly with the first question, it is deeply related to the boundedness of the following maximal operator,

$$M_{\mathfrak{O}}f(x) = \sup_{O\in\mathfrak{O}}\frac{1}{|O|}\int_O |f(x-y)|dy. \tag{1.1.2}$$

One general statement is that when all sets in $\mathfrak{O}$ have bounded eccentricity, $M_{\mathfrak{O}}$ is bounded on L^p if and only if $p > 1$, and weakly bounded on L^1 (see

[65],[75]). Generally, we assume that $\mathfrak{D}$ is generated by finitely many parameters. For example, a family of balls centered at the origin considered in M_{HL} is a one parameter family. The difficulty dealing with $M_{\mathfrak{D}}$ arises when $\mathfrak{D}$ is a multiparameter family. For instance, when $\mathfrak{D}$ is a family of all rectangles centered at the origin in $\mathbb{R}^n$, then $\mathfrak{D}$ is an $n+n(n-1)/2$-parameter family. Indeed, we need n parameters to determine sidelengths of a rectangle, and the dimension of $SO(n)$ is $n(n-1)/2$ which determines orientation. Unfortunately, $M_{\mathfrak{D}}$ is not bounded on any L^p for $p < \infty$. This can be checked by using a fundamental construction due to Besicovitch (see, for example, [75]). Meanwhile, one can easily see that if we consider n-parameter family of rectangles each of whose sides are parallel to the coordinate axis, the corresponding maximal function is bounded on L^p for all $p > 1$. Precisely, we consider the following family of sets.

$$\mathfrak{R}^n_{str} = \{\prod_{i=1}^{n}[-\frac{a_i}{2}, \frac{a_i}{2}] : a_i > 0 \text{ for } 1 \le i \le n\}.$$

Then, the associated maximal operator is defined by

$$\begin{aligned} M_{\mathfrak{R}^n_{str}} f(x) &= \sup_{R \in \mathfrak{R}^n_{str}} \frac{1}{|R|} \int_R |f(x-y)| dy \\ &= \sup_{a_i > 0} \frac{1}{\prod_{i=1}^n a_i} \int_{\prod_{i=1}^n [-\frac{a_i}{2}, \frac{a_i}{2}]} |f(x-y)| dy. \end{aligned}$$

This is called the strong maximal function and many researches were devoted to characterize a function space which ensures the strong maximal function is integrable on any set of finite measure (see [2], [22], [23]). More generally, problems concerning all rectangles with lacunary directions were considered in [17], [19], [56], [77], for instance. Considering all orientation of rectangles with a fixed (large) eccentricity produces one of the core conjectures in harmonic analysis, the Kakeya maximal conjecture, but we do not go further in this direction.

1.2 Maximal averages over submanifolds

The main difficulty in the above multiparameter problems arose since the sets may be "thin". From this viewpoint, we have another natural question. What happens if we consider a family of measure zero sets? We assume that

$\mathfrak{O}$ is a family of submanifolds in $\mathbb{R}^n$. To keep a similar situation when we consider averages, we need to replace dy in (1.1.2) by a suitable submanifold carried measure and $|O|$ by a volume with respect to this measure. Precisely, we are interested in the following one parameter problem. Let $S \subset \mathbb{R}^n$ be a fixed compact submanifold and $d\mu_S$ be the Lebesgue measure on S. Defining a natural normalized measure on a dilation tS by

$$\langle d\mu_S^t, f\rangle = \int_S f(ty)d\mu_S(y),$$

we get an averaging operator

$$A_t^S f(x) = f * d\mu_S^t(x) = \int_S f(x - ty)d\mu_S(y) \tag{1.2.1}$$

and the associated maximal operator

$$M_S f(x) = \sup_{t>0} |A_t^S f(x)|.$$

One can easily find an example that such maximal operator is never bounded on L^p for any $p < \infty$ when the S is completely flat. Thus, it is natural to impose an appropriate curvature condition. Indeed, this assumption implies that the Fourier transform of $d\mu_S$ has a certain power of decay. Of course the boundedness of the maximal operator M_S implies the corresponding convergence property as before.

For the last half century, maximal averaging operators over submanifolds, especially hypersurfaces, have been extensively studied (see [76]). We investigate some history. One remarkable milestone is Stein's work [74] that when S is a sphere centered at the origin, the corresponding spherical maximal operator is bounded on L^p if and only if $p > n/(n-1)$ when $n \geq 3$. However, when $n = 2$, it could not be handled easily since L^2 method is not applicable. Instead, a new idea wihch converts the problem into the study of an associated Fourier multiplier operator was itroduced. Indeed, as (1.2.1), we see that $\widehat{A_t^S f} = \widehat{f}\,\widehat{d\mu_S^t}$ and almost all arguments can be modified considering $\widehat{d\mu_S^t}$ replaced by a multiplier $m(t\cdot)$ which has a suitable decay (see [64], [72], [74]). When the surface varies depending on the location where we take an average, the notion of rotational curvature is needed. It was introduced in [60] which is an equivalent formulation of that in [27]. The nonzero rotational curvature condition says that the corresponding averaging operator can be expressed as a Fourier integral operator of suitable order (see [35]). By assuming the the rotational curvature condition, which essentially means that every surface has nonvanishing Gaussian curvature, it was shown that the maximal averaging operator is bounded on L^p if and only if $p > n/(n-1)$ (see [20], [72], [73]).

1.3 The circular maximal function and L^p improving property

Now we arrive at the main theme of this thesis, Bourgain's circular maximal theorem. The above problem for $n = 2$ was settled by Bourgain [7]. Later Mockenhaupt, Seeger, Sogge [53] also gave an alternative proof. In [53], the authors used an observation that one can obtain extra regularity of $A_t^{\mathbb{S}^1} f(x)$ in comparison with an estimate for a fixed t when we take an average in $t \sim 1$. Precisely, for any fixed $t > 0$, $A_t^{\mathbb{S}^{n-1}}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p_\alpha(\mathbb{R}^n)$ when $\alpha \leq (n-1)/p$. This was proved in [24] for $\alpha < (n-1)/p$, and in [52], [59] for the $\alpha = (n-1)/p$ (see also [69] for a generalization). Averaging in t, we obtain that the averaging operator maps $L^p(\mathbb{R}^n)$ to $L^p_\alpha(\mathbb{R}^n \times [1,2])$ boundedly for some $\alpha > \frac{n-1}{p}$. In [71], it is conjectured that the operator is bounded if and only if $\alpha \leq \max\{n/p, 1/2\}$ for $p \geq 2$. This is called the local smoothing conjecture which is another core conjecture in harmonic analysis since the local smoothing conjecture implies the Bochner-Riesz conjecture, the restriction conjecture, and the Kakeya conjecture (see [79]). For $n = 2$, it was recently solved with ϵ-loss by Guth, Wang, Zhang [30]. However, for $n \geq 3$, it is verified only for $p \geq 2(n+1)/(n-1)$ with ϵ-loss (see [10], [81]). The local smoothing phenomenon has been generalized to various settings (see [6], [34], [45], [54], [62], [68] and references therein). Using the local smoothing estimate, we can observe an interesting feature. When we restrict t in a compact interval away from 0, the associated maximal operator

$$M^c_{\mathbb{S}^{n-1}} f(x) = \sup_{1<t<2} |A_t^{\mathbb{S}^{n-1}} f(x)|$$

may be bounded from L^p to L^q for some $p < q$. This is called the L^p-improving phenomenon and it never occur for M_S, the global operator, by the scaling structure. In [67] and [68], authors characterized the L^p-L^q boundedness of $M^c_{\mathbb{S}^{n-1}}$ except endpoints not only for the circular maximal operator but also variable coefficient analogues. For this purpose, we need another notion of curvature, which is called the cinematic curvature (see [71]). Later, S. Lee [44] proved the boundedness of $M^c_{\mathbb{S}^{n-1}}$ at all endpoints but one point. Using the Littlewood-Paley theory, when $p = q$, the boundedness of $M^c_{\mathbb{S}^{n-1}}$ essentially implies the seemingly stronger boundedness of $M_{\mathbb{S}^{n-1}}$ (see [66]). There are also results in which dilation parameter sets were generalized to sets of fractal dimensions (for example, see [1], [70]).

1.4 Maximal averages on the Heisenberg group

Now we see our first generalization of the circular maximal theorem. We generalize the Euclidean space $\mathbb{R}^n$ to a noncommutative space, the Heisenberg group $\mathbb{H}^n$. $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ under the noncommutative multiplication law

$$(x, x_{2n+1}) \cdot (y, y_{2n+1}) = (x + y, x_{2n+1} + y_{2n+1} + x \cdot Ay),$$

where $(x, x_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}$ and A is the $2n \times 2n$ matrix given by

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with I_n the $n \times n$ identity matrix. We start from the spherical maximal operator in $\mathbb{H}^n$. Let $d\sigma_n$ be the normalized Lebesgue measure supported on $\mathbb{S}^{2n-1} \times \{0\} \in \mathbb{H}^n$. The normalized measure on a dilation $t\mathbb{S}^{n-1} \times \{0\}$ is defined similarly by $\langle d\sigma_n^t, f\rangle = \langle d\sigma_n, f(t\cdot)\rangle$. Because of the noncommutative multiplication law, we define a different averaging operator by convolution.

$$f *_{\mathbb{H}} d\sigma_t(x, x_{2n+1}) = \int_{\mathbb{S}^{2n-1}} f(x - ty, x_{2n+1} - tx \cdot Ay) d\sigma_n(y).$$

Notice that this formula calculates an average of f on an ellipse which is contained in a plane depending on a location. We consider the associated spherical maximal operator

$$M_{\mathbb{H}^n} f(x, x_{2n+1}) = \sup_{t>0} \left| f *_{\mathbb{H}} d\sigma_n^t(x, x_{2n+1}) \right|.$$

This operator has been studied for decades in many papers in the literature. When $n \geq 2$, the boundedness property of $M_{\mathbb{H}^n}$ is already almost completely understood (see Chapter 3). However, the boundedness of $M_{\mathbb{H}^1}$ on any L^p still remains open. It is a variable coefficient generalization of the circular maximal function so that we may apply previous results. However, for $M_{\mathbb{H}^1}$, both the rotational curvature and the cinematic curvature vanish which makes the problem difficult. Meanwhile, Beltran, Guo, Hickman, Seeger [3] restricted the class of functions and obtained the boundedness result of $M_{\mathbb{H}^1}$ for the sharp range $p > 2$ under the condition that the function is Heisenberg radial.

Definition. We say a function $f : \mathbb{H}^1 \to \mathbb{C}$ is Heisenberg radial if $f(x, x_3) = f(Rx, x_3)$ for all $R \in \mathrm{SO}(2)$.

From now on, we simply denote $d\sigma_1^t$ by $d\sigma_t$. Our first main result is the following which completely characterizes L^p improving property of $M^c_{\mathbb{H}^1}$ on Heisenberg radial functions except for some borderline cases. Here, $M^c_{\mathbb{H}^1}$ is defined by

$$M^c_{\mathbb{H}^1} f(x, x_{2n+1}) = \sup_{1<t<2} |f *_{\mathbb{H}} d\sigma_t(x, x_{2n+1})|.$$

Theorem 1.4.1 ([41]). *Let $P_0 = (0,0)$, $P_1 = (1/2, 1/2)$, and $P_2 = (3/7, 2/7)$, and let $\mathbf{T}$ be the closed region bounded by the triangle $\Delta P_0 P_1 P_2$. Suppose $(1/p, 1/q) \in \{P_0\} \cup (\mathbf{T} \setminus (\overline{P_1 P_2} \cup \overline{P_0 P_2}))$. Then, the estimate*

$$\|M^c_{\mathbb{H}^1} f\|_q \lesssim \|f\|_{L^p} \tag{1.4.1}$$

holds for all Heisenberg radial functions f. Conversely, if $(1/p, 1/q) \notin \mathbf{T}$, then the estimate fails.

As in the Euclidean circular maximal operator, the boundedness of $M^c_{\mathbb{H}^1}$ essentially implies the boundedness of $M_{\mathbb{H}^1}$ for Heisenberg radial functions. We will see this implication as well.

1.5 Two parameter maximal averages over tori

Our second main results concern a two-parameter maximal operator over 2-dimensional tori $t\mathbb{S}^1 \times s\mathbb{S}^1$ in $\mathbb{R}^3$ which can be seen as a generalization of the circular maximal operator. Let us set

$$\Phi_t^s(\theta, \phi) = \big((t + s\cos\theta)\cos\phi,\ (t + s\cos\theta)\sin\phi,\ s\sin\theta\big).$$

For $0 < s < t$, we denote $\mathbb{T}_t^s = \big\{\Phi_t^s(\theta,\phi) : \theta, \phi \in [0, 2\pi)\big\}$, which is a parametrized torus in $\mathbb{R}^3$. We consider a measure on $\mathbb{T}_t^s$ which is given by

$$\langle f, d\sigma_t^s \rangle = \int_{[0,2\pi)^2} f(\Phi_t^s(\theta,\phi))\, d\theta d\phi. \tag{1.5.1}$$

Convolution with the measure $d\sigma_t^s$ gives rise to a 2-parameter averaging operator $\mathcal{A}_t^s f := f * d\sigma_t^s$. Let $0 < c_0 < 1$ be a fixed constant. We define the following maximal operator.

$$\mathcal{M}_{\mathbb{T}} f(x) = \sup_{0<s<c_0 t} \big|\mathcal{A}_t^s f(x)\big|$$

Here, the supremum is taken over on the set $\{(t,s) : 0 < s < c_0 t\}$ so that $\mathbb{T}_s^t$ remains to be a torus. Note that when s converges to 0, the operator collapses to the circular maximal operator. We also remark that $\mathbb{T}_s^t$ has a part where Gaussian curvature vanishes so that it is already not possible to obtain a result for the one parameter maximal operator $f \to \sup_{0<t} |\mathcal{A}_t^{c_0 t} f|$ using previous literature of maximal functions. However, Ikromov, Kempe, Müller [37] obtained results for maximal averaging operators over degenerate hypersurfaces which include a torus (see also [15], [16]). According to their result, the one parameter maximal averaging operator $f \to \sup_{0<t} |\mathcal{A}_t^{c_0 t} f|$ is bounded on $L^p(\mathbb{R}^3)$ if and only if $p > 2$. Surprisingly, $\mathcal{M}_{\mathbb{T}}$ has the same boundedness property.

Theorem 1.5.1 ([42])**.** *The maximal operator $\mathcal{M}_{\mathbb{T}}$ is bounded on L^p if and only if $p > 2$.*

We also characterized a typeset of the localized maximal operator

$$\mathcal{M}_{\mathbb{T}}^c f(x) = \sup_{(t,s)\in\mathbb{J}} \big|\mathcal{A}_t^s f(x)\big|.$$

Here $\mathbb{J}$ is a compact subset of $\mathbb{J}_* := \{(t,s) \in \mathbb{R}^2 : 0 < s < t\}$. The next theorem gives L^p–L^q bounds on $\mathcal{M}_{\mathbb{T}}^c$ for a sharp large of p, q.

Theorem 1.5.2 ([42])**.** *Set $P_1 = (5/11, 2/11)$ and $P_2 = (3/7, 1/7)$. Let $\mathcal{Q}$ be the open quadrangle with vertices $(0,0)$, $(1/2,1/2)$, P_1, and P_2 which includes the half open line segment $[(0,0),(1/2,1/2))$. Then, the estimate*

$$\|\mathcal{M}_{\mathbb{T}}^c f\|_{L^q} \lesssim \|f\|_{L^p} \tag{1.5.2}$$

holds if $(1/p,1/q) \in \mathcal{Q}$. Conversely, if $(1/p,1/q) \notin \overline{\mathcal{Q}} \setminus \{(1/2,1/2)\}$, then the estimate (1.5.2) fails.

We also obtained multi-parameter local smoothing estimates for $\mathcal{A}_t^s$. The 2-parameter and 1-parameter local smoothing estimates have extra smoothing of order up to $2/p$ and $1/p$, respectively for suitable p (see Chapter 4 for details).

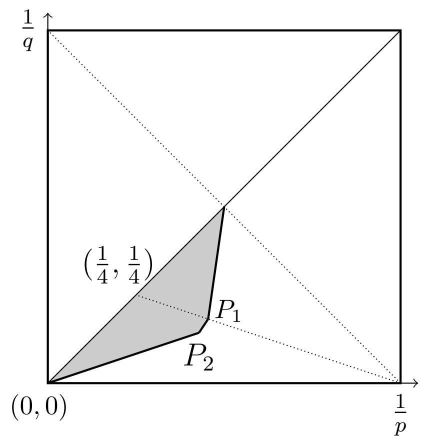

Figure 1.1: The typeset of $\mathcal{M}^c_{\mathbb{T}}$

1.6 Multiparameter maximal averages over ellipses

It is natural to ask that what happens when we consider a strong circular maximal operator as an analogue of the strong maximal operator for rectangles. We have two ways of considering multiparameter circular maximal functions as Stein did for rectangles. First, we may consider a maximal average of f over all ellipses centered at a fixed point. Precisely, this generates a 3-parameter maximal function as follows. Abusing notation, we let $d\sigma$ be the normalized Lebesgue measure on $\mathbb{S}^1$. For $(\theta, t, s) \in \mathbb{T} \times \mathbb{R}^2_+$, $\sigma^\theta_{t,s}$ denotes the measure on the rotated ellipse $\mathbb{E}^\theta_{t,s} := \{R_\theta(t\cos u, s\sin u) : u \in \mathbb{T}\}$ which is given by

$$(f, \sigma^\theta_{t,s}) = \int_{\mathbb{S}^1} f\big(R_\theta(ty_1, sy_2)\big)d\sigma(y).$$

We consider the maximal operator

$$\mathfrak{M}f(x) = \sup_{(\theta,t,s)\in\mathbb{T}\times[1,2]^2} |f * \sigma^\theta_{t,s}(x)|,$$

which was called the *elliptic maximal function* in [21]. Mapping property of $\mathfrak{M}$ was studied by Erdoğan [21], who showed that $\mathfrak{M}$ is bounded from the Sobolev space $W^{4,1/6+\epsilon}(\mathbb{R}^2)$ to $L^4(\mathbb{R}^2)$ for any $\epsilon > 0$. However, the question of whether $\mathfrak{M}$ admits a nontrivial L^p ($p \neq \infty$) bound has remained open. We prove the following result.

Theorem 1.6.1 ([43])**.** *For $p > 12$, there is a constant C such that*

$$\|\mathfrak{M}f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{1.6.1}$$

However, it was shown in [21] that (1.6.1) fails if $p \le 4$. The optimal range of p for which (1.6.2) holds remains open. We now consider a 2-parameter maximal operator

$$\mathcal{M}f(x) = \sup_{(t,s)\in\mathbb{R}^2_+} |f * \sigma^0_{t,s}(x)|.$$

$\mathcal{M}f$ is an circular analogue of the strong maximal function which is known to be bounded on L^p if $p > 1$. So, one may call $\mathcal{M}$ the *strong circular maximal* operator. The next theorem shows existence of a nontrivial L^p bound on $\mathcal{M}$. As far as the author is aware, no such result has been known before.

Theorem 1.6.2 ([43])**.** *For $p > 4$, there is a constant C such that*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}. \tag{1.6.2}$$

A modification of the argument in [21] shows that (1.6.2) fails if $p \le 3$. Whether (1.6.2) holds for $3 < p \le 4$ remains open. We also obtained multiparameter local smoothing estimates for $f * \sigma^\theta_{t,s}$ and $f * \sigma^0_{t,s}$. Following the observation from the case of the torus, we may expect $1/p$ amount of extra smoothing effect for each parameter. However, we remark that 3-parameter average does not give an extra smoothing better than 2-parameter average. These local smoothing estimates are key ingredients in the proof of Theorem 1.6.1 and Theorem 1.6.2.

1.7 Notations

We denote $x = (\bar{x}, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and similarly $\xi = (\bar{\xi}, \xi_3) \in \mathbb{R}^2 \times \mathbb{R}$. In addition to $\widehat{\ }$ and $^\vee$, we occasionally use $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier and inverse Fourier transforms, respectively. We also let $\mathbb{B}^k(p, r)$ denote the ball in $\mathbb{R}^k$ which is centered at p and of radius r. For two given nonnegative quantities A and B, we write $A \lesssim B$ if there is a constant $C > 0$ such that $B \le CA$.

In what follows, we frequently use the Littlewood-Paley decomposition. Let $\varphi \in C^\infty_c((1-2^{-13}, 2+2^{-13}))$ such that $\sum_{j=-\infty}^{\infty} \varphi(s/2^j) = 1$ for $s > 0$. We set $\varphi_j(s) = \varphi(s/2^j)$, $\varphi_{<j}(s) = \sum_{k<j} \varphi_k(s)$, and $\varphi_{>j}(s) = \sum_{k>j} \varphi_k(s)$. Then, define the projection operators

$$\widehat{\mathcal{P}_j g}(\xi) := \varphi_j(|\xi|)\widehat{g}(\xi), \quad \widehat{\mathcal{P}_{<j} g}(\xi) := \varphi_{<j}(|\xi|)\widehat{g}(\xi).$$

For a given f defined on $\mathbb{R}^3$ we define f_j^k and $f_{<j}^{<k}$ by

$$\mathcal{F}(f_j^k) = \varphi_j(|\bar{\xi}|)\varphi_k(|\xi_3|)\widehat{f}(\xi), \quad \mathcal{F}(f_{<j}^{<k}) = \varphi_{<j}(|\bar{\xi}|)\varphi_{<k}(|\xi_3|)\widehat{f}(\xi),$$

and $f_{<j}^{<k}$, $f_{<j}^k$, $f_j^{\geq k}$, $f_{<j}$, and $f^{\geq k}$, etc are similarly defined. In particular, we have $f = \sum_{j,k} f_j^k$.

Chapter 2

Preliminaries

2.1 Decoupling inequalities

"Divide and conquer" is one of main ideas in harmonic analysis. Precisely, we "divide" a function $f = \sum_j f_j$ depending on the context. Then, we usually want to estimate the L^p norm of f so that we focus on "conquering" the L^p norm of each f_j. After estimating each $\|f_j\|_{L^p}$, we need to attach each piece together. One can use the triangle inequality

$$\|\sum_j f_j\|_{L^p} \leq \sum_j \|f_j\|_{L^p}$$

and then the Hölder inequality raising the power of $\|f_j\|_{L^p}$ to combine pieces, but it makes a large constant from the Hölder inequality depending on the number of pieces. For a sharp result, we require a sharp bound but Hölder's inequality usually does not give the best estimate. In this sense, our main aim in dividing a function is obtaining the smallest constant C in the following inequalities.

$$\|\sum_j f_j\|_{L^p} \leq C(\sum_j \|f_j\|_{L^p}^2)^{\frac{1}{2}}, \tag{2.1.1}$$

$$\|\sum_j f_j\|_{L^p} \leq C(\sum_j \|f_j\|_{L^p}^p)^{\frac{1}{p}}. \tag{2.1.2}$$

We call (2.1.1) a l^2L^p-decoupling inequality (or simply l^2-decoupling) and (2.1.2) a l^pL^p-decoupling inequality (or simply l^p-decoupling). For this purpose, we need further structure on f_j. One typical structure is disjointness of Fourier support of f_j. By Plancherel's identity, we have

$$\|\sum_j f_j\|_{L^2} = (\sum_j \|f_j\|_{L^2}^2)^{\frac{1}{2}}$$

so that we do not lose anything as in the application of Hölder. However, when we consider L^p with $p > 2$, the disjointness of the Fourier support is not enough. One can assume that the Fourier supports of f_j are dyadically dispersed so that we can use the Littlewood-Paley theory and the Minkowski inequality to obtain (2.1.1) with C only depending on the dimension. The sharp constant for (2.1.2) is usually obtained by using the Hölder inequality to (2.1.1).

In many problems such as the restriction problem, the Fourier transform of a function is supported in a small neighborhood of a submanifold in a Euclidean space with curvature. We usually want to decompose this function f into a sum of f_j each of whose Fourier support is essentially the largest rectangular box so that the effect of the curvature vanishes. Under this decomposition, Wolff [81] first obtained the l^p-decoupling inequality (2.1.2) with the sharp constant C for a large p when the submanifold is a truncated light cone in $\mathbb{R}^n$. Later, a number of studies developed Wolff's result (see [48], [47], [25], [26], [8]). Finally, Bourgain and Demeter [10] proved the sharp l^2-decoupling inequality for hypersurfaces with positive definite second fundamental form and the truncated light cone. Before the statement of the theorem, we define the decomposition precisely. Let S be a hypersurface in $\mathbb{R}^n$ with positive definite second fundamental form which is a graph of a function Q_S,

$$S = \{(\xi, Q_S(\xi)) \in \mathbb{R}^n : |\xi_i| \le \frac{1}{2} \text{ for } 1 \le i \le n-1\}.$$

For $0 < \delta < 1$, let $\mathcal{N}_\delta(S)$ be the δ neighborhood of any submanifold S. We decompose $\mathcal{N}_\delta(S)$ by (essentially) rectangular boxes with dimension $\delta^{1/2} \times \cdots \times \delta^{1/2} \times \delta$ as follows. For $c \in 2\delta^{\frac{1}{2}}\mathbb{Z}^{n-1} \cap [-\frac{1}{2}, \frac{1}{2}]^{n-1}$, we define

$$\theta_c = \{(\xi, Q_S(\xi) + s) : \xi \in c + [-\delta^{\frac{1}{2}}, \delta^{\frac{1}{2}}]^{n-1},\ |s| \le 4\delta\}.$$

Then, define

$$\mathcal{P}_\delta(S) = \{\theta_c : c \in 2\delta^{\frac{1}{2}}\mathbb{Z}^{n-1} \cap [-\frac{1}{2}, \frac{1}{2}]^{n-1}\}$$

so that $\mathcal{P}_\delta(S)$ is a finitely overlapping partition of $\mathcal{N}_\delta(S)$. Now we state the l^2-decoupling theorem.

Theorem 2.1.1 (Bourgain, Demeter [10])**.** *Let S be a hypersurface in $\mathbb{R}^n$ with positive second fundamental form. If* $\operatorname{supp}\widehat{f} \subset \mathcal{N}_\delta(S)$*, then for* $p \geq \frac{2(n+1)}{n-1}$ *and* $\epsilon > 0$*, we have*

$$\|f\|_{L^p} \lesssim_\epsilon \delta^{-\frac{n-1}{4}+\frac{n+1}{2p}-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta(S)} \|f_\theta\|_{L^p}^2\Big)^{\frac{1}{2}}$$

where f_θ is the Fourier restriction of f to θ.

The decomposition of the truncated light cone

$$C^{n-1} = \{(\xi, |\xi|) : 1 \leq |\xi| \leq 2,\, \xi \in \mathbb{R}^{n-1}\}$$

is slightly different from that of hypersurfaces. Note that our decomposition divides a small neighborhood of a surface by essentially flat pieces, but for C^{n-1}, it is already flat along the radial direction. For the decomposition of $\mathcal{N}_\delta(C^{n-1})$, we use $\mathcal{P}_\delta(\mathbb{S}^{n-1})$. For $\theta \in \mathcal{P}_\delta(\mathbb{S}^{n-1})$, we define

$$\nu_\theta = \{tv : 1 \leq t \leq 2,\, v \in \theta\}.$$

Then, we define a finitely overlapping partition $\mathcal{P}_\delta(C^{n-1})$ of $\mathcal{N}_\delta(C^{n-1})$ by

$$\mathcal{P}_\delta(C^{n-1}) = \{\nu_\theta : \theta \in \mathcal{P}_\delta(\mathbb{S}^{n-1}),\, \theta \cap C^{n-1} \neq \emptyset\}.$$

The following is a consequence of Theorem 2.1.1.

Theorem 2.1.2 (Bourgain, Demeter [10], Wolff [81] for $p > 74$ and $n = 3$)**.** *Suppose* $\operatorname{supp}\widehat{f} \subset \mathcal{N}_\delta(C^{n-1})$*. Then, for* $p \geq \frac{2n}{n-2}$ *and* $\epsilon > 0$*, we have*

$$\|f\|_{L^p} \lesssim_\epsilon \delta^{-\frac{n-2}{4}+\frac{n}{2p}-\epsilon}\Big(\sum_{\nu\in\mathcal{P}_\delta(C^{n-1})} \|f_\nu\|_{L^p}^2\Big)^{\frac{1}{2}}$$

where f_ν is the Fourier restriction of f to ν.

Modifying the interpolation argument, we can also apply the interpolation to the decoupling inequality. Thus, we have the l^2-decoupling inequality for $2 \leq p < \frac{2(n+1)}{n-1}$, $2 \leq p < \frac{2n}{n-2}$ respectively with the δ term replaced by $\delta^{-\epsilon}$. Also, the above theorems are obtained by the endpoint estimate with a trivial L^∞ estimate. In addition, the sharp l^p decoupling inequality is obtained by Hölder's inequality from the l^2-decoupling inequality.

After Bourgain and Demeter's outstanding results, the decoupling inequalities have been applied to numerous problems. For a very small part of it, we refer to the references in [10]. One famous application is the local smoothing estimate for the wave operator as Wolff [81] did. Meanwhile, decoupling estimates for other surfaces are also extensively studied (see [14], [9], [11], [12], [13], [29], [28] and references therein). One interesting result is the decoupling inequality for the moment curve. It is natural to guess that a satisfactory decoupling estimate does not exist when the curve is contained in an affine subspace, for example, a parabola contained in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Thus, we consider a moment curve

$$\Gamma = \{\gamma(t) = (t, t^2, \cdots, t^n) \in \mathbb{R}^n : -1 \leq t \leq 1\}$$

which is never contained in any affine subspace. We may naturally define $\mathcal{N}_\delta(\Gamma)$ as before. However, the partition $\mathcal{P}_\delta(\Gamma)$ is quite different. We have decomposed a δ-neighborhood of surfaces into pieces such that the effect of curvature essentially vanishes in each piece. In the case of γ, the last component t^n has small curvature relative to other components. Thus, a piece of Γ with length $\delta^{-\frac{1}{n}}$ already ignores the curvature of the last component while the other components are still well curved. We define some notations for the decoupling inequality for moment curves. For $c \in 2\delta^{\frac{1}{n}}\mathbb{Z} \cap [-1,1]$, we let π_c be the parallelepiped of dimension $\delta^{\frac{1}{n}} \times \delta^{\frac{2}{n}} \times \cdots \times \delta^1$ whose sides are parallel to $\partial_t\gamma(c), \partial_t^2\gamma(c), \cdots, \partial_t^n\gamma(c)$ respectively, and center is $\gamma(c)$. Then, we define

$$\mathcal{P}_\delta(\Gamma) = \{\pi_c : c \in 2\delta^{\frac{1}{n}}\mathbb{Z} \cap [-1,1]\}.$$

The following is the optimal decoupling inequality for the moment curve from [14].

Theorem 2.1.3 (Bourgain, Demeter, Guth [14])**.** *Let* $0 < \delta < 1$ *and suppose that* $\operatorname{supp} \widehat{f_\pi} \subset \pi$ *for each* $\pi \in \mathcal{P}_\delta(\Gamma)$*. Then, for* $2 \leq p \leq \infty$ *and any* $\epsilon > 0$*, we have*

$$\|\sum_{\pi \in \mathcal{P}_\delta(\Gamma)} f_\pi\|_{L^p} \lesssim_\epsilon \delta^{\max\{0, \frac{1}{2n} - \frac{n+1}{2p}\} - \epsilon} (\sum_{\pi \in \mathcal{P}_\delta(\Gamma)} \|f_\pi\|_{L^p}^2)^{\frac{1}{2}}.$$

Indeed, the above theorem is slightly general in the sense that it implies the decoupling inequality when $\operatorname{supp} \widehat{f} \subset \mathcal{N}_\delta(\Gamma)$. Precisely, for each $\pi \in \mathcal{P}_\delta(\Gamma)$, π contains a δ-neighborhood of Γ restricted to an interval of length $\delta^{\frac{1}{n}}$. The statement in [14] is a little different from Theorem 2.1.3, but they are essentially equivalent. Also, Hölder's inequality to Theorem 2.1.3 gives the sharp l^p-decoupling inequality as before. By the way, in the l^p-decoupling inequality for the moment curve, recall that we lose the effect of curvature in the last component t^n. However, we still have a well curved curve when we project the curve in $\mathbb{R}^{n-1}$. Thus, we may apply the decoupling inequality for lower dimensions to further divide each piece smaller. This is one of the main ideas in Chapter 5.

The essential structure in the decoupling inequality is the scaling structure. It plays an important role not only in the proof of the decoupling inequality, but also implying interesting consequences of the decoupling inequality. The first consequence is the conical extension of the decoupling inequality. Theorem 2.1.2 can be seen as the conical extension of the decoupling for a circle. Modifying an argument in [10], we can conically extend decoupling estimates (see [4]). The second consequence is that we can generalize the decoupling estimates to variable coefficient settings. In this thesis, we are more interested in this part. Beltran, Hickman, Sogge [6] first obtained a variable coefficient variation of the decoupling inequality. We prove a variable coefficient generalization of the decoupling inequality for a conic extension of the moment curve in Chapter 5 using the argument in [6].

2.2 Local smoothing estimates of the wave operator

As mentioned in the previous section, the local smoothing estimate of the wave operator is one of the most famous applications of decoupling. The wave operator in $\mathbb{R}^{n+1}$ is defined by the following.

$$\mathcal{W}_{\pm}f(x,t) = e^{\pm it\sqrt{-\Delta}}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi \pm t|\xi|)}\widehat{f}(\xi)d\xi.$$

This operator is deeply related to $A_t^{\mathbb{S}^{n-1}}$. Indeed,

$$A_t^{\mathbb{S}^{n-1}}f(x) = \int e^{ix\cdot\xi}\widehat{f}(\xi)\widehat{d\sigma_n}(t\xi)d\xi$$

where $d\sigma_n$ is the Lebesgue measure on $\mathbb{S}^{n-1}$. Using the asymptotic formula of the Bessel function,

$$\widehat{d\sigma_n}(\xi) \approx c_+e^{i|\xi|}(1+|\xi|)^{-\frac{n-1}{2}} + c_-e^{-i|\xi|}(1+|\xi|)^{-\frac{n-1}{2}}$$

holds. We see the relation between $A_t^{\mathbb{S}^{n-1}}$ and $\mathcal{W}_\pm$ in detail in later chapters.

Now we define some notations. We denote

$$\mathbb{A}_\lambda = \{\eta \in \mathbb{R}^2 : 2^{-1}\lambda \le |\eta| \le 2\lambda\}, \quad \mathbb{A}_\lambda^\circ = \{\eta \in \mathbb{R}^2 : |\eta| \le 2\lambda\},$$

respectively. Similarly, we set $\mathbb{I} = [1,2]$ and $\mathbb{I}^\circ = [0,2]$, and we denote $\mathbb{I}_\tau = \tau\mathbb{I}$ and $\mathbb{I}_\tau^\circ = \tau\mathbb{I}^\circ$ for $\tau \in (0,1]$. Then, the following conjecture is equivalent to the local smoothing conjecture for $A_t^{\mathbb{S}^{n-1}}$ introduced in Chapter 1.3, up to the endpoints.

Conjecture 1 ([71])**.** *Let $p \ge 2$ and $\lambda \ge 1$. Then,*

$$\|\mathcal{W}_\pm g\|_{L^p(\mathbb{R}^n \times \mathbb{I}^\circ)} \lesssim_\epsilon \lambda^{\max\{\frac{n-1}{2} - \frac{n}{p}, 0\} + \epsilon} \|g\|_{L^p}$$

holds for any $\epsilon > 0$ whenever $\operatorname{supp} \widehat{g} \subset \mathbb{A}_\lambda$.

As mentioned already, Conjecture 1 was solved by Guth, Wang, and Zhang[30] when $n = 2$ while it is known only for $p \ge \frac{2(n+1)}{n-1}$ when $n \ge 3$. Using an interpolation argument, we get the following consequence.

Theorem 2.2.1 (Guth, Wang, Zhang[30], see also [68], [42])**.** *Let $2 \le p \le q$, $1/p + 3/q \le 1$, and $\lambda \ge 1$. Then, the estimate*

$$\big\|\mathcal{W}_\pm g\big\|_{L^q(\mathbb{R}^2 \times \mathbb{I}^\circ)} \le C\lambda^{(\frac{1}{2} + \frac{1}{p} - \frac{3}{q}) + \epsilon} \|g\|_{L^p} \tag{2.2.1}$$

holds for any $\epsilon > 0$ whenever $\operatorname{supp} \widehat{g} \subset \mathbb{A}_\lambda$.

Proof. It is sufficient to show the estimate for $\mathcal{W}_+$ since that for $\mathcal{W}_-$ follows by conjugation and reflection. When the interval $\mathbb{I}^\circ$ is replaced by $\mathbb{I}$, the desired estimate follows from the known estimates and interpolation. Indeed, for $1 \le p \le q \le \infty$ and $1/p + 3/q \le 1$, we have

$$\big\|\mathcal{W}_+ g\big\|_{L^q(\mathbb{R}^2 \times \mathbb{I})} \le C\lambda^{\frac{1}{2} + \frac{1}{p} - \frac{3}{q} + \epsilon} \|g\|_{L^p} \tag{2.2.2}$$

whenever $\operatorname{supp} \widehat{g} \subset \mathbb{A}_\lambda$. This is a consequence of interpolation between the sharp L^p local smoothing estimates for $p = q \ge 4$ ([30]) and the estimate $\|\mathcal{W}_+ g\|_{L^\infty(\mathbb{R}^2 \times \mathbb{I})} \le C\lambda^{\frac{3}{2}} \|g\|_{L^1}$ (e.g., see [72]).

By dyadically decomposing $\mathbb{I}^\circ$ away from 0 and scaling, one can deduce (2.2.1) from (2.2.2). Indeed, since

$$\mathcal{W}_+ g(x, \tau t) = \mathcal{W}_+ g(\tau\, \cdot)(x/\tau, t), \tag{2.2.3}$$

rescaling gives the estimate

$$\big\|\mathcal{W}_+ g\big\|_{L^q(\mathbb{R}^2\times\mathbb{I}_\tau)} \le C\tau^{\frac12-\frac1p}\lambda^{\frac12+\frac1p-\frac3q+\epsilon}\|g\|_{L^p}$$

for any $\epsilon>0$ if $\operatorname{supp}\widehat{g}\subset\mathbb{A}_\lambda$ and $\tau\lambda\gtrsim 1$. When $\tau\sim\lambda^{-1}$, by scaling and an easy estimate we also have $\big\|\mathcal{W}_+ g\big\|_{L^q(\mathbb{R}^2\times\mathbb{I}^\circ_\tau)}\lesssim\lambda^{2/p-3/q}\|g\|_p$. Now, since $p\ge 2$, decomposing $\mathbb{I}^\circ=(\bigcup_{\tau\ge(2\lambda)^{-1}}\mathbb{I}^\circ_\tau)\cup\mathbb{I}^\circ_{\lambda^{-1}}$ and taking sum over the intervals, we get

$$\big\|\mathcal{W}_+ g\big\|_{L^q(\mathbb{R}^2\times\mathbb{I}^\circ)} \le C\max\{\lambda^{\frac12+\frac1p-\frac3q+\epsilon},\lambda^{\frac2p-\frac3q}\}\|g\|_{L^p}\lesssim\lambda^{\frac12+\frac1p-\frac3q+\epsilon}\|g\|_{L^p}$$

for any $\epsilon>0$. □

As a consequence of Theorem 2.2.1 we also have the next lemma, which we use later to obtain estimates for functions whose Fourier supports are included in a conical region with a small angle.

Lemma 2.2.2. *Let $2\le p\le q\le\infty$, $1/p+3/q\le 1$, and $\lambda\ge 1$. Suppose that $\lambda\lesssim h\lesssim\lambda^2$. Then, for any $\epsilon>0$ there is a constant C such that*

$$\big\|\mathcal{W}_\pm g\big\|_{L^q(\mathbb{R}^2\times\mathbb{I}^\circ)} \le C\lambda^{1-\frac1p-\frac3q}h^{\frac2p-\frac12+\epsilon}\|g\|_{L^p} \tag{2.2.4}$$

whenever $\operatorname{supp}\widehat{g}\subset\mathbb{I}_h\times\mathbb{I}^\circ_\lambda$.

Proof. As before, it is sufficient to consider $\mathcal{W}_+$. By interpolation we only need to check the estimate (2.2.4) for $(p,q)=(4,4)$, $(2,6)$, $(2,\infty)$, and (∞,∞). Since $\lambda\le h$, $\operatorname{supp}\widehat{g}\subset\{\eta:|\eta|\sim h\}$. So, the estimate (2.2.4) for $(p,q)=(4,4)$, $(2,6)$ is clear from (2.2.1). Since $\operatorname{supp}\widehat{g}\subset\mathbb{I}_h\times\mathbb{I}^\circ_\lambda$, the estimate (2.2.4) for $(2,\infty)$ follows by the Cauchy-Schwarz inequality and Plancherel's theorem.

It now remains to show (2.2.4) for $p=q=\infty$, that is to say,

$$\|\mathcal{W}_+ g\|_{L^\infty(\mathbb{R}^2\times\mathbb{I}^\circ)}\lesssim\lambda h^{-1/2}\|g\|_{L^\infty}$$

whenever $\operatorname{supp}\widehat{g}\subset\mathbb{I}_h\times\mathbb{I}^\circ_\lambda$. To show this, we cover $\mathbb{I}_h\times\mathbb{I}^\circ_\lambda$ by as many as $C\lambda h^{-1/2}$ boundedly overlapping rectangles of dimension $h\times h^{1/2}$ whose principal axis contains the origin, and consider a partition of unity $\{\tilde\omega_\nu\}$ subordinated to those rectangles such that (α,β)-th derivatives of $\tilde\omega_\nu$ in the directions of the principal and its normal directions is bounded by $Ch^{-\alpha}h^{-\beta/2}$. (In fact, one can also use $\omega_\nu(\eta)$ in the proof of Proposition 4.3.1 below replacing λ by h.) Consequently, we have $\mathcal{W}_+ g=\sum_\nu\mathcal{W}_+\chi_\nu(D)g$. It is easy to see that the kernel of the operator $g\to\mathcal{W}_+\chi_\nu(D)g$ has a uniformly bounded L^1 norm for $t\in\mathbb{I}^\circ,\nu$. Therefore, we get the desired estimate. □

Chapter 3

The circular maximal operators on Heisenberg radial functions

Following the outstanding development for the spherical maximal operators, there was a huge amount of literature concerning various maximal operators. One such attempt is replacing $\mathbb{R}^n$ with some noncommutative spaces. Dealing with fully general spaces is very difficult, but it is available when we consider a relatively simple case, two-step nilpotent groups. The most famous and simple example of the two-step nilpotent group is the Heisenberg group $\mathbb{H}^n$.

As introduced in Chapter 1, we study the operator

$$M_{\mathbb{H}^n} f(x, x_{2n+1}) = \sup_{t>0} |f *_{\mathbb{H}} d\sigma_t(x, x_{2n+1})|$$

when $n = 1$ on the space of the Heisenberg radial functions. Recall that a function $f : \mathbb{H}^1 \to \mathbb{C}$ is Heisenberg radial if $f(x, x_3) = f(Rx, x_3)$ for all $R \in SO(2)$. This type of maximal function was first introduced by Nevo and Thangavelu in [58]. A few years later, Müller and Seeger [55], and Narayanan and Thangavelu [57] independently proved that for $n \geq 2$, $M_{\mathbb{H}^n}$ is bounded on $L^p(\mathbb{H}^n)$ if and only if $p > 2n/(2n-1)$ while Nevo and Thangavelu in [58] only showed a non-optimal range. Indeed, in [55], authors proved analogous estimates for general two-step nilpotent Lie groups (see also [1]). Later, Roos, Seeger, Srivastava [63] obtained sharp L^p-improving estimates for $M_{\mathbb{H}^n}$ up to some endpoints when $n \geq 2$ (see also [38]).

However, the problem becomes very difficult when $n = 1$. There is no result for the boundedness of $M_{\mathbb{H}^1}$ on L^p for any $1 < p < \infty$. As we mentioned already, Beltran, et al [3] proved that $M_{\mathbb{H}^1}$ is bounded on the space of Heisenberg radial functions when $p > 2$. Though the Heisenberg radial assumption significantly simplifies the structure of the averaging operator, the associated defining function of the averaging operator is still lacking of curvature properties. In fact, the defining function has vanishing rotational and cinematic curvatures at some points, see [3] for a detailed discussion. This increases the complexity of the problem. To overcome the issue of vanishing curvatures, Beltran, et al. [3] used the oscillatory integral operators with two-sided fold singularities and the variable coefficient version of local smoothing estimate ([6]) combined with additional localization.

We recall the main theorem of this chapter concerned with L^p-improving estimates for $M^c_{\mathbb{H}^1}$.

Theorem 3.0.1 ([41])**.** *Let $P_0 = (0,0), P_1 = (1/2,1/2)$, and $P_2 = (3/7,2/7)$, and let $\mathbf{T}$ be the closed region bounded by the triangle $\Delta P_0P_1P_2$. Suppose $(1/p,1/q) \in \{P_0\} \cup (\mathbf{T} \setminus (\overline{P_1P_2} \cup \overline{P_0P_2}))$. Then, the estimate*

$$\|M^c_{\mathbb{H}^1} f\|_q \lesssim \|f\|_{L^p} \tag{3.0.1}$$

holds for all Heisenberg radial function f. Conversely, if $(1/p,1/q) \notin \mathbf{T}$, then the estimate fails.

Our approach is quite different from that in [3]. Capitalizing on the Heisenberg radial assumption, we make a change of variables so that the averaging operator on the Heisenberg radial function takes a form close to the circular average. While the defining function of the consequent operator still does not have nonvanishing rotational and cinematic curvatures, via a further change of variables we can apply the L^p–L^q local smoothing estimate of the circular maximal operator in a more straightforward manner by exploiting the apparent connection to the wave operator. Consequently, our approach also provides a simplified proof of the result due to Beltran, et al [3].

Even though we utilize the local smoothing estimate, we do not need to use the full strength of the local smoothing estimate in $d = 2$ since we only need the sharp L^p–L^q local smoothing estimates for (p,q) near $(7/3,7/2)$. Such estimates can also be obtained by interpolation and scaling argument if one uses the trilinear restriction estimates for the cone and the sharp local smoothing estimate for some large p (for example, see [46]).

The estimate (3.0.1) remains open when $(1/p,1/q) \in (\overline{P_1P_2} \cup \overline{P_0P_2}) \setminus \{P_0,P_1\}$. However, we expect that those borderline cases should be true. Most of the corresponding endpoint estimates for the circular maximal function (in $\mathbb{R}^2$) are known to be true ([44]), but to implement the approach in [44] we need the local smoothing estimate without ϵ-loss regularity, which we are not able to establish yet even under the Heisenberg radial assumption.

3.1 Heisenberg radial functions and main estimates

Since f is a Heisenberg radial function, we have $f(x, x_3) = f_0(|x|, x_3)$ for some f_0. Let us set

$$g(s, z) = f_0(\sqrt{2s}, z), \quad s \geq 0.$$

Then, it follows $f(x, x_3) = g(|x|^2/2, x_3)$. Since $f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = \int f(r - ty_1, -ty_2, x_3 - try_2)d\sigma(y) = \int g(\frac{r^2+t^2}{2} - try_1, x_3 - try_2)d\sigma(y)$, we have

$$f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = g * d\sigma_{tr}\Big(\frac{r^2 + t^2}{2}, x_3\Big). \tag{3.1.1}$$

Let us define an operator $\mathcal{A}_t$ by

$$\mathcal{A}_t g(r, x_3) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\frac{r^2+t^2}{2}\xi_1 + x_3\xi_2)} \widehat{d\sigma}(tr\xi)\, \widehat{g}(\xi) d\xi. \tag{3.1.2}$$

Using Fourier inversion, we have

$$f *_{\mathbb{H}} d\sigma_t(r, 0, x_3) = \mathcal{A}_t g(r, x_3). \tag{3.1.3}$$

Since $f *_{\mathbb{H}} d\sigma_t$ is also Heisenberg radial,* $\|M_{\mathbb{H}^1} f\|_q^q = \int |M_{\mathbb{H}^1} f(r, 0, x_3)|^q r dr dx_3$. A computation shows $\|f\|_{L^p_{x,x_3}} = \|g\|_{L^p_{r,x_3}}$. Therefore, we see that the estimate (3.0.1) is equivalent to

$$\Big\| r^{\frac{1}{q}} \sup_{1<t<2} |\mathcal{A}_t g| \Big\|_{L^q_{r,x_3}} \leq C\|g\|_p. \tag{3.1.4}$$

In what follows we show (3.1.4) holds for p, q satisfying

$$p \leq q, \;\; 3/p - 1/q < 1, \;\; 1/p + 2/q > 1. \tag{3.1.5}$$

*This is true because SO(2) is an abelian group. However, SO(n) is not commutative in general, so the property is not valid in higher dimensions.

Then, interpolation with the trivial L^∞ estimate proves Theorem 1.4.1.

To show (3.1.4) we decompose $\mathcal{A}_t$ as follows:

$$\mathcal{A}_t g(r, x_3) = \sum_{k\in\mathbb{Z}} \varphi_k(r)\mathcal{A}_t g(r, x_3).$$

We break g via the Littlewood-Paley decomposition and try to obtain estimates for each decomposed pieces.

Our proof of (3.1.4) mainly relies on the following two propositions, which we prove in Chapter 3.4.

Proposition 3.1.1. *Let $|k| \geq 2$ and $j \geq -k$. Suppose*

$$p \leq q,\ \ 1/p + 1/q \leq 1,\ \ 1/p + 3/q \geq 1. \tag{3.1.6}$$

Then, for $\epsilon > 0$ we have

$$\Big\| \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t\mathcal{P}_j g| \Big\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)+\frac{k}{q}-\frac{2k}{p}} \|g\|_{L^p}, & k \geq 2, \\ 2^{(j+k)(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)+\frac{2k}{q}-\frac{2k}{p}} \|g\|_{L^p}, & k < -2. \end{cases} \tag{3.1.7}$$

The estimate (3.1.7) continues to be valid for the case $k = -1, 0, 1$. However, the range of p, q for which (3.1.7) holds gets smaller.

Proposition 3.1.2. *Let $j \geq -1$ and $k = -1, 0, 1$. Suppose $p \leq q$, $1/p+1/q < 1$ and $1/p + 2/q > 1$. Then, for $\epsilon > 0$ we have*

$$\Big\| \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t\mathcal{P}_j g| \Big\|_{L^q_{r,x_3}} \lesssim 2^{\frac{j}{2}(\frac{3}{p}-\frac{1}{q}-1)+\epsilon j} \|g\|_{L^p}.$$

We frequently use the following elementary lemma (for example, see [44]) which plays the role of the Sobolev imbedding.

Lemma 3.1.3. *Let I be an interval and let F be a smooth function defined on $\mathbb{R}^n \times I$. Then, for $1 \leq p \leq \infty$,*

$$\Big\| \sup_{t\in I} |F(x,t)| \Big\|_{L^p(\mathbb{R}^n)} \lesssim |I|^{-\frac{1}{p}} \|F\|_{L^p(\mathbb{R}^n\times I)} + \|F\|^{\frac{(p-1)}{p}}_{L^p(\mathbb{R}^n\times I)} \|\partial_t F\|^{\frac{1}{p}}_{L^p(\mathbb{R}^n\times I)}.$$

3.2 Local maximal estimates

We prove (3.1.4) handling the three cases $k \le -2$, $|k| \le 1$, and $k \ge 2$, separately. We first consider a change of variables

$$(r, x_3, t) \to (y_1, y_2, \tau) := \left(\frac{r^2 + t^2}{2}, x_3, rt\right), \tag{3.2.1}$$

which plays an important role in what follows. Note that

$$\det \frac{\partial(y_1, y_2, \tau)}{\partial(r, x_3, t)} = r^2 - t^2. \tag{3.2.2}$$

In order to show (3.1.4), we shall use the change of variables (3.2.1) to apply the local smoothing estimate to the averaging operator $\mathcal{A}_t$ (see Proposition 3.4.1). Since $1 < t < 2$, $|\det \partial(y_1, y_2, \tau)/\partial(r, x_3, t)| = |r^2 - t^2| \sim \max(2^{2k}, 1)$ for $|k| \ge 2$. Thus, the cases $|k| \ge 2$ can be handled directly by using local smoothing estimates for the half wave propagator. However, the determinant of the Jacobian may vanish when $|k| \le 1$. This requires further decomposition away from the set $\{r = t\}$. See Chapter 3.6. This is why we need to consider the three cases separately.

Let us set $g_k = \mathcal{P}_{<-k}\, g$ and $g^k = g - \mathcal{P}_{<-k} g$ so that $g = g_k + g^k$. Then, we break

$$\varphi_k(r)\mathcal{A}_t g = \varphi_k(r)\mathcal{A}_t g_k + \varphi_k(r)\mathcal{A}_t g^k. \tag{3.2.3}$$

We use Proposition 3.1.1 and Proposition 3.1.2 to obtain the estimate for $\varphi_k(r)\mathcal{A}_t g^k$, whereas we show the estimate for $\varphi_k(r)\mathcal{A}_t g_k$ by elementary means using (3.1.2).

Case $k \le -2$

We claim that

$$\Big\| r^{\frac{1}{q}} \sum_{k \le -2} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g| \Big\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{3.2.4}$$

holds provided that p, q satisfy $2/p < 3/q$, $3/p - 1/q < 1$, and (3.1.6). Thus (3.2.4) holds for p, q satisfying (3.1.5).

We first consider $\varphi_k(r)\mathcal{A}_t g_k$. We shall show that

$$\Big\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g_k| \Big\|_{L^q_{r,x_3}} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p} \tag{3.2.5}$$

holds for $1 \le p \le q \le \infty$. We recall (3.1.2) and note that $\partial_t(\widehat{d\sigma}(tr\xi))$ is uniformly bounded because $|r\xi| \lesssim 1$. Since $\operatorname{supp}\widehat{g_k} \subset \{\xi : |\xi| \le C2^{-k}\}$ and $\partial_t e^{\frac{r^2+t^2}{2}\xi_1} = t\xi_1 e^{\frac{r^2+t^2}{2}\xi_1}$, we have $\|\varphi_k(r)\partial_t \mathcal{A}_t g_k\|_q \lesssim 2^{-k}\|\varphi_k(r)\mathcal{A}_t g_k\|_q$ by the Mikhlin multiplier theorem. Applying Lemma 4.5.1 to $\varphi_k(r)\mathcal{A}_t g_k$, we see that (3.2.5) follows if we show

$$\|\varphi_k(r)\mathcal{A}_t g_k\|_{L^q_{r,x_3,t}(\mathbb{R}^2\times[1,2])} \lesssim 2^{\frac{3k}{q}-\frac{2k}{p}}\|g\|_{L^p}. \tag{3.2.6}$$

We now make use of the change of variables (3.2.1). Since $k \le -2$ and $t \in [1,2]$, we have $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 1$. Thus the left hand side of (3.2.6) is bounded by

$$C\Big\|\varphi_k(r(y_1,y_2,\tau))\int e^{iy\cdot\xi}\,\widehat{g}(\xi)\widehat{d\sigma}(\tau\xi)\varphi_{<-k}(\xi)d\xi\Big\|_{L^q_{y,\tau}(\mathbb{R}^2\times[2^{-1},2^2])}.$$

Changing variables $\xi \to 2^{-k}\xi$ and $(y,\tau) \to (2^k y, 2^k\tau)$ gives

$$\|\varphi_k(r)\mathcal{A}_t g_k\|_{L^q_{r,x_3,t}(\mathbb{R}^2\times[1,2])} \lesssim 2^{\frac{3k}{q}}\Big\|\int e^{iy\cdot\xi}\,\mathfrak{m}(\xi)\widehat{g(2^k\cdot)}(\xi)d\xi\Big\|_{L^q_{y,\tau}(\mathbb{R}^2\times[2^{-1},2^2])},$$

where $\mathfrak{m}(\xi) = \widehat{d\sigma}(\tau\xi)\varphi_{<0}(\xi)$. Since $\tau \sim 1$ and $\varphi_{<0}(\xi)$ is a smooth function supported in the set $\{\xi : |\xi| \lesssim 1\}$, $\mathfrak{m}(\xi)$ is a smooth multiplier whose derivatives are uniformly bounded. So, the multiplier operator given by $\mathfrak{m}$ is uniformly bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $\tau \in [2^{-1},2^2]$. Thus, via scaling we obtain (3.2.6) and, hence, (3.2.5).

Using the triangle inequality and (3.2.5), we have

$$\Big\|r^{\frac{1}{q}}\sup_{1<t<2}\sum_{k\le -2}|\varphi_k(r)\mathcal{A}_t g_k|\Big\|_{L^q_{r,x_3}} \lesssim \Big(\sum_{k\le -2}2^{\frac{3k}{q}-\frac{2k}{p}}\Big)\|g\|_p \lesssim \|g\|_p$$

because $2/p < 3/q$. We now consider $\varphi_k(r)\mathcal{A}_t g^k$ for which we use Proposition 3.1.1. Since

$$\Big\|r^{\frac{1}{q}}\sup_{1<t<2}\sum_{k\le -2}|\varphi_k(r)\mathcal{A}_t g^k|\Big\|_{L^q_{r,x_3}} \le \sum_{k\le -2}\sum_{j\ge -k}\Big\|r^{\frac{1}{q}}\sup_{1<t<2}|\varphi_k(r)\mathcal{A}_t\mathcal{P}_j g|\Big\|_{L^q_{r,x_3}}$$

and since p, q satisfy $3/p - 1/q < 1$, $2/p < 3/q$, and (3.1.6), using the estimate (3.1.7), we get

$$\Big\|r^{\frac{1}{q}}\sup_{1<t<2}\sum_{k\le -2}|\varphi_k(r)\mathcal{A}_t g^k|\Big\|_{L^q_{r,x_3}} \lesssim \Big(\sum_{k\le -2}2^{\frac{3k}{q}-\frac{2k}{p}}\Big)\|g\|_p \lesssim \|g\|_p.$$

Combining this with the above estimate for $g \to \varphi_k(r)\mathcal{A}_t g^k$ gives (3.2.4) and this proves the claim.

Case $k \geq 2$

In this case we show

$$\Big\| r^{\frac{1}{q}} \sum_{k\geq 2} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g| \Big\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{3.2.7}$$

if $p \leq q$, $3/p - 1/q < 1$, and (3.1.6) holds. So, we have (3.2.7) if (3.1.5) holds.

In order to prove (3.2.7) we first prove the following.

Lemma 3.2.1. *Let $k \geq -1$. If $|t| \lesssim 1$ and $0 \leq s \lesssim 2^{2k}$, then*

$$|\mathcal{A}_t \mathcal{P}_{<-k} g|(\sqrt{2s}, x_3) \lesssim \mathcal{E}_k^N * |g|(s, x_3), \tag{3.2.8}$$

where $\mathcal{E}_\ell^N(y) = 2^{-2\ell}(1 + 2^{-\ell}|y|)^{-N}$.

Proof. We note that

$$\mathcal{A}_t \mathcal{P}_{<-k} g(\sqrt{2s}, x_3) = K * g\big(s + 2^{-1}t^2, x_3\big),$$

where

$$K(y) = \frac{1}{(2\pi)^2} \int e^{iy\cdot\xi} \varphi_{<-k}(\xi) \widehat{d\sigma}(t\sqrt{2s}\xi) d\xi.$$

We note $\partial_\xi^\alpha[\varphi_{<-k}(2^{-k}\xi)\widehat{d\sigma}(2^{-k}t\sqrt{2s}\xi)] = O(1)$ since $s \lesssim 2^{2k}$. Thus, changing variables $\xi \to 2^{-k}\xi$, by integration by parts we have $|K| \lesssim \mathcal{E}_k^N$ for any $N > 0$. Since $|t| \lesssim 1$ and $k \geq -1$, we see $\mathcal{E}_k^N(y_1 + 2^{-1}t^2, y_2) \lesssim \mathcal{E}_k^N(y_1, y_2)$. Therefore, we get (3.2.8). ☐

Proof of (3.2.7). We begin by observing a localization property of the operator $\mathcal{A}_t$. From (3.1.1) we note that

$$\frac{r^2 + t^2}{2} - try_1 \subset I_k := [2^{2k-1}(1 - 10^{-2}), 2^{2k+1}(1 + 10^{-2})]$$

for $r \in \operatorname{supp}\varphi_k$ if k is large enough, i.e., $2^{-k} \leq 10^{-3}$. Thus, from (3.1.1) and (3.1.3) we see that

$$\varphi_k(r)\mathcal{A}_t g(r, x_3) = \varphi_k(r)\mathcal{A}_t([g]_k)(r, x_3) \tag{3.2.9}$$

where $[g]_k(r, x_3) = \chi_{I_k}(r)g(r, x_3)$. Clearly, the intervals I_k are finitely overlapping and so are the supports of φ_k. Since $p \leq q$, by a standard localization argument it is sufficient for (3.2.7) to show

$$\Big\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g| \Big\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p} \tag{3.2.10}$$

for $k \geq 2$.

Using the decomposition (3.2.3), we first consider $\varphi_k(r)\mathcal{A}_t g_k$. Changing variables $r \mapsto \sqrt{2s}$, we have

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}}^q \lesssim \int \varphi_k(\sqrt{2s}) \left(\sup_{1<t<2} |\mathcal{A}_t g_k(\sqrt{2s}, x_3)| \right)^q ds dx_3.$$

Since $1 < t < 2$, $k \geq 2$, and $g_k = \mathcal{P}_{<-k} g$, by Lemma 3.2.1 $|\mathcal{A}_t g_k(\sqrt{2s}, x_3)| \lesssim \mathcal{E}_k^N * |g|(s, x_3)$. Hence,

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim \|\mathcal{E}_k^N * |g|\|_{L^q_{s,x_3}} \lesssim 2^{2k(1/q-1/p)} \|g\|_p \leq \|g\|_p.$$

The second inequality follows by Young's convolution inequality and the third is clear because $k \geq 2$ and $p \leq q$. We now handle $\varphi_k(r)\mathcal{A}_t g^k$. Since

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \leq \sum_{j \geq -k} \left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t \mathcal{P}_j g| \right\|_{L^q_{r,x_3}} \tag{3.2.11}$$

and since $3/p - 1/q < 1$, $p \leq q$, and (3.1.6) holds, using the estimate (3.1.7), we get

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g^k| \right\|_{L^q_{r,x_3}} \lesssim 2^{\frac{2k}{q} - \frac{2k}{p}} \|g\|_p \lesssim \|g\|_p.$$

Therefore, we get (3.2.10). □

3.2.1 Case $|k| \leq 1$

To complete the proof of (3.1.4), the matter is now reduced to obtaining

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_{L^p}, \quad k = -1, 0, 1$$

if p, q satisfy (3.1.5). In order to show this we use Proposition 3.1.2. Using the decomposition (3.2.3), we first consider $\varphi_k(r)\mathcal{A}_t g_k$. Since $1 < t < 2$ and $|k| \leq 1$, by Lemma 3.2.1 we have $\varphi_k(r)|\mathcal{A}_t g_k| \lesssim \mathcal{E}_0^N * |g|$. Hence, it follows that

$$\left\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g_k| \right\|_{L^q_{r,x_3}} \lesssim \|g\|_p$$

for $1 \leq p \leq q \leq \infty$.

We now consider $\varphi_k(r)\mathcal{A}_t g^k$. Note that (3.1.6) is satisfied if (3.1.5) holds. Since $3/p - 1/q < 1$, by (3.2.11) and Proposition 3.1.2 we see

$$\Big\| r^{\frac{1}{q}} \sup_{1<t<2} |\varphi_k(r)\mathcal{A}_t g^k| \Big\|_{L^q_{r,x_3}} \lesssim \sum_{j\geq -k} 2^{\frac{j}{2}(\frac{3}{p}-\frac{1}{q}-1)+\epsilon j} \|g\|_{L^p} \lesssim \|g\|_p$$

taking a small enough $\epsilon > 0$. Therefore we get the desired estimate.

3.3 Global maximal estimates

Using the estimates in the previous section, one can provide a simpler proof of the result due to Beltran et al. [3], i.e.,

$$\|r^{\frac{1}{p}} \sup_{0<t<\infty} |\mathcal{A}_t g|\|_{L^p_{r,x_3}} \leq C\|g\|_p \tag{3.3.1}$$

for $2 < p \leq \infty$. In order to show this we use the following lemma which is a consequence of Proposition 3.1.1 and 3.1.2.

Lemma 3.3.1. *Let $2 \leq p \leq 4$. Then, for some $c > 0$ we have*

$$\big\| r^{\frac{1}{p}} \sup_{1<t<2} |\mathcal{A}_t \mathcal{P}_j g| \big\|_{L^p_{r,x_3}} \leq C 2^{-cj} \|g\|_p \,. \tag{3.3.2}$$

Proof. We briefly explain how one can show (3.3.2). In fact, similarly as before, we decompose

$$\mathcal{A}_t \mathcal{P}_j g = S_1 + S_3 + S_3 + S_4,$$

where

$$S_1 := \sum_{k<-j} \varphi_k(r)\mathcal{A}_t\mathcal{P}_j g,\ S_2 := \sum_{-j\leq k\leq -2} \varphi_k(r)\mathcal{A}_t\mathcal{P}_j g,\ S_3 := \sum_{-1\leq k\leq 1} \varphi_k(r)\mathcal{A}_t\mathcal{P}_j g,$$

and $S_4 = \mathcal{A}_t\mathcal{P}_j g - S_1 - S_2 - S_3$. Then, the estimate (3.3.2) follows if we show $\|r^{\frac{1}{p}} \sup_{1<t<2} |S_\ell|\|_{L^p_{r,x_3}} \leq C2^{-cj}\|g\|_p$, $\ell = 1, 2, 3, 4$ for some $c > 0$. The estimate for S_1 follows from (3.2.5) and summation over $k < -j$. Using the estimate of the second case in (3.1.7), one can easily get the estimate for S_2. The estimate for S_3 is obvious from Proposition 3.1.2. By Proposition 3.1.1 combined with the localization property (3.2.9) we can obtain the estimate for S_4. However, due to the projection operator $\mathcal{P}_j$ we need to modify the previous argument slightly.

From (3.1.1) and (3.1.3) we see

$$\mathcal{A}_t\mathcal{P}_j g(r,x_3) = \iint g(z_1,z_2)K_j\Big(\frac{r^2+t^2}{2} - z_1 - try_1, x_3 - z_2 - try_2\Big)d\sigma(y)dz, \tag{3.3.3}$$

where $K_j = \mathcal{F}^{-1}(\varphi(2^{-j}|\cdot|))$. Note that $|K_j| \lesssim E^N_{-j}$ for any N and $k \geq 2$. If $r \in \operatorname{supp}\varphi_k$, $\sqrt{2z_1} \notin I_k$, and k is large enough, then we have

$$\Big|K_j\Big(\frac{r^2+t^2}{2} - try_1 - z_1, x_3 - try_2 - z_2\Big)\Big| \lesssim 2^{-(2k+j)N}\Big(1+2^j|r^2-2z_1|+2^{-k}|x_3-z_2|\Big)^{-N}$$

for any N since $|2^{-1}r^2 - z_1| \gtrsim 2^{2k}$ and $|rty| \lesssim 2^k$. Hence it follows that

$$\|r^{\frac{1}{p}}\varphi_k(r)\mathcal{A}_t\mathcal{P}_j(1-\chi_{I_k})g\|_p \leq C2^{-(k+j)N}\|g\|_p, \quad 1 \leq p \leq \infty$$

for any N. We break $\mathcal{A}_t\mathcal{P}_j g = \mathcal{A}_t\mathcal{P}_j\chi_{I_k}g + \mathcal{A}_t\mathcal{P}_j(1-\chi_{I_k})g$. Using the last inequality and then Proposition 3.1.1, we obtain

$$\|S_4\|_p \leq \Big(\sum_{k\geq 2}\|r^{\frac{1}{p}}\varphi_k(r)\mathcal{A}_t\mathcal{P}_j\chi_{I_k}g\|_p^p\Big)^{\frac{1}{p}} + \sum_{k\geq 2}2^{-(k+j)N}\|g\|_p \lesssim 2^{-cj}\|g\|_p$$

for some $c > 0$ by taking an N large enough. □

Once we have (3.3.2), using a standard argument which relies on the Littlewood-Paley decomposition and rescaling (for example, see [7, 66, 3]) one can easily show (3.3.1). Indeed, we break the maximal function into high and lower frequency parts:

$$\sup_{0<t<\infty}|\mathcal{A}_t g| \leq \mathcal{A}_{low}\, g + \mathcal{A}_{high}\, g,$$

where

$$\mathcal{A}_{low}\, g = \sup_l \sup_{2^l\leq t<2^{l+1}} |\mathcal{A}_t\mathcal{P}_{<-2l}g|,$$
$$\mathcal{A}_{high}\, g = \sum_{k\geq 0}\sup_l \sup_{2^l\leq t<2^{l+1}} |\mathcal{A}_t\mathcal{P}_{k-2l}g|.$$

For $\mathcal{A}_{low}\, g$ we claim

$$\sup_{2^l \le t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1} r^2, x_3). \tag{3.3.4}$$

This gives $\mathcal{A}_{low}\, g(r, x_3) \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1} r^2, x_3)$. Since $\mathcal{M}_{\mathbb{R}^2}$ is bounded on L^p for $p > 2$, for $2 < p \le \infty$ we get

$$\|r^{\frac{1}{p}} \mathcal{A}_{low}\, g\|_{L^p_{r,x_3}} \le C \|g\|_p.$$

We now proceed to prove (3.3.4). Note that $\sum_{j \le 2l} \varphi(2^{-j}|\cdot|) = \varphi_{<1}(2^{2l}|\cdot|)$ and $\varphi_{<1}$ is a smooth function supported on $[-2^2, 2^2]$. Thus, similarly as in (3.3.3) we note that $\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3) = \iint g(z_1, z_2) \widetilde{K}_l * d\sigma_{tr}(2^{-1}(r^2 + t^2) - z_1, x_3 - z_2) dz$ where $\widetilde{K}_l = \mathcal{F}^{-1}(\varphi_{<1}(2^{2l}|\cdot|))$. Since $\widetilde{K}_l \lesssim \mathcal{E}^N_{2l}$ for any N, for $2^l \le t < 2^{l+1}$ we see

$$|\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \int |g(z_1, z_2)| \mathcal{E}^{2N}_{2l} * d\sigma_{tr}\big(2^{-1} r^2 - z_1, x_3 - z_2\big) dz \tag{3.3.5}$$

because $2^{2l} t^2 \lesssim 1$ and $\mathcal{E}^{2N}_{2l} = 2^{-4l}(1 + 2^{-2l}|y|)^{-2N}$. Hence, taking an N large enough, we note that

$$\mathcal{E}^{2N}_{2l} * d\sigma_{tr}(x) \lesssim \begin{cases} (2^{2l} tr)^{-1}(1 + 2^{-2l}||x| - tr|)^{-N}, & 2^{2l} \ll tr, \\ 2^{-4l}(1 + 2^{-2l}|x|)^{-N}, & 2^{2l} \gtrsim tr, \end{cases} \tag{3.3.6}$$

provided that $2^l \le t < 2^{l+1}$. Indeed, to show this we only have to consider the case $2^{2l} \ll tr$ since the other case is trivial. By scaling $x \to trx$ we may assume that $tr = 1$. Thus, it is enough to show $\int L^{-2}(1 + L^{-1}|x - y|)^{-2N} d\sigma(y) \lesssim L^{-1}(1 + L^{-1}||x| - 1|)^{-N}$ for $L \ll 1$ with an N large enough. However, this is easy to see since $|x - y| \ge ||x| - 1|$ and $\int L^{-1}(1 + L^{-1}|x - y|)^{-N} d\sigma(y) \lesssim 1$.

Therefore, combining (3.3.5) and (3.3.6), one can see

$$\sup_{2^l \le t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{<-2l} g(r, x_3)| \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1} r^2, x_3) + \mathfrak{M}_2 g(2^{-1} r^2, x_3).$$

Here $\mathfrak{M}_2$ denotes the Hardy-Littlewood maximal function on $\mathbb{R}^2$. This proves the claim (3.3.4) since $\mathfrak{M}_2 g \lesssim \mathcal{M}_{\mathbb{R}^2} g$.

So we are reduced to showing $\|r^{\frac{1}{p}} \mathcal{A}_{high}\, g\|_{L^p_{r,x_3}} \le C \|g\|_p$ for $p > 2$. For the purpose it is sufficient to show

$$\|\sup_{2^l \le t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{k-2l} g|\|_p \lesssim 2^{-ck} \|g\|_p \tag{3.3.7}$$

because

$$\mathcal{A}_{high}\, g \le \sum_{k\ge 0}(\sum_{l} | \sup_{2^l \le t < 2^{l+1}} |\mathcal{A}_t \mathcal{P}_{k-2l} g|^p)^{1/p}$$

and

$$(\sum_{l} \|\mathcal{P}_{k-2l} g\|_p^p)^{1/p} \lesssim \|g\|_p.$$

By scaling, using (3.1.2), we can easily see the inequality (3.3.7) is equivalent to (3.3.2) while j replaced by k. So, we have (3.3.7) and this completes the proof of (3.3.1).

3.4 Proof of main estimates

In order to prove Proposition 3.1.1 and 3.1.2, we are led by (3.1.2) to consider $\widehat{d\sigma}(tr\xi)$ for which we use the following well known asymptotic expansion (see, for example, [75]):

$$\widehat{d\sigma}(\xi) = \sum_{j=0}^{N} C_j^{\pm} |\xi|^{-\frac{1}{2}-j} e^{\pm i|\xi|} + E_N(|\xi|), \quad |\xi| \gtrsim 1 \tag{3.4.1}$$

where E_N is a smooth function satisfying

$$|\frac{d^\ell}{dr^\ell} E_N(r)| \lesssim r^{-N} \tag{3.4.2}$$

for $0 \le \ell \le 4$ if $r \gtrsim 1$. The expansion (3.4.1) relates the operator $\mathcal{A}_t$ to the wave propagator. After changing variables, to prove Proposition 3.1.1 and 3.1.2, we can use the local smoothing estimate for the wave operator. The following proposition is directly obtained by Theorem 2.2.4 and interpolation with the trivial $L^2 - L^2$ estimate and the $L^1 - L^\infty$ estimate.

Proposition 3.4.1. *Let* $j \ge 0$*. Suppose* (3.1.6) *holds. Then, for* $\epsilon > 0$ *we have*

$$\left\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \right\|_{L^q_{x,t}(\mathbb{R}^2\times[1,2])} \lesssim 2^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})j+\epsilon j} \|f\|_{L^p} \tag{3.4.3}$$

From Theorem 3.4.1 we can deduce the following estimate via simple rescaling argument.

Corollary 3.4.2. *Let $j \geq -\ell$. Suppose* (3.1.6) *holds. Then, for $\epsilon > 0$ we have*

$$\Big\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \Big\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)(\ell+j)+\left(\frac{3}{q}-\frac{2}{p}\right)\ell+\epsilon(\ell+j)} \|f\|_{L^p}.$$

Proof. Changing variables $(x,t) \to 2^\ell(x,t)$, we see

$$\Big\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \Big\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} = 2^{\frac{3\ell}{q}} \Big\| e^{it\sqrt{-\Delta}} \mathcal{P}_{\ell+j} f(2^\ell \cdot) \Big\|_{L^q_{x,t}(\mathbb{R}^2 \times [1,2])}.$$

Thus, using (3.4.3) we have

$$\Big\| e^{it\sqrt{-\Delta}} \mathcal{P}_j f \Big\|_{L^q_{x,t}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3\ell}{q}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)(\ell+j)+\epsilon(\ell+j)} \|f(2^\ell \cdot)\|_{L^p}.$$

So, rescaling gives the desired inequality. □

3.5 Proof of Proposition 3.1.1

We now recall (3.1.2) and (3.4.1). To show Proposition 3.1.1 we first deal with the contribution from the error part E_N. Let us set

$$\mathcal{E}_t g(r, x_3) = \int e^{i(\frac{r^2+t^2}{2}\xi_1 + x_3\xi_2)} E_N(tr|\xi|)\, \widehat{g}(\xi) d\xi.$$

Lemma 3.5.1. *Let $j \geq -k$. Suppose* (3.1.6) *holds. Then, we have*

$$\Big\| \sup_{1<t<2} |\varphi_k(r) \mathcal{E}_t \mathcal{P}_j g| \Big\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{-(N-3)(j+k)} 2^{k(\frac{1}{q}-\frac{2}{p})} \|g\|_{L^p}, & k \geq -2, \\ 2^{-(N-3)(j+k)} 2^{k(\frac{3}{q}-\frac{2}{p})} \|g\|_{L^p}, & k < -2. \end{cases} \tag{3.5.1}$$

Proof. We first consider the case $k \geq -2$. Using Lemma 4.5.1, we need to estimate $\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g$ and $\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g$ in $L^q_{r,x_3,t}(\mathbb{R}^2 \times [1,2])$. For simplicity we denote $L^q_{r,x_3,t} = L^q_{r,x_3,t}(\mathbb{R}^2 \times [1,2])$. We first consider $\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g$. Changing variables $\frac{r^2}{2} \mapsto s$, we note that

$$\varphi_k(\sqrt{2s})\mathcal{E}_t\mathcal{P}_j g(\sqrt{2s}, x_3) = \varphi_k(\sqrt{2s}) \int \mathcal{K}\big(s - y_1 + 2^{-1}t^2, x_3 - y_2\big) g(y_1, y_2) dy,$$

where

$$\mathcal{K}(s, u) = 2^{2j} \int e^{i2^j(s\xi_1 + u\xi_2)} \varphi_0(\xi) E_N(2^j t\sqrt{2s}|\xi|) d\xi.$$

Since $s \sim 2^{2k}$, using (3.4.2), we have $|\mathcal{K}(s,u)| \lesssim 2^{2j}(1+2^j|(s,u)|)^{-M}2^{-N(j+k)}$ for $1 \le M \le 4$ via integration by parts. Thus, we have $\|\varphi_k(\sqrt{2s})\mathcal{K}(s+\frac{t^2}{2},u)\|_{L^r_{s,u}} \le C2^{-N(j+k)}2^{2j(1-\frac{1}{r})}$ for $1 < t < 2$ with a positive constant C. Young's convolution inequality gives $\|\varphi_k(\sqrt{2s})\mathcal{E}_t\mathcal{P}_j g(\sqrt{2s},x_3)\|_{L^q_{s,x_3,t}} \lesssim 2^{-N(j+k)}2^{2j(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}$. Thus, reversing $s \to r^2/2$, after a simple manipulation we get

$$\Big\|\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g\Big\|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{1}{q}-\frac{2}{p})}\|g\|_{L^p} \tag{3.5.2}$$

for $1 \le p \le q \le \infty$. Indeed, we need only note that $2j(\frac{1}{p}-\frac{1}{q})-\frac{k}{q} \le 2(j+k)+k(\frac{1}{q}-\frac{2}{p})$ because $j \ge -k$ and $\frac{1}{p}-\frac{1}{q}-1<0$.

We now consider $\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g$. Note that

$$\partial_t\mathcal{E}_t g(r,x_3) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2)}\big(t\xi_1 E_N(tr|\xi|)+r|\xi|E_N'(tr|\xi|)\big)\widehat{g}(\xi)d\xi. \tag{3.5.3}$$

Using (3.4.2), we can handle $\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g$ similarly as before. In fact, since $|t\xi_1| \lesssim 2^j$ and $r|\xi| \sim 2^{k+j}$, we see

$$\Big\|\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g\Big\|_{L^q_{r,x_3}} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{1}{q}-\frac{2}{p})}(2^{j+k}+2^j)\|g\|_{L^p}.$$

Hence, combining this and (3.5.2) with Lemma 4.5.1, we get (3.5.1) for $k \ge -2$.

We now consider the case $k < -2$. We first claim that

$$\|\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)}2^{k(\frac{2}{q}-\frac{2}{p})}\|g\|_{L^p}. \tag{3.5.4}$$

We use the transformation (3.2.1). By (3.2.2) we have $|\frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 1$. Therefore,

$$\|\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim \Big(\int\Big|\varphi_k(r(y,\tau))\widetilde{K}(\cdot,\tau)*g(y)\Big|^q dyd\tau\Big)^{\frac{1}{q}},$$

where

$$\widetilde{K}(y,\tau) = \int e^{iy\cdot\xi}\varphi_j(\xi)E_N(\tau|\xi|)d\xi.$$

Note that $\tau \sim 2^k$. Changing $\tau \mapsto 2^k\tau$ and $\xi \mapsto 2^j\xi$, using (3.4.2) and integration by parts, we have $|\widetilde{K}(y,2^k\tau)| \le C2^{2j}(1+2^j|y|)^{-M}2^{-N(j+k)}$ for $1 \le M \le 4$ and $1 < \tau < 2$. Young's convolution inequality gives

$$\|\varphi_k(r)\mathcal{E}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-N(j+k)}2^{2j(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}.$$

Thus, we get (3.5.4). As for $\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g$, we use (3.5.3) and repeat the same argument to see $\|\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-N(j+k)}2^j 2^{2j(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}$ since $|t\xi_1| \lesssim 2^j$, $r|\xi| \sim 2^{k+j}$, and $k < -2$. Thus, we get

$$\|\varphi_k(r)\partial_t\mathcal{E}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)}2^k 2^{k(\frac{2}{q}-\frac{2}{p})}\|g\|_{L^p}.$$

Putting (3.5.4) and this together, by Lemma 4.5.1 we obtain (3.5.1) for $k < -2$. □

By (3.4.1) and Lemma 3.5.1, to prove Proposition 3.1.1 and 3.1.2 we only have to consider contributions from the remaining $C_j^{\pm}|tr\xi|^{-\frac{1}{2}-j}e^{\pm i|tr\xi|}$, $j = 0,\dots,N$. To this end, it is sufficient to consider the major term $C_0^{\pm}|tr\xi|^{-\frac{1}{2}}e^{\pm i|tr\xi|}$ since the other terms can be handled similarly. Furthermore, by reflection $t \to -t$ it is enough to deal with $|tr\xi|^{-\frac{1}{2}}e^{i|tr\xi|}$ since the estimate (3.4.3) clearly holds with the interval $[1,2]$ replaced by $[-2,-1]$.

Let us set

$$\mathcal{U}_t g(r,x_3) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2+tr|\xi|)}|r\xi|^{-\frac{1}{2}}\widehat{g}(\xi)d\xi. \tag{3.5.5}$$

To complete the proof of Proposition 3.1.1, we need to show

$$\Big\|\sup_{1<t<2}|\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g|\Big\|_{L^q_{r,x_3}} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)+\frac{k}{q}-\frac{2k}{p}}\|g\|_{L^p}, & k\ge 2,\\ 2^{(j+k)(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)+\frac{2k}{q}-\frac{2k}{p}}\|g\|_{L^p}, & k\le -2.\end{cases} \tag{3.5.6}$$

Using Lemma 4.5.1, the matter is reduced to obtaining estimates for $\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g$ and $\varphi_k(r)\partial_t\mathcal{U}_t\mathcal{P}_j g$ in $L^q_{r,x_3,t}$. Note that

$$\partial_t\mathcal{U}_t\mathcal{P}_j g(r,x_3,t) = \int e^{i(\frac{r^2+t^2}{2}\xi_1+x_3\xi_2+tr|\xi|)}\widehat{\mathcal{P}_j g}(\xi)\frac{t\xi_1+r|\xi|}{|r\xi|^{1/2}}d\xi. \tag{3.5.7}$$

By the Mikhlin multiplier theorem one can easily see

$$\|\varphi_k(r)\partial_t\mathcal{U}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim \begin{cases} 2^{j+k}\|\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}}, & k\ge 0,\\ 2^j\|\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}}, & k<0,\end{cases}$$

where $L^q_{r,x_3,t}$ denotes $L^q_{r,x_3,t}(\mathbb{R}^2\times[1,2])$. Therefore, by Lemma 4.5.1 it is sufficient for (3.5.6) to prove that

$$\|\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g\|_{L^q_{r,x_3,t}} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}+\epsilon)+\frac{k}{q}-\frac{2k}{p}}\|g\|_{L^p}, & k\ge 2,\\ 2^{(j+k)(\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}+\epsilon)+\frac{3k}{q}-\frac{2k}{p}}\|g\|_{L^p}, & k\le -2.\end{cases}$$

We first consider the case $k \ge 2$. As before, we use the change of variables (3.2.1). Since $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 2^{2k}$ from (3.2.2) and since $\tau = rt$ and $1 < t < 2$, we have

$$\big\|\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g\big\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{2k}{q}-\frac{j+k}{2}}\big\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_j f\big\|_{L^q_{y,\tau}(\mathbb{R}^2\times[2^{k-1},2^{k+2}])}$$

since $|r\xi| \sim 2^{j+k}$. Thus, Corollary 3.4.2 gives the desired estimate (3.5.6) for $k \ge 2$. The case $k \le -2$ can be handled in the exactly same manner. The only difference is that $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 1$. Thus, the desired estimate (3.5.6) immediately follows from Corollary 3.4.2.

3.6 Proof of Proposition 3.1.2

As mentioned already, the determinant of the Jacobian $\partial(y_1, y_2, \tau)/\partial(r, x_3, t)$ may vanish when $|k| \le 1$. So, we need additional decomposition depending on $|r - t|$. We also make decomposition in ξ depending on $|\xi|^{-1}\xi_1 + 1$ to control the size of the multiplier $|t\xi_1 + r|\xi||$ in a more accurate manner (for example, see (3.6.12)).

For $m \ge 0$ let us set

$$\psi_m(\xi) = \varphi\big(2^m\big||\xi|^{-1}\xi_1 + 1\big|\big),$$
$$\psi^m(\xi) = 1 - \sum_{0\le j<m} \psi_j(\xi),$$

so that $\sum_{0\le k<m} \psi_k + \psi^m = 1$. We additionally define

$$\mathcal{P}_{j,m}g = (\varphi_j\psi_m\widehat{g})^\vee, \quad \mathcal{P}^m_j g = (\varphi_j\psi^m\,\widehat{g})^\vee.$$

So it follows that

$$\mathcal{P}_j = \sum_{0\le k<m} \mathcal{P}_{j,k} + \mathcal{P}^m_j. \tag{3.6.1}$$

Proposition 3.6.1. *Let us set $\varphi_{k,l}(r,t) = \varphi_k(r)\varphi(2^l|r-t|)$. Let $j \ge -1$ and $k = -1, 0, 1$. Suppose (3.1.6) holds. Then, for $\epsilon > 0$ we have*

$$\|\varphi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{j}{2}}2^{\frac{l}{q}}2^{(\frac{m}{2}-l)(\frac{1}{p}+\frac{3}{q}-1)+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})+\epsilon j}\|g\|_{L^p}. \tag{3.6.2}$$

In order to prove Proposition 3.6.1, we make the change of variables (3.2.1). Since $|k| \le 1$, we need only to consider (r,t) contained in the set $[2^{-1} - 10^{-2}, 2^2 + 10^2] \times [1,2]$. Set

$$S_l = \big\{(y_1, y_2, \tau) : 2^{-2l-1} \le |y_1 - \tau| \le 2^{-2l+1},\ y_1, \tau \in [2^{-3}, 2^3]\big\}.$$

By (3.2.1) $y_1 - \tau = (r-t)^2/2$. From (3.2.2) we note $|\det \frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 2^{-l}$ if $(y_1,\tau) \in S_l$. Thus, changing variables $(r, x_3, t) \to (y_1, y_2, \tau)$ we obtain

$$\|\varphi_{k,l} \mathcal{U}_t \mathcal{P}_j h\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{1}{2}j} 2^{\frac{l}{q}} \|e^{i\tau\sqrt{-\Delta}} \mathcal{P}_j h\|_{L^q_{y,\tau}(S_l)}. \tag{3.6.3}$$

Therefore, for (3.6.2) it is sufficient to show

$$\|e^{i\tau\sqrt{-\Delta}} \mathcal{P}_{j,m} g\|_{L^q_{y,\tau}(S_l)} \lesssim 2^{(\frac{m}{2}-l)(\frac{1}{p}+\frac{3}{q}-1)+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})+\epsilon j} \|g\|_{L^p} \tag{3.6.4}$$

for p, q satisfying (3.1.6). For the purpose we need the following lemma, which gives an improved L^2 estimate thanks to restriction of the integral over S_l. Indeed, one can remove the localization $y_1, \tau \in [2^{-3}, 2^3]$.

Lemma 3.6.2. *Let $D_l = \{(x_1, x_2, t) : 2^{-2l} \le |x_1 - t| \le 2^{-2l+1}\}$. Then, we have*

$$\Big\| \int e^{i(x\cdot\xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \Big\|_{L^2_{x,t}(D_l)} \lesssim 2^{\frac{m}{2}-l} \|g\|_{L^2}. \tag{3.6.5}$$

Proof. We write $x\cdot\xi + t|\xi| = x_1(\xi_1 + |\xi|) + x_2\xi_2 + (t - x_1)|\xi|$. Then, changing variables $(x, t - x_1) \to (x, t)$ and $\xi \to \eta := \mathcal{L}(\xi) = (\xi_1 + |\xi|, \xi_2)$, we see

$$\Big\| \int e^{i(x\cdot\xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \Big\|_{L^2_{x,t}(D_l)} \le \Big\| \int e^{i(x\cdot\eta + t|\mathcal{L}^{-1}\eta|)} \frac{\widehat{h}(\mathcal{L}^{-1}\eta)}{|\det J\mathcal{L}(\eta)|} d\eta \Big\|_{L^2_{x,t}(\mathbb{R}^2 \times I_l)}$$

where $\widehat{h}(\xi) = \widehat{g}(\xi)\psi_m(\xi)$ and $I_l = [-2^{-2l+1}, -2^{-2l}] \cup [2^{-2l}, 2^{-2l+1}]$. By Plancherel's theorem in the $x-$variable and integrating in t, we have

$$\Big\| \int e^{i(x\cdot\xi + t|\xi|)} \widehat{g}(\xi) \psi_m(\xi) d\xi \Big\|_{L^2_{x,t}(D_l)} \le C 2^{-l} \Big\| \frac{\widehat{h}(\mathcal{L}^{-1}\cdot)}{|\det J\mathcal{L}|} \Big\|_{L^2_x}.$$

A computation shows $\det J\mathcal{L} = 1 + |\xi|^{-1}\xi_1$, so $|\det J\mathcal{L}| \sim 2^{-m}$ on the support of $\widehat{h}$. Thus, by changing variables and Plancherel's theorem we get (3.6.5). $\square$

We also use the following elementary lemma.

Lemma 3.6.3. *For any $1 \le p \le \infty$, j, and m, we have*

$$\|(\varphi_j \psi_m \widehat{g}\,)^\vee\|_{L^p} \lesssim \|g\|_{L^p}, \quad \|(\varphi_j \psi^m \widehat{g}\,)^\vee\|_{L^p} \lesssim \|g\|_{L^p}.$$

Proof. Since $\psi^m - \psi^{m+1} = \psi_m$, it suffices to prove the second inequality only. By Young's inequality we need only to show $\|(\varphi_j \psi^m)^\vee\|_{L^1} \lesssim 1$. By scaling it is clear that $\|(\varphi_j(\xi)\psi^m(\xi))^\vee\|_{L^1} = \|(\varphi_0(\xi)\psi^m(\xi))^\vee\|_{L^1}$. Note that $\mathfrak{m}(\xi) := \varphi_0(\xi)\psi^m(\xi)$ is supported in a rectangular box with dimensions $2^{-m} \times 1$. So, $\mathfrak{m}(\xi_1, 2^{-m}\xi_2)$ is supported in a cube of side length ~ 1 and it is easy to see $\partial_\xi^\alpha(\mathfrak{m}(\xi_1, 2^{-m}\xi_2))$ is uniformly bounded for any α. This gives $\|(\mathfrak{m}(\cdot, 2^{-m}\cdot))^\vee\|_1 \lesssim 1$. Therefore, after scaling we get $\|(\varphi_0(\xi)\psi^m(\xi))^\vee\|_{L^1} \lesssim 1$. □

Proof of (3.6.4). In view of interpolation the estimate (3.6.4) follows for p, q satisfying (3.1.6) if we show the next three estimates:

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^2_{y,\tau}(S_l)} \lesssim 2^{\frac{m}{2}-l}\|g\|_{L^2}, \tag{3.6.6}$$

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^\infty_{y,\tau}(S_l)} \lesssim 2^{\frac{3j}{2}}\|g\|_{L^1}, \tag{3.6.7}$$

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^4_{y,\tau}(S_l)} \lesssim 2^{\epsilon j}\|g\|_{L^4}.$$

The first estimate follows from Lemma 3.6.2. Corollary 3.4.2 and Lemma 3.6.3 give the other two estimates. □

It is possible to improve the estimate (3.6.2) when $j > m$.

Proposition 3.6.4. *Let $j \ge -1$ and $k = -1, 0, 1$. Suppose $1 \le p \le q$, $1/p + 1/q \le 1$, and $j > m$, then*

$$\|\varphi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{j}{2}}2^{\frac{l}{q}}2^{\frac{2}{q}(\frac{m}{2}-l)+\frac{j-m}{2}(1-\frac{1}{p}-\frac{1}{q})+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}.$$

Proof. By (3.6.3) it is sufficient to show

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^q_{y,\tau}(S_l)} \lesssim 2^{\frac{2}{q}(\frac{m}{2}-l)+\frac{j-m}{2}(1-\frac{1}{p}-\frac{1}{q})+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})}\|g\|_{L^p}$$

for p, q satisfying $1 \le p \le q$, $1/p + 1/q \le 1$. In fact, by interpolation with the estimates (3.6.6) and (3.6.7) we only have to show

$$\|e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m}g\|_{L^\infty_{y,\tau}(S_l)} \lesssim 2^{\frac{j-m}{2}}\|g\|_{L^\infty}. \tag{3.6.8}$$

Let us set

$$K_t^{j,m}(x) = \frac{1}{(2\pi)^2} \int e^{i(x\cdot\xi + t|\xi|)} \varphi_j(|\xi|)\psi_m(\xi) d\xi.$$

Then $e^{i\tau\sqrt{-\Delta}}\mathcal{P}_{j,m} g = K_\tau^{j,m} * g$. Therefore, (3.6.8) follows if we show

$$\|K_t^{j,m}\|_{L^1_x} \lesssim 2^{\frac{j-m}{2}} \tag{3.6.9}$$

when $t \sim 1$. Note that $|\xi_2|/|\xi| = \sqrt{1-\xi_1/|\xi|}\sqrt{1+\xi_1/|\xi|} \lesssim 2^{-\frac{m}{2}}$ if $\xi \in \operatorname{supp}\psi_m$. So, $\operatorname{supp}\psi_m$ is contained in a conic sector with angle $\sim 2^{-\frac{m}{2}}$. Let $\mathcal{S}$ be a sector centered at the origin in $\mathbb{R}^2$ with angle $\sim 2^{-\frac{j}{2}}$ and $\varphi_{\mathcal{S}}$ be a cut-off function adapted to $\mathcal{S}$. Then, by integration by parts it follows that

$$\Big\| \int e^{i(x\cdot\xi+t|\xi|)} \varphi_j(|\xi|)\varphi_{\mathcal{S}}(\xi) d\xi \Big\|_{L^1_x} \lesssim 1$$

if $t \sim 1$. (See, for example, [44]). Now (3.6.9) is clear since the support of ψ_m can be decomposed into as many as $C2^{\frac{j-m}{2}}$ such sectors. □

Finally, we prove Proposition 3.1.2 making use of Proposition 3.6.1 and 3.6.4. We recall (3.1.2) and (3.4.1). As mentioned before, by Lemma 3.5.1 we need only to consider $\mathcal{U}_t$ (see (3.5.5)) and it is sufficient to show

$$\big\| \sup_{1<t<2} |\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g| \big\|_{L^q_{r,x_3}} \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q}-1)j+\epsilon j} \|g\|_{L^p} \tag{3.6.10}$$

for p, q satisfying $p \le q$, $1/p + 1/q < 1$ and $1/p + 2/q > 1$.

Proof of (3.6.10). Let us set $\varphi^l(\cdot) = 1 - \sum_{j=0}^{l-1} \varphi(2^j\cdot)$ and $\varphi_k^l(r,t) = \varphi_k(r)\varphi^l(|r-t|)$. Then, we decompose

$$\varphi_k(r) = \sum_{0\le l\le j/2} \varphi_{k,l}(r,t) + \sum_{j/2<l<j} \varphi_{k,l}(r,t) + \varphi_k^j(r,t).$$

Combining this with (3.6.1) and using $\sum_{\frac{j}{2}<l<j} \varphi_{k,l} + \varphi_k^j \le \varphi_k^{[j/2]-1}$, by the triangle inequality we have

$$\big\| \sup_{1<t<2} |\varphi_k(r)\mathcal{U}_t\mathcal{P}_j g| \big\|_{L^q} \le \sum_{i=1}^{5} S_i,$$

where

$$S_1 = \sum_{0\le l\le j/2}\ \sum_{0\le m\le l-1} \big\| \sup_{1<t<2} \varphi_{k,l}|\mathcal{U}_t\mathcal{P}_{j,m}g|\big\|_{L^q},\ S_2 = \sum_{0\le l\le j/2} \| \sup_{1<t<2} \varphi_{k,l}|\mathcal{U}_t\mathcal{P}_j^l g|\|_{L^q},$$

$$S_3 = \sum_{\frac{j}{2}<l<j}\ \sum_{0\le m\le j-1} \| \sup_{1<t<2} \varphi_{k,l}|\mathcal{U}_t\mathcal{P}_{j,m}g|\|_{L^q},\ S_4 = \sum_{0\le m\le j-1} \| \sup_{1<t<2} \varphi_k^j|\mathcal{U}_t\mathcal{P}_{j,m}g|\|_{L^q},$$

$$S_5 = \| \sup_{1<t<2} \varphi_k^{[j/2]-1}|\mathcal{U}_t\mathcal{P}_j^j g|\|_{L^q}.$$

The proof of (3.6.10) is now reduced to showing

$$S_i \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q}-1)j+\epsilon j}\|g\|_{L^p}, \quad 1\le i\le 5, \tag{3.6.11}$$

for p, q satisfying $p\le q$, $1/p+1/q<1$ and $1/p+2/q>1$.

Before we start the proof of (3.6.11), we briefly comment on the decomposition S_i, $i=1,\dots,5$. As for S_4 and S_5, which are easier to handle, the sizes of $r-t$ and $|\xi|^{-1}\xi_1+1$ are sufficiently small on the supports of the associated multipliers, so we can remove the dependence of t by an elementary argument. For S_1, S_2, and S_3, we use Lemma 4.5.1 combined with (3.5.7) to control the maximal operators. Different magnitudes of contribution come from $\partial_t\varphi_{k,l} = O(2^l)$ and $|t\xi_1+r|\xi||$, so we need to compare them. Writing $t\xi_1+r|\xi| = t\big(|\xi|^{-1}\xi_1+1\big)+(r-t)$, we note

$$|t\xi_1+r|\xi|| \lesssim 2^j\max\{2^{-m}, 2^{-l}\}. \tag{3.6.12}$$

The decompositions in S_1, S_2, and S_3 are made according to comparative sizes of $\partial_t\varphi_{k,l}=O(2^l)$ and $|t\xi_1+r|\xi||$ in terms of l, m, and j.

We first consider S_1. Using Lemma 4.5.1, we need to estimate $\varphi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g$ and $\partial_t(\varphi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g)$ in $L^q_{r,x_3,t}(\mathbb{R}^2\times[1,2])$. Note that $\partial_t\varphi_{k,l}=O(2^l)$ and $2^l\lesssim 2^{j-m}$. Thus, recalling (3.5.7), we apply Lemma 4.5.1 and the Mikhlin multiplier theorem to get

$$S_1 \lesssim \sum_{0\le l\le j/2}\sum_{m=0}^{l-1} 2^{\frac{j-m}{q}}\big\|\varphi_{k,l}\mathcal{U}_t\mathcal{P}_{j,m}g\big\|_{L^q}.$$

Thus, by Proposition 3.6.1 it follows that

$$S_1 \lesssim 2^{-\frac{j}{2}+\frac{j}{q}+\frac{3j}{2}(\frac{1}{p}-\frac{1}{q})+\epsilon j}\sum_{0\le l\le j/2} 2^{l(1-\frac{1}{p}-\frac{2}{q})}\sum_{m=0}^{l-1} 2^{\frac{m}{2}(\frac{1}{p}+\frac{1}{q}-1)}\|g\|_{L^p}.$$

Since $1/p + 1/q - 1 < 0$ and $1/p + 2/q > 1$, we obtain (3.6.11) with $i = 1$.

We can show the estimate (3.6.11) with $i = 2$ in the same manner. As before, since $\partial_t \varphi_{k,l} = O(2^l)$ and $2^l \lesssim 2^{j-l}$, using (3.6.12), Lemma 4.5.1, and the Mikhlin multiplier theorem, we have

$$S_2 \lesssim \sum_{0 \le l \le j/2} 2^{\frac{j-l}{q}} \left\| \varphi_{k,l} \mathcal{U}_t \mathcal{P}_j^l g \right\|_{L^q}.$$

Thus, by (3.6.3) and Theorem 3.4.1, we have

$$S_2 \lesssim \sum_{0 \le l \le \frac{j}{2}} 2^{-\frac{j}{2}} 2^{\frac{j}{q} + \frac{3j}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\epsilon}{2} j} \|g\|_{L^p},$$

which gives (3.6.11) with $i = 2$.

We now consider S_3, which we handle as before. Since $j < 2l$, we have that $2^j \max\{2^{-m}, 2^{-l}\} \le 2^l$ if $l + m \ge j$. Similarly, $2^{j-m} \ge 2^j \max\{2^{-m}, 2^{-l}\}$ and $2^{j-m} \ge 2^l$ if $l + m < j$. Using (3.6.12) and (3.5.7), we see

$$S_3 \lesssim \sum_{j/2 < l < j} \Big(\sum_{j-l \le m \le j-1} 2^{\frac{l}{q}} \|\varphi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g\|_{L^q} + \sum_{0 \le m < j-l} 2^{\frac{j-m}{q}} \|\varphi_{k,l} \mathcal{U}_t \mathcal{P}_{j,m} g\|_{L^q} \Big)$$

Since $1/p + 2/q > 1$, using Proposition 3.6.4, we get (3.6.11) for $i = 3$.

We handle S_4 and S_5 in an elementary way without relying on Lemma 4.5.1. Instead, we can control S_4 and S_5 more directly. Concerning S_4 we claim that

$$S_4 \lesssim 2^{\frac{1}{2}(\frac{3}{p} - \frac{1}{q} - 1)j} \|g\|_{L^p} \tag{3.6.13}$$

if $5/q > 1 + 1/p$ and $2 \le p \le q \le \infty$. This clearly gives (3.6.11) with $i = 4$ for p, q satisfying $p \le q$, $1/p + 1/q < 1$ and $1/p + 2/q > 1$. We note that

$$|\varphi_k^j \mathcal{U}_t \mathcal{P}_{j,m} g(r, x_3)| \lesssim 2^{-\frac{1}{2}j} \Big| \varphi_k^j \int e^{i2^j(r^2\xi_1 + x_3\xi_2 + r^2|\xi|)} \mathfrak{m}(\xi) \varphi_0(\xi) \psi_m(\xi) \widehat{g(2^{-j}\cdot)}(\xi) d\xi \Big|,$$

where

$$\mathfrak{m}(\xi) = e^{i2^j(\frac{t^2 - r^2}{2}\xi_1 + (t-r)r|\xi|)} |\xi|^{-\frac{1}{2}} \widetilde{\varphi}_0(\xi),$$

and $\widetilde{\varphi}_0$ is a smooth function supported in $[-\pi, \pi]^2$ such that $\widetilde{\varphi}_0 \varphi_0 = 1$. If $(r, t) \in \operatorname{supp} \varphi_k^j$, then $|t - r| \lesssim 2^{-j}$. Thus, $|\partial_\xi^\alpha m(\xi)| \lesssim 1$ for any α. We remove the dependence of t by using a bound on the coefficient of Fourier series, not the Sobolev embedding. Expanding $\mathfrak{m}$ into Fourier series on $[-\pi, \pi]^2$ we have

$\mathfrak{m}(\xi) = \sum_{k\in\mathbb{Z}^2} C_{\mathbf{k}}(r,t)e^{i\mathbf{k}\cdot\xi}$ while $|C_{\mathbf{k}}(r,t)| \lesssim (1+|\mathbf{k}|)^{-N}$. Since $1 < t < 2$, the estimate (3.6.13) follows after scaling $\xi \to 2^j\xi$ if we obtain

$$\|\mathcal{R}\mathcal{P}_{j,m}g\|_{L^q_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q})j}\|g\|_{L^p},$$

where

$$\mathcal{R}g(r,x_3) = \int e^{i(r^2\xi_1+x_3\xi_2+r^2|\xi|)}\widehat{g}(\xi)d\xi.$$

When $q = 2$, changing variables $r^2 \to r$ and following the argument in the proof of Lemma 3.6.2 we have $\|\mathcal{R}\mathcal{P}_{j,m}g\|_{L^2_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{m/2}\|g\|_{L^2}$. On the other hand, (3.6.8) gives $\|\mathcal{R}\mathcal{P}_{j,m}g\|_{L^\infty_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{(j-m)/2}\|g\|_{L^\infty}$. Interpolation between these two estimates gives

$$\|\mathcal{R}\mathcal{P}_{j,m}g\|_{L^q_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{\frac{m}{q}+\frac{j-m}{2}(1-\frac{2}{q})}\|g\|_{L^q}$$

for $2 \le q \le \infty$. Since the support $\widehat{\mathcal{P}_{j,m}g}(\xi)$ is contained in a rectangular region of dimensions $2^j \times 2^{j-\frac{m}{2}}$, by Bernstein's inequality we have

$$\|\mathcal{R}^j_m g\|_{L^q_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{j(\frac{2}{p}-\frac{3}{q})+m(\frac{5}{2q}-\frac{1}{2}-\frac{1}{2p})}\|g\|_{L^p}$$

for $2 \le p \le q \le \infty$. Since $5/q > 1 + 1/p$, this proves the claimed estimate (3.6.13).

Finally, we show (3.6.11) with $i = 5$. Changing variables $(\xi_1,\xi_2) \to (2^j\xi_1,\xi_2)$, we observe

$$\varphi_k^{[j/2]-1}|\mathcal{U}_t\mathcal{P}_j^j g(r,x_3)| \lesssim 2^{\frac{j}{2}}\varphi_k^{[j/2]-1}\Big|\int e^{i(\frac{(r-t)^2}{2}2^j\xi_1+x_3\xi_2)}\mathfrak{m}(\xi)\widehat{\mathcal{P}_j^j g}(2^j\xi_1,\xi_2)d\xi\Big|,$$

where

$$\widetilde{\mathfrak{m}}(\xi) = e^{i2^jrt(|(\xi_1,2^{-j}\xi_2)|-\xi_1)}|(\xi_1,2^{-j}\xi_2)|^{-\frac{1}{2}}\widetilde{\varphi}_0(|(\xi_1,2^{-j}\xi_2)|)\psi^{j-1}(2^j\xi_1,\xi_2).$$

Note that $\operatorname{supp}\widetilde{\mathfrak{m}} \subset \{\xi_1 \in [2^{-1},2^2], |\xi_2| \le 2^2\}$. Since $|\partial_\xi^\alpha m(\xi)| \lesssim 1$ for any α, expanding $\widetilde{\mathfrak{m}}$ into Fourier series on $[-2\pi,2\pi]^2$, $\widetilde{\mathfrak{m}}(\xi) = \sum_{k\in\mathbb{Z}^2} C_{\mathbf{k}}(r,t)e^{i2^{-1}\mathbf{k}\cdot\xi}$ holds while $|C_{\mathbf{k}}(r,t)| \lesssim (1+|\mathbf{k}|)^{-N}$. Hence, similarly as before, changing variables $(\xi_1,\xi_2) \to (2^{-j}\xi_1,\xi_2)$, to show (3.6.11) with $i = 5$ it is sufficient to obtain

$$\Big\|\sup_{1<t<2}\mathcal{P}_j^j g\Big(\frac{(r-t)^2}{2},x_3\Big)\Big\|_{L^q_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{\frac{1}{2}(\frac{3}{p}-\frac{1}{q})j}\|g\|_{L^p} \qquad (3.6.14)$$

for $1 \le p \le q \le \infty$. Clearly, the left hand side is bounded by $\|\mathcal{P}_j^j g(x_1, x_3)\|_{L^q_{x_3}(L^\infty_{x_1})}$. $\widehat{\mathcal{P}_j^j g}$ is supported on the rectangle $\{\xi_1 \in [2^{j-1}, 2^{j+2}], |\xi_2| \le 2^{j+2}\}$. Thus, using Bernstein's inequality in x_1, we get

$$\Big\| \sup_{1<t<2} \mathcal{P}_j^j g\Big(\frac{(r-t)^2}{2}, x_3\Big)\Big\|_{L^q_{r,x_3}([2^{-2},2^3]\times\mathbb{R})} \lesssim 2^{-\frac{j}{2}+\frac{j}{q}} \|\mathcal{P}_j^j g\|_{L^q}$$

for $1 \le q \le \infty$. Another use of Bernstein's inequality gives (3.6.14) for $1 \le p \le q \le \infty$. This completes the proof of (3.6.10). □

3.7 Sharpness of the range of p, q

We show (3.0.1) implies $(1/p, 1/q) \in \mathbf{T}$, that is to say,

$$(\mathbf{a})\ p \le q, \quad (\mathbf{b})\ 1 + 1/q \ge 3/p, \quad (\mathbf{c})\ 3/q \ge 2/p.$$

To see (**a**), let f_R be the characteristic function of a ball of radius $R \gg 1$, centered at 0. Then, $M_{\mathbb{H}^1} f_R$ is also supported in a ball B of radius $\sim R$ and $M_{\mathbb{H}^1} f_R \gtrsim 1$ on B. Thus, $\sup_{R>1} \|M_{\mathbb{H}^1} f_R\|_q / \|f_R\|_p$ is finite only if $p \le q$. For (**b**) let g_r be the characteristic function of a ball of radius $r \ll 1$ centered at 0. Then, $|M_{\mathbb{H}^1} g_r(x, x_3)| \gtrsim r$ when (x, x_3) is contained in a $c_0 r-$neighborhood of $\{(x, x_3) : 1 < |x| < 2, x_3 = 0\}$ for a small constant $c_0 > 0$. Thus, (3.0.1) implies $r^{1+1/q} \lesssim r^{3/p}$, which gives $1 + 1/q \ge 3/p$ if we let $r \to 0$. Finally, to show (**c**) we consider h_r which is the characteristic function of an $r-$neighborhood of $\{(x, x_3) : |x| = 1, x_3 = 0\}$ with $r \ll 1$. Then, $|M_{\mathbb{H}^1} h_r(x, x_3)| \gtrsim c > 0$ when (x, x_3) is in an $r-$ball centered at 0. Thus, (3.0.1) gives $r^{3/q} \lesssim r^{2/p}$, which yields $3/q \ge 2/p$.

The maximal estimate (3.0.1) for general L^p functions has a smaller range of p, q. Let h_r be a characteristic function of the set $\{(x, x_3) : |x_1 - 1| < r^2, |x_2| < r, |x_3| < r\}$ for a sufficiently small $r > 0$. Then $M_{\mathbb{H}^1} h_r(x, x_3) \sim r$ if $-1 \le x_1 \le 0, |x_2| < cr, |x_3| < cr$ for a small constant $c > 0$ independent of r. Thus, (3.0.1) implies $r^{1+2/q} \lesssim r^{4/p}$. It seems to be plausible to conjecture that (3.0.1) holds for general f modulo some endpoint cases as long as $1 + 2/q - 4/p \ge 0$, $3/q \ge 2/p$, and $1/q \le 1/p$. The range of p, q is properly contained in $\mathbf{T}$.

Chapter 4

Two parameter averages over tori

As in the M_{HL} case, people also have considered strong maximal averaging operators over lower dimensional submanifolds. Erdoğan [21] and Pramanik, Seeger [60] tried to obtain a result for two parameter maximal averages over curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, but it was not an L^p estimate we are interested in since both results require regularity of a function f. Following the schematized proof of the one parameter maximal operators, Cho [18] and Heo [33] obtained boundedness results for multiparameter maximal operators built on the L^2 method which requires sufficient decay of the Fourier transform of the associated surface measures or associated multiplier in the abstract setting. Two-parameter maximal functions associated with homogeneous surfaces were studied by Marletta, Ricci [49], and Marletta, Ricci, Zienkiewicz [50], who obtained their boundedness on the sharp range. In those works, homogeneity makes it possible to deduce L^p boundedness from that of a one-parameter maximal operator. Not much is known so far about the maximal functions which are genuine multiparameter operators. In this chapter we mainly prove Theorem 1.5.1 and Theorem 1.5.2 which concerns a two parameter maximal operator.

4.1 Comparison with one parameter maximal average

We begin our discussion with the maximal operator

$$f \to \sup_{0<t} |\mathcal{A}_t^{c_0 t} f|,$$

which is generated by the averages over (isotropic) dilations of the torus $\mathbb{T}_1^{c_0}$. While we mentioned that this operator is bounded on $L^p(\mathbb{R}^3)$ if and only if $p > 2$ by Ikromov, Kempe, Müller [37], it is not difficult to see prove the same result directly. Indeed, writing $f * \sigma_t^{c_0 t} = \int f * \mu_t^\phi \, d\varphi$, where μ_t^ϕ is the measure on the circle $\{t\Phi_1^{c_0}(\phi, \theta) : \theta \in [0, 2\pi)\}$. Since these circles are subsets of 2-planes containing the origin, L^p boundedness of $f \to \sup_{t>0} |f * \mu_t^\phi|$ for $p > 2$ can be obtained using the circular maximal theorem [7]. In fact, we need L^p boundedness of the maximal function given by the convolution averages in $\mathbb{R}^2$ over the circles $\mathrm{C}((t/c_0)e_1, t)$, which are not centered at the origin. Here, $\mathrm{C}(y, r)$ denotes the circle $\{x \in \mathbb{R}^2 : |x - y| = r\}$. However, such a maximal estimate can be obtained by making use of the local smoothing estimate for the wave operator (see, for example, [54]). Failure of L^p boundedness of $f \to \sup_{0<t} |\mathcal{A}_t^{c_0 t} f|$ for $p \le 2$ follows if one takes $f(x) = \tilde{\chi}(x)|x_3|^{-1/2}|\log|x_3||^{-1/2-\epsilon}$ for a small $\epsilon > 0$, where $\tilde{\chi}$ is a smooth positive function supported in a neighborhood of the origin.

In the study of the averaging operator defined by hypersurface, nonvanishing curvature of the underlying surface plays a crucial role. However, the torus $\mathbb{T}_1^{c_0}$ has vanishing curvature. More precisely, the Gaussian curvature $K(\theta, \phi)$ of $\mathbb{T}_1^{c_0}$ at the point $\Phi_1^{c_0}(\theta, \phi)$ is given by

$$K(\theta, \phi) = \frac{\cos\theta}{c_0(1 + c_0 \cos\theta)}.$$

Notice that K vanishes on the circles $\Phi_1^{c_0}(\pm\pi/2, \phi)$, $\phi \in [0, 2\pi)$. Decomposing $\mathbb{T}_1^{c_0}$ into the parts which are away from and near those circles, we can show, in an alternative way, L^p boundedness of $f \to \sup_{0<t} |\mathcal{A}_t^{c_0 t} f|$ for $p > 2$. The part away from the circles has nonvanishing curvature. Thus, the associated maximal function is bounded on L^p for $p > 3/2$ ([74]). Meanwhile, the other parts near the circles can be handled by the result in [37]. Unlike the one-parameter maximal function, (nontrivial) L^p estimates on $\mathcal{M}$ cannot be obtained by the same argument as above which relies L^p boundedness of a related circular maximal function in $\mathbb{R}^2$. In fact, to carry out the same argument, one needs L^p boundedness of the maximal function given by the (convolution) averages over the circles $\mathrm{C}(se_1, t)$ while supremum is taken over $0 < s < c_0 t$. However, Talagrand's construction [78] (also see [32, Corollary A.2]) shows that this (two-parameter) maximal function can not be bounded on any L^p, $p \neq \infty$.

4.2 Local smoothing estimates of averages over tori

Smoothing estimates for averaging operators have a close connection to the associated maximal functions. Especially, the local smoothing estimate for the wave operator was used by Mockenhaupt, Seeger, and Sogge [53] to provide an alternative proof of the circular maximal theorem as introduced in the introduction. Recent progress [40, 5, 39] on the maximal functions associated with the curves in higher dimensions were also achieved by relying on local smoothing estimates (also see [61]). Analogously, our proofs of Theorem 1.5.1 and 1.5.2 are also based on 2-parameter local smoothing estimates for the averaging operator $\mathcal{A}_t^s$, which are of independent interest. In the following, we obtain the sharp two-parameter local smoothing estimate for $\mathcal{A}_t^s$.

Theorem 4.2.1. *Let $p \geq 2$ and ψ be a smooth function with its support contained in $\mathbb{J}_*$. Set $\tilde{\mathcal{A}}_t^s f(x) = \psi(t,s)\mathcal{A}_t^s f(x)$. Then, the estimate*

$$\|\tilde{\mathcal{A}}_t^s f\|_{L^p_\alpha(\mathbb{R}^5)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \tag{4.2.1}$$

holds if $\alpha < \min\{1/2, 4/p\}$.

The result in Theorem 4.2.1 is sharp in that $\tilde{\mathcal{A}}_t^s$ can not be bounded from L^p to L^p_α if $\alpha > \min\{1/2, 4/p\}$ (see Chapter 4.8 below). Using the estimate (4.2.1), one can deduce results concerning the dimension of a union of tori $x + \mathbb{T}_t^s$, $(x,t,s) \in E \subset \mathbb{R}^3 \times \mathbb{J}_*$. See [31].

We also obtain the sharp local smoothing estimate for the one-parameter operator $f \to \mathcal{A}_t^{c_0 t} f$.

Theorem 4.2.2. *Let $\chi_0 \in C_c^\infty(0,\infty)$. Let $p \geq 2$ and $0 < c_0 < 1$. Then, for $\alpha < \min\{1/2, 3/p\}$, we have*

$$\|\chi_0(t)\mathcal{A}_t^{c_0 t} f\|_{L^p_\alpha(\mathbb{R}^4)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{4.2.2}$$

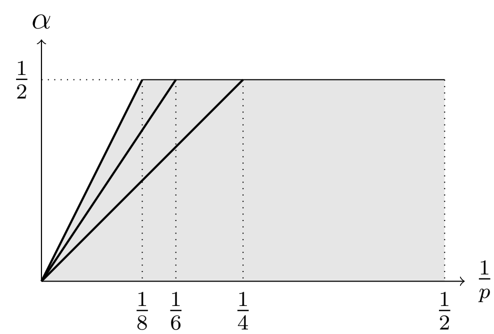

Figure 4.1: Smoothing orders of the estimates (4.2.1), (4.2.2), and (4.2.3)

The estimate above is sharp since $f \to \chi_0(t)\mathcal{A}_t^{c_0 t} f$ fails to be bounded from L_x^p to $L_\alpha^p(\mathbb{R}^4)$ if $\alpha > \min\{1/2, 3/p\}$ (Chapter 4.8). The next theorem gives the sharp regularity estimate for $\mathcal{A}_t^s$ when s, t fixed.

Theorem 4.2.3. *Let* $0 < s < t$. *If* $\alpha < \min\{1/2, 2/p\}$, *then we have*

$$\|\mathcal{A}_t^s f\|_{L_\alpha^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{4.2.3}$$

If $\alpha > \min\{1/2, 2/p\}$, $\tilde{\mathcal{A}}_t^s$ is not bounded from $L^p(\mathbb{R}^3)$ to $L_\alpha^p(\mathbb{R}^3)$ (Chapter 4.8). One can compare the local smoothing estimates in Theorem 4.2.1 and 4.2.2 with the regularity estimate in Theorem 4.2.3. The 2-parameter and 1-parameter local smoothing estimates have extra smoothing of order up to $2/p$ and $1/p$, respectively, when $p > 8$ (see Figure 4.1).

For $p < 2$, it is easy to show that there is no additional smoothing (local smoothing) for the operators $\tilde{\mathcal{A}}_t^s$ and $\chi_0(t)\mathcal{A}_t^{c_0 t}$ when compared with the estimates with fixed s, t (Theorem 4.2.3). That is to say, $\tilde{\mathcal{A}}_t^s$ fails to be bounded from $L^p(\mathbb{R}^3)$ to $L_\alpha^p(\mathbb{R}^5)$ and so does $\chi_0(t)\mathcal{A}_t^{c_0 t}$ from $L^p(\mathbb{R}^3)$ to $L_\alpha^p(\mathbb{R}^4)$ if $\alpha > \min(2/p', 1/2)$ and $1 \le p \le 2$. We remark that the result for two-parameter 2-dimensional tori can be extended to multiparameter tori in higher dimensions.

4.3 Two parameter propagator

We define an operator $\mathcal{U}$ by

$$\mathcal{U}f(x,t,s) = \int e^{i(x\cdot\xi + t|\bar{\xi}| + s|\xi|)} \widehat{f}(\xi) d\xi,$$

which is closely related to the averaging operator $\mathcal{A}_t^s$ and the wave operator $\mathcal{W}_+$. In fact, we obtain the estimates for $\mathcal{U}$ making use of those for $\mathcal{W}_+$.

Let $\mathbb{J}_0 = \{(t,s) : 0 < s < c_0 t\}$ and $\mathbb{J}_\tau = (\mathbb{I} \times \mathbb{I}_\tau) \cap \mathbb{J}_0$. To obtain the required estimates for our purpose, we consider the estimates over $\mathbb{R}^3 \times \mathbb{J}_\tau$ for small τ. This is the key estimate in this chapter.

Proposition 4.3.1. *Let $2 \le p \le q \le \infty$ satisfy $1/p + 3/q \le 1$, and let $0 < \tau \le 1$ and $\lambda \ge \tau^{-1}$. (a) If $\lambda \lesssim h \lesssim \tau\lambda^2$, then for any $\epsilon > 0$ the estimate*

$$\|\mathcal{U}f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{(\frac{1}{2}-\frac{1}{p})}\lambda^{\frac{3}{2}-\frac{1}{p}-\frac{5}{q}}h^{-\frac{1}{2}+\frac{2}{p}+\epsilon}\|f\|_{L^p} \tag{4.3.1}$$

holds whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. *Moreover, (b) if* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, *then we have the estimate* (4.3.1) *with $h = \lambda$. (c) If $h \gtrsim \tau\lambda^2$, then we have*

$$\|\mathcal{U}f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}}\lambda^{\frac{1}{2}+\frac{1}{p}-\frac{3}{q}+\epsilon}h^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p} \tag{4.3.2}$$

whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$.

For a bounded measurable function m, we denote by $m(D)$ the multiplier operator defined by $\mathcal{F}(m(D)f)(\xi) = m(\xi)\widehat{f}(\xi)$. In what follows, we occasionally use the following lemma.

Lemma 4.3.2. *Let $\xi = (\xi', \xi'') \in \mathbb{R}^k \times \mathbb{R}^{d-k}$. Let χ be an integrable function on $\mathbb{R}^k$ such that $\widehat{\chi}$ is also integrable. Suppose $\|m(D)f\|_q \le B\|f\|_p$ for a constant $B > 0$, then we have $\|m(D)\chi(D')f\|_q \le B\|\widehat{\chi}\|_1\|f\|_p$.*

This lemma follows from the identity

$$m(D)\chi(D')f(x) = (2\pi)^{-k}\int_{\mathbb{R}^k}\widehat{\chi}(y)(m(D)f)(x'+y, x'')dy,$$

which is a simple consequence of the Fourier inversion. The desired inequality is immediate from Minkowski's inequality and translation invariance of L^p norm.

Proof of Proposition 4.3.1. We make use of the decoupling inequality for the cone ([10]) and the sharp local smoothing estimate (Lemma 2.2.2) for $\mathcal{W}_+$.

We first show the case (a) where $\lambda \lesssim h \lesssim \tau\lambda^2$. To this end, we prove the estimate (4.3.1) under the additional assumption that $q \ge 6$. We subsequently extend the range by interpolation between the consequent estimates and (4.3.1) for $(p,q) = (4,4)$, which we prove later.

Fixing x_3 and s, we define an operator $\mathcal{T}^s_{x_3}$ by setting

$$\widehat{\mathcal{T}^s_{x_3} F}(\bar\xi) = \int e^{i(x_3\xi_3+s|\xi|)} \widehat{F}(\bar\xi, \xi_3) d\xi_3, \quad \xi = (\bar\xi, \xi_3).$$

Then, we observe that

$$\mathcal{U} f(x,t,s) = \mathcal{W}(\mathcal{T}^s_{x_3} f)(\bar x, t).$$

Let $\mathfrak{V}_\lambda \subset \mathbb{S}^1$ be a collection of $\sim \lambda^{-1/2}$-separated points. By $\{w_\nu\}_{\nu\in\mathfrak{V}_\lambda}$ we denote a partition of unity on the unit circle $\mathbb{S}^1$ such that w_ν is supported in an arc centered at ν of length about $\lambda^{-1/2}$ and $|(d/d\theta)^k w_\nu| \lesssim \lambda^{k/2}$. For each $\nu \in \mathfrak{V}_\lambda$, we set $\omega_\nu(\bar\xi) = w_\nu(\bar\xi/|\bar\xi|)$ and

$$\mathcal{W}_\nu g(\bar x, t) = \int e^{i(\bar x\cdot\bar\xi + t|\bar\xi|)} \omega_\nu(\bar\xi) \widehat{g}(\bar\xi) d\bar\xi.$$

Let $\tilde\chi \in \mathcal{S}(\mathbb{R})$ such that $\tilde\chi \ge 1$ on $\mathbb{I}$ and $\operatorname{supp} \mathcal{F}(\tilde\chi) \subset [-1/2, 1/2]$. Note that the Fourier transform of $\tilde\chi(t)\mathcal{W}_\nu g(\bar x, t)$ is supported in the set $\{(\bar\xi, \tau) : |\tau - |\bar\xi|| \lesssim 1, \bar\xi/|\bar\xi| \in \operatorname{supp}\omega_\nu, |\bar\xi| \sim \lambda\}$ if $\operatorname{supp}\widehat{g} \subset \mathbb{A}_\lambda$. Thus, by Bourgain–Demeter's l^2 decoupling inequality ([10]) followed by Hölder's inequality, we have

$$\Big\| \sum_{\nu\in\mathfrak{V}_\lambda} \mathcal{W}_\nu g \Big\|_{L^q_{\bar x,t}(\mathbb{R}^2\times\mathbb{I})} \lesssim \lambda^{\frac12 - \frac1{2p} - \frac3{2q} + \epsilon} \Big(\sum_{\nu\in\mathfrak{V}_\lambda} \big\| \tilde\chi(t) \mathcal{W}_\nu g \big\|^p_{L^q_{\bar x,t}(\mathbb{R}^3)} \Big)^{1/p} \tag{4.3.3}$$

for any $\epsilon > 0$, $q \ge 6$, and $p \ge 2$, provided that $\operatorname{supp}\widehat{g} \subset \mathbb{A}_\lambda$. Note that $\mathcal{U} f(x,t,s) = \sum_\nu \mathcal{W}_\nu(\mathcal{T}^s_{x_3} f)(\bar x, t)$ and $\mathcal{W}_\nu(\mathcal{T}^s_{x_3} f)(\bar x, t) = \mathcal{U}\omega_\nu(\bar D) f(x,t,s)$. Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, freezing s, x_3, we apply the inequality (4.3.3), followed by Minkowski's inequality, to get

$$\|\mathcal{U} f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \lambda^{\frac12 - \frac1{2p} - \frac3{2q} + \epsilon} \Big(\sum_{\nu\in\mathfrak{V}_\lambda} \big\| \tilde\chi(t) \mathcal{U} f_\nu \big\|^p_{L^q_{x,t,s}(\mathbb{R}^4\times\mathbb{I}_\tau)} \Big)^{1/p} \tag{4.3.4}$$

for $q \ge 6$ where $f_\nu = \omega_\nu(\bar D) f$. We now claim that

$$\|\tilde\chi(t)\mathcal{U} f_\nu\|_{L^q(\mathbb{R}^4\times\mathbb{I}_\tau)} \lesssim \tau^{(\frac12 - \frac1p)} \lambda^{1 - \frac1{2p} - \frac7{2q}} h^{\frac2p - \frac12 + \epsilon} \|f_\nu\|_{L^p} \tag{4.3.5}$$

holds for $1/p + 3/q \le 1$. Note that $(\sum_\nu \|f_\nu\|_p^p)^{1/p}$ for $1 \le p \le \infty$. Thus, from (4.3.4) and (4.3.5) the estimate (4.3.1) follows for $q \ge 6$.

To obtain (4.3.5), we begin by showing

$$\big\|\tilde\chi(t)\mathcal{U}f_\nu(\cdot,s)\big\|_{L^q_{x,t}(\mathbb{R}^4)} \le C\big\|e^{is|D|}f_\nu\big\|_{L^q_x(\mathbb{R}^3)}. \tag{4.3.6}$$

To do this, we apply the argument used to show Lemma 4.3.2. Let us set

$$\tilde\chi_\nu(t,\bar\xi) = e^{it(|\bar\xi|-\bar\xi\cdot\nu)}\widetilde\omega_\nu(\bar\xi)\varphi(\bar\xi/\lambda)$$

so that $\tilde\chi_\nu(t,\bar\xi)\widehat{f_\nu}(\xi) = e^{it(|\bar\xi|-\bar\xi\cdot\nu)}\widehat{f_\nu}(\xi)$. Here $\widetilde\omega_\nu(\bar\xi)$ is a angular cutoff function given in the same manner as $\omega_\nu(\bar\xi)$ such that $\widetilde\omega_\nu\omega_\nu = \omega_\nu$. Then, a computation shows that

$$|(\nu\cdot\nabla_{\bar\xi})^k(\nu_*\cdot\nabla_{\bar\xi})^l\tilde\chi_\nu(t,\bar\xi)| \lesssim (1+|t|)^{k+l}\lambda^{-k}\lambda^{-\frac{l}{2}}(1+\lambda^{-1}|\nu\cdot\bar\xi|)^{-N}(1+\lambda^{-\frac12}|\nu_*\cdot\bar\xi|)^{-N}$$

for any N where ν_* denotes a unit vector orthogonal to ν. Indeed, this can be easily seen via rotation and scaling (i.e., setting $\nu = e_1$ and scaling $\xi_1 \to \lambda\xi_1$ and $\xi_2 \to \lambda^{1/2}\xi_2$). Thus, using the above inequality for $0 \le k, l \le 2$ and integration by parts, we see $\|(\tilde\chi_\nu(t,\cdot))^\vee\|_1 \le C(1+|t|)^4$ for a constant $C > 0$. Since $\mathcal{U}f_\nu(x,t,s) = \mathcal{F}^{-1}(e^{i(t\nu\cdot\bar\xi+s|\xi|)}\tilde\chi_\nu(t,\bar\xi)\widehat{f_\nu}(\xi))$, we have

$$\mathcal{U}f_\nu(x,t,s) = \int (\tilde\chi_\nu(t,\cdot))^\vee(\eta)\, e^{is|D|}f_\nu(\bar x-\eta+t\nu, x_3)d\eta.$$

By Minkowski's inequality and changing variables $\bar x \to \bar x + \eta - t\nu$ we see that the left hand side of (4.3.6) is bounded by $C\|\tilde\chi(t)(1+|t|)^4\|_{L^q_t(\mathbb{R}^1)}\big\|e^{is|D|}f_\nu\big\|_{L^q_x(\mathbb{R}^3)}$. Therefore, we get the desired inequality (4.3.6).

Let us set

$$\chi_s(\xi) = e^{is(|\xi|-|\xi^\nu|)}\widetilde\omega_\nu(\bar\xi)\varphi(\bar\xi/\lambda)\varphi(\xi_3/h),$$

where $\xi^\nu := (\bar\xi\cdot\nu, \xi_3)$. Since $\lambda \lesssim h$, similarly as before, one can easily see $\|\widehat{\chi_s}\|_1 \le C$ for a constant. Thus, by Lemma 4.3.2 we have $\|e^{is|D|}f_\nu\|_{L^q_x} \lesssim \|e^{is|D^\nu|}f_\nu\|_{L^q_x}$. Combining this and (4.3.6) yields

$$\big\|\mathcal{U}f_\nu\big\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \|e^{is|D^\nu|}f_\nu\|_{L^q_{x,s}(\mathbb{R}^3\times\mathbb{I}_\tau)} \lesssim \lambda^{\frac{1}{2p}-\frac{1}{2q}}\|e^{is|D^\nu|}f_\nu\|_{L^p_{\bar x'_\nu}(L^q_{\bar x_\nu,x_3,s}(\mathbb{R}^2\times\mathbb{I}_\tau))},$$

where $\bar x_\nu = \nu\cdot\bar x$ and $\bar x'_\nu = \nu_*\cdot\bar x$. For the second inequality we use Bernstein's inequality (see, for example, [82, Ch.5]) and Minkowski's inequality together with the fact that the projection of supp $\widehat{f_\nu}(\cdot,\xi_3) \in \mathbb{R}^2$ to span$\{\nu_*\}$ is contained in an interval of length $\lesssim \lambda^{1/2}$.

Note that the projection $\operatorname{supp} \widehat{f}$ to $\operatorname{span}\{\nu, e_3\}$ is contained in the rectangle $\mathbb{I}_\lambda \times \mathbb{I}_h$. By rotation the matter is reduced to obtaining estimate for the 2-d wave operator. That is to say, the inequality (4.3.5) follows for $q \geq 6$ if we show

$$\|\mathcal{W}_+ g\|_{L^q(\mathbb{R}^2 \times \mathbb{I}_\tau)} \lesssim \tau^{\frac{1}{2}-\frac{1}{p}} \lambda^{1-\frac{1}{p}-\frac{3}{q}} h^{\frac{2}{p}-\frac{1}{2}+\epsilon} \|g\|_{L^p}$$

for $1/p + 3/q \leq 1$ whenever $\operatorname{supp} \widehat{g} \subset \mathbb{I}_h \times \mathbb{I}_\lambda^\circ$. This inequality is an immediate consequence of (2.2.4) and scaling. Indeed, as before, after scaling (i.e., (2.2.3)) we apply Lemma 2.2.4 with $\operatorname{supp} \mathcal{F}(g(\tau\,\cdot)) \subset \mathbb{I}_{\tau h} \times \mathbb{I}_{\tau\lambda}^\circ$. To this end, we use the condition $h \leq \tau\lambda^2$, equivalently, $\tau h \leq (\tau\lambda)^2$.

We now have the estimate (4.3.1) for $6 \leq q$, $2 \leq p$, and $1/p + 3/q \leq 1$. In order to prove it in the full range, by interpolation we only have to show (4.3.1) for $p = q = 4$.

Let us define $f_\pm$ by setting $\widehat{f_\pm}(\xi) = \chi_{(0,\infty)}(\pm\xi_2)\widehat{f}(\xi)$ where χ_E denotes the character function of a set E. Then, changing variables $\xi_2 \to \pm\sqrt{\rho^2 - \xi_1^2}$, we write

$$\mathcal{U}f(x,t,s) = \textstyle\sum_\pm \int e^{i(x_3\xi_3 + t\rho + s\sqrt{\rho^2+\xi_3^2})} \mathcal{F}(\mathcal{S}_\pm^{\bar{x}} f_\pm)(\rho, \xi_3)\, d\rho\, d\xi_3,$$

where

$$\mathcal{F}(\mathcal{S}_\pm^{\bar{x}} f_\pm)(\rho, \xi_3) = \pm \int e^{i(x_1\xi_1 \pm x_2\sqrt{\rho^2-\xi_1^2})} \widehat{f_\pm}(\xi_1, \pm\sqrt{\rho^2 - \xi_1^2}, \xi_3) \frac{\rho}{\sqrt{\rho^2 - \xi_1^2}}\, d\xi_1.$$

We observe the following, which is a consequence of the estimate (2.2.2) with $p = q = 4$ and the finite speed of propagation of the wave operator:

$$\|\mathcal{W}_+ g\|_{L^4_{x_3,t,s}(\mathbb{R}\times\mathbb{I}\times\mathbb{I}_\tau)} \lesssim \tau^{\frac{1}{4}} (\tau h)^\epsilon \|g\|_{L^4_{x_3,t}(\mathbb{R}\times\mathbb{I}_2^\circ)} + h^{-N} \|t^{-N} g\|_{L^4_{x_3,t}(\mathbb{R}\times(\mathbb{I}_2^\circ)^c)} \tag{4.3.7}$$

for any N whenever $\operatorname{supp} g \subset \{\bar{\xi} : |\bar{\xi}| \sim h\}$. Indeed, to show this we decompose $g = g_1 + g_2 := g\chi_{\mathbb{I}_2^\circ}(y_2) + g\chi_{(\mathbb{I}_2^\circ)^c}(y_2)$. By finite speed of propagation (in fact, by straightforward kernel estimate) we have $\|\mathcal{W}_+ g_2\|_{L^4(\mathbb{R}\times\mathbb{I}\times\mathbb{I}_\tau)} \lesssim h^{-N} \| |y_2|^{-N} g\|_{L^4(\mathbb{R}\times(\mathbb{I}_2^\circ)^c)}$. Meanwhile, by scaling and (2.2.2) with $p = q = 4$, we have $\|\mathcal{W}_+ g_1\|_{L^4(\mathbb{R}\times\mathbb{I}\times\mathbb{I}_\tau)} \lesssim \tau^{\frac{1}{4}} (\tau h)^\epsilon \|g\|_{L^4(\mathbb{R}\times\mathbb{I}_2^\circ)}$. Combining those two estimates, we obtain (4.3.7).

We now note that $\mathcal{U}f(x,t,s) = \sum_\pm \mathcal{W}_+(\mathcal{S}_\pm^{\bar{x}} f_\pm)(x_3, t, s)$ and $\operatorname{supp} \mathcal{F}(\mathcal{S}_\pm^{\bar{x}} f_\pm) \subset \{\bar{\xi} : |\bar{\xi}| \sim h\}$ since $\lambda \leq h$. Here, we regard (x_3, t) and s as the spatial and temporal variables, respectively. Applying (4.3.7) to $\mathcal{W}_+(\mathcal{S}_\pm^{\bar{x}} f_\pm)$ with $g = \mathcal{S}_\pm^{\bar{x}} f_\pm$, we obtain

$$\|\mathcal{U}f\|_{L^4_{x,t,s}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \textstyle\sum_\pm \Big(\tau^{\frac{1}{4}} h^\epsilon \|\mathcal{S}_\pm^{\bar{x}} f\|_{L^4_{x,t}(\mathbb{R}^3\times\mathbb{I}_2^\circ)} + h^{-N} \|t^{-N} \mathcal{S}_\pm^{\bar{x}} f\|_{L^4_{x,t}(\mathbb{R}^3\times(\mathbb{I}_2^\circ)^c)} \Big).$$

Reversing the change of variables $\xi_2 \to \pm\sqrt{\rho^2 - \xi_1^2}$, we note that $\mathcal{S}^{\bar{x}}_{\pm} f(x_3, t) = \mathcal{W}_+ f_{\pm}(\cdot, x_3)(\bar{x}, t)$. Recalling $\operatorname{supp} \mathcal{F}f \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, we see that the second term in the right hand side is bounded by a constant times $h^{-N/2}\|f\|_{L^4}$. Since $\operatorname{supp} \mathcal{F}(f(\cdot, x_3)) \subset \mathbb{A}_\lambda$ for all x_3, using Lemma 2.2.2 for $p = q = 4$, we obtain (4.3.1) for $p = q = 4$. This completes the proof of (a).

The case (b) in which $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}^\circ_\lambda$ can be handled without change. We only need to note that the Fourier support of f_ν is included in $\{\xi : |(\xi \cdot \nu, \xi_3)| \sim \lambda\}$, instead of $\{\xi : |(\xi \cdot \nu, \xi_3)| \sim h\}$, if $f_\nu \neq 0$.

We now consider the case (c) where $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ with $\tau\lambda^2 \le h$. The estimate (4.3.2) is easier to show. We note that the Fourier transform of

$$e^{is(|\xi|-|\xi_3|)}\varphi(\bar{\xi}/\lambda)\varphi(\xi_3/h)$$

has uniformly bounded L^1 norm. One can easily verify this using $\partial^\alpha_\xi s(|(\lambda\bar{\xi}, h\xi_3)| - |h\xi_3|) = O(1)$ on $\mathbb{A}^\circ_1 \times \mathbb{I}_1$ if $\tau\lambda^2 \le h$. Thus, by Lemma 4.3.2 we have $\|\mathcal{U}f(\cdot, t, s)\|_{L^q} \lesssim \|e^{it|\bar{D}|}f\|_{L^q}$ uniformly in s. So, taking integration in t, s, we get

$$\|\mathcal{U}f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}}\|e^{it|\bar{D}|}f\|_{L^q(\mathbb{R}^3\times\mathbb{I})} \lesssim \tau^{\frac{1}{q}}h^{\frac{1}{p}-\frac{1}{q}}\|e^{it|\bar{D}|}f\|_{L^p_{x_3}(L^q_{\bar{x},t}(\mathbb{R}^2\times\mathbb{I}))}.$$

For the second inequality we use Bernstein's and Minkowski's inequalities. Using Proposition 2.2.1 in $\bar{x}, t$, we obtain the estimate (4.3.2) for $2 \le p \le q \le \infty$ satisfying $1/p + 3/q \le 1$. □

Remark 1. Following the argument in the proof of Proposition 4.3.1 and using Theorem 2.2.1 and Lemma 2.2.2, one can see without difficulty that $f \to \mathcal{U}f(x, -t, s)$ satisfies the same estimates in Proposition 4.3.1 in place of $\mathcal{U}$. Then, by conjugation and reflection it follows that the estimates also hold for $f \to \mathcal{U}f(x, \pm t, -s)$.

4.4 Estimates for the averaging operator $\mathcal{A}^s_t$

Making use of the estimates for $\mathcal{U}$ in Chapter 4.3 (Proposition 4.3.1), we obtain estimates for the averaging operator $\mathcal{A}^s_t$ while assuming the input function is localized in the Fourier side. These estimates are to play crucial roles in proving Theorem 1.5.1, 1.5.2, and 4.2.1.

We relate $\mathcal{A}^s_t$ to $\mathcal{U}$ via asymptotic expansion of the Fourier transform of $d\sigma^s_t$. Note that

$$\widehat{d\sigma^s_t}(\xi) = \int_0^{2\pi} e^{-is\sin\theta\cdot\xi_3}\widehat{d\mu}((t + s\cos\theta)\bar{\xi})d\theta, \tag{4.4.1}$$

where $d\mu$ denotes the normalized arc length measure on the unit circle. We recall the well known asymptotic expansion of the Bessel function (for example, see [75]):

$$\widehat{d\mu}(\bar{\xi}) = \sum_{\pm,\, 0\le j\le N} C_j^{\pm} |\bar{\xi}|^{-\frac{1}{2}-j} e^{\pm i|\bar{\xi}|} + E_N(|\bar{\xi}|), \qquad |\bar{\xi}| \gtrsim 1 \tag{4.4.2}$$

for some constants $C_j^{\pm}$ where E_N is a smooth function satisfying

$$\big|(d/dr)^l E_N(r)\big| \le C r^{-l-(N+1)/4}, \quad 0 \le l \le N', \tag{4.4.3}$$

for $r \gtrsim 1$ and a constant $C > 0$ where $N' = [(N+1)/4]$. We use (4.4.2) by taking N large enough.

Combining (4.4.1) and (4.4.2) gives an asymptotic expansion for $\mathcal{F}(d\sigma_t^s)$, which we utilize by decomposing f in the Fourier side. We consider the cases $\operatorname{supp} \widehat{f} \subset \{\xi : |\bar{\xi}| > 1/\tau\}$ and $\operatorname{supp} \widehat{f} \subset \{\xi : |\bar{\xi}| \le 1/\tau\}$, separately.

When $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda^\circ \times \mathbb{R}$**,** $\lambda \le 1/\tau$

If $\operatorname{supp} \widehat{f} \subset \mathbb{A}_{1/\tau}^\circ \times \mathbb{I}_{1/\tau}^\circ$, the sharp estimates are easy to obtain.

Lemma 4.4.1. *Let* $1 \le p \le q \le \infty$ *and* $\tau \in (0,1]$. *Suppose* $\operatorname{supp} \widehat{f} \subset B(0, 1/\tau) := \{x : |x| < 1/\tau\}$. *Then, for a constant* $C > 0$ *we have*

$$\|\mathcal{A}_t^s f\|_{L^q_{x,t,s}(\mathbb{R}^3 \times \mathbb{J}_\tau)} \le C \tau^{\frac{4}{q}-\frac{3}{p}} \|f\|_{L^p}. \tag{4.4.4}$$

Proof. Since $\mathcal{A}_t^s$ is a convolution operator and $\operatorname{supp} \widehat{f} \subset B(0, \tau^{-1})$, Bernstein's inequality gives $\|\mathcal{A}_t^s f\|_{L^q_x} \lesssim \tau^{\frac{3}{q}-\frac{3}{p}} \|\mathcal{A}_t^s f\|_{L^p_x}$ for any $s, t \in \mathbb{R}$. Thus, we have

$$\|\mathcal{A}_t^s f\|_{L^q_x} \lesssim \tau^{\frac{3}{q}-\frac{3}{p}} \|f\|_{L^p}, \quad \forall s, t \in \mathbb{R}. \tag{4.4.5}$$

The inequality (4.4.4) follows by integration in t, s over $\mathbb{J}_\tau$. □

Proposition 4.4.2. *Let* $1 \le p \le q \le \infty$, $\tau \lesssim 1$, *and* $h \gtrsim 1/\tau$. *Suppose* $\operatorname{supp} \widehat{f} \subset \mathbb{A}_1^\circ \times \mathbb{I}_h$. *Then, we have*

$$\|\mathcal{A}_t^s f\|_{L^q_{x,t,s}(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{1/q} (\tau h)^{-\frac{1}{2}} h^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p}. \tag{4.4.6}$$

Proof. To prove (4.4.6) it is sufficient to show, for a positive constant C,

$$\|\mathcal{A}_t^s f\|_{L_x^q} \le C(\tau h)^{-\frac{1}{2}} h^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p}, \quad \forall (t,s) \in \mathbb{J}_\tau. \tag{4.4.7}$$

In fact, integration over $\mathbb{J}_\tau$ yields (4.4.6).

For simplicity, we denote $\mathbf{v}_\phi = (\cos\phi, \sin\phi)$, and we note that

$$\mathcal{A}_t^s f(x) = (2\pi)^{-3} \int\int e^{i((\bar{x}-t\mathbf{v}_\phi)\cdot\bar{\xi}+x_3\xi_3-s(\mathbf{v}_\phi\cdot\bar{\xi},\xi_3)\cdot\mathbf{v}_\theta)} \widehat{f}(\xi) d\phi d\theta d\xi.$$

Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_1^\circ \times \mathbb{I}_h$, we may disregard the factor $e^{-it\mathbf{v}_\phi\cdot\bar{\xi}}$ using Lemma 4.3.2. Indeed, let $\rho \in C_c(\mathbb{A}_2^\circ)$ such that $\rho = 1$ on $\mathbb{A}_1$. Setting $\rho_t^\phi(\bar{\xi}) = \rho(\bar{\xi})e^{it\mathbf{v}_\phi\cdot\bar{\xi}}$, we see $\|\mathcal{F}(\rho_t^\phi)\|_1 \le C$ for a constant $C > 0$ and $|t| \lesssim 1$. Thus, by Minkowski's inequality and Lemma 4.3.2 we have

$$\|\mathcal{A}_t^s f\|_{L_x^q} \lesssim \sup_\phi \Big\| \int e^{ix\cdot\xi} \int_0^{2\pi} e^{-is(\mathbf{v}_\phi\cdot\bar{\xi},\xi_3)\cdot\mathbf{v}_\theta} d\theta \widehat{f}(\xi) d\xi \Big\|_{L_x^q}$$

for $|t| \lesssim 1$. We denote $\xi_\phi = (\mathbf{v}_\phi\cdot\bar{\xi}, \xi_3)$, and notice that $|s\xi_\phi| \gtrsim 1$ since $h\tau \ge 1$. So, usng (4.4.2), we have

$$\textstyle\int e^{-is\xi_\phi\cdot\mathbf{v}_\theta} d\theta = \sum_{\pm,\, 0\le j\le N} C_j^\pm |s\xi_\phi|^{-\frac{1}{2}-j} e^{\pm is|\xi_\phi|} + E_N(s|\xi_\phi|).$$

To show (4.4.7), we obtain only the estimates for the operators $m_s^\pm(D)$, $E_N(s|D_\phi|)$ whose multipliers are given by

$$m_s^\pm(\xi) := |s\xi_\phi|^{-1/2} e^{\pm is|\xi_\phi|}, \quad E_N(s|\xi_\phi|).$$

Contributions from the multiplier operators associated with the other terms can be handled similarly but those are easier. Since $|\bar{\xi}| < 2$ and $|\xi_3| \sim h \ge 1/\tau$, we use the Mikhlin multiplier theorem and Lemma 4.3.2 to see

$$\big\|m_s^\pm(D)f\big\|_{L_x^q} \lesssim (\tau h)^{-\frac{1}{2}} \Big\| \int e^{i(x\cdot\xi\pm s|\xi_3|)} \widehat{f}(\xi) d\xi \Big\|_{L_x^q} \le (\tau h)^{-\frac{1}{2}} \|f\|_{L_x^q}.$$

Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_1^\circ \times \mathbb{I}_h$, by Bernstein's lemma we have $\|f\|_{L^q} \lesssim h^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p}$. This gives the desired estimates for $m_s^\pm(D)$. For the multiplier operator $E_N(s|D_\phi|)$, note from (4.4.3) that $\partial_{\xi_\phi}^\alpha(|s\xi_\phi|^{N'} E_N(|s\xi_\phi|) \le C(|s\xi_\phi|^{-|\alpha|})$ for $|\alpha| \le N'$ and a constant $C > 0$. Using the Mikhlin multiplier theorem again, we have

$$\big\|E_N(s|D_\phi|)f\big\|_{L_x^q} \lesssim \Big\| \int e^{ix\cdot\xi} |s\xi_3|^{-N'} \widehat{f}(\xi) d\xi d\theta \Big\|_{L_x^q}.$$

Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_1^\circ \times \mathbb{I}_h$, we see, as before, that the right hand side is bounded by $C(h\tau)^{-N'} h^{1/p-1/q}\|f\|_{L^p}$. Thus, the desired estimate for $E_N(s|D_\phi|)$ follows. □

When $\lambda \gtrsim 1$, to handle the case $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ we need more than the estimates with fixed t, s. We need the smoothing estimates obtained in Chapter 4.3.

Proposition 4.4.3. *Let $2 \le p \le q \le \infty$, $1/p + 1/q \le 1$, and $1 \lesssim \lambda \lesssim 1/\tau \lesssim h$. Suppose $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. Then, for any $\epsilon > 0$ we have the following:*

$$\|\mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}}(\tau h)^{-\frac{1}{2}} h^{\frac{1}{p}-\frac{1}{q}} \lambda^{\frac{1}{p}-\frac{3}{q}+\epsilon}\|f\|_{L^p}, \qquad 1/p + 3/q \le 1, \tag{4.4.8}$$

$$\|\mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}}(\tau h)^{-\frac{1}{2}} h^{\frac{1}{p}-\frac{1}{q}} \lambda^{-\frac{1}{2}+\frac{3}{2p}-\frac{3}{2q}+\epsilon}\|f\|_{L^p}, \qquad 1/p + 3/q > 1. \tag{4.4.9}$$

To show Proposition 4.4.3, as mentioned above, we use the asymptotic expansion of the Fourier transform of $d\sigma_t^s$. Let us set

$$m_l^\pm(\xi, t, s) = \int e^{-i(s\xi_3 \sin\theta \mp s|\bar\xi|\cos\theta)} a_l(\theta, t, s)\, d\theta,$$

where $a_l(\theta, t, s) = (t + s\cos\theta)^{-(2l+1)/2}$. Putting (4.4.1) and (4.4.2) together, we have

$$\widehat{d\sigma_t^s}(\xi) = \textstyle\sum_{\pm, 0\le l\le N} M_l^\pm(\xi, t, s) + \mathcal{E}(\xi, t, s) \tag{4.4.10}$$

for $|\bar\xi| \gtrsim 1$ where

$$M_l^\pm(\xi, t, s) = C_l|\bar\xi|^{-l-\frac{1}{2}} e^{\pm it|\bar\xi|} m_l^\pm(\xi, t, s), \qquad l = 0, \dots, N, \tag{4.4.11}$$

$$\mathcal{E}(\xi, t, s) = \int e^{-is\xi_3\sin\theta} E_N((t + s\cos\theta)|\bar\xi|)\, d\theta. \tag{4.4.12}$$

Proof. We first show (4.4.8). From (4.4.10) we need to obtain estimates for the operators associated to the multipliers $M_l^\pm$ and $\mathcal{E}$. The major contributions are from $M_l^\pm(D, t, s)$. We claim that

$$\left\|M_l^\pm(D, t, s) f\right\|_{L^q_{x,t}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}}(\tau h)^{-\frac{1}{2}} h^{\frac{1}{p}-\frac{1}{q}} \lambda^{\frac{1}{p}-\frac{3}{q}-l+\epsilon}\|f\|_{L^p} \tag{4.4.13}$$

holds for $p \le q$ and $1/p + 3/q \le 1$. To show this, we consider the operator $e^{\pm it|\bar{D}|} m_l^{\pm}(D,t,s)$. Note that $m_l^{\pm}(\xi,t,s) = \int e^{-is(\mp|\bar{\xi}|,\xi_3)\cdot \mathbf{v}_\theta} a_l(\theta,t,s)d\theta$. By the stationary phase method, we have

$$m_l^{\pm}(\xi,t,s) = \sum_{\pm,0\le j\le N} B_j^{\pm}|s\xi|^{-\frac{1}{2}-j}e^{\pm i|s\xi|} + \tilde{E}_N^{\pm}(s|\xi|), \quad (t,s)\in \mathbb{J}_\tau \tag{4.4.14}$$

for $|s\xi| \gtrsim 1$. Here, $B_l^{\pm}$ and $\tilde{E}_N^{\pm}$ depend on t, s. However, $(\partial/\partial_\theta)^k a_l$ is uniformly bounded since $s < c_0 t$, i.e., $(t,s) \in \mathbb{J}_0$, so $B_l^{\pm}$ are uniformly bounded and $\tilde{E}_N^{\pm}$ satisfies (4.4.3) in place of E_N as long as $(t,s)\in \mathbb{J}_\tau$.

For the error term $\tilde{E}_N^{\pm}(s|\xi|)$, we can replace it, similarly as before, by $|s\xi|^{-N'}$ using the Mikhlin multiplier theorem. Thus, using (2.2.2) and Bernstein's inequality in x_3 (see, for example, [82, Ch.5]), we obtain

$$\big\|\chi_{\mathbb{J}_\tau}(t,s)e^{\pm it|\bar{D}|}\tilde{E}_N^{\pm}(s|D|)f\big\|_{L^q_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim (\tau h)^{-N'}h^{\frac{1}{p}-\frac{1}{q}}\lambda^{\frac{1}{2}+\frac{1}{p}-\frac{3}{q}+\epsilon}\|f\|_{L^p} \tag{4.4.15}$$

for p, q satisfying $1/p + 3/q \le 1$ since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, $s \in \mathbb{I}_\tau$, and $\tau h \gtrsim 1$. Recalling (4.4.14), we consider the multiplier operator given by

$$a_{l,t,s}^{\pm}(\xi) = \textstyle\sum_{\pm,0\le j\le N} B_j^{\pm}|s\xi|^{-\frac{1}{2}-j}.$$

Since $\lambda \lesssim 1/\tau \lesssim h$, using the same argument as before (e.g., Lemma 4.3.2), we may replace $e^{\pm i|s\xi|}$ with $e^{\pm i|s\xi_3|}$. By the Mikhlin multiplier theorem, we have

$$\big\|\chi_{\mathbb{J}_\tau}(t,s)e^{\pm i(t|\bar{D}|+s|D|)}a_{l,t,s}^{\pm}(D)f\big\|_{L^q_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim (\tau h)^{-\frac{1}{2}}\Big\|\chi_{\mathbb{J}_\tau}(t,s)e^{\pm it|\bar{D}|}f\big\|_{L^q_{x,t}(\mathbb{R}^3\times\mathbb{I})}.$$

Applying (2.2.4) and Bernstein's inequality as before, we have the left hand side bounded by $(\tau h)^{-\frac{1}{2}}h^{\frac{1}{p}-\frac{1}{q}}\lambda^{\frac{1}{2}+\frac{1}{p}-\frac{3}{q}+\epsilon}\|f\|_{L^p}$ for $1/p + 3/q \le 1$. Combining this and (4.4.15), we obtain

$$\big\|\chi_{\mathbb{J}_\tau}(t,s)M_l^{\pm}(D,t,s)f\big\|_{L^q_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim (\tau h)^{-\frac{1}{2}}h^{\frac{1}{p}-\frac{1}{q}}\lambda^{\frac{1}{p}-\frac{3}{q}-l+\epsilon}\|f\|_{L^p}.$$

Thus, taking integration in s gives (4.4.13).

We now consider the contribution of the error term $\mathcal{E}$ in (4.4.10), whose contribution is less significant. It can be handled by using the estimates for fixed $(t,s)\in\mathbb{J}_\tau$. Recalling (4.4.10), we set

$$E_N^0(\theta) := E_N^0(\theta,s,t,\bar{\xi}) = |\bar{\xi}|^{N'}E_N((t+s\cos\theta)|\bar{\xi}|).$$

We have $|\partial_\theta^n E_N^0(\theta)| \lesssim 1$ uniformly in n, θ for $(t,s) \in \mathbb{J}_\tau$ since $(t + s\cos\theta) \gtrsim 1 - c_0$ for $(t,s) \in \mathbb{J}_\tau$. By the stationary phase method [36, Theorem 7.7.5] one can obtain a similar expansion as before:

$$\int e^{-is\xi_3 \sin\theta} E_N^0(\theta) d\theta = \sum_{\pm, 0 \le w \le M} D_w^\pm |s\xi_3|^{-\frac{1}{2}-w} e^{\pm is\xi_3} + E'_M(|s\xi_3|) \quad (4.4.16)$$

for $(t,s) \in \mathbb{J}_\tau$. Here, E'_M satisfies the same bounds as E_N (i.e., (4.4.3)) and $M \le N/4$. $D_w^\pm$ and E'_M depend on t, ξ, but they are harmless as can be seen by the Mikhlin multiplier theorem. The contribution from E'_M can be directly controlled by the Mikhlin multiplier theorem. Since supp $f \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, Bernstein's inequality gives

$$\Big\| \int e^{-isD_3 \sin\theta} E_N((t + s\cos\theta)|D|) d\theta f \Big\|_{L_x^q} \lesssim (\tau h)^{-\frac{1}{2}} \lambda^{-N'} (\lambda^2 h)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p}$$

for $(t,s) \in \mathbb{J}_\tau$. Note that the implicit constant here does not depend on t, s. Thus, integration in s, t gives

$$\|\mathcal{E}(D,t,s)f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \le C\tau^{\frac{1}{q}} (\tau h)^{-\frac{1}{2}} h^{\frac{1}{p} - \frac{1}{q}} \lambda^{2-N'} \|f\|_p \quad (4.4.17)$$

for $1 \le p \le q \le \infty$. So, the contribution of $\mathcal{E}(D,t,s)f$ is acceptable. Therefore, from (4.4.10) and (4.4.13), we obtain (4.4.8).

Putting (4.4.10), (4.4.11), (4.4.12), and (4.4.14) together, by Plancherel's theorem one can easily see $\|\mathcal{A}_t^s f\|_{L_x^2} \lesssim (\tau h)^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \|f\|_2$. Thus, integration in s, t gives

$$\|\mathcal{A}_t^s f\|_{L^2(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim h^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \|f\|_2, \quad (4.4.18)$$

which is (4.4.9) for $p = q = 2$. Interpolation between this and the estimate (4.4.8) for p, q satisfying $1/p + 3/q = 1$ gives (4.4.9) for $1/p + 3/q > 1$. □

When supp $\widehat{f} \subset \mathbb{A}_\lambda^\circ \times \mathbb{R}$ **and** $\lambda \gtrsim 1/\tau$

We have the following estimate.

Proposition 4.4.4. *Let* $2 \le p \le q \le \infty$ *satisfy* $1/p + 1/q \le 1$. *(a) If* $1/\tau \lesssim \lambda \lesssim h \lesssim \tau\lambda^2$, *then for any* $\epsilon > 0$ *we have the estimates*

$$\|\mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{\frac{3}{2q} - \frac{1}{2} - \frac{1}{2p}} h^{-\frac{1}{2} + \frac{3}{2p} - \frac{3}{2q} + \epsilon} \lambda^{\frac{1}{2p} - \frac{1}{2q} - \frac{1}{2}} \|f\|_{L^p} \quad (4.4.19)$$

for $1/p+3/q>1$*, and*

$$\|\mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{-\frac{1}{p}} h^{-1+\frac{2}{p}+\epsilon} \lambda^{1-\frac{1}{p}-\frac{5}{q}} \|f\|_{L^p} \tag{4.4.20}$$

for $1/p+3/q \le 1$ *whenever* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. (*b*) *If* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, *the estimates* (4.4.19) *and* (4.4.20) *hold with* $h=\lambda$. (*c*) *Suppose* $1/\tau \lesssim \lambda$ *and* $h \gtrsim \lambda^2\tau$*, then the estimates* (4.4.8) *and* (4.4.9) *hold whenever* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$.

We can prove Proposition 4.4.4 in the same manner as Proposition 4.4.3, using the expansions (4.4.10) and (4.4.14). By (4.4.17) we may disregard the contribution from $\mathcal{E}$. Thus, we only need to handle $M_l^\pm$. Moreover, one can easily see the contribution from the multiplier operator $\tilde{E}_N^\pm(s|D|)$ is acceptable. In fact, we have the following.

Lemma 4.4.5. *Let* $2 \le p \le q \le \infty$ *and* $1/p+1/q \le 1$. *If* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ *and* $h \gtrsim \lambda$, *then we have the estimates*

$$\big\||\bar{D}|^{-\frac{1}{2}} e^{\pm it|\bar{D}|} \tilde{E}_N^\pm(s|D|) f\big\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}} (\tau h)^{-N'} h^{\frac{1}{p}-\frac{1}{q}} \lambda^{\frac{1}{p}-\frac{3}{q}+\epsilon} \|f\|_{L^p} \tag{4.4.21}$$

for $1/p+3/q \le 1$*, and*

$$\big\||\bar{D}|^{-\frac{1}{2}} e^{\pm it|\bar{D}|} \tilde{E}_N^\pm(s|D|) f\big\|_{L^q(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}} (\tau h)^{-N'} h^{\frac{1}{p}-\frac{1}{q}} \lambda^{\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}+\epsilon} \|f\|_{L^p} \tag{4.4.22}$$

for $1/p+3/q>1$. *If* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, (4.4.21) *and* (4.4.22) *hold with* $h=\lambda$.

Proof. We first consider the case $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ and $h \gtrsim \lambda$. The estimate (4.4.21) is easy to show by using (2.2.1) and Bernstein's inequality (for example, see (4.4.15)). Note that (4.4.22) with $p=q=2$ follows by Plancherel's theorem. Thus, interpolation between this estimate and (4.4.21) for $1/p+3/q=1$ gives (4.4.22) for $1/p+3/q>1$. If $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, the estimates (4.4.21) and (4.4.22) with $h=\lambda$ follow in the same manner. We omit the detail. □

Proof of Proposition 4.4.4. Recalling (4.4.14) and comparing the estimates (4.4.21) and (4.4.19), we notice that it is sufficient to consider the estimates for the multiplier operators defined by $B_j^\pm |s\xi|^{-\frac{1}{2}-j} e^{\pm i|s\xi|}$. Therefore, the matter is reduced to obtaining, instead of $\mathcal{A}_t^s$, the estimates for the operators

$$\mathcal{C}_\pm^\kappa f(x,t,s) := |\bar{D}|^{-\frac{1}{2}} |sD|^{-\frac{1}{2}} \mathcal{U} f(x,\kappa t,\pm s), \quad \kappa=\pm, \tag{4.4.23}$$

which constitute the major part. We first consider the case (a): $1/\tau \lesssim \lambda \lesssim h \lesssim \tau\lambda^2$ and $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. Note that

$$\|\mathcal{C}_\pm^\kappa f(\cdot, s, t)\|_{L^q(\mathbb{R}^3)} \lesssim (\tau\lambda h)^{-\frac{1}{2}} \|\mathcal{U} f(\cdot, \kappa t, \pm s)\|_{L^q(\mathbb{R}^3)}$$

for $\kappa = \pm$. Thus, by (4.3.1) and Remark 1 we get

$$\|\mathcal{C}_\pm^\kappa f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{-\frac{1}{p}} h^{-1+\frac{2}{p}+\epsilon} \lambda^{1-\frac{1}{p}-\frac{5}{q}} \|f\|_{L^p}, \quad \kappa = \pm$$

for $1/p + 3/q \le 1$. Therefore, we obtain (4.4.20). So, (4.4.19) follows from interpolation with (4.4.18).

If $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, by the estimate (4.3.1) with $\lambda = h$ ((b) in Lemma 4.3.1) we get the desired estimates (4.4.20) and (4.4.19) with $h = \lambda$. This proves (b).

If $1/\tau \lesssim \lambda$, $h \gtrsim \lambda^2\tau$, and $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, the estimate (4.4.8) follows by (4.3.2). As a result, we get (4.4.9) by interpolation between (4.4.18) and (4.4.8). $\square$

Since the main contribution to the estimate for $\mathcal{A}_t^s f$ is from $\mathcal{C}_t^s f$, by the same argument in the proof of Proposition 4.4.4 one can easily obtain the next.

Corollary 4.4.6. *Let* $\alpha, \beta \in \mathbb{N}_0$. (a) *If* $1/\tau \lesssim \lambda \lesssim h \lesssim \tau\lambda^2$, *then for any* $\epsilon > 0$

$$\|\partial_t^\alpha \partial_s^\beta \mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{\frac{3}{2q}-\frac{1}{2}-\frac{1}{2p}} h^{\beta-\frac{1}{2}+\frac{3}{2p}-\frac{3}{2q}+\epsilon} \lambda^{\alpha+\frac{1}{2p}-\frac{1}{2q}-\frac{1}{2}} \|f\|_{L^p}, \quad 1/p + 3/q > 1,$$
$$\|\partial_t^\alpha \partial_s^\beta \mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{-\frac{1}{p}} h^{\beta-1+\frac{2}{p}+\epsilon} \lambda^{\alpha+1-\frac{1}{p}-\frac{5}{q}} \|f\|_{L^p}, \quad 1/p + 3/q \le 1,$$

hold whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. (b) *If* $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, *we obtain the above two estimates with* $h = \lambda$. (c) *When* $1/\tau \lesssim \lambda$ *and* $h \gtrsim \lambda^2\tau$, *for any* $\epsilon > 0$ *we have*

$$\|\partial_t^\alpha \partial_s^\beta \mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}} (\tau h)^{-\frac{1}{2}} h^{\beta+\frac{1}{p}-\frac{1}{q}} \lambda^{\alpha+\frac{1}{p}-\frac{3}{q}+\epsilon} \|f\|_{L^p}, \quad 1/p + 3/q \le 1,$$
$$\|\partial_t^\alpha \partial_s^\beta \mathcal{A}_t^s f\|_{L^q(\mathbb{R}^3 \times \mathbb{J}_\tau)} \lesssim \tau^{\frac{1}{q}} (\tau h)^{-\frac{1}{2}} h^{\beta+\frac{1}{p}-\frac{1}{q}} \lambda^{\alpha-\frac{1}{2}+\frac{3}{2p}-\frac{3}{2q}+\epsilon} \|f\|_{L^p}, \quad 1/p + 3/q > 1,$$

whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$.

Remark 2. By (4.4.10) and (4.4.14) it follows that

$$|\widehat{d\sigma_t^s}(\xi)| \lesssim (1 + |\xi_3|)^{-1/2} (1 + |\bar{\xi}|)^{-1/2}.$$

Furthermore, if $|\bar{\xi}| \lesssim 1$, we have $|\widehat{d\sigma_t^s}(\xi)| \sim |\xi|^{-1/2}$ for $|\xi|$ large enough. Therefore, by Plancherel's theorem one can see that the L^2 to $L^2_{1/2}$ estimate for $\mathcal{A}_t^s$ is optimal. One can also see that the part of the surface $\mathbb{T}_t^s$ near the sets $\{\Phi_s^t(\pm\pi/2, \phi) : \phi \in [0, 2\pi)\}$ is responsible for the worst decay while the Fourier transform of the part (of the surface) away from the sets enjoys better decay.

4.5 Global maximal estimates

Now we prove our main theorems in this chapter. First, we recall an elementary lemma, which enables us to relate the local smoothing estimate to the estimate for the maximal function and also a generalization of Lemma 3.1.3.

Lemma 4.5.1. *Let $1 \le p \le \infty$, and let I and J be closed intervals of length 1 and ℓ, respectively. Suppose G be a smooth function on the rectangle $R = I \times J$. Then, for any $\lambda, h > 0$, we have*

$$\sup_{(t,s)\in I\times J} |G(t,s)| \lesssim (1+\lambda^{\frac{1}{p}})(\ell^{-\frac{1}{p}} + h^{\frac{1}{p}})\|G\|_{L^p(R)} + (\ell^{-\frac{1}{p}} + h^{\frac{1}{p}})\lambda^{-\frac{1}{p'}}\|\partial_t G\|_{L^p(R)}$$
$$+ (1+\lambda^{\frac{1}{p}})h^{-\frac{1}{p'}}\|\partial_s G\|_{L^p(R)} + \lambda^{-\frac{1}{p'}}h^{-\frac{1}{p'}}\|\partial_t\partial_s G\|_{L^p(R)}.$$

Proof. We first recall the inequality

$$\sup_{t\in I'} |F(t)| \lesssim |I'|^{-1/p}\|F\|_{L^p(I')} + \|F\|_{L^p(I')}^{(p-1)/p}\|\partial_t F\|_{L^p(I')}^{1/p},$$

which holds whenever F is a smooth function defined on an interval I' (for example, see [44]). By Young's inequality we have

$$\sup_{t\in I'} |F(t)| \lesssim |I'|^{-1/p}\|F\|_{L^p(I')} + \lambda^{1/p}\|F\|_{L^p(I')} + \lambda^{-1/p'}\|\partial_t F\|_{L^p(I')}.$$

for any $\lambda > 0$. We use this inequality with $F = G(\cdot, s)$ and $I' = I$ to get

$$\sup_{(t,s)\in I\times J} |G(t,s)| \lesssim (1+\lambda^{1/p})\|\sup_{s\in J}|G(t,s)|\|_{L^p(I)} + \lambda^{-1/p'}\|\sup_{s\in J}|\partial_t G(t,s)|\|_{L^p(I)}.$$

Then, we apply the above inequality again to $G(t,\cdot)$ and $\partial_t G(t,s)$ with $I' = J$ taking $\lambda = h$. □

By a standard argument using scaling, it is sufficient to show L^p boundedness of a localized maximal operator

$$\mathfrak{M}f(x) = \sup_{0<s<c_0 t<1} \big|\mathcal{A}_t^s f(x)\big|.$$

Furthermore, we only need to show that $\mathfrak{M}$ is bounded on L^p for $2 < p \le 4$ since the other estimates follow by interpolation with the trivial L^∞ bound. To this end, we consider

$$\mathfrak{M}_n f(x) = \sup_{(t,s)\in\mathbb{J}_{2^{-n}}} \big|\mathcal{A}_t^s f(x)\big|, \quad n \ge 0. \tag{4.5.1}$$

In order to obtain estimates for $\mathfrak{M}_n$, we consider $\mathfrak{M}_n f_j^k$ for each j, k. The correct bounds in terms of n, not to mention j, k, are also important for our purpose.

Lemma 4.5.2. *Let $k, j \ge n$. ($\tilde{a}$) If $j \le k \le 2j - n$, we have*

$$\|\mathfrak{M}_n f_j^k\|_{L^q} \lesssim \begin{cases} 2^{n(\frac{1}{2}+\frac{1}{2p}-\frac{3}{2q})+j(\frac{1}{2p}+\frac{1}{2q}-\frac{1}{2})+k(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)}\|f\|_{L^p}, & \frac{1}{p}+\frac{3}{q}\ge 1, \\ 2^{\frac{n}{p}+j(1-\frac{1}{p}-\frac{4}{q})+k(\frac{2}{p}+\frac{1}{q}-1+\epsilon)}\|f\|_{L^p}, & \frac{1}{p}+\frac{3}{q}< 1. \end{cases} \tag{4.5.2}$$

($\tilde{b}$) For $\mathfrak{M}_n f_j^{<j}$, the same bounds hold with $k = j$. ($\tilde{c}$) If $2j - n \le k$, then we have

$$\|\mathfrak{M}_n f_j^k\|_{L^q} \lesssim \begin{cases} 2^{n(\frac{1}{2}-\frac{1}{q})+j(\frac{3}{2p}-\frac{1}{2q}-\frac{1}{2}+\epsilon)+k(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}, & \frac{1}{p}+\frac{3}{q}\ge 1, \\ 2^{n(\frac{1}{2}-\frac{1}{q})+j(\frac{1}{p}-\frac{2}{q}+\epsilon)+k(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}, & \frac{1}{p}+\frac{3}{q}< 1. \end{cases} \tag{4.5.3}$$

Proof. Let n_0 be the smallest integer such $2^{-n_0+1} \le c_0$. If $n \ge n_0$, then $\mathbb{J}_{2^{-n}} = \mathbb{I} \times \mathbb{I}_{2^{-n}}$. Since $n \le k, j$, using Lemma 4.5.1, one can obtain ($\tilde{a}$), ($\tilde{b}$), and ($\tilde{c}$) from (a), (b), and (c) in Corollary 4.4.6, respectively. For $n < n_0$, we can not directly apply Lemma 4.5.1. However, this can be easily overcome by a simple modification. Indeed, we cover $\bigcup_{n=0}^{n_0-1} \mathbb{J}_{2^{-n}}$ with essentially disjoint closed dyadic cubes Q of side length $L \in (2^{-7}(1-c_0), 2^{-6}(1-c_0)]$ so that $\bigcup Q \subset \mathbb{J}_0' := \{(t,s) : 2^{1-n_0} \le s < 2^{-1}(1+c_0)t, 1 \le t \le 2\}$. Thus, we note

$$\big\| \sup_{(t,s)\in\mathbb{J}_{2^{-n}}} |\mathcal{A}_t^s g| \big\|_{L^q} \lesssim \sum_Q \big\| \sup_{(t,s)\in Q} |\mathcal{A}_t^s g| \big\|_{L^q}.$$

for $n < n_0$. We may now apply Lemma 4.5.1 to $\mathcal{A}_t^s g$ and Q. Since $\bigcup Q \subset \mathbb{J}_0'$, we clearly have the same maximal bounds up to a constant multiple for $n < n_0$. $\square$

We denote $\mathrm{Q}_l^m = \mathbb{J}_0 \cap (\mathbb{I}_{2^{-l}} \times \mathbb{I}_{2^{-m}})$ for simplicity. Then, it follows that

$$\mathfrak{M}f(x) = \sup_{m \ge l \ge 0} \sup_{(t,s) \in \mathrm{Q}_l^m} |\mathcal{A}_t^s f|.$$

Decomposing $f = \sum_{j,k} f_j^k$, we have

$$\mathfrak{M}f(x) \le \mathfrak{N}^1 f + \mathfrak{N}^2 f + \mathfrak{N}^3 f + \mathfrak{N}^4 f,$$

where

$$\mathfrak{N}^1 f = \sup_{m \ge l \ge 0} \sup_{(t,s) \in \mathrm{Q}_l^m} |\mathcal{A}_t^s f_{\le l}^{\le m}|, \qquad \mathfrak{N}^2 f = \sup_{m \ge l \ge 0} \sup_{(t,s) \in \mathrm{Q}_l^m} |\mathcal{A}_t^s f_{\le l}^{> m}|,$$
$$\mathfrak{N}^3 f = \sup_{m \ge l \ge 0} \sup_{(t,s) \in \mathrm{Q}_l^m} |\mathcal{A}_t^s f_{> l}^{\le m}|, \qquad \mathfrak{N}^4 f = \sup_{m \ge l \ge 0} \sup_{(t,s) \in \mathrm{Q}_l^m} |\mathcal{A}_t^s f_{> l}^{> m}|.$$

The maximal operators $\mathfrak{N}^1, \mathfrak{N}^2$ and $\mathfrak{N}^3$ can be handled by using the L^p bounds on the Hardy-Littlewood maximal and the circular maximal functions.

We first handle $\mathfrak{N}^1 f$. We set $\bar{K} = \mathcal{F}^{-1}(\varphi_{\le 1}(|\bar{\xi}|))$ and $K_3 = \mathcal{F}^{-1}(\varphi_{\le 1}(|\xi_3|))$. Since $\mathcal{F}(f_{\le l}^{\le m})(\xi) = \varphi_{\le l}(\bar{\xi})\varphi_{\le m}(\xi_3)\widehat{f}(\xi)$ and $\varphi_{\le m}(t) = \varphi_{\le 1}(2^{-m}t)$, we have

$$f_{\le l}^{\le m}(x) = 2^{2l+m} \int f(x-y)\bar{K}(2^l \bar{y}) K_3(2^m y_3) dy.$$

Hence, it follows that

$$\mathcal{A}_t^s f_{\le l}^{\le m}(x) = 2^{2l+m} \int_{\mathbb{T}_t^s} \int f(x-y)\bar{K}(2^l(\bar{y}-\bar{z})) K_3(2^m(y_3 - z_3)) dy \, d\sigma_t^s(z).$$

If $(t,s) \in \mathrm{Q}_l^m$, $|\bar{K}(2^l(\bar{y}-\bar{z}))K_3(2^m(y_3-z_3)| \le C(1+2^l|\bar{y}|)^{-M}(1+2^m|y_3|)^{-M}$ for any M. By a standard argument using dyadic decomposition, we see

$$\mathfrak{N}^1 f(x) \lesssim \bar{H} H_3 f(x),$$

where $\bar{H}$ and H_3 denote the 2-d and 1-d Hardy-Littlewood maximal operators acting on $\bar{x}$ and x_3, respectively. The right hand side is bounded by the strong maximal function. Thus, $\mathfrak{N}^1$ is bounded on L^p whenever $p > 1$.

Next, we consider $\mathfrak{N}^2$. Since $f_{\le l}^{> m}(x) = 2^{2l}\big(f^{>m}(\cdot, x_3) * \bar{K}(2^l \cdot)\big)(\bar{x})$, we have

$$\mathcal{A}_t^s f_{\le l}^{> m} = 2^{2l} \int f^{>m}(\bar{x}-\bar{y}, x_3 - s\sin\theta)\bar{K}(2^l(\bar{y} - (t + s\cos\theta)\mathbf{v}_\phi)) d\theta d\phi d\bar{y}.$$

Note that $s < c_0 t \lesssim 2^{-l}$, so we have $|\bar{K}(2^l(\bar{y} - (t + s\cos\theta)\mathbf{v}_\phi))| \lesssim C(1 + 2^l|\bar{y}|)^{-M}$ for any M. Similarly as above, this gives

$$|\mathcal{A}_t^s f_{\le l}^{>m}(x)| \lesssim \int_0^{2\pi} \bar{H} f^{>m}(\bar{x}, x_3 - s\sin\theta)d\theta \lesssim \int_0^{2\pi} \bar{H} H_3 f(\bar{x}, x_3 - s\sin\theta)d\theta$$

For the second inequality, we use $f^{>m} = f - f^{\le m}$ and $|f|, |f^{\le m}| \le H_3 f$. As a result, we have

$$\mathfrak{N}^2 f(x) \lesssim \sup_{s>0} \int_0^{2\pi} \bar{H} H_3 f(\bar{x}, x_3 - s\sin\theta)d\theta.$$

To handle the consequent maximal operator, we use the following simple lemma.

Lemma 4.5.3. *For $p > 2$, we have the estimate*

$$\Big\| \sup_{0<s<1} \Big| \int g(x_3 - s\sin\theta)d\theta \Big| \Big\|_{L^p_{x_3}} \lesssim \|g\|_{L^p}.$$

Proof. Let us define $\widetilde{g}$ on $\mathbb{R}^2$ by setting $\widetilde{g}(z, x_3) = g(x_3)$ for $x_3 \in \mathbb{R}$ and $-10 \le z \le 10$, and $\widetilde{g}(z, x_3) = 0$ if $|z| > 10$. Note that $\int g(x_3 - s\cos\theta)d\theta = \int \widetilde{g}(z - s\cos\theta, x_3 - s\sin\theta)d\theta$ for $|z| \le 1, 0 < s < 1$. So, $\sup_{0<s<1} |\int g(x_3 - s\sin\theta)d\theta| \lesssim M_{cr}\widetilde{g}(z, x_3)$ for $|z| \le 1$, where M_{cr} denotes the circular maximal operator. By the circular maximal theorem [7], $\|\sup_{0<s<1} |\int g(x_3 - s\sin\theta)d\theta\|_{L^p_{x_3}}$ is bounded above by a constant times $\|\widetilde{g}\|_{L^p_{x_3,z}} = 20^{1/p}\|g\|_{L^p_{x_3}}$ for $p > 2$. □

Therefore, by Lemma 4.5.3 and L^p boundedness of $\bar{H}$ and H_3 we see that $\mathfrak{N}^2$ is bounded on L^p for $p > 2$.

$\mathfrak{N}^3$ can be handled similarly. Since $f_{>l}^{\le m} = 2^m\big(f_{>l}(\bar{x}, \cdot) * K_3(2^m \cdot)\big)(x_3)$, we get

$$\mathcal{A}_t^s f_{>l}^{\le m}(x) = 2^m \int f_{>l}(\bar{x} - (t + s\cos\theta)\mathbf{v}_\phi, x_3 - y_3)K_3(2^m(y_3 - s\sin\theta))d\theta d\phi dy_3.$$

Since $s \lesssim 2^{-m}$, $|K_3(2^m(y_3 - s\sin\theta))| \lesssim (1 + 2^m|y_3|)^{-N}$. Hence, using $f_{>l} = f - f_{\le l}$ and $|f|, |f_{\le l}| \le \bar{H} f$, we have

$$|\mathcal{A}_t^s f_{>l}^{\le m}(x)| \lesssim \int_0^{2\pi} H_3 \bar{H} f(\bar{x} - (t + s\cos\theta)\mathbf{v}_\phi, x_3)d\phi \lesssim M_{cr}[(H_3 \bar{H} f)(\cdot, x_3)](\bar{x}).$$

Thus, $\mathfrak{N}_3 f(x) \lesssim M_{cr}[(H_3\bar{H}f)(\cdot, x_3)](\bar{x})$. Using the circular maximal theorem, we see that $\mathfrak{N}^3$ is bounded on L^p for $p > 2$.

Finally, we consider $\mathfrak{N}^4$. For simplicity, we set

$$\mathfrak{A}_{l,j}^{m,k} f = \sup_{(t,s)\in \mathrm{Q}_l^m} |\mathcal{A}_t^s f_j^k|.$$

Decomposing $\sum_{j\ge l,k\ge m} = \sum_{m\le k\le j} + \sum_{j<k\le 2j-m} + \sum_{l\le j,\, m\vee(2j-m)<k}$, we have

$$\mathfrak{N}^4 f \le \sup_{m\ge l\ge 0} \mathfrak{S}_1^{m,l} f + \sup_{m\ge l\ge 0} \mathfrak{S}_2^{m,l} f + \sup_{m\ge l\ge 0} \mathfrak{S}_3^{m,l} f,$$

where

$$\mathfrak{S}_1^{m,l} f = \sum_{m\le k\le j} \mathfrak{A}_{l,j}^{m,k} f, \quad \mathfrak{S}_2^{m,l} f = \sum_{j<k\le 2j-m} \mathfrak{A}_{l,j}^{m,k} f, \quad \mathfrak{S}_3^{m,l} f = \sum_{l\le j,\, m\vee(2j-m)<k} \mathfrak{A}_{l,j}^{m,k} f.$$

Here, $a \vee b$ denotes $\max(a, b)$. Thus, the matter is reduced to showing, for $\kappa = 1, 2, 3,$

$$\big\| \sup_{m\ge l\ge 0} \mathfrak{S}_\kappa^{m,l} f \big\|_{L^p} \lesssim C\|f\|_p, \quad p \in (2, 4]. \tag{4.5.4}$$

We consider $\mathfrak{S}_1^{m,l}$ first. Recalling (4.5.1), by scaling we have

$$\mathfrak{A}_{l,j}^{m,k} f(x) = \mathfrak{M}_{m-l}(f_j^k(2^{-l}\cdot))(2^l x) = \mathfrak{M}_{m-l}[f(2^{-l}\cdot)]_{j-l}^{k-l}(2^l x). \tag{4.5.5}$$

So, reindexing $k \to k + l$ and $j \to j + l$ gives

$$\mathfrak{S}_1^{m,l} f(x) \le \sum_{m-l\le k\le j} \mathfrak{M}_{m-l}[f(2^{-l}\cdot)]_j^k(2^l x).$$

Thus, the imbedding $\ell^p \subset \ell^\infty$ and Minkowski's inequality yield

$$\| \sup_{m\ge l\ge 0} \mathfrak{S}_1^{m,l} f\|_{L^p}^p \le \sum_{m\ge l\ge 0} \Big(\sum_{m-l\le k\le j} \big\|\mathfrak{M}_{m-l}[f(2^{-l}\,\cdot)]_j^k(2^l\,\cdot)\big\|_{L^p}\Big)^p.$$

We now use $(\tilde{b})$ in Lemma 4.5.2 (with $n = m - l$) for $\mathfrak{M}_{m-l}[f(2^{-l}\,\cdot)]_j^k(2^l\,\cdot)$. Thus, by the first estimate in (4.5.2) with $k = j$, we have

$$\| \sup_{m\ge l\ge 0} \mathfrak{S}_1^{m,l} f\|_{L^p}^p \lesssim \sum_{m\ge l\ge 0} 2^{(m-l)p(\frac12-\frac1p)} \Big(\sum_{m-l\le j} 2^{-2j(\frac12-\frac1p)} 2^{\epsilon j} \|f_{j+l}\|_{L^p}\Big)^p$$

for any $\epsilon > 0$ for $2 < p \le 4$. Taking $\epsilon > 0$ small enough, we have

$$\| \sup_{m\ge l\ge 0} \mathfrak{S}_1^{m,l} f\|_{L^p}^p \lesssim \sum_{m\ge l\ge 0} \sum_{m-l\le j} 2^{-a(m-l)} 2^{-bj} \|f_{j+l}\|_{L^p}^p$$

for some positive numbers a, b for $2 < p \le 4$. Changing the order of summation, we see the right hand side is bounded above by $C\sum_{j\ge 0}^{\infty} 2^{-bj}\sum_{l\ge 0}\|f_{j+l}\|_{L^p}^p$, which is bounded by $C\|f\|_p^p$, as can be seen, for example, using the Littlewood-Paley inequality. Consequently, we obtain (4.5.4) for $\kappa = 1$.

We now consider $\mathfrak{S}_2^{m,l}$. As before, by the imbedding $\ell^p \subset \ell^\infty$, Minkowski's inequality, (4.5.5), and reindexing $k \to k + l$ and $j \to j + l$, we get

$$\big\|\sup_{m\ge l\ge 0}\mathfrak{S}_2^{m,l}f\big\|_{L^p}^p \le \sum_{m\ge l\ge 0}\Big(\sum_{j<k\le 2j-(m-l)}\big\|\mathfrak{M}_{m-l}[f(2^{-l}\,\cdot)]_j^k(2^l\,\cdot)\big\|_{L^p}\Big)^p.$$

The first inequality in (4.5.2) with $n = m - l$ gives

$$\big\|\sup_{m\ge l\ge 0}\mathfrak{S}_2^{m,l}f\big\|_{L^p}^p \le \sum_{m\ge l\ge 0}2^{(m-l)p(\frac12-\frac1p)}\Big(\sum_{j<k\le 2j-(m-l)}2^{-(j+k)(\frac12-\frac1p)}2^{\epsilon k}\|f_{j+l}\|_{L^p}\Big)^p$$

for any $\epsilon > 0$ for $2 < p \le 4$. Note that $m - l < j$ for the inner sum, which is bounded by a constant times $\sum_{m-l\le j}2^{-2j(1/2-1/p)}2^{\epsilon j}\|f_{j+l}\|_{L^p}$ by taking sum over k with an $\epsilon > 0$ small enough. Since $p > 2$, similarly, we have

$$\|\sup_{m\ge l\ge 0}\mathfrak{S}_2^{m,l}f\|_{L^p}^p \lesssim \sum_{m\ge l\ge 0}\sum_{m-l\le j}2^{-a(m-l)}2^{-bj}\|f_{j+l}\|_{L^p}^p$$

for some $a, b > 0$ and $2 < p \le 4$. Thus, the right hand is bounded above by $C\|f\|_{L^p}^p$. This proves (4.5.4) for $\kappa = 2$.

Finally, we consider $\mathfrak{S}_3^{m,l}f$, which we can handle in the same manner as before. Via the imbedding $\ell^p \subset \ell^\infty$, (4.5.5), and reindexing after applying Minkowski's inequality we have

$$\|\sup_{m\ge l\ge 0}\mathfrak{S}_2^{m,l}f\|_{L^p}^p \lesssim \sum_{m\ge l\ge 0}\Big(\sum_{0\le j,n\vee(2j-n)<k}\big\|\mathfrak{M}_n[f(2^{-l}\,\cdot)]_j^k(2^l\,\cdot)\big\|_{L^p}\Big)^p,$$

where $n := m-l$. Breaking $\sum_{0\le j,n\vee(2j-n)<k} = \sum_{0\le j\le n\le k} + \sum_{n<j,(2j-n)<k}$, we apply the first estimate in (4.5.3) to get

$$\|\sup_{m\ge l\ge 0}\mathfrak{S}_2^{m,l}f\|_{L^p}^p \lesssim \sum_{m\ge l\ge 0}2^{np(\frac12-\frac1p)}(\mathrm{S}_1^p + \mathrm{S}_2^p)$$

for any $\epsilon > 0$ and $2 < p \le 4$, where

$$\mathrm{S}_1 := \sum_{0\le j\le n\le k}2^{(j+k)(\frac1p-\frac12)}2^{\epsilon j}\|f_{j+l}^{k+l}\|_{L^p}, \quad \mathrm{S}_2 := \sum_{n<j,(2j-n)<k}2^{(j+k)(\frac1p-\frac12)}2^{\epsilon j}\|f_{j+l}^{k+l}\|_{L^p}.$$

For the second sum S_2, we note that $k > j > n$. Thus, taking $\epsilon > 0$ small enough, we get

$$\sum_{m\ge l\ge 0} 2^{np(\frac{1}{2}-\frac{1}{p})}\mathrm{S}_2^p \lesssim \sum_{m\ge l\ge 0}\sum_{m-l\le j} 2^{-a(m-l)}2^{-bj}\|f_{j+l}\|_{L^p}^p$$

for some $a, b > 0$ since $p > 2$. Thus, the right hand side is bounded by $C\|f\|_{L^p}^p$. To handle S_1, note that $(\sum_{0\le j\le n\le k} 2^{(j+k)(\frac{1}{p}-\frac{1}{2})})^{p/p'} \lesssim 2^{n(p-1)(\frac{1}{p}-\frac{1}{2})}$. Thus, by Hölder's inequality we have

$$\mathrm{S}_1^p \lesssim 2^{n(p-1)(\frac{1}{p}-\frac{1}{2})} \sum_{0\le j\le n\le k} 2^{(j+k)(-\frac{1}{2}+\frac{1}{p})}2^{\epsilon pj}\|f_{j+l}^{k+l}\|_{L^p}^p.$$

Hence, changing the order of summation, we get

$$\sum_{m\ge l\ge 0} 2^{np(\frac{1}{2}-\frac{1}{p})}\mathrm{S}_1^p \lesssim \sum_{0\le j} 2^{j(\frac{1}{p}-\frac{1}{2}+\epsilon p)}\mathrm{S}_{1,j}^p,$$

where

$$\mathrm{S}_{1,j}^p = \sum_{m\ge l\ge 0}\sum_{m-l\le k} 2^{(m-l)(\frac{1}{2}-\frac{1}{p})}2^{k(-\frac{1}{2}+\frac{1}{p})}\|f_{j+l}^{k+l}\|_{L^p}^p.$$

Therefore, since $2 < p \le 4$, taking a sufficiently small $\epsilon > 0$, we obtain the desired inequality $\sum_{m\ge l\ge 0} 2^{np(\frac{1}{2}-\frac{1}{p})}\mathrm{S}_1^p \lesssim \|f\|_{L^p}^p$ if we show that $\mathrm{S}_{1,j}^p \lesssim \|f\|_{L^p}^p$ for $0 \le j$. To this end, rearranging the sums, we observe

$$\mathrm{S}_{1,j}^p = \sum_{0\le k}\sum_{0\le l}\sum_{l\le m\le l+k} 2^{(m-l)(\frac{1}{2}-\frac{1}{p})}2^{k(-\frac{1}{2}+\frac{1}{p})}\|f_{j+l}^{k+l}\|_{L^p}^p \lesssim \sum_{0\le k}\sum_{0\le l}\|f_{j+l}^{k+l}\|_{L^p}^p.$$

Since $\sum_{0\le k}\|f_{j+l}^{k+l}\|_{L^p}^p \lesssim \|f_{j+l}\|_{L^p}^p$, by the same argument as above it follows that $\mathrm{S}_{1,j}^p \le C\|f\|_{L^p}^p$. Consequently, we obtain (4.5.4) for $\kappa = 3$. □

4.6 Local maximal estimates

Since $\mathbb{J}$ is a compact subset of $\mathbb{J}_*$, there are constants $c_0 \in (0, 1)$, and $m_1, m_2 > 0$ such that

$$\mathbb{J} \subset \{(t, s) : m_1 \le s \le m_2, s < c_0 t\}.$$

Therefore, via finite decomposition and scaling it is sufficient to show that the maximal operator

$$\mathfrak{M}_c f(x) := \sup_{(t,s)\in\mathbb{J}_0} |\mathcal{A}_t^s f(x)|$$

is bounded from L^p to L^q for $(1/p, 1/q) \in \operatorname{int} \mathcal{Q}$. To do this, we decompose $f = f_{\geq 0} + f^{\geq 0}_{<0} + f^{<0}_{<0}$ to have

$$\mathfrak{M}_c f \lesssim \mathfrak{M}_c f_{\geq 0} + \mathfrak{M}_c f^{\geq 0}_{<0} + \mathfrak{M}_c f^{<0}_{<0}.$$

The last two operators are easy to deal with. As before, we have $\mathfrak{M}_c f^{<0}_{<0}(x) \lesssim (1+|\cdot|)^{-M} * |f|(x)$, hence $\|\mathfrak{M}_c f^{<0}_{<0}\|_{L^q} \lesssim \|f\|_{L^p}$ for $1 \leq p \leq q \leq \infty$. Concerning $\mathfrak{M}_c f^{\geq 0}_{<0}$, we use Lemma 4.5.1 and (4.4.6) to get

$$\|\mathfrak{M}_c f^k_{<0}\|_{L^q} \lesssim 2^{k(-\frac{1}{2}+\frac{1}{p})} \|f\|_{L^p}, \quad 1 \leq p \leq q \leq \infty,$$

for $k \geq 0$. So, it follows that $\|\mathfrak{M}_c f^{\geq 0}_{<0}\|_{L^q} \lesssim \|f\|_{L^p}$ for $2 < p \leq q$. Thus, we only need to show that $\mathfrak{M}_c f_{\geq 0}$ is bounded from L^p to L^q for $(1/p, 1/q) \in \operatorname{int} \mathcal{Q}$.

Decomposing $f_{\geq 0} = \sum_{j \geq 0} (f^{<j}_j + \sum_{j \leq k \leq 2j} f^k_j + \sum_{k > 2j} f^k_j)$, we have

$$\mathfrak{M}_c f_{\geq 0} \leq \textstyle\sum_{j \geq 0} (\mathfrak{S}^1_j f + \mathfrak{S}^2_j f),$$

where

$$\mathfrak{S}^1_j f = \mathfrak{M}_c f^{<j}_j + \textstyle\sum_{j \leq k \leq 2j} \mathfrak{M}_c f^k_j, \qquad \mathfrak{S}^2_j f = \textstyle\sum_{k > 2j} \mathfrak{M}_c f^k_j.$$

We first show L^p–L^q bound on $\mathfrak{M}_c f_{\geq 0}$ for $(1/p, 1/q)$ contained in the interior of the triangle $\mathfrak{T}$ with vertices $(1/4, 1/4)$, P_1, and $(1/2, 1/2)$ (see Figure 1.5). The first estimate in (4.5.2) with $2^n \sim 1$ gives

$$\|\mathfrak{M}_c f^k_j\|_{L^q} \lesssim 2^{j(-\frac{1}{2}+\frac{1}{2p}+\frac{1}{2q})} 2^{k(-\frac{1}{2}+\frac{3}{2p}-\frac{1}{2q}+\epsilon)} \|f\|_{L^p}, \quad 1/p + 3/q \geq 1,$$

for $0 \leq j \leq k \leq 2j$. $\mathfrak{M}_c f^{<j}_j$ satisfies the same bound with $k = j$. Note that $-3/2 + 7/(2p) - 1/(2q) < 0$, $-1 + 2/p < 0$, and $1/p + 3/q > 1$ if $(1/p, 1/q) \in \operatorname{int} \mathfrak{T}$ (Figure 1.5). Thus, using those estimates, we get

$$\textstyle\sum_{j \geq 0} \|\mathfrak{S}^1_j f\|_{L^p} \lesssim \sum_{j \geq 0} \big(2^{j(-\frac{3}{2}+\frac{7}{2p}-\frac{1}{2q}+\epsilon)} + 2^{j(-1+\frac{2}{p}+\epsilon)} \big) \|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $(1/p, 1/q) \in \operatorname{int} \mathfrak{T}$. We now consider $\sum_{j \geq 0} \mathfrak{S}^2_j f$. By the first estimate in (4.5.3) with $2^n \sim 1$, we have

$$\textstyle\sum_{j \geq 0} \|\mathfrak{S}^2_j f\|_{L^p} \lesssim \sum_{0 \leq j, 2j < k} 2^{j(-\frac{1}{2}+\frac{3}{2p}-\frac{1}{2q}+\epsilon)} 2^{k(-\frac{1}{2}+\frac{1}{p})} \|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $(1/p, 1/q) \in \operatorname{int} \mathfrak{T}$. Thus, $\mathfrak{M}_c f_{\geq 0}$ is bounded from L^p to L^q for $(1/p, 1/q) \in \operatorname{int} \mathfrak{T}$.

Next, we show L^p–L^q bound on $\mathfrak{M}_c f_{\ge 0}$ for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$ where $\mathcal{Q}'$ is the quadrangle with vertices $(1/4, 1/4)$, $(0, 0)$, P_1, and P_2 (see Figure 1.5). Note that $1/p + 3/q < 1$ if $(p, q) \in \text{int}\, \mathcal{Q}'$. By the second estimate of (4.5.2) with $2^n \sim 1$, we have

$$\|\mathfrak{M}_c f_j^k\|_{L^q} \lesssim 2^{j(1-\frac{1}{p}-\frac{4}{q})} 2^{k(-1+\frac{2}{p}+\frac{1}{q}+\epsilon)} \|f\|_{L^p}, \quad 1/p + 3/q < 1$$

for $0 \le j \le k \le 2j$. $\mathfrak{M}_c f_j^{<j}$ satisfies the same bound with $k = j$. Thus,

$$\textstyle\sum_{j\ge 0} \|\mathfrak{S}_j^1 f\|_{L^p} \lesssim \sum_{j\ge 0} (2^{j(\frac{1}{p}-\frac{3}{q}+\epsilon)} + 2^{j(\frac{3}{p}-\frac{2}{q}-1+2\epsilon)}) \|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$ since $1/p - 3/q < 0$ and $3/p - 2/q < 1$ for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$. Similarly, the second estimate of (4.5.3) with $2^n \sim 1$ gives

$$\textstyle\sum_{j\ge 0} \|\mathfrak{S}_j^2 f\|_{L^p} \lesssim \sum_{k>2j\ge 0} 2^{j(\frac{1}{p}-\frac{2}{q}+\epsilon)} 2^{k(-\frac{1}{2}+\frac{1}{p})} \|f\|_{L^p} \lesssim \sum_{j\ge 0} 2^{j(-1+\frac{3}{p}-\frac{2}{q}+\epsilon)} \|f\|_{L^p}$$

for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$. Note that $-1 + 3/p - 2/q < 0$ for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$, so it follows that $\sum_{j\ge 0} \|\mathfrak{S}_j^2 f\|_{L^p} \lesssim \|f\|_{L^p}$ for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$. Thus, $f \to \mathfrak{M}_c f_{\ge 0}$ is bounded from L^p to L^q for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}'$.

Consequently, $f \to \mathfrak{M}_c f_{\ge 0}$ is bounded from L^p to L^q for $(1/p, 1/q) \in \text{int}\, \mathfrak{T} \cup \text{int}\, \mathcal{Q}'$. Thus, via interpolation $f \to \mathfrak{M}_c f_{\ge 0}$ is bounded from L^p to L^q for $(1/p, 1/q) \in \text{int}\, \mathcal{Q}$. This complete the proof of Theorem 1.5.2.

4.7 Proof of smoothing estimates

In this section, we prove Theorem 4.2.1, 4.2.2 and 4.2.3.

4.7.1 Two parameter smoothing estimate

We set $\mathbb{D}_\tau = \mathbb{R}^3 \times \mathbb{J}_\tau$. By $L^p_{\alpha,x}$ we denote the L^p Sobolev space of order α in x, and set $\mathcal{L}^p_\alpha(\mathbb{D}_\tau) = L^p_{s,t}(\mathbb{J}_\tau; L^p_{\alpha,x}(\mathbb{R}^3))$. We prove Theorem 4.2.1 making use of the next lemma.

Proposition 4.7.1. *Let $\tau \in (0, 1]$ and $8 \le p < \infty$. If $\alpha < 4/p$, then we have*

$$\|\tilde{\mathcal{A}}_t^s f\|_{\mathcal{L}^p_\alpha(\mathbb{D}_\tau)} \lesssim \tau^{-\frac{3}{p}} \|f\|_{L^p}.$$

It is not difficult to see that the bound $\tau^{-3/p}$ is sharp up to a constant by using a frequency localized smooth function. Assuming Proposition (4.7.1) for the moment, we prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Since $\psi \in C_c^\infty(\mathbb{J}_*)$, as before, there are constants $c_0 \in (0,1)$, and $m_1, m_2 > 0$ such that $\operatorname{supp}\psi \subset \{(t,s) : m_1 \le s \le m_2, s < c_0 t\}$. By finite decomposition and scaling, we may assume $\operatorname{supp}\psi \subset \{(t,s) : 1 \le s \le 2, s < c_0 t\}$.

We now consider the Fourier transform of the function $(x,t,s) \to \tilde{\mathcal{A}}_t^s f(x)$:

$$F(\zeta) = S(\zeta)\widehat{f}(\xi) := \iiiint e^{-i(t\tau + s\sigma + \Phi_t^s(\theta,\phi)\cdot\xi)}\psi(t,s)\, d\theta d\phi ds dt\, \widehat{f}(\xi),$$

where $\zeta = (\xi,\tau,\sigma)$. Let us set $m^\alpha(\zeta) = (1+|\zeta|^2)^{\alpha/2}$, $\varphi_\circ = \varphi_{<0}(|\cdot|)$, and $\tilde{\varphi}_\circ = 1 - \varphi_\circ$. To prove Theorem 4.2.1, we need to show $\|\mathcal{F}^{-1}(m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}$. Since $\|\mathcal{F}^{-1}(\varphi_\circ m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}$, we only have to show

$$\|\mathcal{F}^{-1}(\tilde{\varphi}_\circ m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}.$$

For a large positive constant C, we set $\varphi_*(\zeta) = \varphi_{<0}(|\tau|/C|\xi|)$ and $\varphi^*(\zeta) = \varphi_{<0}(|\sigma|/C|\xi|)$. We also set $\tilde{\varphi}_* = 1 - \varphi_*$ and $\tilde{\varphi}^* = 1 - \varphi^*$. Thus, we have

$$\varphi_*\varphi^* + \tilde{\varphi}_*\varphi^* + \varphi_*\tilde{\varphi}^* + \tilde{\varphi}_*\tilde{\varphi}^* = 1.$$

If $|\tau| \ge C|\xi|$, integration by parts in t gives $|S(\zeta)| \lesssim (1+|\tau|)^{-N}$ for any N. Since $|\tau| \ge C|\xi|$ and $|\sigma| \le C|\xi|$ on the support of $\tilde{\varphi}_*\varphi^*$, one can easily see $\|\mathcal{F}^{-1}(\tilde{\varphi}_*\varphi^*\tilde{\varphi}^\circ m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}$ for any α. The same argument also shows that $\|\mathcal{F}^{-1}(\varphi_*\tilde{\varphi}^*\tilde{\varphi}^\circ m^\alpha F)\|_{L^p}, \|\mathcal{F}^{-1}(\tilde{\varphi}_*\tilde{\varphi}^*\tilde{\varphi}^\circ m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}$ for any α. Now, we note that $|\tau| \le C|\xi|$ and $|\sigma| \le C|\xi|$ on the support of $\varphi_*\varphi^*$. Thus, by the Mikhlin multiplier theorem

$$\|\mathcal{F}^{-1}(\varphi_*\varphi^*\tilde{\varphi}^\circ m^\alpha F)\|_{L^p} \lesssim \|\mathcal{F}^{-1}(\bar{m}^\alpha F)\|_{L^p},$$

where $\bar{m}^\alpha(\zeta) = (1+|\xi|^2)^{\alpha/2}$. Since $\operatorname{supp}\psi \subset \{(t,s) : 1 \le s \le 2, s < c_0 t\}$, the right hand side is bounded above by $\|\tilde{\mathcal{A}}_t^s f\|_{\mathcal{L}_\alpha^p(\mathbb{D}_1)}$. Therefore, using Proposition 4.7.1, we get $\|\mathcal{F}^{-1}(\varphi_*\varphi^*\tilde{\varphi}^\circ m^\alpha F)\|_{L^p} \lesssim \|f\|_{L^p}$. □

In what follows, we prove Proposition 4.7.1 using the estimates obtained in Chapter 4.4.

Proof of Proposition 4.7.1. Let n be an integer such that $2^n \le 1/\tau < 2^{n+1}$. Then, we decompose

$$\mathcal{A}_t^s f = \mathcal{A}_t^s f_{<n}^{<n} + \sum_{k\ge n} \mathcal{A}_t^s f_{<0}^k + \sum_{0\le j<n\le k} \mathcal{A}_t^s f_j^k + \mathrm{I}_t^s f + \mathrm{II}_t^s f, \tag{4.7.1}$$

where

$$\mathrm{I}_t^s f = \sum_{j\ge n,\, k>2j-n} \mathcal{A}_t^s f_j^k, \qquad \mathrm{II}_t^s f = \sum_{n\le j\le k\le 2j-n} \mathcal{A}_t^s f_j^k + \sum_{n\le j} \mathcal{A}_t^s f_j^{<j}.$$

Note that $\|\mathcal{A}_t^s f_{<n}^{<n}\|_{L_x^{p,\alpha}} \lesssim \tau^{-\alpha}\|\mathcal{A}_t^s f\|_{L_x^p}$. So,

$$\|\mathcal{A}_t^s f_{<n}^{<n}\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{-\alpha+1/p}\|f\|_{L^p} \lesssim \tau^{-3/p}\|f\|_{L^p}$$

since $\alpha < 4/p$. Similarly, using (4.4.6), we have

$$\|\mathcal{A}_t^s f_{<0}^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{1/p-1/2}2^{(\alpha-1/2)k}\|f\|_{L^p}$$

for $k \ge n$. Taking sum over k gives

$$\|\textstyle\sum_{k\ge n} \mathcal{A}_t^s f_{<0}^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \sum_{k\ge n} 2^{(\alpha-\frac12)k}\tau^{\frac1p-\frac12}\|f\|_{L^p} \lesssim \tau^{-3/p}\|f\|_{L^p}$$

since $\alpha < 4/p$ and $p > 8$. When $0 \le j < n \le k$, by (4.4.8) it follows that $\|\mathcal{A}_t^s f_j^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac1p-\frac12}2^{j(-\frac2p+\epsilon)+k(\alpha-\frac12)}\|f\|_{L^p}$ for $p \ge 4$. Thus, we see that

$$\|\textstyle\sum_{0\le j<n\le k} \mathcal{A}_t^s f_j^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac1p-\alpha}\|f\|_{L^p} \lesssim \tau^{-\frac3p}\|f\|_{L^p}.$$

Therefore, it remains to show the estimates for the operators I_t^s and II_t^s. Using (c) and (a) in Proposition 4.4.4, we obtain, respectively,

$$\begin{aligned}
&\|\mathcal{A}_t^s f_j^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac1p-\frac12}2^{j(-\frac2p+\epsilon)}2^{k(\alpha-\frac12)}\|f\|_{L^p}, && j\ge n,\ k>2j-n,\\
&\|\mathcal{A}_t^s f_j^k\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{-\frac1p}2^{j(1-\frac6p)+k(\alpha+\frac2p-1+\epsilon)}\|f\|_{L^p}, && n\le j\le k\le 2j-n
\end{aligned}$$

for any $\epsilon > 0$ and $p \ge 4$. Besides, (b) in Proposition 4.4.4 ((4.4.20) with $h=\lambda$) gives $\|\mathcal{A}_t^s f_j^{<j}\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{-1/p}2^{j(\alpha-4/p)}\|f\|_{L^p}$ for $p \ge 4$. Therefore, recalling $p > 8$ and $\alpha < 4/p$, we get

$$\begin{aligned}
&\|\mathrm{I}_t^s f\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{\frac1p-\frac12}\textstyle\sum_{j\ge n,\, k>2j-n} 2^{j(-\frac2p+\epsilon)}2^{k(\alpha-\frac12)}\|f\|_{L^p} \lesssim \tau^{-\frac3p}\|f\|_{L^p},\\
&\|\mathrm{II}_t^s f\|_{\mathcal{L}^{p,\alpha}(\mathbb{R}^3\times\mathbb{J}_\tau)} \lesssim \tau^{-\frac1p}\textstyle\sum_{n\le j\le k\le 2j-n} 2^{j(1-\frac6p)+k(\alpha+\frac2p-1+\epsilon)}\|f\|_{L^p} \lesssim \tau^{-\frac3p}\|f\|_{L^p}.
\end{aligned}$$

This completes the proof. □

4.7.2 One parameter smoothing estimate

In order to prove Theorem 4.2.2, we make use of local smoothing estimate for the operator $f \to \mathcal{U}f(x,t,c_0t)$. For the two-parameter propagator $\mathcal{U}$, we can handle the associated operators $e^{it|\bar{D}|}$ and $e^{is|D|}$ separately so that the sharp smoothing estimates are obtained by utilizing the decoupling and local smoothing inequalities for the cone in $\mathbb{R}^{2+1}$. However, for the sharp estimate for $f \to \mathcal{U}f(x,t,c_0t)$ a similar approach does not work. Instead, we make use of the decoupling inequality for the conic surface $(\xi, |\bar{\xi}| + c_0|\xi|)$ in $\mathbb{R}^{3+1}$ (see [10] and Theorem 2.1 of [6]).

Proposition 4.7.2. *Set* $\tilde{\mathcal{U}}_\pm f(x,t) = \mathcal{U}f(x,t,\pm c_0 t)$. *Let* $1 \le \lambda \le h \le \lambda^2$. *Then, if* $6 \le p \le \infty$, *for any* $\epsilon > 0$ *we have*

$$\|\tilde{\mathcal{U}}_\pm f\|_{L^p_{x,t}(\mathbb{R}^3\times[1,2])} \lesssim \lambda^{\frac{3}{2}-\frac{5}{p}} h^{\frac{2}{p}-\frac{1}{2}+\epsilon} \|f\|_{L^p} \tag{4.7.2}$$

whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. *Also, the same bound with* $h = \lambda$ *holds for* $4 \le p \le \infty$ *whenever* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}^\circ_\lambda$.

Proof. When $p = \infty$, the estimate (4.7.2) is already shown in the previous section (see (4.3.1)). Thus, we focus on the estimates (4.7.2) for $p = 4, 6$, and the other estimates follow by interpolation.

We first consider the case $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}^\circ_\lambda$, for which (4.7.2) hold on a larger range $4 \le p \le \infty$. To show (4.7.2), we make use of the decoupling inequality associated to the conic surfaces

$$\Gamma_\pm = \{(\xi, P_\pm(\xi)), \quad \xi \in \mathbb{A}_1 \times \mathbb{I}^\circ_1\}$$

where $P_\pm(\xi) := |\bar{\xi}| \pm c_0|\xi|$. In fact, we use the ℓ^p decoupling inequality for the conic surfaces [10, 6]. To this end, we first check that the Hessian matrix of $P_\pm$ is of rank 2. Indeed, a computation shows that

$$\operatorname{Hess} P_\pm(\xi) = \frac{1}{|\bar{\xi}|^3}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & 0 \\ -\xi_1\xi_2 & \xi_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pm \frac{c_0}{|\xi|^3}\begin{pmatrix} \xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & \xi_1^2+\xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_1^2+\xi_2^2 \end{pmatrix}.$$

Note that $\operatorname{Hess} P_\pm(\xi)\xi = 0$, so Γ has a vanishing principal curvature in the direction of ξ. By rotational symmetry in $\bar{\xi}$, to compute the eigenvalues of $\operatorname{Hess} P_\pm(\xi)$ it is sufficient to consider the case $\xi_1 = 0$ and $\xi_2 = |\bar{\xi}| \neq 0$.

Consequently, one can easily see that the matrix Hess $P_{\pm}(\xi)$ has two nonzero eigenvalues
$$|\bar{\xi}|^{-1} \pm c_0|\xi|^{-1}, \quad \pm c_0|\xi|^{-1}.$$

Let us denote by $\mathfrak{V}^\lambda$ a collection of points which are maximally $\sim \lambda^{-1/2}$ separated in the set $\mathbb{S}^2 \cap \{\xi : |\bar{\xi}| \geq 2^{-2}\xi_3\}$. Let $\{W_\mu\}_{\mu\in\mathfrak{V}^\lambda}$ denote a partition of unity subordinated to a collection of finitely overlapping spherical caps centered at μ of diameter $\sim \lambda^{-1/2}$ which cover $\mathbb{S}^2 \cap \{\xi : |\bar{\xi}| \geq 2^{-2}\xi_3\}$ such that $|\partial^\alpha W_\mu| \lesssim \lambda^{|\alpha|/2}$. Denote $\Omega_\mu(\xi) = W_\mu(\xi/|\xi|)$. Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, we have $f = \sum_{\mu\in\mathfrak{V}^\lambda} f_\mu$ where $f_\mu = \mathcal{F}^{-1}(\Omega_\mu\widehat{f}\,)$. So, we can write

$$\tilde{\mathcal{U}}_{\pm} f(x,t) = \sum_{\mu\in\mathfrak{V}^\lambda} \tilde{\mathcal{U}}_{\pm} f_\mu(x,t) = \sum_{\mu\in\mathfrak{V}^\lambda} \int e^{i(x\cdot\xi + tP_{\pm}(\xi))}\widehat{f}_\mu(\xi)d\xi.$$

Since $\Gamma_\pm$ are conic surfaces with two nonvanishing curvatures in $\mathbb{R}^4$, we have the following l^p-decoupling inequality:

$$\|\tilde{\chi}(t)\tilde{\mathcal{U}}_{\pm} f\|_{L^p_{x,t}} \lesssim \lambda^{1-\frac{3}{p}+\epsilon}\Big(\sum_{\mu\in\mathfrak{V}^\lambda} \|\tilde{\chi}(t)\tilde{\mathcal{U}}_{\pm} f_\mu\|^p_{L^p_{x,t}}\Big)^{1/p} \tag{4.7.3}$$

for $p \geq 4$ (see [13] and [6, Theorem 1.4]). Here $\tilde{\chi} \in \mathcal{S}(\mathbb{R})$ such that $\tilde{\chi} \geq 1$ on $\mathbb{I}$ and $\operatorname{supp}\mathcal{F}(\tilde{\chi}) \subset [-1/2, 1/2]$. Using Lemma 4.3.2 as before, we see $\|\tilde{\chi}(t)\tilde{\mathcal{U}}_{\pm} f_\mu\|_{L^p_{x,t}} \lesssim \|\tilde{\chi}(t)e^{t(\bar{D}\cdot(\bar{\mu}/|\bar{\mu}|)\pm c_0 D\cdot\mu)} f_\mu\|_{L^p_{x,t}}$ where $\mu = (\bar{\mu}, \mu_3)$. Thus, a change of variables gives $\|\tilde{\chi}(t)\tilde{\mathcal{U}}_{\pm} f_\mu\|_{L^p_{x,t}} \lesssim \|f_\mu\|_{L^p}$ for $1 \leq p \leq \infty$. Since $(\sum_\mu \|f_\mu\|_p^p) \lesssim \|f\|_p$ for $p \geq 2$, combining the estimates and (4.7.3) with $p = 4$, we obtain
$$\|\mathcal{U}_{\pm} f\|_{L^4_{x,t}} \lesssim \lambda^{\frac{1}{4}+\epsilon}\|f\|_{L^4}.$$
Interpolation with the easy L^∞ estimate ((4.3.1) with $p = q = \infty$) gives (4.7.2) with $h = \lambda$ for $4 \leq p \leq \infty$.

Now, we consider the case $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ with $\lambda \leq h \leq \lambda^2$. Recall the partition of unity $\{w_\nu\}_{\nu\in\mathfrak{V}_\lambda}$ on the unit circle $\mathbb{S}^1$ and $f_\nu = \omega_\nu(\bar{D})f$. Note that $\tilde{\mathcal{U}}_{\pm} f_\nu(\cdot, x_3, t)$, $\nu \in \mathfrak{V}_\lambda$ have Fourier supports contained in finitely overlapping rectangles of dimension $\lambda \times \lambda^{1/2}$. So, we have

$$\|\textstyle\sum_{\nu\in\mathfrak{V}_\lambda} \tilde{\mathcal{U}}_{\pm} f_\nu(\cdot, x_3, t)\|_p \lesssim \lambda^{1/2-1/p}(\textstyle\sum_{\nu\in\mathfrak{V}_\lambda} \|\tilde{\mathcal{U}}_{\pm} f_\nu(\cdot, x_3, t)\|_p^p)^{1/p}$$

for $2 \leq p \leq \infty$, which is a simple consequence of the Plancherel theorem and interpolation (for example, see Lemma 6.1 in [80]). Integration in x_3 and t

gives

$$\|\tilde{\mathcal{U}}_{\pm} f\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{\nu\in\mathfrak{V}_\lambda} \|\tilde{\mathcal{U}}_{\pm} f_\nu\|^p_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})}\Big)^{1/p}, \quad 2\le p\le\infty. \qquad (4.7.4)$$

We proceed to obtain estimates for $\|\tilde{\mathcal{U}}_{\pm} f_\nu\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})}$. Using Lemma 4.3.2 and changing variables $x\to x-(\nu,0)t$, we see $\|\tilde{\mathcal{U}}_{\pm} f_\nu\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \|e^{\pm itc_0|D|} f_\nu\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})}$. Similarly, we also have $\|e^{\pm itc_0|D|} f_\nu\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \|\tilde{\mathcal{U}}^\nu_{\pm} f_\nu\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})}$, where

$$\tilde{\mathcal{U}}^\nu_{\pm} h(x,t) = \int e^{i\big(x\cdot\xi\pm c_0 t\sqrt{(\nu\cdot\bar{\xi})^2+\xi_3^2}\big)} \widehat{h}(\xi)d\xi.$$

Therefore, from (4.7.4) it follows that

$$\|\tilde{\mathcal{U}}_{\pm} f\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{\nu\in\mathfrak{V}_\lambda} \|\tilde{\mathcal{U}}^\nu_{\pm} f_\nu\|^p_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})}\Big)^{1/p}, \quad 2\le p\le\infty. \qquad (4.7.5)$$

Note that Fourier transform of f is contained in $\{\xi : |\xi|\sim h\}$ because $\lambda\le h$. To estimate $\tilde{\mathcal{U}}^\nu_{\pm} f_\nu$, freezing $\nu^*\cdot\bar{x}$, we use the ℓ^2 decoupling inequality [10] (i.e., (4.3.3) with $p=2$, $q=6$, and $\lambda=h$) with respect to $\nu\cdot\bar{x}, x_3$ variables. Thus, by the decoupling inequality followed by Minkowski's inequality, we get

$$\|\tilde{\mathcal{U}}^\nu_{\pm} f_\nu\|_{L^6_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim h^\epsilon \Big(\sum_{\tilde{\nu}\in\mathfrak{V}_h} \|\tilde{\chi}(t)\tilde{\mathcal{U}}^\nu_{\pm} f^{\tilde{\nu}}_\nu\|^2_{L^6_{x,t}}\Big)^{1/2},$$

where $\mathcal{F}(f^{\tilde{\nu}}_\nu)(\xi) = \omega_{\tilde{\nu}}(\nu\cdot\bar{\xi},\xi_3)\widehat{f}_\nu(\xi)$. Since $\#\{\tilde{\nu} : f^{\tilde{\nu}}_\nu \ne 0\} \lesssim \lambda h^{-1/2}$, by Hölder's inequality it follows that

$$\|\tilde{\mathcal{U}}^\nu_{\pm} f_\nu\|_{L^6_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim h^\epsilon (\lambda h^{-1/2})^{\frac{1}{3}} \Big(\sum_{\tilde{\nu}\in\mathfrak{V}_h} \|\tilde{\chi}(t)\tilde{\mathcal{U}}^\nu_{\pm} f^{\tilde{\nu}}_\nu\|^6_{L^6_{x,t}}\Big)^{1/6}.$$

Lemma 4.3.2 and a similar argument as before yield $\|\tilde{\chi}(t)\tilde{\mathcal{U}} f^{\tilde{\nu}}_\nu\|_{L^6_{x,t}} \lesssim \|f^{\tilde{\nu}}_\nu\|_6$. Hence, $\|\tilde{\mathcal{U}}^\nu_{\pm} f_\nu\|^6_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \lambda^2 h^{-1+6\epsilon} \sum_{\tilde{\nu}\in\mathfrak{V}_h} \|f^{\tilde{\nu}}_\nu\|^6_{L^6_{x,t}} \lesssim \lambda^2 h^{-1+6\epsilon} \|f_\nu\|^6_{L^6}$. Therefore, combining this and (4.7.5) with $p=6$, we obtain (4.7.2) for $p=6$. □

We denote $\mathcal{L}^p_\alpha(\mathbb{R}^3\times\mathbb{I}) = L^p_t(\mathbb{I}; L^p_{\alpha,x}(\mathbb{R}^3))$. By an argument similar to the proof of Theorem 4.2.1 it is sufficient to show that

$$\|\tilde{\mathcal{A}}^{c_0 t}_t f\|_{\mathcal{L}^p_\alpha(\mathbb{R}^3\times\mathbb{I})} \lesssim \|f\|_{L^p(\mathbb{R}^3)}, \quad \alpha < 3/p$$

for a constant $c_0 \in (0,1)$. We use the decomposition (4.7.1) with $s = c_0 t$ and $n = 0$ to have

$$\mathcal{A}_t^{c_0 t} f = \mathcal{A}_t^{c_0 t} f_{<0}^{<0} + \sum_{k\geq 0} \mathcal{A}_t^{c_0 t} f_{<0}^{k} + \mathrm{I}_t^{c_0 t} f + \mathrm{II}_t^{c_0 t} f.$$

The estimates for $\mathcal{A}_t^{c_0 t} f_{<0}^{<0}$ and $\sum_{k\geq 0} \mathcal{A}_t^{c_0 t} f_{<0}^{k}$ follow from (4.4.5) and (4.4.7) for fixed t, s. Indeed, we have $\|\mathcal{A}_t^{c_0 t} f_{<0}^{<0}\|_{\mathcal{L}^{p,3/p}(\mathbb{R}^3\times\mathbb{I})} \lesssim \|f\|_p$ and

$$\sum_{k\geq 0} \|\mathcal{A}_t^{c_0 t} f_{<0}^{k}\|_{\mathcal{L}^{p,3/p}(\mathbb{R}^3\times\mathbb{I})} \lesssim \sum_{k\geq 0} 2^{(3/p-1/2)k}\|f\|_p \lesssim \|f\|_p$$

for $p > 6$.

We obtain the estimates for $\mathrm{I}_t^{c_0 t}$ and $\mathrm{II}_t^{c_0 t}$ using the next proposition.

Proposition 4.7.3. *(a) If $1 \leq \lambda \leq h \leq \lambda^2$, then for any $\epsilon > 0$ we have*

$$\|\mathcal{A}_t^{c_0 t} f\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \lambda^{1-\frac{5}{p}} h^{-1+\frac{2}{p}+\epsilon}\|f\|_{L^p} \tag{4.7.6}$$

for $6 \leq p \leq \infty$ whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. *(b) If* $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$*, the estimate* (4.7.6) *holds with $h = \lambda$ for $4 \leq p \leq \infty$. (c) If $1 \leq \lambda$ and $\lambda^2 \leq h$, we have*

$$\|\mathcal{A}_t^{c_0 t} f\|_{L^p_{x,t}(\mathbb{R}^3\times\mathbb{I})} \lesssim \lambda^{-\frac{2}{p}+\epsilon} h^{-\frac{1}{2}}\|f\|_{L^p}$$

for $4 \leq p \leq \infty$ whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$.

Assuming this for the moment, we finish the proof of Theorem 4.2.2. By (a) and (b) in Proposition 4.7.3 we have

$$\|\mathrm{II}_t^{c_0 t} f\|_{\mathcal{L}^p_\alpha(\mathbb{R}^3\times\mathbb{I})} \lesssim \sum_{j\geq 0} 2^{(1-\frac{5}{p})j} \sum_{j\leq k\leq 2j} 2^{k(-1+\frac{2}{p}+\alpha+\epsilon)}\|f\|_{L^p}.$$

Since $p > 6$ and $\alpha < 3/p$, taking $\epsilon > 0$ small enough, we have the right hand side bounded above by $C\|f\|_{L^p}$. Finally, using (c) in Proposition 4.7.3 we obtain

$$\|\mathrm{I}_t^{c_0 t} f\|_{\mathcal{L}^p_\alpha(\mathbb{R}^3\times\mathbb{I})} \lesssim \sum_{j\geq 0}\sum_{k\geq 2j} 2^{j(-\frac{2}{p}+\epsilon)+k(-\frac{1}{2}+\alpha)}\|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $p > 6$ and $\alpha < 3/p$.

To complete the proof, it remains to prove Proposition 4.7.3. For the purpose we closely follow the proof of Proposition 4.4.4.

Proof of Proposition 4.7.3. We recall (4.4.10), (4.4.11), and (4.4.12). As seen in the proof of Proposition 4.4.4, using the Mikhlin multiplier theorem, we can handle $\mathcal{E}(\xi, t, c_0 t)$ as if it is $|\bar{\xi}|^{-N'}|\xi_3|^{-1}$ (see (4.4.16)). Likewise, we can replace $\tilde{E}_N(c_0 t|\xi|)$ by $(c_0 t|\xi|)^{-N'}$. Thus, the matter is reduced to obtaining estimates for the operators

$$\tilde{\mathcal{C}}^{\kappa}_{\pm} f(x,t) := |\bar{D}|^{-\frac{1}{2}}|sD|^{-\frac{1}{2}} e^{i(\kappa t|\bar{D}| \pm c_0 t|D|)} f(x), \quad \kappa = \pm$$

(cf. (4.4.23)). Thus, it is sufficient to show that the desired bounds on $\mathcal{A}^{c_0 t}_t$ also hold on $\tilde{\mathcal{C}}^{\kappa}_{\pm}$.

We first consider the case (a). Note

$$\|\tilde{\mathcal{C}}^{\kappa}_{\pm} f\|_{L^p_x(\mathbb{R}^3)} \lesssim (\lambda h)^{-1/2} \|e^{i(\kappa t|\bar{D}| \pm c_0 t|D|)} f\|_{L^p_x(\mathbb{R}^3)}$$

since $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. By Proposition 4.7.2 we get

$$\|\tilde{\mathcal{C}}^{\kappa}_{\pm} f\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{I})} \lesssim \lambda^{1-\frac{5}{p}} h^{\frac{2}{p}-1+\epsilon} \|f\|_{L^p}, \quad \kappa = \pm$$

for $6 \le p \le \infty$ as desired. In fact, the estimates for $e^{i(-t|\bar{D}| \pm c_0 t|D|)} f$ follow by conjugation and reflection as before (cf. Remark 1). Also, note that $\|\tilde{\mathcal{C}}^{\kappa}_{\pm} f\|_{L^p_x} \lesssim \lambda^{-2} \|e^{i(-t|\bar{D}| \pm c_0 t|D|)}\|_{L^p_x}$ when $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}^\circ_\lambda$. Thus, we get the estimate in the case (b) in the same manner.

Finally, we consider the case (c). Since $\operatorname{supp} \widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ and $\lambda^2 \le h$, applying Mikhlin's multiplier theorem and Lemma 4.3.2 successively, we see $\|\tilde{\mathcal{C}}^{\kappa}_{\pm} f\|_{L^p_x} \lesssim (\lambda h)^{-1/2} \|e^{i(\kappa t|\bar{D}| \pm c_0 t|D|)} f\|_{L^p_x} \lesssim (\lambda h)^{-1/2} \|e^{i(\kappa t|\bar{D}| \pm c_0 D_3)} f\|_{L^p_x}$. Thus, by a change of variables we have

$$\|\tilde{\mathcal{C}}^{\kappa}_{\pm} f\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{I})} \lesssim (\lambda h)^{-1/2} \|e^{i\kappa t|\bar{D}|} f\|_{L^p_{x,t}(\mathbb{R}^3 \times \mathbb{I})}$$

for $1 \le p \le \infty$ and $\kappa = \pm$. Therefore, for $4 \le p \le \infty$, the desired estimate follows from (2.2.1). □

4.7.3 Sobolev regularity estimate

In this subsection we prove Theorem 4.2.3. We consider estimates for $\mathcal{A}^s_t$ with fixed $0 < s < t$.

Lemma 4.7.4. *Let $1 \le p \le \infty$, $0 < s < t$ and $h \ge \lambda \sim 1$. Suppose $\operatorname{supp} \widehat{f} \subset \mathbb{A}^\circ_\lambda \times \mathbb{I}_h$. Then, we have $\|\mathcal{A}^s_t f\|_{L^p_x} \lesssim_{s,t} h^{-\frac{1}{2}} \|f\|_{L^p}$.*

Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda^\circ \times \mathbb{I}_h$, recalling the function $\varphi_{\leq 1}$ from the Notation section, observe that $\mathcal{A}_t^s f = f*(\mathcal{K}_h*\sigma_t^s)$ where $\mathcal{K}_h = \mathcal{F}^{-1}(\varphi_{\leq 1}(|\,\bar{\xi}\,|/\lambda)\varphi_{\leq 1}(|\xi_3|/h))$. Thus, Lemma 4.7.4 follows if we show $\|\mathcal{K}_h*\sigma_t^s\|_{L_x^1} \lesssim h^{-1/2}$. This is clear since, for fixed s, t, $\|\mathcal{K}_h * \sigma_t^s\|_{L_x^\infty} \lesssim h^{-1/2}$ and $\mathcal{K}_h * \sigma_t^s$ is essentially supported in a $O(1)$ neighborhood of Γ_t^s.

Lemma 4.7.5. *Let $0 < s < t$ and $p \geq 2$. (a) If $1 \leq \lambda \leq h \leq \lambda^2$, then for any $\epsilon > 0$*

$$\|\mathcal{A}_t^s f\|_{L_x^p} \lesssim \lambda^{1-\frac{3}{p}} h^{-1+\frac{1}{p}+\epsilon}\|f\|_{L^p} \tag{4.7.7}$$

holds whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$. (b) If $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_\lambda^\circ$, we have the estimate (4.7.7) with $h = \lambda$. (c) If $1 \leq \lambda$ and $\lambda^2 \leq h$, then for any $\epsilon > 0$

$$\|\mathcal{A}_t^s f\|_{L_x^p} \lesssim \lambda^{-\frac{1}{p}} h^{-\frac{1}{2}+\epsilon}\|f\|_{L^p}$$

holds whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$.

Proof. As before, it is sufficient to show that $\mathcal{C}_\pm^\kappa$ (4.4.23) satisfies the above estimates in place of $\mathcal{A}_t^s$. Note that

$$\|\mathcal{C}_\pm^\kappa f\|_{L_x^q} \lesssim (\lambda h)^{-1/2}\|\mathcal{U}f(\cdot, \kappa t, \pm s)\|_{L_x^q}.$$

For all the cases (a), (b), and (c), the desired estimates for $p = 2$ follows by Plancherel's theorem. Thus, we only need to show the estimates for $p = \infty$. For the cases (a) and (b) the estimates for $p = \infty$ follow from (4.3.1) of the corresponding cases (a) and (b) with $p = q = \infty$ (Remark 1). Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$ and $1 \leq \lambda$ and $\lambda^2 \leq h$, by Lemma 4.3.2 we note that $\|\mathcal{U}f(\cdot, \kappa t, \pm s)f\|_{L_x^\infty} \lesssim \|e^{i(\kappa t|\bar{D}|\pm s|D_3|)}f\|_{L_x^\infty} \lesssim \sum_\pm \|e^{it|\bar{D}|}f_\pm\|_{L_x^\infty}$ where $\widehat{f}_\pm(\xi) = \chi_{(0,\infty)}(\pm\xi_2)\widehat{f}(\xi)$. Since $\operatorname{supp}\widehat{f} \subset \mathbb{A}_\lambda \times \mathbb{I}_h$, the estimate for $p = \infty$ in the case (c) follows from (2.2.1). $\square$

Proof of Theorem 4.2.3. Since $\mathcal{A}_t^s f$ is bounded from L^2 to $L_{1/2}^2$, it is sufficient to show $\mathcal{A}_t^s f$ is bounded from L^p to L_α^p for $p > 4$ and $\alpha > 2/p$.

We use the decomposition (4.7.1) with $2^n \sim 1$. Note that $\|\mathcal{A}_t^s f_{<0}^{\leq 0}\|_{L_{\alpha,x}^p} \lesssim \|\mathcal{A}_t^s f_{<0}^{\leq 0}\|_{L_x^p}$ and $\|\mathcal{A}_t^s f_{<0}^k\|_{L_{\alpha,x}^p} \lesssim 2^{\alpha k}\|\mathcal{A}_t^s f_{<0}^k\|_{L_x^p}$. By Lemma 4.7.4 we have

$$\|\mathcal{A}_t^s f_{<0}^{\leq 0}\|_{L_{\alpha,x}^p} + \textstyle\sum_{k\geq 0}\|\mathcal{A}_t^s f_{<0}^k\|_{L_{\alpha,x}^p} \lesssim \sum_{k\geq 0} 2^{(\alpha-1/2)k}\|f\|_{L^p} \lesssim \|f\|_p$$

for $\alpha < 2/p$ and $p > 4$. Since $\alpha < 2/p$, using (a) and (b) in Lemma 4.7.5 with an ϵ small enough, we have

$$\|\mathrm{II}_t^s f\|_{L_{\alpha,x}^p} \lesssim \textstyle\sum_{0\leq j\leq k\leq 2j} 2^{j(1-\frac{3}{p})}2^{k(\alpha-1+\frac{1}{p}+\epsilon)}\|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $p \geq 2$. Similarly, using (c) in Lemma 4.7.5, we obtain

$$\|\mathrm{I}_t^s f\|_{L^p_{\alpha,x}} \lesssim \sum_{j\geq 0} \sum_{k\geq 2j} 2^{(\alpha-\frac{1}{2}+\epsilon)k} 2^{-\frac{1}{p}j} \|f\|_{L^p} \lesssim \|f\|_{L^p}$$

for $p > 4$ and $\alpha < 2/p$. $\square$

4.8 Optimality of the estimates

In this section, considering specific examples, we show sharpness of the estimates in Theorem 1.5.2, 4.2.1, 4.2.2, and 4.2.3 except for some endpoint cases.

Necessary conditions on (p, q) for (1.5.2) to hold

We show that if (1.5.2) holds, then the following hold true:

$$(\mathbf{a})\, p \leq q, \qquad (\mathbf{b})\, 3 + 1/q \geq 7/p, \qquad (\mathbf{c})\, 1 + 2/q \geq 3/p, \qquad (\mathbf{d})\, 3/q \geq 1/p.$$

This shows that (1.5.2) fails unless $(1/p, 1/q)$ is contained in the closure of $\mathcal{Q}$.

To show (**a**)–(**d**), it is sufficient to consider $\mathfrak{M}_0$ (see (4.5.1)) instead of $\mathcal{M}^c_{\mathbb{T}}$ with $\mathbb{J}_1 = \{(t, s) \in [1, 2]^2 : s < c_0 t\}$. The condition (**a**) is clear since $\mathcal{A}^s_t$ is an translation invariant operator, which can not be bounded from L^p to L^q if $p > q$. It can also be seen by a simple example. Indeed, let f_R be the characteristic function of a ball of radius $R \gg 1$ which is centered at the origin. Then, $\mathfrak{M}_0 f_R(x) \sim 1$ for $|x| \leq R/2$, so we have $\|\mathfrak{M}_0 f_R\|_{L^q}/\|f_R\|_{L^p} \gtrsim R^{3/q-3/p}$. Thus, $\mathfrak{M}_0$ can be bounded from L^p to L^q only if $p \leq q$.

To show (**b**), let f_r denote the characteristic function of the set

$$\{(x_1, x_2, x_3) : |x_1| < r^2,\ |x_2| < r,\ |x_3| < r^4\}$$

for a small $r > 0$. One can easily see that $\mathfrak{M}_0 f_r(x) \approx r^3$ if $x_1 \sim 1$, $|x_2| \lesssim r$, and $x_3 \sim 1$. This gives

$$\|\mathfrak{M}_0 f_r\|_{L^q}/\|f_r\|_{L^p} \gtrsim r^{3+\frac{1}{q}-\frac{7}{p}}.$$

Therefore, letting $r \to 0$ shows that the maximal operator is bounded from L^p to L^q only if (**b**) holds. Now, for (**c**) we consider the characteristic function of

$$\{(\bar{x}, x_3) : ||\bar{x}| - 1| < r,\ |x_3| < r^2\},$$

which we denote by $\tilde{f}_r$. Note that $\mathfrak{M}_0\tilde{f}_r \sim r$ if $|\bar{x}| \lesssim r$ and $x_3 \sim 1$. So, we have

$$\|\mathfrak{M}_0\tilde{f}_r\|_{L^q}/\|\tilde{f}_r\|_{L^p} \gtrsim r^{1+\frac{2}{q}-\frac{3}{p}},$$

which gives (**c**) by taking $r \to 0$. Finally, to show (**d**), let $\bar{f}_r$ be the characteristic function of the r-neighborhood of $\mathbb{T}_1^{c_0}$. Then, $|\mathfrak{M}_0\bar{f}_r(x)| \approx 1$ if $|x| \lesssim r$. Thus, it follows that $\|\mathfrak{M}_0\bar{f}_r\|_{L^q}/\|\bar{f}_r\|_{L^p} \gtrsim r^{\frac{3}{q}-\frac{1}{p}}$. So, letting $r \to 0$, we obtain (**d**).

Sharpness of smoothing estimates

Let $c_0 \in (0, 8/9)$, and let ψ be a smooth function supported in $[1/2, 2] \times [(1-2^{-4})c_0, (1+2^{-3})c_0]$ such that $\psi = 1$ if $(t, s) \in [3/4, 7/4] \times [(1-2^{-5})c_0, (1+2^{-5})c_0]$. Then, we consider

$$\tilde{\mathcal{A}}_t^s f(x) = \psi(t, s)\mathcal{A}_t^s f(x).$$

We first claim that the estimates (4.2.1), (4.2.2), and (4.2.3) imply $\alpha \le 4/p$, $\alpha \le 3/p$, and $\alpha \le 2/p$, respectively.

Let ζ_0 be a function such that $\operatorname{supp}\widehat{\zeta_0} \subset [-10^{-2}, 10^{-2}]$ and $\zeta_0(s) > 1$ if $|s| < c_1$ for a small constant $0 < c_1 \ll c_0$. Let $\zeta_* \in C_c([-2, 2])$ such that $\zeta_* = 1$ on $[-1, 1]$. Note that $\tilde{\mathbb{T}}_1^{c_0} := \mathbb{T}_1^{c_0} \cap \{x : ||\bar{x}| - 1| < 10c_1, x_3 > 0\}$ can be parametrized by a smooth radial function ϕ. That is to say,

$$\tilde{\mathbb{T}}_1^{c_0} = \{(\bar{x}, \phi(\bar{x})) : ||\bar{x}| - 1| < 10c_1\}.$$

For a large $R \gg 1$, we consider

$$f_R(x) = e^{iR(x_3+\phi(\bar{x}))}\zeta_0\big(R(x_3 + \phi(\bar{x}))\big)\zeta_*(||\bar{x}| - 1|/c_1).$$

Then, we claim that

$$|\mathcal{A}_t^s f_R(x)| \gtrsim 1, \quad (x, t, s) \in S_R, \tag{4.8.1}$$

where $S_R = \{(x, t, s) : |x| \le 1/(CR), |t - 1| \le 1/(CR), |s - c_0| \le 1/(CR)\}$ for a large constant $C > 0$. Indeed, note that

$$\mathcal{A}_t^s f(x) = \int_{\mathbb{T}_t^s} e^{iR(x_3+\phi(\bar{y}-\bar{x})-y_3)}\zeta_0(R(x_3 + \phi(\bar{y} - \bar{x}) - y_3))\zeta_*(\frac{||\bar{x} - \bar{y}| - 1|}{c_1})d\sigma_t^s(y).$$

If $|x| \le 1/(CR)$ and $||\bar{y}| - 1| \le 2c_1$, we have $|\phi(\bar{y} - \bar{x}) - y_3| \lesssim 1/(CR)$ and $|x_3+\phi(\bar{y}-\bar{x})-y_3| \lesssim 1/(CR)$ when $y_3 = \phi(y)$, i.e., $y \in \tilde{\mathbb{T}}_1^{c_0}$. Thus, $|\mathcal{A}_1^{c_0} f(x)| \sim$

1 if $|x| \le 1/(CR)$. Furthermore, if $|t-1| \le 1/(CR)$ and $|s-c_0| \le 1/(CR)$, the integration is actually taken over a surface which is $O(1/(CR))$ perturbation of the surface $\tilde{\mathbb{T}}_1^{c_0}$. Thus, taking C large enough, we see that (4.8.1) holds.

By Mikhlin's theorem it follows that $\|\tilde{\mathcal{A}}_t^s g\|_{L^p_\alpha(\mathbb{R}^5)} \gtrsim \|(1+|D_3|^2)^{\alpha/2}\tilde{\mathcal{A}}_t^s g\|_{L^p_\alpha(\mathbb{R}^5)}$. Note that $\widehat{f_R}(\xi) = 0$ if $\xi_3 \notin [(1-10^{-2})R, (1+10^{-2})R]$. Since $\mathcal{F}(\mathcal{A}_t^s f)(\xi) = \widehat{f}(\xi)\mathcal{F}(d\sigma_t^s)(\xi)$, we see

$$\|\tilde{\mathcal{A}}_t^s f_R\|_{L^p_\alpha(\mathbb{R}^5)} \gtrsim R^\alpha \|\mathcal{A}_t^s f_R\|_{L^p(\mathbb{R}^5)} \gtrsim R^\alpha \|\mathcal{A}_t^s f_R\|_{L^p(S_R)} \gtrsim R^{\alpha-5/p}.$$

For the last inequality we use (4.8.1). Since $\|f_R\|_{L^p} \sim R^{-1/p}$, (4.2.1) implies that $\alpha \le 4/p$. Fixing $t=1$ and $s=c_0$, by (4.8.1) we similarly have $\|\mathcal{A}_1^{c_0} f_R\|_{L^p_{\alpha,x}} \gtrsim R^{\alpha-3/p}$. Thus, (4.2.3) holds only if $\alpha \le 2/p$. Concerning $\mathcal{A}_t^{c_0 t}$, by (4.8.1) it follows that $|\mathcal{A}_t^{c_0 t} f_R(x)| \gtrsim 1$ if $|t-1| \le /CR$ and $|x| \le 1/CR$ for C large enough. Thus, $\|\mathcal{A}_t^{c_0 t} f_R\|_{L^{p,\alpha}_{x,t}} \gtrsim R^\alpha \|\mathcal{A}_t^{c_0 t} f_R\|_{L^p_{x,t}} \gtrsim R^{\alpha-4/p}$. Therefore, (4.2.2) implies $\alpha \le 3/p$. This proves the claim.

Therefore, to show sharpness of the estimates (4.2.1)–(4.2.3), we only need to show that each of the estimates (4.2.1), (4.2.2), and (4.2.3) holds only if $\alpha \le 1/2$. To do this, we consider

$$g_R(x) = e^{iR(x_3+c_0)}\zeta_0(R(x_3+c_0))\zeta(|x|).$$

Then, we have

$$|\mathcal{A}_t^s g_R(x)| \gtrsim R^{-\frac{1}{2}} \tag{4.8.2}$$

if $(x,t,s) \in \tilde{S}_R := \{(x,t,s) : |x|, |t-1|, |s-c_0| \le 1/C, |x_3+c_0-s| \le 1/CR\}$ for a large constant $C \gg c_0$. Indeed, note that

$$\mathcal{A}_t^s g_R(x) = \int_{\mathbb{T}_t^s} e^{iR(x_3+c_0-y_3)}\zeta_0(CR(x_3+c_0-y_3))\zeta(|x-y|)d\sigma_t^s(\bar{y}).$$

Recalling (1.5.1), we see that the integral is nonzero only if $|R(x_3+c_0-s\sin\theta)| \le 2/CR$. Since $|x_3+c_0-s| \le 1/CR$, the integral is taken over the set $\tilde{\mathbb{T}} := \{\Phi_t^s(\theta,\phi) : |1-\sin\theta| \lesssim 1/R\}$. Note that the surface area of $\tilde{\mathbb{T}}$ is about $R^{-1/2}$, thus (4.8.2) follows. Since $\widehat{g_R}(\xi) = 0$ if $\xi_3 \notin [(1-10^{-2})R, (1+10^{-2})R]$, following the same argument as above, from (4.8.2) we obtain $\|\mathcal{A}_t^s g_R\|_{L^{p,\alpha}_{x,t,s}} \gtrsim R^\alpha R^{-1/2-1/p}$. Hence, (4.2.1) implies that $\alpha \le 1/2$.

Regarding (4.2.2), we consider $\tilde{S}_R' := \{(x,t,s) : |x|, |t-1| \le 1/C, |x_3+c_0-c_0 t| \le 1/CR\}$ for a large constant $C \gg c_0$. Then, we have $|\mathcal{A}_t^{c_0 t} g_R(x)| \gtrsim R^{-1/2}$ for $(x,t) \in \tilde{S}_R'$, thus we see (4.2.2) implies $\alpha \le 1/2$.

Finally, for (4.2.3), fixing $t = 1$ and $s = c_0$, we consider $\bar{S}_R := \{x : |x| \leq 1/C, |x_3| \leq 1/CR\}$ for a constant $C > 0$. Then, it is easy to see $|A_1^{c_0} g_R(x)| \gtrsim R^{-1/2}$ for $x \in \bar{S}_R$ if we take C large enough. Similarly as before, we have $\|A_1^{c_0} g_R\|_{L^p_{\alpha,x}} \gtrsim R^\alpha R^{-1/2-1/p}$. Therefore, (4.2.3) implies $\alpha \leq 1/2$ because $\|g_R\|_{L^p} \sim R^{-1/p}$.

Chapter 5

Multiparameter averages over ellipses

As introduced before, maximal operators generated by averages over ellipses are natural multiparameter operators which generalize the circular maximal operator. Even though it is natural, the L^p-boundedness of the corresponding operator has been unknown for a long time. In this chapter we prove the boundedness result of $\mathfrak{M}$ and $\mathcal{M}$, Theorem 1.6.1 and Theorem 1.6.2, respectively. We recall the definition of the operators.

$$\mathfrak{M}f(x) = \sup_{(\theta,t,s)\in\mathbb{T}\times[1,2]^2} |f * \sigma^{\theta}_{t,s}(x)|,$$

$$\mathcal{M}f(x) = \sup_{(t,s)\in\mathbb{R}^2_+} |f * \sigma^{0}_{t,s}(x)|.$$

As a consequence of the maximal estimate (1.6.1) one can deduce some measure theoretical results concerning collections of the rotated ellipses (see, for example, [51]). In analogue to the results concerning the circular maximal function [67, 68, 44], L^p improving property of $\mathfrak{M}$ is also of interest. Using the estimates in what follows, one can easily see that $\mathfrak{M}$ is bounded from L^p to L^q for some $p < q$. However, we do not pursue the matter here.

One can notice that $\mathfrak{M}$ takes a supremum in a compact set $[0, 1]^2$ while $\mathcal{M}$ takes a supremum in a global domain of t, s. Let $\mathbb{J}$ be an interval which is a subset of $\mathbb{R}_+ := (0, \infty)$. Our approach and a standard argument relying on the Littlewood–Paley decomposition ([7, 67]) also show that the (global)maximal operator

$$\bar{\mathfrak{M}}f(x) = \sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{R}^2_+ :\, t/s\in\mathbb{J}} |f * \sigma^{\theta}_{t,s}(x)|$$

is bounded on L^p for $p > 12$ if $\mathbb{J}$ is a compact subset of $\mathbb{R}_+$. However, as eccentricity of the ellipse $\mathbb{E}^\theta_{t,s}$ increases, $\mathbb{E}^\theta_{t,s}$ gets close to a line. Using Besicovitch's construction (see, for example, [75]) and taking rotation into account, it is easy to see that (1.6.2) fails for any $p \neq \infty$ if $\mathbb{J}$ is unbounded or the closure of $\mathbb{J}$ contains the point zero.

5.1 Local smoothing estimates for averaging operators over ellipses

As is well known in the study on the circular maximal function, the L^p maximal bounds are closely related to the local smoothing estimate for the operator $f \mapsto f * d\sigma^0_{t,t}$ ([53, 72]). One may try to combine the (one-parameter) sharp local smoothing estimate for the 2-d wave operator ([30]) and the Sobolev imbedding to get bounds on $\mathfrak{M}$ and $\mathcal{M}$. However, the local smoothing estimate of (smoothing) order $2/p - \epsilon$ is not strong enough to generate any maximal bound. More specifically, in this way, one can only get $L^p_{1/p+\epsilon}$–L^p, L^p_ϵ–L^p estimates for $\mathfrak{M}$, $\mathcal{M}$, respectively. To get L^p bound, we make use of additional smoothing effect which is associated with averages along more than one parameter.

Our proofs of Theorem 1.6.1 and 1.6.2, in fact, rely on some sharp multi-parameter local smoothing estimates (see (5.1.2) and (5.1.3) in Theorem 5.1.1 below). It seems that no such smoothing estimate has appeared in literature until now. For $\xi \in \mathbb{R}^2$ and $(t,s) \in \mathbb{R}^2_+$, let $\xi_{t,s} = (t\xi_1, s\xi_2)$ and

$$\Phi^\theta_\pm(x,t,s,\xi) = x \cdot \xi \pm |(R^*_\theta \xi)_{t,s}|.$$

Here, R^*_θ denotes the transpose of R_θ. Let $B(x,r)$ denote the ball centered at x with radius r in this chapter. The asymptotic expansion of the Fourier transform of $d\sigma$ (see (4.4.2) below) naturally leads us to consider the operators

$$\mathcal{U}^\theta_\pm f(x,t,s) = a(x,t,s) \int e^{i\Phi^\theta_\pm(x,t,s,\xi)} \widehat{f}(\xi) d\xi, \tag{5.1.1}$$

where $a \in \mathrm{C}^\infty_c(B(0,2) \times (2^{-1},2^2) \times (2^{-1},2^2))$. The following are our main estimates, which play crucial roles in proving the maximal estimates.

Theorem 5.1.1. *If $p \geq 12$ and $\alpha > 1/2 - 3/p$, then the estimate*

$$\|\mathcal{U}^0_\pm f\|_{L^p_{x,t,s}} \leq C\|f\|_{L^p_\alpha} \tag{5.1.2}$$

holds. Let us set $\Delta = \{(t,s) \in (2^{-1}, 2^2)^2 : s = t\}$. *Additionally, suppose that* $\operatorname{supp} a(x,\cdot) \cap \Delta = \emptyset$ *for all* $x \in B(2,0)$. *Then, if* $p \geq 20$ *and* $\alpha > 1/2 - 4/p$, *we have the estimate*

$$\|\mathcal{U}^{\theta}_{\pm} f\|_{L^p_{x,t,s,\theta}} \leq C\|f\|_{L^p_\alpha}. \tag{5.1.3}$$

Compared with the local smoothing estimate for the 2-d wave operator $f \mapsto \mathcal{U}^0_{\pm} f(\cdot, t, t)$, the estimates (5.1.2) and (5.1.3) have additional smoothing of order up to $1/p$ and $2/p$, respectively, which results from averages in s, t; and s, t, and θ. The smoothing orders in (5.1.2) and (5.1.3) are sharp (see Chapter 5.6) in that (5.1.2), (5.1.3) fail if $\alpha < 1/2 - 3/p$, $\alpha < 1/2 - 4/p$, respectively. However, there is no reason to believe that so are the ranges of p where (5.1.2) and (5.1.3) hold true. It is clear that the condition $\operatorname{supp} a(x,\cdot) \cap \Delta = \emptyset$ is necessary for (5.1.3) to hold with all $\alpha > 1/2 - 4/p$. Indeed, when t get close to s, the ellipse $\mathbb{E}^{\theta}_{t,s}$ becomes close to the circle $\mathbb{E}^{\theta}_{s,s}$, which is invariant under rotation, so that average in θ does not yield in any further regularity gain.

An immediate consequence of the estimate (5.1.2) is that the two-parameter averaging operator $f \mapsto a(f * \sigma^0_{t,s})$ is bounded from L^p to L^p_α for $\alpha < 3/p$. From these Sobolev estimates, following the argument in [31], one can obtain results regarding dimensions of unions of ellipses.

Key observation

The main ingredients for the proof of the estimates (5.1.2) and (5.1.3) are decoupling inequalities for the operators $\mathcal{U}^0_{\pm}$ and $\mathcal{U}^{\theta}_{\pm}$ (see, for example, Lemma 5.4.1 and 5.4.3 below). Those inequalities are built on our striking observation that the immersions

$$\xi \mapsto \nabla_{x,t,s} \Phi^0_{\pm}(x,t,s,\xi), \tag{5.1.4}$$

$$\xi \mapsto \nabla_{x,t,s,\theta} \Phi^{\theta}_{\pm}(x,t,s,\xi) \tag{5.1.5}$$

(fixing (x,t,s) and (x,t,s,θ) with $s \neq t$, respectively) give rise to submanifolds which are conical extensions of a finite type curve in $\mathbb{R}^3$ and a nondegenerate curve in $\mathbb{R}^4$, respectively. By this observation, we are naturally led to regard the operators $\mathcal{U}^{\theta}_{\pm}, \mathcal{U}^0_{\pm}$ as variable coefficient generalizations of the associated conic surfaces.

Meanwhile, the decoupling inequalities for the extension (adjoint restriction) operators given by these conic surfaces, which are constant coefficient counterparts of the abovementioned operators, are already known (see Theorem 5.3.1 below and [4]). Those inequalities were, in fact, deduced from the decoupling inequality for the nondegenerate curve due to Bourgain, Demeter, and Guth [14]. To obtain such inequalities for $\mathcal{U}^0_\pm$ and $\mathcal{U}^\theta_\pm$, we combine the known inequalities for the extension operators and the argument in [6] to get a desired decoupling inequalities in a variable coefficient setting (see Theorem 5.3.2).

Decoupling inequalities of different forms have been extensively used in the recent studies on maximal and smoothing estimates for averaging operators. We refer the reader to [60, 39, 40, 3] and references therein for related works.

5.2 Proof of maximal bounds

In this section we prove the maximal estimates while assuming the smoothing estimates. We begin by recalling an elementary lemma, which is a 3-parameter analogue of Lemma 4.5.1.

Lemma 5.2.1. *Let $1 \le p \le \infty$, and J_1, J_2, and J_3 be closed intervals of length ~ 1. Let $\mathfrak{R} = J_1 \times J_2 \times J_3$ and $G \in C^1(\mathfrak{R})$. Then, there is a constant $C > 0$ such that*

$$\sup_{(t,s,\theta)\in\mathfrak{R}} |G(t,s,\theta)| \le C(\lambda_1\lambda_2\lambda_3)^{\frac{1}{p}} \sum_{\beta\in\{0,1\}^3} (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})^\beta \|\partial^\beta_{t,s,\theta} G\|_{L^p(\mathfrak{R})}$$

holds for any $\lambda_1, \lambda_2, \lambda_3 \ge 1$. Here $\beta = (\beta_1, \beta_2, \beta_3)$ denotes a triple multi-index.

By the Fourier inversion formula, we write

$$f * \sigma^\theta_{t,s}(x) = (2\pi)^{-2} \int e^{ix\cdot\xi}\, \widehat{f}(\xi)\, \widehat{d\sigma}\big((R_\theta\xi)^*_{t,s}\xi\big)\, d\xi. \tag{5.2.1}$$

We now recall the asymptotic formula of the Bessel function (4.4.2). Fixing a sufficiently large N, we may disregard the contribution from E_N. Thus, it suffices to consider the contribution from the main part $j = 0$ since the remaining parts can be handled similarly but more easily. Using (2.2.1), one can get the following estimate, which is useful later:

$$\|\mathcal{U}^\theta_\pm f\|_{L^4_{x,t,s}} \lesssim 2^{\epsilon j} \|f\|_{L^4} \tag{5.2.2}$$

for any $\epsilon > 0$ provided that $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$. Indeed, note that

$$\tilde{\mathcal{U}}_{\pm}^{\theta,s} f(x,t) := \mathcal{U}_{\pm}^{\theta} f(x,t,ts) = a(x,t,ts) \int e^{i(x\cdot\xi \pm t|(R_\theta^*\xi)_{1,s}|)} \widehat{f}(\xi) d\xi. \tag{5.2.3}$$

By a change of variables and the L^4 local smoothing estimate (2.2.1) for $\mathcal{W}_\pm$, one can easily see that $\|\tilde{\mathcal{U}}_{\pm}^{\theta,s} f\|_{L^4_{x,t}} \le C2^{\epsilon j}\|f\|_{L^4}$ for any (θ, s) whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$. Taking integration in s, we get (5.2.2).

5.2.1 2-parameter maximal function $\mathcal{M}f$: Proof of Theorem 1.6.2

To show Theorem 1.6.2 we make use of the following, which we prove in Chapter 5.4.

Proposition 5.2.2. *Let $4 \le p \le \infty$. For any $\epsilon > 0$, we have*

$$\|\mathcal{U}_{\pm}^0 f\|_{L^p_{x,t,s}} \lesssim \begin{cases} 2^{(\frac{3}{8}-\frac{3}{2p}+\epsilon)j}\|f\|_{L^p}, & 4 \le p \le 12, \\ 2^{(\frac{1}{2}-\frac{3}{p}+\epsilon)j}\|f\|_{L^p}, & 12 < p \le \infty \end{cases} \tag{5.2.4}$$

whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$.

To prove the estimate (1.6.2), we consider a local maximal operator

$$\mathcal{M}_{loc} f(x) = \sup_{(t,s)\in(0,2]^2} |f * \sigma_{t,s}^0(x)|.$$

By scaling it is sufficient to show

$$\|\mathcal{M}_{loc} f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}, \quad p > 4. \tag{5.2.5}$$

We recall the following decomposition. For any n, m we have

$$f = f_{<k}^{<n} + f_{\ge k}^{<n} + f_{<k}^{\ge n} + f_{\ge k}^{\ge n}. \tag{5.2.6}$$

Proof of (5.2.5). Denoting $Q_k^n = [2^{-k}, 2^{-k+1}] \times [2^{-n}, 2^{-n+1}]$, we set

$$\mathcal{M}_1 f = \sup_{k,n\ge0} \sup_{(t,s)\in Q_k^n} |f_{<k}^{<n} * \sigma_{t,s}^0|,$$

$$\mathcal{M}_2 f = \sup_{k,n\ge0} \sup_{(t,s)\in Q_k^n} |f_{\ge k}^{<n} * \sigma_{t,s}^0|,$$

$$\mathcal{M}_3 f = \sup_{k,n\ge0} \sup_{(t,s)\in Q_k^n} |f_{<k}^{\ge n} * \sigma_{t,s}^0|,$$

$$\mathcal{M}_4 f = \sup_{k,n\ge0} \sup_{(t,s)\in Q_k^n} |f_{\ge k}^{\ge n} * \sigma_{t,s}^0|.$$

Since $\mathcal{M}_{loc}f(x) = \sup_{k,n\ge 0}\sup_{(t,s)\in Q_k^n} |f * \sigma_{t,s}^0(x)|$, from (5.2.6) it follows that

$$\mathcal{M}_{loc}f(x) \le \textstyle\sum_{j=1}^4 \mathcal{M}_j f(x).$$

The maximal operators $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ can be handled easily as follows. We note that $f_{<k}^{<n} = f * K$ with a kernel K satisfying

$$|K(x)| \lesssim \mathrm{K}_k^n(x) := 2^{k+n}(1+2^k|x_1|)^{-N}(1+2^n|x_2|)^{-N}$$

for any large N. So, it follows that $|f_{<k}^{<n} * \sigma_{t,s}^0(x)| \lesssim \mathrm{K}_k^n * |f|(x)$ if $(t,s)\in Q_k^n$. This gives $\mathcal{M}_1 f(x) \lesssim M_s f(x)$ where $M_{\mathfrak{R}^2_{str}}$ denotes the 2-d strong maximal operator. Therefore, we get $\|\mathcal{M}_1 f\|_p \lesssim \|f\|_p$ for $1 < p \le \infty$.

We denote by H the one dimensional Hardy-Littlewood maximal operator. For the maximal operator $\mathcal{M}_2$, note that $\mathcal{F}(f_{\ge k}^{<n}) = \widehat{f}(\xi)\varphi_{<n}(|\xi_2|) - \widehat{f}(\xi)\varphi_{<k}(|\xi_1|)\varphi_{<n}(|\xi_2|)$. Thus, as before, we observe that

$$|f_{\ge k}^{<n} * \sigma_{t,s}^0(x)| \lesssim \iint \frac{2^n|f(x_1 - ty_1, x_2)|}{(1+2^n|x_2 - z_2|)^N} d\sigma(y)dz_2 + \mathcal{M}_1 f(x)$$

for $s \sim 2^{-n}$. This yields

$$\mathcal{M}_2 f(x) \lesssim H(M_c f(x_1,\cdot))(x_2) + \mathcal{M}_1 f(x),$$

where $M_c h(x_1) = \sup_{0<t<2}\int h(x_1 - ty_1)d\sigma(y)$. Using Bourgain's circular maximal theorem, Lemma 4.5.3, it is easy to see that M_c is bounded on $L^p(\mathbb{R})$ for $p > 2$. Consequently, L^p boundedness of $M_{\mathfrak{R}^2_{str}}$ and H yields $\|\mathcal{M}_2 f\|_p \lesssim \|f\|_p$ for $2 < p \le \infty$. A symmetric argument also shows that $\|\mathcal{M}_3 f\|_p \lesssim \|f\|_p$ for $2 < p \le \infty$.

Finally, we consider $\mathcal{M}_4 f$, which constitutes the main part. We note that $\mathcal{M}_4 f \le \sup_{k,n\ge 0}\sum_{j,l\ge 0}\sup_{(t,s)\in Q_k^n} |f_{j+k}^{l+n} * d\sigma_{s,t}^0|$. The embedding $\ell^p \hookrightarrow \ell^\infty$, followed by Minkowski's inequality, gives

$$\|\mathcal{M}_4 f\|_p \le \sum_{j,l\ge 0}\Big(\sum_{k,n\ge 0}\Big\|\sup_{(t,s)\in Q_k^n}\big|f_{j+k}^{l+n} * \sigma_{t,s}^0\big|\Big\|_p^p\Big)^{\frac{1}{p}}.$$

We now claim that

$$\Big\|\sup_{(t,s)\in Q_k^n}\big|f_{j+k}^{l+n} * \sigma_{t,s}^0\big|\Big\|_p \lesssim 2^{-\delta\max(j,l)}\|f_{j+k}^{l+n}\|_p, \quad j,l\ge 0 \tag{5.2.7}$$

for some $\delta > 0$ if $p > 4$. Once we have this, it is easy to show that $\mathcal{M}_4$ is bounded on L^p for $p > 4$. Indeed, note that $(\sum_{k,n} \|f_{j+k}^{l+n}\|_{L^p}^p)^{1/p} \le C\|f\|_p$ for $2 \le p \le \infty$, which follows by interpolation between the estimates for $p = 2$ and $p = \infty$(see, for example, [80, Lemma 6.1]). Combining (5.2.7) and this inequality gives

$$\|\mathcal{M}_4 f\|_{L^p} \lesssim \textstyle\sum_{j,l\ge 0} 2^{-\delta \max(j,l)} \|f\|_p \lesssim \|f\|_p.$$

It remains to show (5.2.7). By rescaling we note that the two operators $f \to \sup_{(t,s)\in Q_0^0} |f_j^l * \sigma_{t,s}^0|$ and $f \to \sup_{(t,s)\in Q_k^n} |f_{j+k}^{l+n} * \sigma_{t,s}^0|$ have the same bounds on L^p. Thus, it suffices to show (5.2.7) for $k = n = 0$. To this end, by the finite speed of propagation and translation invariance, it is enough to prove that

$$\Big\| \sup_{(t,s)\in Q_0^0} \big|f_j^l * \sigma_{t,s}^0\big| \Big\|_{L^p(B(0,1))} \lesssim 2^{-\delta \max(j,l)} \|f_j^l\|_p, \quad j, l \ge 0. \tag{5.2.8}$$

Note that the Fourier support of f_j^l is included in $\mathbb{A}_{\max(j,l)}$. We recall (5.2.1) and (4.4.2). So, it is enough to consider, instead of $f \to f_j^l * \sigma_{t,s}^0$, the operators

$$\mathcal{A}_\pm f(x,t,s) = \int |\xi_{t,s}|^{-\frac{1}{2}} e^{\pm i|\xi_{t,s}|} \widehat{f_j^l}(\xi)\, d\xi.$$

Contributions from other terms in (4.4.2) can be handled similarly but they are less significant. Therefore, the matter is reduced to obtaining the estimate

$$\Big\| \sup_{(t,s)\in Q_0^0} \big|\mathcal{A}_\pm f\big| \Big\|_{L^p(B(0,1))} \lesssim 2^{-\delta \max(j,l)} \|f_j^l\|_p \tag{5.2.9}$$

for $j, l \ge 0$. Since $\partial_t |\xi_{t,s}| = t\xi_1^2/|\xi_{t,s}|$ and $\partial_s |\xi_{t,s}| = s\xi_2^2/|\xi_{t,s}|$, applying Lemma 5.2.1 (with $\lambda_1 = \lambda_2 = 2^{\max(j,l)}$ and $\lambda_3 = 1$) to $\mathcal{A}_\pm f$ and Mikhlin's multiplier theorem, we have

$$\Big\| \sup_{(t,s)\in Q_0^0} \big|\mathcal{A}_\pm f\big| \Big\|_{L^p(B(0,1))} \lesssim 2^{(\frac{2}{p}-\frac{1}{2})\max(j,l)} \big\|\mathcal{U}_\pm^0 f_j^l\big\|_{L^p_{x,t,s}}.$$

Since the Fourier support of f_j^l is included in $\mathbb{A}_{\max(j,l)}$, by Proposition 5.2.2 it follows that (5.2.9) holds for some $\delta > 0$ as long as $p > 4$. □

5.2.2 3-parameter maximal function $\mathfrak{M}f$: Proof of Theorem 1.6.1

The proof basically relies on the estimate (5.1.3). However, to control the averages when t, s are close to each other, we need to make an additional decomposition:

$$\mathcal{U}^\theta_\pm f(x,t,s) = \sum_k \mathcal{U}^\theta_{\pm,k} f(x,t,s) := \sum_k \psi_k(t,s)\mathcal{U}^\theta_\pm f(x,t,s), \tag{5.2.10}$$

where $\psi_k(t,s) = \varphi(2^k|s-t|)$. Note that $\mathcal{U}^\theta_{\pm,k} = 0$ if $k \le -3$.

Proposition 5.2.3. *Let $4 \le p \le \infty$ and $0 \le k \le j$. For any $\epsilon > 0$, we have*

$$\|\mathcal{U}^\theta_{\pm,k} f\|_{L^p_{x,t,s,\theta}} \lesssim \begin{cases} 2^{(\frac{3}{8}-\frac{3}{2p}+\epsilon)j} 2^{\frac{k}{p}} \|f\|_{L^p}, & 4 \le p \le 20, \\ 2^{(\frac{1}{2}-\frac{4}{p}+\epsilon)j} 2^{\frac{k}{p}} \|f\|_{L^p}, & 20 < p \le \infty \end{cases} \tag{5.2.11}$$

whenever $\operatorname{supp} \widehat{f} \in \mathbb{A}_j$.

Once we have Proposition 5.2.3, the proof of Theorem 1.6.1 proceeds in a similar manner as that of Theorem 1.6.2. Note that

$$\sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |f_{<1} * \sigma^\theta_{t,s}(x)| \lesssim K_N * |f|(x)$$

for any N where $K_N(x) := (1+|x|)^{-N}$. Thus, it suffices to consider $f \mapsto \tilde{\mathfrak{M}} f := \sum_{j\ge1} \sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |f_j * \sigma^\theta_{t,s}|$. We make decomposition in s, t using ψ_k to get

$$\tilde{\mathfrak{M}} f \le \mathfrak{M}' f + \mathfrak{M}'' f := \sum_{j\ge1}\sum_{k\le j} \mathfrak{M}_k f_j + \sum_{j\ge1} \sup_{k>j} \mathfrak{M}_k f_j,$$

where

$$\mathfrak{M}_k f(x) = \sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |\psi_k(t,s)\, f * \sigma^\theta_{t,s}(x)|.$$

The operator $\mathfrak{M}''$ can be handled by using the bound on the circular maximal function. Indeed, observe that

$$2^{2j}(K_N(2^j\cdot) * \sigma^\theta_{t,s})(x) \lesssim 2^j(1 + 2^j||(R^*_\theta x)_{1,t/s}| - t|)^{-N+2}$$

for $t, s \in \mathbb{I}$. This gives $2^{2j}|\psi_k(t,s)|(K_N(2^j\cdot) * \sigma^\theta_{t,s})(x) \lesssim 2^j(1+2^j||x|-t|)^{-N+2}$ for $k \ge j$ because $|t-s| \lesssim 2^{-j}$. Note $|f_j * \sigma^\theta_{t,s}| \lesssim |f_j| * 2^{2j} K_N(2^j\cdot) * \sigma^\theta_{t,s}$. So, combining these inequalities and taking N sufficiently large, we see that

$$\sup_{k>j} \mathfrak{M}_k f_j \lesssim \bar{\mathcal{M}} f_j(x) + 2^{-10j} K_{10} * |f_j|(x),$$

where $\bar{\mathcal{M}}g(x) = \sup_{t\in(2^{-1},2^2)} |g * \sigma^0_{t,t}(x)|$. It is well known that $\|\bar{\mathcal{M}}f_j\|_p \le 2^{-cj}\|f\|_p$ for some $c > 0$ if $p > 2$ (see [53, 44]). Therefore, $\mathfrak{M}''$ is bounded on L^p for $p > 2$.

To show L^p bound on $\mathfrak{M}'$, as before, we only need to show the local estimate $\|\mathfrak{M}'f\|_{L^p(B(0,1))} \lesssim \|f\|_p$ for $p > 12$. This is immediate once we have

$$\|\mathfrak{M}_k f_j\|_{L^p(B(0,1))} \lesssim 2^{-\epsilon_0 j}\|f\|_{L^p}, \quad 1 \le k \le j$$

for any $p > 12$ and some $\epsilon_0 > 0$. By (5.2.1) and (4.4.2), the estimate follows if we show

$$\Big\| \sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |\mathcal{U}^\theta_{\pm,k} f_j| \Big\|_p \lesssim 2^{(\frac{3}{8}+\frac{3}{2p}+\epsilon)j}\|f\|_p, \quad 4 \le p \le 20. \tag{5.2.12}$$

Proof of (5.2.12). We use Lemma 5.2.1. To do so, we observe that

$$\begin{aligned}
\partial_t|(R^*_\theta\xi)_{t,s}| &= m_1 := t(R^*_\theta\xi)_1^2|(R^*_\theta\xi)_{t,s}|^{-1},\\
\partial_s|(R^*_\theta\xi)_{t,s}| &= m_2 := s(R^*_\theta\xi)_2^2|(R^*_\theta\xi)_{t,s}|^{-1},\\
\partial_\theta|(R^*_\theta\xi)_{t,s}| &= m_3 := (t^2 - s^2)(R^*_\theta\xi)_1(R^*_\theta\xi)_2|(R^*_\theta\xi)_{t,s}|^{-1}.
\end{aligned}$$

It is clear that $|\partial^\alpha_\xi m_l| \lesssim |\xi|^{1-|\alpha|}$, $l = 1, 2$, and $|\partial^\alpha_\xi m_3| \lesssim 2^{-k}|\xi|^{1-|\alpha|}$. Note that $|\nabla_{t,s}\psi_k(t, s)| \lesssim 2^k \le 2^j$. Recalling (5.1.1) and (5.2.10), we apply Lemma 5.2.1 to $\sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |\mathcal{U}^\theta_{k,\pm} f_j|$ with $\lambda_1 = \lambda_2 = 2^j$ and $\lambda_3 = 2^{j-k}$. Thus, by Mikhlin's multiplier theorem, we have

$$\Big\| \sup_{(\theta,t,s)\in\mathbb{T}\times\mathbb{I}^2} |\mathcal{U}^\theta_{k,\pm} f_j| \Big\|_p \lesssim 2^{(3j-k)/p} \big\|\mathcal{U}^\theta_{k,\pm} f_j\big\|_{L^p_{x,t,s,\theta}}.$$

By Proposition 5.2.3 the estimate (5.2.12) follows. □

5.3 Variable coefficient decoupling inequalities

In this section, we discuss the decoupling inequalities which we need to prove Proposition 5.2.2 and 5.2.3.

Definition. Let I be an interval and $\gamma : I \to \mathbb{R}^d$ be a smooth curve. We say γ is nondegenerate if $\det(\gamma'(u), \cdots, \gamma^{(d)}(u)) \ne 0$ for all $u \in I$.

For a curve γ defined on $\mathbb{I}_0 := [-1,1]$, we set $\mathfrak{C}(\gamma) = \big\{r(1,\gamma(u)) : u \in \mathbb{I}_0,\ r \in \mathbb{I}\big\}$, which we call the *conical extension of* γ. Consider an adjoint restriction operator

$$E^\gamma g(z) := \iint_{\mathbb{I}_0\times\mathbb{I}} e^{iz\cdot r(1,\gamma(u))} g(u,r)dudr, \quad z \in \mathbb{R}^{d+1}, \tag{5.3.1}$$

which is associated with $\mathfrak{C}(\gamma)$. By $\mathcal{J}(\delta)$ we denote a collection of disjoint intervals of length $l \in (2^{-1}\delta, 2\delta)$ which are included in $\mathbb{I}_0$. For a given function g on $\mathbb{I}_0 \times \mathbb{I}$ and $J \in \mathcal{J}(\delta)$, we set

$$g_J(u,r) = \chi_J(u)g(u,r).$$

We denote $\operatorname{supp}_u g = \{u : \operatorname{supp} f(u,\cdot) \neq \emptyset\}$ so that $\operatorname{supp}_u g_J$ is included in J.

Using the decoupling inequality for the nondegenerate curve [14] and the argument in [10] (see also [4]), we have the following decoupling inequality for E^γ.

Theorem 5.3.1. *Let $p \geq d(d+1)$ and $\alpha_d(p) := (2p-d^2-d-2)/(2dp)$. Let $0<\delta<1$ and $\mathcal{J}(\delta^{1/d})$ be a collection of disjoint intervals given as above. Let B denote a ball of radius δ^{-1} in $\mathbb{R}^{d+1}$. Suppose that γ is nondegenerate. Then, for any $\epsilon > 0$ we have*

$$\|E^\gamma(\sum_{J\in\mathcal{J}(\delta^{1/d})} g_J)\|_{L^p(\omega_B)} \lesssim_\epsilon \delta^{-\alpha_d(p)-\epsilon}\Big(\sum_{J\in\mathcal{J}(\delta^{1/d})} \|E^\gamma g_J\|^p_{L^p(\omega_B)}\Big)^{1/p}.$$

Here, $\omega_B(x) = (1+R_B^{-1}|x-c_B|)^{-N}$ with a sufficiently large $N \geq 100(d+1)$ and c_B, R_B denoting the center of B, the radius of B, respectively.

However, the phase function $\Phi^\theta_\pm(x,t,s,\xi)$ is not linear in t,s,θ. So, for our purpose of proving the smoothing estimate, we need a variable coefficient generalization of Theorem 5.3.1.

5.3.1 Variable coefficient decoupling

Let

$$\mathbb{D} = \mathbb{B}^{d+1}(0,2)\times(-1,1)\times(1/2,2). \tag{5.3.2}$$

Let $\Phi : \overline{\mathbb{D}} \to \mathbb{R}$ be a smooth function and A be a smooth function with $\operatorname{supp} A \subset \mathbb{D}$. For $\lambda \geq 1$, we consider

$$\mathcal{E}_\lambda g(z) = \iint e^{i\lambda r\Phi(z,u)} A(z,u,r) g(u,r)dudr.$$

The following is a variable coefficient generalization of Theorem 5.3.1. Let

$$\mathcal{T}(\Phi)(z,u) = (\Phi(z,u), \partial_u \Phi(z,u), \cdots, \partial_u^d \Phi(z,u)).$$

Theorem 5.3.2. *Let $p \geq d(d+1)$. Suppose that*

$$\operatorname{rank} D_z \mathcal{T}(\Phi) = d+1 \tag{5.3.3}$$

on supp a. *Then, for any $\epsilon > 0$ and $M > 0$, we have*

$$\|(\sum_{J \in \mathcal{J}(\lambda^{-1/d})} \mathcal{E}_\lambda g_J)\|_{L^p} \lesssim_{\epsilon,M} \lambda^{\alpha_d(p)+\epsilon} \Big(\sum_{J \in \mathcal{J}(\lambda^{-1/d})} \|\mathcal{E}_\lambda g_J\|_{L^p}^p \Big)^{1/p} + \lambda^{-M} \|g\|_2. \tag{5.3.4}$$

Here, we allow discrepancy between amplitude functions in the left hand and right hand sides, that is to say, the amplitude functions on the both sides are not necessarily the same.

We refer to the inequality (5.3.4) as a decoupling of $\mathcal{E}_\lambda$ at scale $\lambda^{-1/d}$. As is clear, the implicit constant in (5.3.4) is independent of particular choices of $\mathcal{J}(\lambda^{-1/d})$. The role of the amplitude function A is less significant. In fact, changes of variables $z \to \mathcal{Z}(z)$ and $u \to \mathcal{U}(u)$ separately in z and u do not have effect on the decoupling inequality as long as $\mathcal{Z}$, $\mathcal{Z}^{-1}$, $\mathcal{U}$, and $\mathcal{U}^{-1}$ are smooth with uniformly bounded derivatives up to some large order. The decoupling for the original operator can be recovered by undoing the changes of variables. This makes it possible to decouple an operator by using the decoupling inequality in a normalized form. For our purpose it is enough to consider the amplitude of the form $A(z,u,r) = A_1(z)A_2(u,r)$. This can be put together with those aforementioned changes of variables to deduce decoupling inequalities.

Theorem 5.3.2 can be shown through routine adaptation of the argument in [6], where the authors obtained a variable coefficient generalization of decoupling inequality for conic hypersurfaces, that is to say, Fourier integral operators. However, we include a proof of Theorem 5.3.2 for convenience of the readers (see Chapter 5.5).

5.3.2 Decoupling with a degenerate phase

To show Proposition 5.2.2 we also need to consider an operator which does not satisfy the nondegenerate condition (5.3.3). In particular, we make use of the following for this purpose.

Corollary 5.3.3. *Let $\mathcal{J}(\lambda^{-1/d})$ be a collection of disjoint intervals such that $J \subset (-\epsilon_0, \epsilon_0)$ for $J \in \mathcal{J}(\lambda^{-1/d})$. Suppose that* $\det D_z\mathcal{T}(\Phi)(z,0) = 0$ *and*

$$\det(\nabla_z\Phi(z,0), \partial_u\nabla_z\Phi(z,0), \cdots, \partial_u^{d-1}\nabla_z\Phi(z,0), \partial_u^{d+1}\nabla_z\Phi(z,0)) \neq 0 \quad (5.3.5)$$

for $z \in \operatorname{supp}_z A$. Then, if ϵ_0 is small enough, for $\epsilon > 0$ and $M > 0$ we have

$$\|(\sum_{J\in\mathcal{J}(\lambda^{-1/d})} \mathcal{E}_\lambda g_J)\|_{L^p} \lesssim_{\epsilon,M} \lambda^{\alpha_d(p)+\epsilon}\Big(\sum_{J\in\mathcal{J}(\lambda^{-1/d})} \|\mathcal{E}_\lambda g_J\|_{L^p}^p\Big)^{1/p} + \lambda^{-M}\|g\|_2.$$

A typical example of the phase which satisfies (5.3.5) is

$$\tilde{\Phi}_0(z,u) := z\cdot\big(1, u, \cdots, u^{d-1}/(d-1)!, u^{d+1}/(d+1)!\big).$$

Such a phase becomes nondegenerate away from the origin. This fact can be exploited using dyadic decomposition and a standard rescaling argument.

Let j_0 be the largest integer satisfying $2^{j_0} \le \lambda^{1/(d+1)-\epsilon}$. For $j < j_0$, we set

$$g_j = \sum_{1\le 2^j \operatorname{dist}(0,J)<2} g_J.$$

Thus, $\sum_J g_J = \sum_{1\le j<j_0} g_j + \sum_{\operatorname{dist}(0,J)<2^{-j_0}} g_J$. Let $A_j(z,u,r) = A(z, 2^{-j}u, r)$ and $\tilde{g}_j = 2^{-j}g_j(2^{-j}\cdot,\cdot)$. For $0 \le j < j_0$, changing variables $u \to 2^{-j}u$, we get

$$\textstyle\sum_J \mathcal{E}_\lambda g_J = \sum_{0\le j<j_0} \mathcal{E}_{A_j}^{\lambda\Phi(\cdot,2^{-j}\cdot)}\tilde{g}_j + \sum_{\operatorname{dist}(0,J)<2^{-j_0}} \mathcal{E}_A^{\lambda\Phi} g_J,$$

Here and afterwards, for given Ψ and b, we denote

$$\mathcal{E}_b^\Psi g(z) = \iint e^{ir\Psi(z,u)}b(z,u,r)g(u,r)dudr.$$

We set $\tilde{\Phi}(z,u) = \sum_{k=0}^{d+1}\partial_u^k\Phi(z,0)u^k/k!$. Using Taylor's expansion, we have

$$\Phi(z,u) = \tilde{\Phi}(z,u) + \mathcal{R}(z,u),$$

where $\mathcal{R}(z,u) = \int_0^u \partial_u^{d+2}\Phi(z,s)(u-s)^{d+1}ds/(d+1)!$. From the condition (5.3.5) we note that the vectors $\nabla_z\Phi(z,0), \partial_u\nabla_z\Phi(z,0), \cdots, \partial_u^{d-1}\nabla_z\Phi(z,0)$ are linearly independent. Meanwhile, since $\det D_z\mathcal{T}(\Phi)(z,0) = 0$, $\nabla_z\Phi(z,0)$, $\cdots, \partial_u^{d-1}\nabla_z\Phi(z,0), \partial_u^d\nabla_z\Phi(z,0)$ are linearly dependent. Thus, there are smooth functions $r_0, \dots, r_{d-1}$, such that $\partial_u^d \Phi(z,0) = \sum_{k=0}^{d-1} r_k(z)\partial_u^k \Phi(z,0)/k!$. This yields

$$\tilde{\Phi}(z,u) = \sum_{k=0}^{d-1}(1 + u^{d-k} r_k(z))\partial_u^k \Phi(z,0)\frac{u^k}{k!} + \partial_u^{d+1}\Phi(z,0)\frac{u^{d+1}}{(d+1)!}.$$

Let $\mathcal{L}$ denote the inverse of the map $z \mapsto (\Phi(z,0), \cdots, \partial_u^{d-1}\Phi(z,0), \partial_u^{d+1}\Phi(z,0))$. Setting $T_j(z) = \mathcal{L}(2^{-(d+1)j}z_1, \cdots, 2^{-2j}z_d, z_{d+1})$, we have

$$2^{(d+1)j}\tilde{\Phi}(T_j(z), 2^{-j}u) = \sum_{k=0}^{d-1}(1 + (2^{-j}u)^{d-k} r_k(T_j(z)))\frac{z_{k+1}u^k}{k!} + \frac{z_{d+1}u^{d+1}}{(d+1)!}.$$

It is clear that $\partial^\alpha(2^{(d+1)j}\mathcal{R}(T_j(\cdot), 2^{-j}\cdot)) = O(2^{-j})$ and $\partial_{z,u}^\alpha(2^{(d+1)j}\tilde{\Phi}(T_j(z), 2^{-j}u) - \tilde{\Phi}_0(z,u)) = O(2^{-j})$ for any α. Therefore,

$$[\Phi_j](z,u) := 2^{(d+1)j}\Phi(T_j(z), 2^{-j}u),$$

which is close to $\tilde{\Phi}_0(z,u)$, satisfies the nondegeracy condition (5.3.3) for $|u| \sim 1$ if ϵ_0 is small enough. Changing variables $z \to T_j(z)$, we have

$$\mathcal{E}_{A_j}^{\lambda\Phi(\cdot,2^{-j}\cdot)}\tilde{g}_j(T_j(z)) = \mathcal{E}_{A_j\circ T_j}^{\lambda 2^{-(d+1)j}[\Phi_j]}\tilde{g}_j(z).$$

Decomposing $A_j \circ T_j$ into smooth functions which are supported in a ball of radius~ 1, we may apply Theorem 5.3.2. By putting together the resultant inequalities on each ball, this gives decoupling of $\mathcal{E}_{A_j\circ T_j}^{\lambda 2^{-(d+1)j}[\Phi_j]}\tilde{g}_j$ at scales $\lambda^{-1/d}2^{(d+1)j/d}$. Here, it should be note that the constants in the decoupling inequality can be taken uniformly since the phases $[\Phi_j]$ are close to $\tilde{\Phi}_0$.

After undoing the change of variables and rescaling it in turn gives decoupling of $\mathcal{E}_\lambda g_j$ at scales $\lambda^{-1/d}2^{j/d}$. Now, in order to obtain decoupling at scale $\lambda^{-1/d}$, we make use of the trivial decoupling.* Since there are as many as $\sim 2^{j/d}$ intervals J, it produces a factor of $O(2^{j(1-2/p)/d})$ in its bound. Putting everything together, we see that $\|\mathcal{E}_\lambda g_j\|_{L^p}$ is bounded above by a constant times

$$(\lambda 2^{-(d+1)j})^{\alpha_d(p)+\epsilon}2^{j(\frac{1}{d}-\frac{2}{pd})}\Big(\sum_{1\le 2^j \operatorname{dist}(0,J)<2}\|\mathcal{E}_\lambda g_J\|_p^p\Big)^{\frac{1}{p}} + (2^{(d+1)j}/\lambda)^M\|g\|_2.$$

Terms with $\operatorname{dist}(0,J) < 2^{-j_0}$ can be handled easily. Since $-(d+1)\alpha_d(p) + (1-2/p)/d < 0$, taking summation along $1 \le j \le j_0$, we get the desired inequality.

*$\|(\sum_{J\in\mathcal{J}}\mathcal{E}_\lambda f_J)\|_{L^p} \le (\#\mathcal{J})^{1-2/p}(\sum_{J\in\mathcal{J}}\|\mathcal{E}_\lambda f_J)\|_{L^p}^p)^{1/p}$

5.4 Proof of local smoothing estimates

In this section, we prove Proposition 5.2.2 and 5.2.3 making use of the key observation that the immersions (5.1.4) and (5.1.5) give conic extensions of finite type curves. Using suitable forms of decoupling inequalities, we first decompose the averaging operators so that the consequent operators have their Fourier supports in narrow angular sectors. For each of those operators, fixing some variables, we make use of the local smoothing estimate for the 2-d wave propagator in $\mathbb{R}^{2+1}$ (for example, see (5.2.2)), or lower dimensional decoupling inequality.

Throughout this section, we assume

$$\operatorname{supp} \widehat{f} \subset \mathbb{A}_j.$$

To exploit the decoupling inequalities, we decompose f into functions whose Fourier supports are contained in angular sectors. For $\kappa \in (0,1)$, let $\{\Theta_m^\kappa\}_{m=1}^N$ denote a collection of disjoint arcs of length $L \in (2^{-1}\kappa, 2\kappa)$ such that $\bigcup_{m=1}^N \Theta_m^\kappa = \mathbb{S}^1$. Let $\{\zeta_m^\kappa\}_{j=1}^N$ be a partition of unity on $\mathbb{S}^1$ satisfying $\operatorname{supp} \zeta_m^\kappa \subset \Theta_{m-1}^\kappa \cup \Theta_m^\kappa \cup \Theta_{m+1}^\kappa$ for $1 \le m \le N$ (here, we identify $\Theta_0^\kappa = \Theta_N^\kappa$ and $\Theta_{N+1}^\kappa = \Theta_1^\kappa$) and $|(d/d\theta)^l \zeta_m^\kappa| \lesssim \kappa^{-l}$ for $l \ge 0$. We denote

$$\mathfrak{S}(\kappa) = \{\zeta_m^\kappa\}_{j=1}^N.$$

For each $\nu \in \mathfrak{S}(\kappa)$, set

$$\widehat{f_\nu}(\xi) = \widehat{f}(\xi)\nu(\xi/|\xi|).$$

5.4.1 2-parameter case: Proof of Proposition 5.2.2

We only consider the estimate for $\mathcal{U}^0 := \mathcal{U}_+^0$. The estimate for $\mathcal{U}_-^0$ follows by the same argument. We begin with the next lemma, which we obtain by using Corollary 5.3.3.

Lemma 5.4.1. *Let $p \ge 12$ and $j \ge 0$. Suppose that* $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$. *Then, for any $\epsilon > 0$ and $M > 0$, we have*

$$\|\mathcal{U}^0 f\|_{L^p} \lesssim 2^{(\frac{1}{3} - \frac{7}{3p} + \epsilon)j} \Big(\sum_{\nu \in \mathfrak{S}(2^{-j/3})} \|\mathcal{U}^0 f_\nu\|_{L^p}^p \Big)^{1/p} + 2^{-Mj} \|f\|_{L^p}. \tag{5.4.1}$$

Proof. By decomposing f on the Fourier side and symmetry, we assume that $\operatorname{supp}\widehat{f}$ is additionally included in the set $\{\xi : |\xi_2| \le 2\xi_1\}$. We make changes of variables $\xi \to 2^j\xi$ and $(\xi_1, \xi_2) \to (r, ru)$, successively, to obtain

$$\mathcal{U}^0 f(x,t,s) = a(x,t,s)\int e^{i2^j r\Phi(x,t,s,u)}\widehat{f(2^{-j}\cdot)}(r,ru)rdrdu,$$

where $\Phi(x,t,s,u) = x_1 + x_2 u + |(1,u)_{t,s}|$. Let us set

$$h(u) = (\rho^2+u^2)^{-1/2}, \quad \rho = t/s. \tag{5.4.2}$$

Then, a computation shows that

$$\nabla_{x,t,s}\Phi(x,t,s,u) = \bar\gamma(u) := (1,\gamma(u)) := \big(1, u, \rho h(u), u^2 h(u)\big).$$

Lemma 5.4.2. *Let $t, s \in \mathbb{I}$. Then, we have*

$$|\det(\bar\gamma(u), \bar\gamma'(u), \bar\gamma''(u), \bar\gamma'''(u))| \sim |u|, \tag{5.4.3}$$

$$|\det(\bar\gamma(u), \bar\gamma'(u), \bar\gamma''(u), \bar\gamma''''(u))|_{u=0} \sim 1. \tag{5.4.4}$$

Proof. Note that

$$\gamma^{(k)}(u) = \big(0,\ \rho h^{(k)}(u),\ 2(2k-3)h^{(k-2)}(u) + 2kuh^{(k-1)}(u) + u^2 h^{(k)}(u)\big)$$

for $k = 2, 3$. Since $\det(\bar\gamma(u), \bar\gamma'(u), \bar\gamma''(u), \bar\gamma'''(u)) = \det(\gamma'(u), \gamma''(u), \gamma'''(u))$,

$$\det(\bar\gamma(u), \bar\gamma'(u), \bar\gamma''(u), \bar\gamma'''(u)) = 2\rho\det\begin{pmatrix} h'' & h + 2uh' \\ h''' & 3h' + 3uh'' \end{pmatrix}.$$

After a computation one can easily check the following:

$$\begin{aligned} h'(u) &= -uh^3(u), & h''(u) &= (2u^2-\rho^2)h^5(u), \\ h'''(u) &= 3(3\rho^2 u - 2u^3)h^7(u), & h''''(u) &= 3(8u^4 - 24\rho^2u^2 + 3\rho^4)h^9(u). \end{aligned} \tag{5.4.5}$$

Using this, we obtain $\det(\gamma'(u), \gamma''(u), \gamma'''(u)) = -6\rho^5 u(u^2+\rho^2)^{-5}$. This gives (5.4.3) since $\rho \sim 1$. Furthermore, differentiating both sides of the equation, we also have $\det(\gamma'(u), \gamma''(u), \gamma''''(u)) = 6\rho^5(u^2+\rho^2)^{-6}(9u^2-\rho^2)$, which shows (5.4.4). □

Lemma 5.4.2 shows that $\nabla_{x,t,s}\Phi(x,t,s,u)$ satisfies the assumption in Corollary 5.3.3 for $d=3$. Thus, if u is away from $u=0$, then $\nabla_{x,t,s}\Phi(x,t,s,u)$ fulfills the nondegeneracy condition (5.3.3). Therefore, decomposing the integral $\mathcal{U}^0 f$ into two parts, one near $u=0$ and one away from $u=0$, we apply Corollary 5.3.3 and Theorem 5.3.2 to the former and the latter, respectively, so that we can get decoupling at scale $2^{-j/3}$ for the both parts. Note that the u-support of $g_\nu(u,r) := \mathcal{F}f_\nu(2^{-j}\cdot)(r,ru)$, $\nu\in\mathfrak{S}(2^{-j/3})$ are contained in boundedly overlapping intervals of length $\sim 2^{-j/3}$, so we have the decoupling inequality

$$\|\sum_{\nu\in\mathfrak{S}(2^{-j/3})}\mathcal{E}_{2^j}g_\nu\|_p \lesssim 2^{(\frac{1}{3}-\frac{7}{3p}+\epsilon)j}(\sum_{\nu\in\mathfrak{S}(2^{-j/3})}\|\mathcal{E}_{2^j}g_\nu\|_{L^p}^p)^{1/p}.$$

Therefore, undoing the changes of variables $\xi\to 2^j\xi$ and $(\xi_1,\xi_2)\mapsto(r,ru)$, we get the desired inequality (5.4.1). $\square$

To complete the proof of Proposition 5.2.2 it is sufficient to show (5.2.4) for $p>12$ since the estimate for $4\le p\le 12$ follows by interpolation with the estimate (5.2.2). By the inequality (5.4.1), we only have to prove that

$$\Big(\sum_{\nu\in\mathfrak{S}(2^{-j/3})}\|\mathcal{U}^0 f_\nu\|_{L^p}^p\Big)^{1/p} \lesssim 2^{(\frac{1}{6}-\frac{2}{3p}+\epsilon)j}\|f\|_{L^p}$$

for $p>12$. Since $\operatorname{supp}\widehat{f}\subset\mathbb{A}_j$, one can easily see that $(\sum_\nu\|f_\nu\|_p^p)^{1/p}\lesssim\|f\|_p$ for $2\le p\le\infty$. This in fact follows by interpolation between the estimates for $p=2$ and $p=\infty$. So, the matter is reduced to showing that

$$\|\mathcal{U}^0 f_\nu\|_{L^p_{x,t,s}} \lesssim 2^{(\frac{1}{6}-\frac{2}{3p}+\epsilon)j}\|f_\nu\|_{L^p} \tag{5.4.6}$$

for $p\ge 12$. Recalling (5.2.3) with $\theta=0$ and changing variables $\xi_2\to\xi_2/s$, we can use the local smoothing estimate for the wave operator. Since $s\sim 1$, the support of $\widehat{f_\nu}(\xi_1,\xi_2/s)$ is included in an angular sector of angle $\sim 2^{-j/3}$. Applying Lemma 2.2.2 with $\lambda=2^j$ and $b\sim 2^{-j/3}$, we obtain

$$\|\mathcal{U}^0 f_\nu(x,t,ts)\|_{L^p_{x,t}} \le C2^{(\frac{1}{6}-\frac{2}{3p}+\epsilon)j}\|f_\nu\|_{L^p}$$

for any $\epsilon>0$. Integrating in s gives the desired estimate (5.4.6).

5.4.2 3-parameter case: Proof of Proposition 5.2.3

As before, we only consider the estimate for

$$\mathcal{U}_k^\theta := \mathcal{U}_{+,k}^\theta$$

given by (5.2.10). That for $\mathcal{U}_{-,k}^\theta$ can be obtained in the same manner. We use the decoupling inequality in Theorem 5.3.2 to obtain estimates for $\mathcal{U}_k^\theta$. We start with the next lemma.

Lemma 5.4.3. *Let $p \ge 20$ and $0 \le k \le j$, and set $z = (x, t, s, \theta)$. Suppose* $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$. *Then, for any $\epsilon > 0$ and $M > 0$, we have*

$$\|\mathcal{U}_k^\theta f\|_{L_z^p} \lesssim_{\epsilon, M} 2^{(1-\frac{11}{p}+\epsilon)\frac{j-k}{4}} \Big(\sum_{\nu \in \mathfrak{S}(2^{(k-j)/4})} \|\mathcal{U}_k^\theta f_\nu\|_{L_z^p}^p \Big)^{1/p} + 2^{-M(j-k)} \|f\|_p. \tag{5.4.7}$$

Proof. By rotational symmetry, we may assume that θ is restricted near $\theta = 0$. Thus, we only need to consider

$$\tilde{\mathcal{U}} f(z', \theta) = \int e^{i\Psi(z', \theta, \xi)} \tilde{a}(z', \theta) \widehat{f}(\xi) d\xi, \quad z' = (x, t, s),$$

where

$$\Psi(z', \theta, \xi) = x \cdot \xi + |(R_\theta^* \xi)_{t,s}|, \quad \tilde{a}(z', \theta) = a(x, t, s) \phi_{<0}(\theta/\epsilon_0) \psi(2^k |t - s|)$$

for a small $\epsilon_0 > 0$. Changing variables $z' \to 2^{-k} z'$ and $\xi \to 2^j \xi$, we have

$$\tilde{\mathcal{U}} f(z', \theta) = \int e^{i 2^{j-k} \Psi(z', \theta, \xi)} \tilde{a}(2^{-k} z', \theta) \widehat{f(2^{-j} \cdot)}(\xi) d\xi.$$

We decompose $\tilde{a}(2^{-k} z', \theta) = \sum_n a_n(z', \theta)$ such that $\operatorname{supp}_{z'} a_n$ are included in finitely overlapping balls of radius 1 and the derivatives of a_n are uniformly bounded. We are now reduced to obtaining the decoupling inequality for the operator

$$\mathcal{E}(\lambda \Psi, a_n) g(z', \theta) := \int e^{i \lambda \Psi(z', \theta, \xi)} a_n(z', \theta) g(\xi) d\xi \tag{5.4.8}$$

with $\lambda := 2^{j-k}$ and $g := \mathcal{F}[f(2^{-j} \cdot)]$ whose support is included in $\mathbb{A}_0$.

We now intend to apply Theorem 5.3.2. However, the cutoff a_n is no longer supported in a fixed bounded set, so the constants appearing the decoupling inequality for $\mathcal{E}(\lambda\Psi, a_n)g$ may differ. To guarantee the the constants are uniformly bounded, one may consider a slightly modified operator. Let $z_0' \in \operatorname{supp}_{z'} a_n$. Changing variables $z' \to z' + z_0'$, we may replace a_n, g, Ψ by $\tilde{a}_n(z', \theta) := a_n(z' + z_0', \theta)$, $\tilde{g}(\xi) := e^{i\lambda\Psi(z_0', 0, \xi)} g(\xi)$,

$$\tilde{\Psi}_n(z', \theta, \xi) := \Psi(z' + z_0', \theta, \xi) - \Psi(z_0', 0, \xi),$$

respectively. For our purpose it is enough to consider $\mathcal{E}(\lambda\tilde{\Psi}_n, \tilde{a}_n)\tilde{g}$. One can easily check that the derivatives of $\tilde{\Psi}_n$ and $\tilde{a}_n$ are uniformly bounded on $B(0,2) \times \mathbb{A}_0$ for each n.

In order to apply Theorem 5.3.2 to $\mathcal{E}(\lambda\tilde{\Psi}_n, \tilde{a}_n)\tilde{g}$, we need to verify that the assumption of Theorem 5.3.2 is satisfied after suitable decomposition and allowable transformation. Let $\mathbb{A}' = \{(\xi_1, \xi_2) \subset \mathbb{A}_0 : |\xi_2| < 2\xi_1\}$ and $\mathbb{A}'' = \{(\xi_1, \xi_2) \subset \mathbb{A}_0 : |\xi_1| < 2\xi_2\}$. Decomposing g, we separately consider the following four cases:

$$\operatorname{supp} g \subset \mathbb{A}', \quad \operatorname{supp} g \subset \mathbb{A}'', \quad \operatorname{supp} g \subset -\mathbb{A}', \quad \operatorname{supp} g \subset -\mathbb{A}''. \tag{5.4.9}$$

We first handle the case $\operatorname{supp} g \subset \mathbb{A}'$. Writing $\tilde{\Psi}_n(z', \theta, \xi) = \xi_1 \tilde{\Psi}_n(z', \theta, 1, \xi_2/\xi_1)$, we set

$$\Phi_n(z, u) = \tilde{\Psi}_n(z, 1, u), \quad z = (z', \theta).$$

As in the proof of Lemma 5.4.1, the desired decoupling inequality follows if we obtain a decoupling inequality for $\mathcal{E}^{\lambda\Phi_n}_{\tilde{a}_n}$ of scale $\lambda^{-1/4}$.

Even though we have translated $z' \to z' + z_0$, it is more convenient to do computation before the translation on $\operatorname{supp} a_n$, that is to say, $|s - t| \sim 1$ and $t, s \sim 2^k$. Note that

$$\nabla_z \Psi(z', \theta, \xi) := \Big(\xi, \frac{t(R_\theta^*\xi)_1^2}{|(R_\theta^*\xi)_{t,s}|}, \frac{s(R_\theta^*\xi)_2^2}{|(R_\theta^*\xi)_{t,s}|}, \frac{2(t^2 - s^2)(R_\theta^*\xi)_1(R_\theta^*\xi)_2}{|(R_\theta^*\xi)_{t,s}|}\Big),$$

where $R_\theta^*\xi = \big((R_\theta^*\xi)_1, (R_\theta^*\xi)_2\big)$. To show that the condition (5.3.3) holds, it is sufficient to consider $\theta = 0$ since $\operatorname{supp}_\theta a \subset (-\epsilon_0, \epsilon_0)$. We set

$$\Upsilon(u) = \big(u, \rho h(u), u^2 h(u), 2s^{-1}(t^2 - s^2) u h(u)\big). \tag{5.4.10}$$

Then, recalling (5.4.2), we see that $\nabla_z \Psi(z', 0, u) = (1, \Upsilon(u))$. To verify (5.3.3) we have only to show that Υ is nondegenerate, i.e.,

$$H(u) := \det(\Upsilon'(u), \Upsilon''(u), \Upsilon'''(u), \Upsilon''''(u)) \neq 0.$$

To show this, we note that $sH(u)/2\rho(t^2-s^2)$ equals

$$\det\begin{pmatrix} h'' & 2h+4uh'+u^2h'' & 2h'+uh'' \\ h''' & 6h'+6uh''+u^2h''' & 3h''+uh''' \\ h'''' & 12h''+8uh'''+u^2h'''' & 4h'''+uh'''' \end{pmatrix} = \det\begin{pmatrix} h'' & 2h & 2h' \\ h''' & 6h' & 3h'' \\ h'''' & 12h'' & 4h''' \end{pmatrix}.$$

Therefore, using (5.4.5), one can readily see

$$H(u) = W(u,t,s)\det\begin{pmatrix} 2u^2-\rho^2 & 1 & -2u \\ -2u^3+3\rho^2u & -u & 2u^2-\rho^2 \\ 8u^4-24\rho^2u^2+3\rho^4 & 4u^2-2\rho^2 & -8u^3+12\rho^2u \end{pmatrix},$$

where $W(u,t,s) = 36\rho(t^2-s^2)s^{-1}h^{15}(u)$. A computation shows that the determinant equals ρ^6, so we get $H(u) = 36\rho^7(t^2-s^2)s^{-1}h^{15}(u)$. Since $t,s\sim 2^k$ and $|t-s|\sim 1$ on $\operatorname{supp} a_n$, we have $|H(u)|\ge c$ for a constant $c>0$ on $\operatorname{supp} a_n$. This shows that Φ satisfies the nondegeneracy condition (5.3.3) on $\operatorname{supp}\tilde a_n\times(-2,2)$ (uniformly for each n).

Therefore, by Theorem 5.3.2 with $d=4$ we get decoupling of $\mathcal{E}^{\lambda\Phi_n}_{\tilde a_n}$. In fact, we get ℓ^p decoupling of $\mathcal{E}(\lambda\Phi,\tilde a_n)\tilde g$ into $\mathcal{E}(\lambda\Phi,\tilde a_n)(\tilde g\nu(\cdot/|\cdot|))$, $\nu\in\mathfrak{S}(\lambda^{-1/4})$. Putting the inequality for each n together and reversing all changes to recover $\mathcal{U}^\theta_k f_\nu$, we obtain (5.4.7) when $\operatorname{supp}\widehat f\subset\mathbb{A}'$.

For the other cases it is sufficient to show that the nondegeneracy condition is fulfilled after allowable transformations. For the case $\operatorname{supp} g\subset\mathbb{A}''$ we write $\tilde\Psi(z',\theta,\xi) = \xi_2\tilde\Psi(z',\theta,\xi_1/\xi_2,1)$ and set $\Phi(z,u)=\tilde\Psi(z,u,1)$. Then, the matter is reduced to decoupling of the operator $\mathcal{E}(\lambda\Phi,\tilde a_n)$. Note that

$$\nabla_z\Phi(z,u) = \big(u,1,u^2\tilde h(u),\tilde\rho\tilde h(u),2t^{-1}(t^2-s^2)u\tilde h(u)\big),$$

where $\tilde\rho = 1/\rho$ and $\tilde h(u) = (\tilde\rho^2+u^2)^{-1/2}$. Changing coordinates, we only need to show that the curve

$$\tilde\Upsilon(u) := (u,u^2\tilde h(u),\tilde\rho\tilde h(u),2t^{-1}(t^2-s^2)u\tilde h(u))$$

is nondegenerate on $\operatorname{supp}\tilde a_n\times(-2,2)$, i.e., $\det(\tilde\Upsilon',\tilde\Upsilon'',\tilde\Upsilon''',\tilde\Upsilon'''')\neq 0$. This can be easily shown by a similar computation as above. Therefore, Φ satisfies (5.3.3).

The remaining two cases $\operatorname{supp} g\subset -\mathbb{A}'$, $\operatorname{supp} g\subset -\mathbb{A}''$ can be handled similarly. So, we omit the details. $\square$

Lemma 5.4.3 is not enough for our purpose since the size of sectors is too large. Since the nondegeneracy condition with $d = 3$ is satisfied after fixing the variable s or t, we may apply a lower dimensional decoupling inequality, which allows us to decompose further the angular sectors. To do this, we focus on a piece $\mathcal{U}^\theta f_\nu$ with $\nu \in \mathfrak{S}(2^{(k-j)/4})$.

Lemma 5.4.4. *Let $p \geq 12$ and $0 \leq k \leq j$. Let $F = f_\nu$ for some $\nu \in \mathfrak{S}(2^{(k-j)/4})$. Then, for any $\epsilon > 0$ and $M > 0$, we have*

$$\|\mathcal{U}_k^\theta F\|_{L_z^p} \lesssim_{\epsilon,M} 2^{(1-\frac{7}{p}+\epsilon)\frac{j-k}{12}} \Big(\sum_{\nu' \in \mathfrak{S}(2^{(k-j)/3})} \|\mathcal{U}_k^\theta F_{\nu'}\|_{L_z^p}^p \Big)^{\frac{1}{p}} + 2^{-M(j-k)} \|F\|_p.$$

Proof. We fix s and, then, apply Theorem 5.3.2 and a rescaling argument, i.e., Lemma 5.5.1 below with $d = 3$, $\lambda = 2^{j-k}$, and $\mu = 2^{(k-j)/4}$. Following the same lines of argument as in the proof of Lemma 5.4.3, we need to consider the operator given in (5.4.8) while fixing s, that is to say, $\mathcal{E}(\lambda\Psi_s, \tilde{a}_n^s)g$ where $\tilde{a}_n^s(x,t,\theta) := \tilde{a}_n(x,t,s,\theta)$ and

$$\tilde{\Psi}_s(x,t,\theta,\xi) := \tilde{\Psi}(x,t,s,\theta,\xi).$$

As before, we may assume $\operatorname{supp}_\theta \tilde{a}_n^s \subset (-\epsilon_0, \epsilon_0)$, and we separately handle the four cases in (5.4.9). It is enough to consider the first case since the other cases can be handled similarly as in the proof of Lemma 5.4.3. We consider $\Phi(x,t,\theta,u) := \tilde{\Psi}_s(x,t,\theta,1,u)$. To show the nondegeneracy condition for Φ, from (5.4.10) we only have to show that the curve

$$(u, \rho h(u), 2s^{-1}(t^2 - s^2)uh(u))$$

is nondegenerate. This is clear from a similar computation as in the proof of Lemma 5.4.3.

As mentioned above, we now apply the rescaling argument: Lemma 5.5.1 below with $\mu = \lambda^{-1/4}$ and $R = \lambda^{1-\delta}$ for a sufficiently small $\delta = \delta(\epsilon) > 0$. In fact, Theorem 5.3.2 gives $\mathfrak{D}_{R\mu^d}^{\lambda\mu^d,\epsilon} \lesssim 1$. Thus, we combine this and Lemma 5.5.1 to obtain the desired inequality using the trivial decoupling inequality. □

We now complete the proof of Proposition 5.2.3. As mentioned above, we only consider $\mathcal{U}_k^\theta := \mathcal{U}_{+,k}^\theta$. The proof is similar with that of Proposition 5.2.2 since we now have all the necessary decoupling inequalities. It is sufficient to show (5.2.11) for $p \geq 20$ thanks to the estimate $\|\mathcal{U}_k^\theta f\|_{L^4_{x,s,t,\theta}} \lesssim 2^{\epsilon j}\|f\|_{L^4}$

which follows from (5.2.2) by taking integration in θ. Interpolation gives (5.2.11) for $4 \le p < 20$. Combining Lemma 5.4.3 and 5.4.4 gives

$$\|\mathcal{U}_k^\theta f\|_{L_z^p} \lesssim 2^{(\frac{1}{3}-\frac{10}{3p}+\epsilon)(j-k)}\Big(\sum_{\nu\in\mathfrak{S}(2^{(k-j)/3})} \|\mathcal{U}_k^\theta f_\nu\|_{L_z^p}^p\Big)^{\frac{1}{p}} + 2^{-M(j-k)}\|f\|_p$$

for $p > 20$. Since $\sum_{\nu\in\mathfrak{S}(2^{(k-j)/3})} \|f_\nu\|_p^p \lesssim \|f\|_p^p$, it suffices to show

$$\|\mathcal{U}_k^\theta f_\nu\|_{L_z^p} \lesssim 2^{(1-\frac{4}{p}+\epsilon)\frac{j+2k}{6}-\frac{k}{p}}\|f_\nu\|_{L^p}, \quad \nu \in \mathfrak{S}(2^{(k-j)/3}).$$

Since $t, s \sim 1$, changing variables $s \to ts$ and recalling (5.2.3), we note that

$$\|\mathcal{U}_k^\theta f_\nu\|_{L_z^p}^p \lesssim \iint_{|s-1|\lesssim 2^{-k}} \iint |\tilde{\mathcal{U}}_+^{\theta,s} f_\nu(x,t)|^p dxdtdsd\theta.$$

Since supp $\widehat{f_\nu}$ is included in an angular sector of angle $\sim 2^{(k-j)/3}$, a similar argument as before and Lemma 2.2.2 give $\|\tilde{\mathcal{U}}_\pm^{\theta,s} f\|_p \lesssim 2^{(1-\frac{4}{p}+\epsilon)\frac{(j+2k)}{6}}\|f_\nu\|_{L^p}$. This yields the desired estimate (5.2.11) for $p \ge 20$.

5.5 Proof of Theorem 5.3.2

To prove Theorem 5.3.2, we closely follow the argument in [6]. Let us denote

$$\gamma_\circ(u) = (1, u, u^2/2!, \cdots, u^d/d!).$$

After suitable decomposition and scaling, it is enough to consider a class of phase functions which are close to $z \cdot \gamma_\circ(u)$. More precisely, exploiting the assumption (5.3.3), we can normalize the phase function such that

$$\begin{aligned} |\partial_u^k \nabla_z \Phi - \partial_u^k \gamma_\circ| &\le \epsilon_0, \qquad 0 \le k \le d, \\ |\partial_u^k \partial_z^\beta \Phi| &\le \epsilon_0, \qquad d+1 \le k \le 4N,\ 1 \le |\beta| \le 4N \end{aligned} \tag{5.5.1}$$

for a small $\epsilon_0 > 0$ and some large N. Indeed, decomposing the amplitude function A, we assume that

$$\operatorname{supp} A \subset \mathbb{B}^{d+1}(w,\rho) \times \mathbb{B}^1(v,\rho) \times (1/2, 2)$$

for some w, v and $\rho \ge \lambda^{-1/d}$. Changing variables $(z,u) \to (z+w, u+v)$, we replace

$$\Phi_w^v(z,u) := \Phi(z+w, u+v) - \Phi(w, u+v),$$

$A^v_w(z,u) := A(z+w, u+v)$, and $g^v_w(u,r) := e^{ir\Phi(w,u+v)}g(u+v,r)$ for Φ, A, and g, respectively. This is harmless because the decoupling inequality for $\mathcal{E}^{\lambda\Phi^v_w}_{A^v_w} g^v_w$ gives the corresponding one for $\mathcal{E}_\lambda g$ as soon as we undo the procedure.

Note that $\mathcal{T}(\Phi^v_w)(0,u) = 0$. Thanks to (5.3.3), taking ρ to be small enough, we may also assume by the inverse function theorem that the map $z \mapsto \mathcal{T}(\Phi^v_w)(z,u)$ has a smooth local inverse

$$z \mapsto \mathcal{I}^v_w(z,u)$$

in a neighborhood of the origin. Using Taylor's theorem, we have

$$\Phi^v_w(z,u) = \sum_{k=0}^{d} \frac{\partial^k_u \Phi^v_w(z,0)}{k!} u^k + \frac{1}{d!}\int_0^u \partial^{d+1}_u \Phi^v_w(z,s)(u-s)^d ds. \tag{5.5.2}$$

Setting $\mathrm{D}_\mu z = (\mu^d z_1, \mu^{d-1}z_2, \cdots, \mu z_d, z_{d+1})$ for $\mu > 0$, we have

$$[\Phi^v_w]_\rho(z,u,r) := \rho^{-d}\Phi^v_w\big(\mathcal{I}^v_w(\mathrm{D}_\rho z, 0), \rho u\big) = z\cdot\gamma_\circ(u) + \mathcal{R}^v_w(z,u)$$

where

$$\mathcal{R}^v_w(z,u) = \frac{\rho}{d!}\int_0^u \partial^{d+1}_u \Phi^v_w\big(\mathcal{I}^v_w(\mathrm{D}_\rho z,0), \rho s\big)(u-s)^d ds.$$

Thus, it follows that

$$\mathcal{E}^{\lambda\Phi^v_w}_{A^v_w} g(\mathcal{I}^v_w(\mathrm{D}_\rho z, 0)) = \mathcal{E}^{\lambda\rho^d[\Phi^v_w]_\rho}_{[A^v_w]_\rho}([g^v_w]_\rho)(z),$$

where $[A^v_w]_\rho(z,u,r) := A^v_w(\mathcal{I}^v_w(\mathrm{D}_\rho z, 0), \rho u, r)$ and $[g^v_w]_\rho(u,r) := \rho g^v_w(\rho u, r)$. Taking ρ small enough, we have $|\partial^k_u \partial^\beta_z \mathcal{R}^v_w| \le \epsilon_0$ for $0 \le k, |\beta| \le 4N$ on $\operatorname{supp}[A^v_w]_\rho$. Therefore, making additional decomposition of $[A^v_w]_\rho$ and translation, we note that $\mathcal{E}^{\lambda\Phi^v_w}_{A^v_w} g(\mathcal{I}^v_w(\mathrm{D}_\rho z, 0))$ can be expressed as a finite sum of the operators

$$\mathcal{E}^{\lambda\rho^d\tilde\Phi}_{\tilde A}\tilde g$$

with $\tilde\Phi$ satisfying (5.5.1) and $\tilde A \in C^\infty_c(\mathbb{D})$ where $\mathbb{D}$ is defined at (5.3.2). Replacing $\lambda\rho^d$ with λ, we only need to prove the decoupling inequality for the operator of the above form. For the rest of this section we assume that (5.5.1) holds for Φ.

In order to show (5.3.4), we make use of Theorem 5.3.1. For this purpose we set

$$\Phi_\lambda(z,u) = \lambda\Phi(z/\lambda, u), \quad A_\lambda(z,u,r) = A(z/\lambda, u, r).$$

For $1 \le R \le \lambda$, we denote by $\mathfrak{D}_R^{\lambda,\epsilon}$ the infimum over all $\mathfrak{D}$ for which

$$\|\mathcal{E}_{A_\lambda}^{\Phi_\lambda} g\|_{L^p(B)} \le \mathfrak{D} R^{\alpha_d(p)+\epsilon}\Big(\sum_{J\in\mathcal{J}(R^{-1/d})} \|\mathcal{E}_{\tilde{A}_\lambda}^{\Phi_\lambda} g_J\|_{L^p(\omega_B)}^p\Big)^{\frac{1}{p}} + R^{2d}\big(\frac{\lambda}{R}\big)^{-\frac{\epsilon N}{8d}}\|g\|_2$$

holds for any ball B of radius R included in $\mathbb{B}^{d+1}(0,2\lambda)$, all Φ satisfying (5.5.1), and $A \in \mathrm{C}_c^\infty(\mathbb{D})$ with some $\tilde{A} = \tilde{A}(A) \in \mathrm{C}_c^\infty(\mathbb{D})$ satisfying $\|\tilde{A}\|_{\mathrm{C}^N} \le \|A\|_{\mathrm{C}^N}$.

5.5.1 Rescaling

By a rescaling argument, we have following.

Lemma 5.5.1. *Let $R^{-1/d} < \mu < 1 \le R \le \lambda$. Let B be a ball of radius R included in $\mathbb{B}^{d+1}(0,2\lambda)$. Suppose $\{J\} \subset \mathcal{J}(R^{-1/d})$ and $J \subset \mathbb{B}^1(v,\mu)$ for some $v \in [-1,1]$. If μ is sufficiently small, then*

$$\|\sum_J \mathcal{E}^{\Phi_\lambda} g_J\|_{L^p(\omega_B)} \lesssim \mathfrak{D}_{R\mu^d}^{\lambda\mu^d,\epsilon}(R\mu^d)^{\alpha_d(p)+\epsilon}\big(\sum_J \|\mathcal{E}^{\Phi_\lambda} g_J\|_{L^p(\omega_B)}^p\big)^{\frac{1}{p}} + \mu^2 R^{2d}\big(\tfrac{\lambda}{R}\big)^{-\frac{\epsilon N}{8d}}\|g\|_2.$$

We occasionally drop the amplitude functions, which are are generically assumed to be admissible.

Proof. To prove Lemma 5.5.1, we only need to consider $\|\mathcal{E}^{\Phi_\lambda} g\|_{L^p(B)}$ instead of $\|\mathcal{E}^{\Phi_\lambda} g\|_{L^p(\omega_B)}$. Since ω_B is bounded by a rapidly decreasing sum of characteristic functions, the bounds on $\|\mathcal{E}^{\Phi_\lambda} g\|_{L^p(B)}$ imply those for $\|\mathcal{E}^{\Phi_\lambda} g\|_{L^p(\omega_B)}$.

Let $B = \mathbb{B}^{d+1}(\lambda w, R)$ for some w. We make a slightly different form of scaling from the previous one to ensure that the consequent phase satisfies (5.5.1). Recalling (5.5.2), we have

$$\lambda\Phi_w^v(\mathcal{I}_w^v(D'_\mu \tfrac{z}{\lambda},0),\mu u) = z\cdot\gamma_\circ(u) + \frac{\mu^{d+1}}{d!}\int_0^u \partial_u^{d+1}\Phi_w^v\big(\mathcal{I}_w^v(D'_\mu \tfrac{z}{\lambda},0),\mu s\big)(u-s)^d ds,$$

where $D'_\mu z = (z_1, \mu^{-1}z_2, \cdots, \mu^{-d}z_{d+1})$. Setting

$$\tilde{\Phi}(z,u,r) = z\cdot\gamma_\circ(u) + \frac{\mu}{d!}\int_0^u \partial_u^{d+1}\Phi_w^v(D_\mu z,\mu s)(u-s)^d ds,$$

we have $\lambda\Phi_w^v(\mathcal{I}_w^v(\mathrm{D}'_\mu \tfrac{z}{\lambda},0),\mu u) = (\tilde{\Phi})_{\lambda\mu^d}(z,u)$. This gives

$$\mathcal{E}_{A_w^v}^{\Phi_w^v} g(\lambda\mathcal{I}_w^v(\mathrm{D}'_\mu z/\lambda,0)) = \mathcal{E}_{\tilde{A}}^{\tilde{\Phi}_{\lambda\mu^d}}([g_w^v]_\mu)(z/\lambda)$$

where $\tilde{A}(z,u,r) = A^v_w(\mathcal{I}^v_w(D'_\mu z/\lambda),0),\mu u,r)$. Changing variables $(z,u) \to (z+w,\mu u+v)$ and $z \to \mathcal{I}(z) := \lambda \mathcal{I}^v_w(\mathrm{D}'_\mu z/\lambda,0)$ gives

$$\|\sum_J \mathcal{E}^{\Phi_\lambda}_A g_J\|_{L^p(B)} \lesssim \mu^{-\frac{d^2+d}{2p}} \|\sum_J \mathcal{E}^{\tilde{\Phi}_{\lambda\mu^d}}_{\tilde{A}} \tilde{g}_J\|_{L^p(\mathcal{I}^{-1}(B))}, \tag{5.5.3}$$

where $\tilde{g}_J = [(g_J)^v_w]_\mu$. We cover $\mathcal{I}^{-1}(B)$ by a collection $\mathbf{B}$ of finitely overlapping $R\mu^d$-balls. So, we have

$$\|\sum_J \mathcal{E}^{\tilde{\Phi}_{\lambda\mu^d}}_{\tilde{A}} \tilde{g}_J\|_{L^p(\mathcal{I}^{-1}(B))} \lesssim \big(\sum_{B'\in\mathbf{B}} \|\sum_J \mathcal{E}^{\tilde{\Phi}_{\lambda\mu^d}}_{\tilde{A}} \tilde{g}_J\|^p_{L^p(B')}\big)^{\frac{1}{p}}.$$

Here, we note that $\mathrm{supp}_z\, \tilde{a}$ may be not included in $B(0,\mu^d R)$. However, by a harmless translation in z we may assume that $\mathrm{supp}_z\, \tilde{A} \subset B(0,\mu^d R)$ by replacing the phase and amplitude functions with $[\tilde{\Phi}]^0_{w'}$ and $[\tilde{A}]^0_{w'}$ for some w' since undoing the translation recovers the desired decoupling inequality.

Note that $\tilde{\Phi}$ satisfies (5.5.1) if μ is small enough. Since $\mathrm{supp}_u\, \tilde{g}_J$ are included in disjoint intervals of length $\sim \mu^{-1}R^{-1/d}$, we now have

$$\|\sum_J \mathcal{E}^{\tilde{\Phi}_{\lambda\mu^d}}_{\tilde{A}} \tilde{g}_J\|_{L^p(B')} \le \mathfrak{D}^{\lambda\mu^d,\epsilon}_{R\mu^d}(R\mu^d)^{\alpha_d(p)+\epsilon}\Big(\sum_J \|\mathcal{E}^{\tilde{\Phi}_{\lambda\mu^d}}_{\tilde{A}} \tilde{g}_J\|^p_{L^p(\omega_{B'})}\Big)^{1/p} + \mathcal{R},$$

where $\mathcal{R} = (R\mu^d)^{2d}(\lambda/R)^{-\epsilon N/8d}\|\sum_J \tilde{g}_J\|_2$. We put together the inequalities over each B' and then reverse the various changes of variables so far to recover the original operator $\mathcal{E}^{\Phi_\lambda}$. Note that we may incur a different amplitude function however, the decoupling state is not changed. Since $\#\mathbf{B} \lesssim \mu^{-d(d+1)/2}$, we can conclude that $\|\mathcal{E}^{\Phi_\lambda} g\|_{L^p(\omega_B)}$ is bounded by a constant times

$$\mathfrak{D}^{\lambda\mu^d,\epsilon}_{R\mu^d}(R\mu^d)^{\alpha_d(p)+\epsilon}\big(\sum_J \|\mathcal{E}^{\Phi_\lambda} g_J\|^p_{L^p(\omega_B)}\big)^{\frac{1}{p}} + \mu^{-\frac{d^2+d}{p}+\frac{1}{2}}(R\mu^d)^{2d}\big(\tfrac{\lambda\mu^d}{R}\big)^{-\frac{\epsilon N}{8d}}\|\sum_J g_J\|_p.$$

Finally, using $(d^2+d)/p \le 2d^2-2$, we can get the desired result. $\square$

5.5.2 Linearization of the phase

Let Φ be a smooth phase satisfying (5.5.1). For simplicity, denote $\partial_k = \partial_{z_k}$, $k = 1,\dots,d+1$. From (5.5.1), we have $\partial_u(\partial_2\Phi/\partial_1\Phi) - 1 = O(\epsilon_0)$. Thus, there exists the map η_z such that $(\partial_2\Phi/\partial_1\Phi)(z,\eta_z(u)) = u$. Let

$$\Gamma_z(u) = \frac{\nabla_z\Phi(z,\eta_z(u))}{\partial_1\Phi(z,\eta_z(u))}.$$

Also note from (5.5.1) that $\partial_1\Phi - 1 = O(\epsilon_0)$. Furthermore, we have $\Gamma_z \cdot e_1 = 1$ and $\Gamma_z \cdot e_2 = u$ where $\{e_1, \dots, e_{d+1}\}$ is the standard basis in $\mathbb{R}^{d+1}$.

Let $\lambda w \in \mathbb{B}^{d+1}(0, 2\lambda)$. By a Taylor expansion and changing variables $u \to \eta_w^{-1}(u)$ we have

$$\Phi_\lambda(z + \lambda w, u) - \Phi_\lambda(\lambda w, u) = \partial_1\Phi(w, u)\Gamma_w(\eta_w^{-1}(u)) \cdot z + \mathcal{R}_w^\lambda(z, u),$$

where

$$\mathcal{R}_w^\lambda(z, u) = \frac{1}{\lambda}\int_0^1 (1-\tau)\big\langle \mathbf{Hess}_z\Phi\big(\lambda^{-1}\tau z + w, \eta_w^{-1}(u)\big)z, z\big\rangle d\tau.$$

Let us set

$$\Omega_z(u, r) = (\eta_z(u), r/\partial_1\Phi(z, \eta_z(u))).$$

Then, (5.5.1) ensures that Ω_z is smooth. Changing variables $(u, r) \to \Omega_w(u, r)$, we see that $\mathcal{E}_{A_\lambda}^{\Phi_\lambda} g(z + \lambda w)$ is equal to

$$\iint e^{irz\cdot\Gamma_w(u)} A_w(z, \Omega_w(u, r))(g_w \circ \Omega_w)(u, r)\frac{\eta_w'(u)dudr}{\partial_1\Phi(w, \eta_w(u))}, \tag{5.5.4}$$

where

$$A_w(z, u, r) = e^{ir\mathcal{R}_w^\lambda(z,u)}\, A(\lambda^{-1}z + w, u, r), \quad g_z(u, r) = e^{ir\lambda\Phi(z,u)}g(u, r).$$

For this operator we could directly apply Theorem 5.3.1 if it were not for the extra factor $e^{ir\mathcal{R}_w^\lambda(z,\eta_w(u))}$. This is not generally allowed. However, if $|z| \le \lambda^{1/2}$, expanding it into Fourier series in (u, r), we may disregard it as a minor error.

More precisely, from (5.5.1) we note that $\partial_1\Phi - 1 = O(\epsilon_0)$ and $\eta_z' - 1 = O(\epsilon_0)$. With a sufficiently small ϵ_0 we may assume that $g_w \circ \Omega_w$ is supported in $(-1, 1)\times[1, 2]$. Using (5.5.1), we have $|\partial_u^k \mathcal{R}_w^\lambda(z, u)| \le C|z|^2/\lambda$ for $0 \le k \le 4N$. Consequently, if $|z| \le \lambda^{1/2}$,

$$|\partial_u^k\big(A_w(z, \Omega_w(u/r, r)\big)| \le C, \quad 0 \le k \le 4N. \tag{5.5.5}$$

Thus, expanding $A_w(z, \Omega_w(u/r, r))$ into Fourier series, we have $A_w(z, \Omega_w(u, r)) = \sum_{\ell\in\mathbb{Z}^2} b_\ell(z)e^{ir\ell\cdot(1,u)}$ with $|b_\ell(z)| \lesssim_N (1 + |l|)^{-N}$. From (5.5.4) we have

$$|\mathcal{E}_{A_\lambda}^{\Phi_\lambda} g(z + \lambda w)| \le \sum_{\ell\in\mathbb{Z}^2}(1 + |\ell|)^{-N}|E^{\Gamma_w}(\tilde{g}_w)(z + v_\ell)|, \tag{5.5.6}$$

for $|z| \lesssim \lambda^{1/2}$ where $v_\ell := \ell_1 e_1 + \ell_2 e_2$ and

$$\tilde{g}_w(u,r) = (g_w \circ \Omega_w)(u,r)\eta_w'(u)/\partial_1\Phi(w,\eta_w(u)).$$

This almost allows us to obtain the first part of the next Lemma, which is basically the same as Lemma 2.6 in [6]. We recall (5.3.1).

Lemma 5.5.2. *Let* $0 < \delta \le 1/2$ *and* $1 \le \rho \le \lambda^{1/2-\delta}$. *Let* $B := \mathbb{B}^{d+1}(\lambda w, \rho) \subset \mathbb{B}^{d+1}(0, 3\lambda/4)$ *and* $B_0 := \mathbb{B}^{d+1}(0,\rho)$. *Suppose that* Φ *satisfies* (5.5.1). *Then*

$$\|\mathcal{E}^{\Phi_\lambda}_{A_\lambda} g\|_{L^p(\omega_B)} \lesssim \|E^{\Gamma_w}(\tilde{g}_w)\|_{L^p(\omega_{B_0})} + \lambda^{-\delta N/2}\|g\|_2. \tag{5.5.7}$$

Additionally, assume that $|w| \le \lambda^{1-\delta'}$. *Then, for some admissible* $\tilde{A}$, *we have*

$$\|E^{\Gamma_w}(\tilde{g}_w)\|_{L^p(\omega_{B_0})} \lesssim \|\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda} g\|_{L^p(\omega_B)} + \lambda^{-\min\{\delta,\delta'\}N/2}\|g\|_2, \tag{5.5.8}$$

Proof. For (5.5.7), we separately consider two cases $|z-\lambda w| \le \lambda^{1/2}$ and $|z-\lambda w| > \lambda^{1/2}$. We first consider the case $|z-\lambda w| > \lambda^{1/2}$. So, we have $\omega_B(z) \lesssim \lambda^{-\delta(N-d-2)}(1+\rho^{-1}|z-\lambda w|)^{-d-2}$. Combining this and a trivial inequality $|\mathcal{E}^{\Phi_\lambda}_{A_\lambda} g| \lesssim \|g\|_2$, we have

$$\|\chi_{B(\lambda w,\lambda^{1/2})^c}\mathcal{E}^{\Phi_\lambda}_{A_\lambda} g\|_{L^p(\omega_B)} \lesssim \lambda^{-\delta N/2}\|g\|_2, \tag{5.5.9}$$

for a sufficiently large N. Next, we handle the remaining part $\chi_{B(\lambda w,2\lambda^{1/2})}\mathcal{E}^{\Phi_\lambda}_{A_\lambda} g$. Using (5.5.6) and Hölder's inequality in l, one can obtain

$$\|\chi_{B(\lambda w,2\lambda^{1/2})}\mathcal{E}^{\Phi_\lambda}_{A_\lambda} g\|_{L^p(\omega_B)} \lesssim \Big\|E^{\Gamma_w}(\tilde{g}_w)\sum_\ell \frac{\omega^{1/p}_{B(v_l,\rho)}}{(1+|\ell|)^N}\Big\|_{L^p} \lesssim \|E^{\Gamma_w}(\tilde{g}_w)\|_{L^p(\omega_{B_0})}.$$

The second inequality follows from the fact $\sum_{\ell\in\mathbb{Z}^2}(1+|\ell|)^{-N}\omega^{1/p}_{B(v_\ell,\rho)} \lesssim \omega^{1/p}_{B(0,\rho)}$. By the above inequality and (5.5.9), we conclude that (5.5.7) holds.

To show (5.5.8), we use a similar argument. By the same reason as in the proof of (5.5.7), we have $\|E^{\Gamma_w}(\tilde{g}_w)(1-\chi_{B(0,2\lambda^{1/2})})\|_{L^p(\omega_{B_0})} \lesssim \lambda^{-\delta N/2}\|g\|_2$ for a sufficiently large N. For the integral over the set $B(0,2\lambda^{1/2})$, we now undo the changes of variables including $(u,r) \mapsto \Omega_w(u,r)$ which are performed to get (5.5.4). Consequently, we have

$$E^{\Gamma_w}(\tilde{g}_w)(z) = \iint e^{ir\Phi_\lambda(z,u)}\tilde{A}_w(z,u,r)g(u,r)dudr,$$

where $\tilde{A}_w(z,u,r) = e^{-ir\mathcal{R}_w^\lambda(z,u)}\, a(\lambda^{-1}z + w, u, r)$. As before, we can expand the function $\tilde{A}_w(z, \Omega_w(u/r, r))$ (cf. (5.5.5)) into Fourier series in u, r. Thus, if $z \in B(0, 2\lambda^{1/2})$,

$$|E^{\Gamma_w}(\tilde{g}_w)(z)| \le C_N \textstyle\sum_{\ell\in\mathbb{Z}^2}(1+|\ell|)^{-2N}|\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}(g_\ell)(z)|,$$

for a suitable symbol $\tilde{A}$ where

$$g_\ell := e^{i\ell\cdot\tilde{\Omega}_w^{-1}(u,r)}g.$$

We again perform the previous linearization procedure again for $\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}(g_\ell)$. Since $\tilde{\Omega}_w^{-1}\circ\Omega_w(u,r) = (ru, r)$, by (5.5.7) we have

$$\|\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}(g_\ell)\|_{L^p(\omega_B)} \lesssim \|E^{\Gamma_w}(\tilde{g}_w)\|_{L^p(\omega_{B(v_\ell,\rho)})} + \lambda^{-\delta N/2}\|g\|_2.$$

By this inequality we have $S := \sum_{|\ell|\ge M}(1+|\ell|)^{-2N}\|\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}(g_\ell)\|_{L^p(\omega_B)}$ bounded by a constant times $M^{-N}\|E^{\Gamma_w}(\tilde{g}_w)\|_{L^p(\omega_{B_0})} + \lambda^{-\delta N/2}\|g\|_2$. If we choose a sufficiently large M, the part S can be absorbed in the left hand side of (5.5.8). Thus, we obtain

$$\|E^{\Gamma_w}\tilde{g}_w\|_{L^p(\omega_{B_0})} \lesssim \sum_{|\ell|<M}\|\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}g_\ell\|_{L^p(\omega_{B(w,\rho)})} + \lambda^{-\delta N/2}\|g\|_2.$$

We note that $\mathcal{E}^{\Phi_\lambda}_{\tilde{A}_\lambda}g_\ell = \mathcal{E}^{\Phi_\lambda}_{\tilde{A}_{\lambda,\ell}}g$ where $\tilde{A}_{\lambda,\ell} := \tilde{A}_\lambda e^{i\ell\cdot\tilde{\Omega}_w^{-1}(u,r)}$. Expanding $\tilde{A}_{\lambda,\ell}$ in a Taylor series one can get amplitude functions which are independent of a particular B. From those one can find an operator which has the desired property by pigeonholing. See [6] for details. □

5.5.3 Proof of Theorem 5.3.2

Assume that Φ satisfies (5.5.1) and

$$1 \le K \le R \le \lambda^{1-\epsilon/d}.$$

Let $\mathcal{J} := \mathcal{J}(R^{-1/d})$ be a collection of disjoint intervals. For simplicity we set $g = \sum_{J\in\mathcal{J}(R^{-1/d})} g_J$. Partition $\mathcal{J}(R^{-1/d})$ in such a way that there is a collection $\mathcal{J}'$ of disjoint intervals J' of length $\sim K^{-1/d}$ which include each interval in $\mathcal{J}(R^{-1/d})$. So, we have

$$g = \textstyle\sum_{J'\in\mathcal{J}'} g_{J'} = \sum_{J'\in\mathcal{J}'}\sum_{J\in\mathcal{J}:J\subset J'} g_J.$$

We consider a ball B of radius R included in $B(0,\lambda)$ and a collection $\mathcal{B}_K$ of finitely overlapping balls B' of radius $K=\lambda^{1/4}$ which covers B. Since $R\le\lambda^{1-\epsilon/d}$, one may assume that the center of $B'\in\mathcal{B}_K$ lies in $B(0,\lambda^{1-\epsilon/d})$ after a translation. Using (5.5.7), we have

$$\|\mathcal{E}_\lambda g\|_{L^p(B)}\lesssim\big(\sum_{B'\in\mathcal{B}_K}\|E^{\Gamma_{c_{B',\lambda}}}\tilde g_{c_{B',\lambda}}\|^p_{L^p(\omega_{B(0,K)})}\big)^{\frac1p}+\big(\tfrac{R}{K}\big)^{d+1}\lambda^{-N/8}\|g\|_2.$$

Here $c_{B',\lambda}=\lambda^{-1}c_{B'}$ and $c_{B'}$ denotes the center of B'. We apply Theorem 5.3.1 to each $B'\in\mathcal{B}_K$ and (5.5.8) subsequently to get decoupling at scale $K^{-1/d}$. Consequently, combining the inequality on each B', we obtain

$$\|\mathcal{E}_\lambda g\|_{L^p(B_R)}\lesssim K^{\alpha_d(p)+\epsilon}\big(\textstyle\sum_{J'\in\mathcal{J}'}\|\mathcal{E}_\lambda g_{J'}\|^p_{L^p(\omega_{B_R})}\big)^{\frac1p}+K^{-1}R^{2d}\big(\tfrac{\lambda}{R}\big)^{-\frac{\epsilon N}{8d}}\|g\|_2.$$

Using Lemma 5.5.1, we get

$$\|\mathcal{E}_\lambda g\|_{L^p(\omega_B)}\lesssim\mathfrak{D}^{\lambda K^{-1},\epsilon}_{RK^{-1}}R^{\alpha_d(p)+\epsilon}\big(\textstyle\sum_{J\in\mathcal{J}}\|\mathcal{E}_\lambda g_J\|^p_{L^p(\omega_B)}\big)^{\frac1p}+K^{-\frac{\epsilon}{d}}R^{2d}(\tfrac{\lambda}{R})^{-\frac{\epsilon N}{8d}}\|g\|_2.$$

Thus, for a sufficiently large λ, we have $\mathfrak{D}^{\lambda,\epsilon}_R\le\mathfrak{D}^{\lambda^{3/4},\epsilon}_{R\lambda^{-1/4}}$. Iteratively applying this inequality, one can show $\mathfrak{D}^{\lambda,\epsilon}_{\lambda^{1-\epsilon}}\lesssim\lambda^\delta$ for any $\delta>0$, which completes the proof of Theorem 5.3.2.

5.6 Optimality of the estimates

We close this dissertation by making some remarks regarding the local smoothing estimates (5.1.2) and (5.1.3). Once one has the estimates (5.2.4) and (5.2.11), the proofs of the estimates (5.1.2) and (5.1.3) are straightforward. So, we omit them.

As mentioned before, the smoothing orders in the estimates (5.1.2) and (5.1.3) are sharp except the endpoints cases. To see this, we only consider the operator $\mathcal{U}^\theta_+$. The other $\mathcal{U}^\theta_-$ can be handled similarly. The following arguments are almost similar with that of Chapter 4.8. Let g be a function given by $\widehat{g}(\xi)=\varphi(2^{-j}|\xi_{1,2}|)e^{-i|\xi_{1,2}|}$. It is easy to see that $\|g\|_{L^p_\alpha}\lesssim 2^{(\alpha+3/2-1/p)j}$. Note that

$$\mathcal{U}^\theta_+g(x,t,s)=2^{2j}\int e^{2^j(x\cdot\xi+|(R^*_\theta\xi)_{s,t}|-|\xi_{1,2}|)}\varphi(|\xi|)d\xi.$$

Thus, we have $|\mathcal{U}^0g(x,t,s)|\gtrsim 2^{2j}$ if $|x|,|t-1|,|s-1|\le 2^{-j}/100$. So, if the estimate (5.1.2) holds true, then $2^{(2-4/p)j}\lesssim 2^{(\alpha+3/2-1/p)j}$. Letting $j\to\infty$

shows that (5.1.2) holds only if $\alpha \geq 1/2 - 3/p$. Similarly, for (5.1.3) we note that $|\mathcal{U}^\theta f(x,t,s)| \gtrsim 2^{2j}$ if $|x|, |\theta|, |t-1|, |s-2|, \leq 2^{-j}/100$. So, (5.1.3) gives $2^{(2-5/p)j} \lesssim 2^{(\alpha+3/2-1/p)j}$. Therefore, (5.1.3) holds only if $\alpha \geq 1/2 - 4/p$.

Besides those upper bounds on the smoothing orders, one can find other upper bounds testing the estimates (5.1.2) and (5.1.3) with different type of examples. However, we are far from being able to prove the estimates of smoothing orders up to any of such bounds. This problem seems to be very challenging.

Bibliography

[1] T. C. Anderson, K. Hughes, J. Roos, A. Seeger, *L^p–L^q bounds for spherical maximal operators*, Math. Z. **297** (2021), 1057–1074.

[2] R. Bagby, *A note on the strong maximal function*, Proc. Amer. Math. Soc. **88** (1983), no. 4, 648–650.

[3] D. Beltran, S. Guo, J, Hickman, A. Seeger, *The circular maximal operator on Heisenberg radial functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci(2021), https://doi.org/10.2422/2036-2145.202001-006.

[4] D. Beltran, S. Guo, J. Hickman, A. Seeger, *Sobolev improving for averages over curves in $\mathbb{R}^4$*, Adv. Math. **393** (2021) 108089.

[5] D. Beltran, S. Guo, J. Hickman, A. Seeger, *Sharp L^p bounds for the helical maximal function*, arXiv:2102.08272.

[6] D. Beltran, J. Hickman, C. D. Sogge, *Variable coefficient Wolff-type inequalities and sharp local smoothing estimates for wave equations on manifolds*, Anal. PDE. **13** (2020) 403–433.

[7] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Anal. Math. **47** (1986), 69–85.

[8] J. Bourgain, *Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces*, Israel J. Math. **193** (2013), no. 1, 441–458.

[9] J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, J. Amer. Math. Soc. **30** (2017), no. 1, 205–224.

[10] J. Bourgain, C. Demeter, *The proof of the ℓ^2 decoupling conjecture,* Ann. of Math. **182** (2015), 351–389.

[11] J. Bourgain, C. Demeter, *Decouplings for surfaces in* $\mathbb{R}^4$, J. Funct. Anal. **270** (2016), no. 4, 1299–1318.

[12] J. Bourgain, C. Demeter, *Mean value estimates for Weyl sums in two dimensions*, J. Lond. Math. Soc. (2) **94** (2016), no. 3, 814–838.

[13] J. Bourgain, C. Demeter, *Decouplings for curves and hypersurfaces with nonzero Gaussian curvature*, J. Anal. Math. **133** (2017), 279–311.

[14] J. Bourgain, C. Demeter, L. Guth, *Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three*, Ann. of Math. **184** (2016), 633–682.

[15] S. Buschenhenke, S. Dendrinos, I. A. Ikromov, D. Müller, *Estimates for maximal functions associated to hypersurfaces in* $\mathbb{R}^3$ *with height* $h < 2$*: Part I*, Trans. Amer. Math. Soc. **372** (2019), 1363–1406.

[16] S. Buschenhenke, S. Dendrinos, I. A. Ikromov, D. Müller, *Estimates for maximal functions associated to hypersurfaces in* $\mathbb{R}^3$ *with height* $h < 2$ *: Part II – A geometric conjecture and its proof for generic 2-surfaces*, arXiv:2209.07352.

[17] A. Carbery, *Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem*, Ann. Inst. Fourier (Grenoble) **38** (1988), no. 1, 157–168.

[18] Y. Cho, *Multiparameter maximal operators and square functions on product spaces*, Indiana Univ. Math. J. **43** (1994), 459–491.

[19] A. Córdoba, R. Fefferman, *On differentiation of integrals*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 6, 2211–2213.

[20] M. Cowling, G. Mauceri, *Oscillatory integrals and Fourier transforms of surface carried measures*, Trans. Amer. Math. Soc. **304** (1987), no. 1, 53–68.

[21] B. M. Erdoğan, *Mapping properties of the elliptic maximal function*, Rev. Mat. Iberoamericana **19** (2003), 221–234.

[22] N. Fava, *Weak type inequalities for product operators*, Studia Math. **42** (1972), 271–288.

[23] N. Fava, E. Gatto, C. Gutiérrez, *On the strong maximal function and Zygmund's class $L(\log^+ L)^n$*, Studia Math. **69** (1980/81), no. 2, 155–158.

[24] C. Fefferman, E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.

[25] G. Garrigós, A. Seeger, *On plate decompositions of cone multipliers*, Proc. Edinb. Math. Soc. (2) **52** (2009), no. 3, 631–651.

[26] G. Garrigós, A. Seeger, *A mixed norm variant of Wolff's inequality for paraboloids*, Harmonic analysis and partial differential equations, 179–197, Contemp. Math., **505**, Amer. Math. Soc., Providence, RI, 2010.

[27] V. Guillemin, S. Sternberg, *Geometric asymptotics*, Mathematical Surveys, No. 14. American Mathematical Society.

[28] S. Guo, C. Oh, R. Zhang, P. Zorin-Kranich, *Decoupling inequalities for quadratic forms*, Duke Math. J. **172** (2023), no. 2, 387–445.

[29] S. Guo, P. Zorin-Kranich, *Decoupling for moment manifolds associated to Arkhipov-Chubarikov-Karatsuba systems*, Adv. Math. **360** (2020), 106889, 56 pp.

[30] L. Guth, H. Wang, R. Zhang, *A sharp square function estimate for the cone in $\mathbb{R}^3$*, Ann. of Math. **192** (2020), 551–581.

[31] S. Ham, H. Ko, S. Lee, S. Oh, *Remarks on dimension of unions of curves*, Nonlinear Anal. **229** (2023), Paper No. 113207, 14 pp.

[32] W. Hansen, *Littlewood's one-circle problem, revisited,* Expo. Math. **26** (2008), 365–374.

[33] Y. Heo, *Multi-parameter maximal operators associated with finite measures and arbitrary sets of parameters*, Integral Equations Operator Theory **86** (2016), 185—208.

[34] Y. Heo, F. Nazarov, and A. Seeger, *Radial Fourier multipliers in high dimensions*, Acta Math. **206**, (2011), no. 1, 55–92 .

[35] L. Hörmander, *Fourier integral operators. I*, Acta Math. **127** (1971), no. 1-2, 79–183.

[36] L. Hörmander, *The analysis of linear partial differential operators I: Distribution Theory and Fourier Analysis*, 2nd ed., Springer-Verlag, 1990.

[37] I. A. Ikromov, M. Kempe, D. Müller, *Estimates for maximal functions associated with hypersurfaces in* $\mathbb{R}^3$ *and related problems of harmonic analysis*, Acta Math. **204** (2010), 151–271.

[38] J. Kim, *Annulus maximal averages on variable hyperplanes,* arxiv:1906.03797.

[39] H. Ko, S. Lee, S. Oh, *Maximal estimates for averages over space curves*, Invent. Math. **228** (2022), 991–1035.

[40] H. Ko, S. Lee, S. Oh, *Sharp smoothing properties of averages over curves*, Forum of Mathematics, Pi **11** (2023), doi:10.1017/fmp.2023.2.

[41] J. Lee, S. Lee, L^p–L^q *estimates for the circular maximal operator on Heisenberg radial functions*, Math. Ann. **385**, 1–24 (2023).

[42] J. Lee, S. Lee, L^p *maximal bound and Sobolev regularity of two-parameter averages over tori*, arXiv:2210.13377.

[43] J. Lee, S. Lee, S. Oh, *The elliptic maximal function*, arXiv:2305.16221.

[44] S. Lee, *Endpoint estimates for the circular maximal function*, Proc. Amer. Math. Soc. **131** (2003), 1433–1442.

[45] S. Lee, A. Seeger, *Lebesgue space estimates for a class of Fourier integral operators associated with wave propagation*, Math. Nachr. **286** (2013), no. 7, 743–755.

[46] S. Lee, A. Vargas, *On the cone multiplier in* $\mathbb{R}^3$, J. Funct. Anal. **263** (2012), 925–940.

[47] I. Łaba, M. Pramanik, *Wolff's inequality for hypersurfaces*, Collect. Math. 2006, Vol. Extra, 293–326.

[48] I. Łaba, T. Wolff, *A local smoothing estimate in higher dimensions*, Dedicated to the memory of Tom Wolff. J. Anal. Math. **88** (2002), 149–171.

[49] G. Marletta, F. Ricci, *Two-parameter maximal functions associated with homogeneous surfaces in* $\mathbb{R}^n$, Studia Math. **130** (1998), 53–65.

[50] G. Marletta, F. Ricci, J. Zienkiewicz, *Two-parameter maximal functions associated with degenerate homogeneous surfaces in* $\mathbb{R}^3$, Studia Math. **130** (1998), 67–75.

[51] J.M. Marstrand, *Packing circles in the plane*, Proc. London Math. Soc. **55** (1987), no. 1, 37–58.

[52] A. Miyachi, *On some estimates for the wave equation in* L^p *and* H^p, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 331–354.

[53] G. Mockenhaupt, A. Seeger, C. Sogge, *Wave front sets, local smoothing and Bourgain's circular maximal theorem*, Ann. of Math. **136** (1992), 207–218.

[54] G. Mockenhaupt, A. Seeger, C. Sogge, *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. **6** (1993), no. 1, 65–130.

[55] D. Müller, A. Seeger, *Singular spherical maximal operators on a class of two step nilpotent Lie groups,* Israel J. Math. **141** (2004), 315–340.

[56] A. Nagel, E. Stein, S. Wainger, *Differentiation in lacunary directions*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), no. 3, 1060–1062.

[57] E. Narayanan, S. Thangavelu, *An optimal theorem for the spherical maximal operator on the Heisenberg group*, Israel J. Math. **144** (2004), 211–219.

[58] A. Nevo, S. Thangavelu, *Pointwise ergodic theorems for radial averages on the Heisenberg group*, Adv. Math. **127** (1997), no. 2, 307–334.

[59] J. Peral, L^p *estimates for the wave equation*, J. Functional Analysis **36** (1980), no. 1, 114–145.

[60] D. Phong, E. Stein, *Hilbert integrals, singular integrals, and Radon transforms. I*, Acta Math. **157** (1986), no. 1-2, 99–157.

[61] M. Pramanik, A. Seeger, L^p *regularity of averages over curves and bounds for associated maximal operators*, Amer. J. Math. **129** (2007), 61–103.

[62] K. Rogers, *A local smoothing estimate for the Schrödinger equation*, Adv. Math. **219** (2008), no. 6, 2105–2122.

[63] J. Roos, A. Seeger, R. Srivastava, *Lebesgue space estimates for spherical maximal functions on Heisenberg groups*, Int. Math. Res. Not. IMRN 2022, no. 24, 19222–19257.

[64] J. Rubio de Francia, *Maximal functions and Fourier transforms*, Duke Math. J. **53** (1986), no. 2, 395–404.

[65] W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill Book Co., New York, 1987.

[66] W. Schlag, *L^p-L^q estimates for the circular maximal function*, Thesis (Ph.D.)–California Institute of Technology. 1996. 79 pp.

[67] W. Schlag, *A generalization of Bourgain's circular maximal theorem*, J. Amer. Math. Soc. **10** (1997), no. 1, 103–122.

[68] W. Schlag, C. Sogge, *Local smoothing estimates related to the circular maximal theorem*, Math. Res. Lett. **4** (1997), no. 1, 1–15.

[69] A. Seeger, C. Sogge, E. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. (2) **134** (1991), no. 2, 231–251.

[70] A. Seeger, S. Wainger, J. Wright, *Spherical maximal operators on radial functions*, Math. Nachr. **187**, 95–105 (1997).

[71] C. Sogge, *Propagation of singularities and maximal functions in the plane*, Invent. Math. **104** (1991), no. 2, 349–376.

[72] C. Sogge, E. Stein, *Averages of functions over hypersurfaces in $\mathbb{R}^n$*, Invent. Math. **82** (1985), no. 3, 543–556.

[73] C. Sogge, E. Stein, *Averages over hypersurfaces. Smoothness of generalized Radon transforms*, J. Analyse Math. **54** (1990), 165–188.

[74] E. Stein, *Maximal functions: spherical means*, Proc. Nat. Acad. Sci. USA **73** (1976), 2174–2175.

[75] E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[76] E. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1239–1295.

[77] J. Strömberg, *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. of Math. (2) **107** (1978), no. 2, 399–402.

[78] M. Talagrand, *Sur la mesure de la projection d'un compact et certaines familles de cercles*, Bull. Sci. Math. **104** (1980), 225–231.

[79] T. Tao, *The Bochner-Riesz conjecture implies the restriction conjecture*, Duke Math. J. **96** (1999), no. 2, 363–375.

[80] T. Tao, A. Vargas, L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc. **11** (1998), 967–1000.

[81] T. Wolff, *Local smoothing type estimates on L^p for large p*, Geom. Funct. Anal. **10** (2000), no. 5, 1237–1288.

[82] T. Wolff, *Lectures on harmonic analysis*, University Lecture Series, 29. American Mathematical Society, Providence, RI, 2003.

국문초록

극대 함수에 대한 계측은 편미분방정식, 기하학적 측도이론, 조화해석학과 같은 수리해석학의 여러 분야의 문제에서 중요한 역할을 한다. 1950년대 이후, 평균으로 정의된 극대 함수는 고전적 조화해석학 분야에서 광범위하게 연구되어왔고, 현재 이 주제의 연구에 관련한 방대한 문헌이 존재한다. 1976년에 스타인은 '3 이상의 모든 차원에서 구면 극대 함수의 L^p 계측'을 규명하는 개창적 결과를 증명하였다. 2차원 문제에 해당하는 원 극대 함수의 유계성은, 고전적인 L^2 방법의 한계로 인하여 매우 어려운 것으로 알려져 있었다. 그러나 1986년에 부르갱은 '원 극대 연산자는 p가 2보다 클 때 L^p에서 유계이다'라는 그의 유명한 원 극대 함수 정리를 증명함으로써 이 문제에 마침표를 찍었다. 이 학위 논문에서는 부르갱 원 극대 함수 정리를 더욱 강화하는 세 가지 결과를 증명한다. 첫째, 하이젠베르그 군 위에서의 원 대칭 함수에 대해서 원 극대 연산자의 L^p–L^q 유계를 최적 p, q 영역에서 얻는다. 둘째, 원환체 위의 평균에 의해 정의된 2개의 매개변수를 가지는 극대 연산자의 최적 L^p–L^q 유계성을 규명한다. 마지막으로, 타원에 의해 정의되는 다중변수 극대 연산자인 타원 극대 연산자의 L^p 계측을 증명한다.

주요어휘: 평균 연산자, 극대 유계, 소볼레프 정칙성, 국소적 평활화
학번: 2016-29031

감사의 글

먼저, 항상 저에게 가르침을 주시는 이상혁 선생님께 감사의 말씀을 올립니다. 학위과정에 있는 긴 시간동안 학문적 가르침 외에도 연구자가 가져야 할 모범적인 자세를 보고 느끼며 많은 깨달음을 얻었습니다. 선생님의 대학원 생활과 연구 활동에 대한 아낌없는 지원과 조언들은 앞으로도 큰 도움이 될 것입니다. 또, 이후에 어떠한 일이 있더라도 수학적 순수함을 잃지 않을 것을 약속드립니다. 바쁘신 와중에도 제 논문심사를 맡아주시고 논문발표에 직접 참여해주신 김준일 선생님, 서인석 선생님, 이훈희 선생님, Neal Bez 선생님께 깊이 감사드립니다.

대학원 입학동기인 호식, 성윤, 성해, 태형이 형에게도 어린 저를 스스럼없이 대해주며 즐겁게 생활하며 공부 할 수 있게 해주어 감사의 말을 전합니다. 학부시절부터 10년에 가까운 시간동안 함께 수학을 공부하며 서로에게 모범이 되어준 내훈, 재현, 형민, 신명, 건호, 우주, 재원, 영호, 상훈에게도 고맙다는 말을 전합니다. 함께 조화해석학을 공부하며 여러 가지 크고 작은 도움을 주었던 조주희 박사님, 함세헌 박사님, 권예현 박사님, 양창훈 박사님, 정은희 박사님, 고혜림 박사님, 이진봉 박사님, 오세욱 박사님, 유재현 박사님, 홍석창 박사님, Kalachand Shuin 박사님께도 크게 감사드립니다.

지금까지 아무 걱정 없이 공부에만 집중 할 수 있게 뒤에서 묵묵히 지원해주신 부모님께도 감사의 말씀을 올립니다. 마지막으로, 어린 시절 저를 키워주시고, 항상 저의 편이 되어주신 외할머니께 가장 큰 감사의 말을 올립니다. 또, 미처 언급하지 못하였지만 제가 이 자리에 있기까지 도움을 주신 모든 분들께도 감사드립니다.