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이학석사 학위논문

Finite-time blow-up to hyperbolic Keller-Segel system
of consumption type with logarithmic sensitivity

2023년 8월

서울대학교 대학원

수리과학부

나정경

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**Finite-time blow-up to hyperbolic Keller-Segel system
of consumption type with logarithmic sensitivity**

**A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Master of Science
to the faculty of the Graduate School of
Seoul National University**

by

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Abstract

This paper deals with a hyperbolic Keller-Segel system of consumption type with the logarithmic sensitivity

$$\partial_t \rho = -\chi \nabla \cdot (\rho \nabla \log c), \quad \partial_t c = -\mu c \rho \quad (\chi, \mu > 0)$$

in \mathbb{R}^d ($d \geq 1$) for nonvanishing initial data. This system is closely related to tumor angiogenesis, an important example of chemotaxis. We firstly show the local existence of smooth solutions corresponding to nonvanishing smooth initial data. Next, through Riemann invariants, we present some sufficient conditions of this initial data for finite-time singularity formation when $d = 1$. We then prove that for any $d \geq 1$, some nonvanishing C^∞ -data can become singular in finite time. Moreover, we derive detailed information about the behaviors of solutions when the singularity occurs. In particular, this information tells that singularity formation from some initial data is not because c touches zero (which makes $\log c$ diverge) but due to the blowup of $C^1 \times C^2$ -norm of (ρ, c) . As a corollary, we also construct initial data near any constant equilibrium state which blows up in finite time for any $d \geq 1$. Our results are the extension of finite-time blow-up results in [9], where initial data is required to satisfy some vanishing conditions. Furthermore, we interpret our results in a way that some kinds of damping or dissipation of ρ are necessarily required to ensure the global existence of smooth solutions even though initial data are small perturbations around constant equilibrium states.

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Chapter 1

Introduction

Chemotaxis refers to the motion of biological cells toward a higher (or lower) concentration of some chemical substance. Understanding and analyzing its mechanism is crucial since it describes many ubiquitous biological and ecological phenomena. After its first mathematical modeling by Patlak ([26]) and Keller-Segel ([12, 13]), there have been many variations in their models to explain realistic phenomena more precisely. One of the greatly important examples of chemotaxis is tumor angiogenesis, the new blood vessel formation induced by tumor cells. To be precise, by releasing vascular endothelial growth factor (VEGF), tumor cells induce endothelial cells to make new blood vessels toward them. In order to model the mechanism of tumor angiogenesis, the authors of [17] proposed the following system:

$$\begin{cases} \partial_t \rho = \kappa \Delta \rho - \chi \nabla \cdot (\rho \nabla \log c), \\ \partial_t c = -\mu c \rho. \end{cases} \quad (\text{P-KS})$$

Here, ρ denotes the density of endothelial cells, and c represents the concentration of VEGF that is consumed by the cells. $\kappa > 0$ denotes the diffusion coefficient of the cells, and $\chi, \mu > 0$ represent the intensity of chemotaxis and the consumption rate of VEGF, respectively.

In this paper, we derive finite-time singularity formation of the following

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Cauchy problem for a hyperbolic counterpart of (P-KS):

$$\begin{cases} \partial_t \rho = -\chi \nabla \cdot (\rho \nabla \log c), \\ \partial_t c = -\mu c \rho, \\ \rho(x, 0) = \rho_0(x), \quad c(x, 0) = c_0(x), \end{cases} \quad (\text{H-KS})$$

for $(x, t) \in \mathbb{R}^d \times (0, T)$ with $d \geq 1$. Our finite-time blow-up is not attributed to $c \rightarrow 0$, which makes $\log c \rightarrow \infty$, but due to divergence of $C^1 \times C^2$ -norm of (ρ, c) . Moreover, our result considers initial data not touching zero, where in the case of initial data vanishing at some point, finite-time blow-up can be shown using propagation of certain degeneracy ([9]). To effectively discuss the motivation for dropping $\Delta \rho$ and the significant meanings of our results, we first review some related previous results.

1.1 Previous works

(P-KS). There have been a lot of studies on the long-time dynamics of (P-KS). In addition to its biological significance, (P-KS) contains a mathematically interesting structure: logarithmic sensitivity, $\log c$. Since $\log c$ diverges at $c = 0$, this sensitivity has given difficulties in analyzing (P-KS). To overcome these, the Cole-Hopf transformation

$$q = -\frac{\nabla c}{c} = -\nabla \log c, \quad (1.1)$$

and scalings

$$x \rightarrow \frac{\sqrt{\chi\mu}}{\kappa}x, \quad t \rightarrow \frac{\chi\mu}{\kappa}t, \quad q \rightarrow \sqrt{\frac{\chi}{\mu}}q$$

have been used to transform (P-KS) into

$$\begin{cases} \partial_t \rho = \Delta \rho + \nabla \cdot (\rho q), \\ \partial_t q = \nabla \rho. \end{cases} \quad (\text{P-KS}')$$

We briefly review the main results of (P-KS'). Global existence of classical solutions to (P-KS') in a one-dimensional bounded domain was obtained for small initial data in [30], and for large initial data in [27] and [20].

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The paper [20] also showed that the solutions converge to their boundary data at an exponential rate as time goes to infinity. In the one-dimensional whole line \mathbb{R} , the authors of [7] established the global existence of classical solutions to (P-KS') for large initial data. Furthermore, the authors of [19] showed that the global classical solutions for large initial perturbations around constant equilibrium states in \mathbb{R} converge to the equilibrium states as time approaches infinity. The authors of [23] and [24] proved the non-linear stability of traveling wave solutions, while the authors of [21] showed the stability of composite waves. In multidimensional space, global existence and long-time behaviors of classical solutions to (P-KS') were obtained for initial data near a constant equilibrium state. In two or three-dimensional bounded domain Ω , the paper [22] obtained the global existence and large-time asymptotic behavior of classical solutions for initial data (ρ_0, q_0) satisfying $\|(\rho_0 - \text{avg}(\rho_0), q_0)\|_{(H^3 \times H^3)(\Omega)} \ll 1$, where $\text{avg}(\rho_0)$ denotes the average of ρ_0 over Ω . The authors of [18] showed global well-posedness and long time behavior of classical solutions in \mathbb{R}^d ($d \geq 2$) when $\|(\rho_0 - \bar{\rho}, q_0)\|_{(H^s \times H^s)(\mathbb{R}^d)} \ll 1$ with $s > \frac{d}{2} + 1$ for some constant $\bar{\rho} > 0$. The paper [8] generalized this results to the critical Besov space $B^{\frac{d}{2}-2}(\mathbb{R}^d) \times (B^{\frac{d}{2}-2, \frac{d}{2}-1}(\mathbb{R}^d))^d$ with $d \geq 2$ under the assumption that $\|(\rho_0 - \bar{\rho}, q_0)\|_{B^{\frac{d}{2}-2} \times (B^{\frac{d}{2}-2, \frac{d}{2}-1})^d} \ll 1$ is sufficiently small for some constant $\bar{\rho} > 0$. In \mathbb{R}^3 , the authors of [5] established global well-posedness of classical solutions when $\|(\rho_0 - \bar{\rho}, q_0)\|_{(L^2 \times H^1)(\mathbb{R}^3)} \ll 1$ for some constant $\bar{\rho} > 0$. Moreover, for the initial data satisfying $\|(\rho_0 - \bar{\rho}, q_0)\|_{(H^2 \times H^1)(\mathbb{R}^3)} \ll 1$, the paper [5] further derived decay property of the solutions. Later, the authors of [28] obtained asymptotic decay rates of classical solutions in \mathbb{R}^d with $d = 2, 3$ under assumptions that $(\rho_0 - \bar{\rho}, q_0) \in ((H^2 \cap \dot{H}^{-s}) \times (H^2 \cap \dot{H}^{-s}))(\mathbb{R}^d)$ for some $s \in (0, \frac{d}{2})$ and $\|(\rho_0 - \bar{\rho}, q_0)\|_{H^1 \times H^1} \ll 1$.

(cP-KS). We can compare (P-KS) with the following classical counterpart:

$$\begin{cases} \partial_t \rho = \kappa \Delta \rho - \chi \nabla \cdot (\rho \nabla \log c), \\ \partial_t c = \mu c^\lambda \rho, \end{cases} \quad (\text{cP-KS})$$

where the main difference from (P-KS) is that ρ does not consume but produces c . Depending on the range of λ , finite-time blow-up or global well-posedness of (cP-KS) have been studied by many authors. Since our main concerns are focused on the consumption type, we only list some references in

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this direction. See [15, 16, 4] for finite-time singularity formation when $\lambda = 1$, and [4] for global regularity when $\lambda = 0$. Some blow-up criteria and asymptotic behaviors of solutions can be found in [11, 1], and references therein. Refer to [25] for numerical investigation.

(H-KS). The first attempt to drop $\Delta\rho$ was from [3]. To be more precise, the author of [3] considered

$$\begin{cases} \partial_t \rho = \nabla \cdot (\rho q) + \rho(1 - \rho), \\ \partial_t q = \nabla \rho, \end{cases} \quad (\text{dH-KS})$$

where a term $\rho(1 - \rho)$ represents a logistic growth restriction. In other words, the author considered the case with damping rather than dissipation of ρ . In this system, the author of [3] obtained the existence of global smooth solutions in \mathbb{R}^2 when $\|(\rho_0 - 1, q_0)\|_{(H^3 \times H^3)(\mathbb{R}^2)} \ll 1$ and $\nabla \times q_0 = 0$. Recently, in [9] (see also [2]), the authors proved the finite-time singularity formation of (H-KS) in \mathbb{R}^d in the case when initial data satisfies some vanishing condition. To be precise, they showed that for initial data (ρ_0, c_0) satisfying $\rho_0(x_0) = 0$ and $\nabla \rho_0(x_0) = \nabla c_0(x_0) = \mathbf{0}$ for some $x_0 \in \mathbb{R}^d$, there exists a time $T^* > 0$ such that

$$\lim_{t \rightarrow T^*} \left(\|\rho(\cdot, t)\|_{W^{2,\infty}(\mathbb{R}^d)} + \|c(\cdot, t)\|_{W^{2,\infty}(\mathbb{R}^d)} \right) = \infty. \quad (1.2)$$

1.2 Main results and Discussion

Our primary motivation for studying the hyperbolic version (H-KS) is that finite-time singularity formation of (H-KS) could lead to rapid norm growth of solutions to (P-KS) when the diffusion coefficient κ is small. In fact, in [17] where (P-KS) was initially derived, very small κ was used, and numerical simulations showed large norm growth. This kind of result can be found in other hyperbolic-elliptic type Keller-Segel equations (see [29, 10]).

The previous works covered in the last section raise three interesting questions:

- Q1.** Can we delete the vanishing condition of initial data in [9]? In other words, is there any nonvanishing initial data leading to finite time singularity formation?

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Q2. If some nonvanishing initial data blows up at some finite time, then is it because c touches zero at some point, which makes logarithmic sensitivity $\log c$ blow up? Furthermore, can we get more detailed behaviors of solutions than (1.2) at the blow-up time?

Q3. Compared with global regularity results near constant equilibrium states for (P-KS) and (dH-KS), what happens to solutions of (H-KS) near constant equilibrium states?

To the best of the author's knowledge, there are no known answers to these questions. The aim of this paper is to answer them. To begin with, our physical motivation for **Q1** is to consider non-vacuum states. Moreover, considering nonvanishing data, we can have an opportunity to deal with **Q3**. In order to explain how difficult to answer **Q1**, we need to compare **Q1** with vanishing conditions in [9]: $\rho_0(x_0) = 0$, $\nabla \rho_0(x_0) = \nabla c_0(x_0) = \mathbf{0}$ for some $x_0 \in \mathbb{R}^d$. The authors of [9] employed propagation of these conditions to prove the finite-time blow-up. Heuristically, if initial cells don't exist at x_0 but initial chemicals are abundant at x_0 , cells would be concentrated fast in the neighborhood of the point to consume the chemicals. But due to the propagation of the vanishing conditions, cell density remains zero at the point at which solutions blow up within some finite time. But this scenario does not work in nonvanishing initial data, so we need to consider other scenarios to deal with it.

To do so, in this note, we shall assume that initial data (ρ_0, c_0) satisfies

$$\begin{cases} (\rho_0, c_0) \in (L^\infty \times L^\infty)(\mathbb{R}^d), \\ \rho_0(x) \geq \beta_1 > 0, \quad c_0(x) \geq \beta_2 > 0 \quad \text{for some constants } \beta_1, \beta_2. \end{cases} \quad (1.1)$$

Under this assumption, the dynamics of (H-KS) guarantees $\rho, c > 0$, and $\|c(t)\|_{L^\infty} \leq \|c_0\|_{L^\infty} < \infty$.

In order to investigate whether solutions to (H-KS) blow up at a finite time, we firstly have to establish local well-posedness of (H-KS) for nonvanishing initial data in \mathbb{R}^d with $d \geq 1$.

Theorem 1 (Local well-posedness for nonvanishing data). *Let $d \geq 1$. Assume that the initial data (ρ_0, c_0) satisfies (1.1), and*

$$(\nabla \rho_0, \nabla c_0) \in (H^{m-1} \times H^m)(\mathbb{R}^d) \quad \text{for some } m > \frac{d}{2} + 1. \quad (1.2)$$

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Then there exist a time $T > 0$, time-dependent positive constants $\beta_1(t)$, $\beta_2(t)$, and a unique solution (ρ, c) to (H-KS) such that

$$\begin{cases} \rho(x, t) \geq \beta_1(t), & c(x, t) \geq \beta_2(t) \quad \text{on } [0, T], \\ (\rho, c) \in L^\infty([0, T]; (L^\infty \times L^\infty)(\mathbb{R}^d)), \\ (\nabla \rho, \nabla c) \in L^\infty([0, T]; (H^{m-1} \times H^m)(\mathbb{R}^d)). \end{cases} \quad (1.3)$$

Furthermore, if initial data satisfies $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R}^d)$, then the unique solution satisfies $(\nabla \rho, \nabla c) \in L^\infty([0, T]; (H^\infty \times H^\infty)(\mathbb{R}^d))$, where $H^\infty(\mathbb{R}^d) := \bigcap_{k \geq 0} H^k(\mathbb{R}^d)$.

This local well-posedness result is not trivial but surprising because the equation of ρ in (H-KS) contains two derivatives of $\log c$. In order to overcome the loss of regularity, we compare $\nabla^m \rho$ with $\sqrt{\rho} \nabla^{m+1} \log c$ (rather than $\nabla^{m+1} \log c$) and observe a cancellation of certain quantity which involves highest derivatives. This idea was motivated by [9]. We shall prove Theorem 1 in Chapter 2.

Next, in Chapter 3, we present some sufficient conditions of nonvanishing initial data for finite-time singularity formation when $d = 1$.

Theorem 2 (Sufficient conditions of data for finite-time blow-up in \mathbb{R}).
Suppose that initial data (ρ_0, c_0) satisfies (1.1) with $d = 1$,

$$\bullet \quad (\partial_x \rho_0, \partial_x c_0) \in (H^\infty \times H^\infty)(\mathbb{R}), \quad (1.4a)$$

$$\bullet \quad \rho_0, c_0, \text{ and } \partial_x c_0 \text{ attain their maximums and minimums at some points in } \mathbb{R}, \quad (1.4b)$$

$$\bullet \quad \partial_x \rho_0(x_0) \leq 0 \text{ and } c_0(x_0) \partial_{xx} c_0(x_0) < (\partial_x c_0(x_0))^2 \text{ at some } x_0 \in \mathbb{R}. \quad (1.4c)$$

Then there exists some finite time $T^* > 0$ such that the unique solution (ρ, c) to (H-KS) satisfies

$$\sup_{t \in [0, T^*)} \left(\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R})} + \|c(\cdot, t)\|_{W^{1,\infty}(\mathbb{R})} + \|\partial_x \log c(\cdot, t)\|_{L^\infty(\mathbb{R})} \right) < \infty \quad (1.5)$$

and

$$\lim_{t \rightarrow T^*} \left(\|\partial_x \rho(\cdot, t)\|_{L^\infty(\mathbb{R})} + \|\partial_{xx} c(\cdot, t)\|_{L^\infty(\mathbb{R})} \right) = \infty. \quad (1.6)$$

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Remark 1.1. *There is a large class of functions satisfying (1.1) and (1.4a)-(1.4c). For instance, let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth bump function such that*

$$\psi(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2). \end{cases}$$

Then we can find $x_0 \in \mathbb{R}$ satisfying $\partial_x \psi(x_0) \leq 0$ and $\partial_{xx} \psi(x_0) < 0$. Hence for any constants $\bar{\rho}, \bar{c} > 0$, if we define

$$\rho_0 := \bar{\rho} + \psi \quad \text{and} \quad c_0 := \bar{c} + \psi, \tag{1.7}$$

then we can check that (ρ_0, c_0) satisfies (1.1) and (1.4a)-(1.4c).

Theorem 2 answers to our **Q1** and **Q2** in one dimension. Regarding **Q1**, nonvanishing initial data given in (1.7) leads to the finite-time blow-up. Furthermore, for **Q2**, noticing $\|\partial_x \log c(\cdot, t)\|_{L^\infty(\mathbb{R})} < \infty$ on $[0, T^*]$ as we can see in (1.5), we conclude that our singularity formation is not because c touches zero but derivatives of ρ and c blow up as (1.6). To the best of the author's knowledge, it is the first time to show that the finite-time blow-up of systems with logarithmic sensitivity is not attributed to the singularity from $c \rightarrow 0$.

The main difficulty of showing Theorem 2 lies in finding proper variables to control the equation for ρ containing many derivatives. A key tool to solve this problem is a diffeomorphic transformation $w = (w_1, w_2) : \Omega \rightarrow w(\Omega)$ with $\Omega := \{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0\}$. This w is called Riemann invariants (refer to [14] or [6] for instance). In addition to the general definition of Riemann invariants, we consider w satisfying certain extra properties. Combining these extra properties with our assumptions (1.1), (1.4a)-(1.4c), we deduce the stated finite-time singularity formation by using newly defined variables: $w_1(\rho, \log c)$, $w_2(\rho, \log c)$.

With the above one-dimensional result at hand, we finally prove the existence of a set of nonvanishing smooth initial data in \mathbb{R}^d with $d \geq 1$ making the solution singular in finite time. Our result is divided into \mathbb{R} and \mathbb{R}^d ($d \geq 2$) cases, respectively:

Theorem 3 (Finite-time blow-up in \mathbb{R}^d).

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• Let $d = 1$. Then for any interval $I \subset \mathbb{R}$, there exist some finite time $T^* > 0$ and initial data (ρ_0, c_0) satisfying (1.1) and $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R})$ such that the unique solution (ρ, c) to (H-KS) satisfies (1.5) and

$$\lim_{t \rightarrow T^*} \left(\|\partial_x \rho(\cdot, t)\|_{L^\infty(I)} + \|\partial_{xx} c(\cdot, t)\|_{L^\infty(I)} \right) = \infty.$$

• Let $d \geq 2$. Then there exist some finite time $T^* > 0$ and initial data (ρ_0, c_0) satisfying (1.1) and $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R}^d)$ such that the unique solution (ρ, c) to (H-KS) satisfies

$$\lim_{t \rightarrow T^*} \left(\|\rho(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c(\cdot, t)\|_{W^{2,\infty}(\mathbb{R}^d)} \right) = \infty.$$

We need some remarks to interpret our results as well as to answer our questions **Q1** and **Q2**.

Remark 1.2. Our result in \mathbb{R} tells us that we can control the region where the finite-time blow-up occurs as we please. Hence we can obtain the same blow-up result in torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ by making the singularity formed within the interval $(0, 1)$ and extending the solution periodically.

Remark 1.3. Concerning \mathbb{R}^d with $d \geq 2$, our result does not give as much information about blow-up as the 1D case. This is because we cannot make use of Riemann invariants technique in the proof of the multi-dimension case. However, noticing the previous remark, we can obtain the following result in torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ for any $d \geq 1$:

There exist some finite time $T^* > 0$ and initial data $(\rho_0, c_0) \in (C^\infty \times C^\infty)(\mathbb{T}^d)$ such that the unique solution (ρ, c) to (H-KS) satisfies

$$\sup_{t \in [0, T^*)} \left(\|\rho(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} + \|c(\cdot, t)\|_{W^{1,\infty}(\mathbb{T}^d)} + \|\nabla \log c(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} \right) < \infty$$

and

$$\lim_{t \rightarrow T^*} \left(\|\nabla \rho(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} + \|\nabla^2 c(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} \right) = \infty.$$

Indeed, since Remark 1.2 guarantees a 1D solution $(\rho^1(x_1, t), c^1(x_1, t))$ defined on \mathbb{T} which blows up at some finite time, just defining $(\rho(x, t), c(x, t)) := (\rho^1(x_1, t), c^1(x_1, t))$ for $x \in \mathbb{T}^d$, we can check that $(\rho(x, t), c(x, t))$ is a solution to (H-KS) which undergoes the same blow-up.

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The key of the proof of Theorem 3 is (1.5) which is responsible for the finite propagation speed (Lemma 4.1). We prove Theorem 3 in Chapter 4.

Moreover, proceeding in the same manner as the proof of the previous theorem, we can construct initial data near any constant equilibrium state which blows up in finite time.

Corollary 3.1 (Finite-time blow-up near any constant equilibrium state in \mathbb{R}^d). *Let $\bar{\rho} > 0$ be any constant equilibrium state. Then for any $d \geq 1$, $\epsilon > 0$, and integer $m \geq 0$, there exist some finite time $T^* > 0$ and initial data (ρ_0, c_0) satisfying (1.1), $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R}^d)$, and*

$$\|\rho_0 - \bar{\rho}\|_{H^m(\mathbb{R}^d)} + \|\nabla \log c_0\|_{H^m(\mathbb{R}^d)} \leq \epsilon \quad (1.8)$$

such that the unique solution (ρ, c) to (H-KS) satisfies

$$\lim_{t \rightarrow T^*} \left(\|\rho(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c(\cdot, t)\|_{W^{2,\infty}(\mathbb{R}^d)} \right) = \infty.$$

This corollary gives us a surprising answer to our last question, **Q3**. Comparing this corollary with the aforementioned global regularity results near constant equilibrium states for (P-KS) and (dH-KS), we conclude that some kinds of damping or dissipation of ρ are essential to extend the local classical solutions to global ones. The proof can be found in Chapter 4.

Notation

We employ the letter $C = C(a, b, \dots)$ to denote any constant depending on a, b, \dots , which may change from line to line in a given computation. We sometimes use $A \approx B$ and $A \lesssim B$, which mean $A = CB$ and $A \leq CB$, respectively, for some constant C .

Chapter 2

Local well-posedness

In this chapter, we show Theorem 1. Under the transformation (1.1) and the scalings

$$t \rightarrow \sqrt{\chi\mu}t, \quad q \rightarrow \sqrt{\frac{\chi}{\mu}}q,$$

(H-KS) becomes

$$\begin{cases} \partial_t \rho = \nabla \cdot (\rho q), \\ \partial_t q = \nabla \rho. \end{cases} \quad (\text{H-KS}')$$

We shall divide the proof into two steps, which correspond to a priori estimates and the existence and uniqueness of a solution.

2.1 A priori estimates

This section is devoted to a priori estimate for a solution (ρ, c) of (H-KS), which is assumed to be sufficiently smooth, so that the following computation can be justified. The equation of ρ in (H-KS') gives

$$\begin{aligned} \frac{d}{dt} \|\rho\|_{L^\infty} &\lesssim \|\nabla \rho\|_{L^\infty} \|q\|_{L^\infty} + \|\rho\|_{L^\infty} \|\nabla q\|_{L^\infty} \\ &\lesssim (\|\rho\|_{L^\infty} + \|\nabla \rho\|_{H^{m-1}}) \|q\|_{H^m}, \end{aligned} \quad (2.1)$$

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where in the last inequality, we used the Sobolev embedding, $H^{m-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. It is clear that the equation of c in (H-KS) implies

$$\frac{d}{dt} \|c\|_{L^\infty} \lesssim \|\rho\|_{L^\infty} \|c\|_{L^\infty}. \quad (2.2)$$

From the equation of ρ in (H-KS'), we can derive

$$\partial_t (\rho^{-1}) - q \cdot \nabla (\rho^{-1}) = -(\nabla \cdot q) (\rho^{-1}).$$

Evaluating along the characteristics defined by $-q$, we obtain

$$\frac{d}{dt} \|\rho^{-1}\|_{L^\infty} \lesssim \|\nabla q\|_{L^\infty} \|\rho^{-1}\|_{L^\infty} \lesssim \|q\|_{H^m} \|\rho^{-1}\|_{L^\infty}. \quad (2.3)$$

On the other hand, since the equation of c in (H-KS) gives

$$\partial_t (c^{-1}) = \mu \rho c^{-1},$$

we have

$$\frac{d}{dt} \|c^{-1}\|_{L^\infty} \lesssim \|\rho\|_{L^\infty} \|c^{-1}\|_{L^\infty}. \quad (2.4)$$

From (H-KS'), we also obtain

$$\frac{1}{2} \frac{d}{dt} \|q\|_{L^2}^2 \lesssim (\|\nabla \rho\|_{L^2}^2 + \|q\|_{L^2}^2). \quad (2.5)$$

Applying ∇^m to the equation of ρ in (H-KS') and taking L^2 inner product with $\nabla^m \rho$, we compute

$$\frac{1}{2} \frac{d}{dt} \|\nabla^m \rho\|_{L^2}^2 = \text{I}_1 + \text{I}_2 + \int (\rho \nabla^{m+1} \cdot q) \cdot \nabla^m \rho$$

with

$$\text{I}_1 = \int (\nabla^{m+1} \rho \cdot q) \cdot \nabla^m \rho,$$

and

$$\text{I}_2 \approx \int \left(\sum_{1 \leq |\alpha| \leq m} \sum_{|\alpha|+|\beta|=m+1} \nabla^\alpha \rho \cdot \nabla^\beta q \right) \cdot \nabla^m \rho.$$

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For I_1 , the integration by parts gives

$$I_1 = - \int (\nabla^m \rho \nabla \cdot q) \cdot \nabla^m \rho - I_1,$$

so that we obtain

$$I_1 = -\frac{1}{2} \int (\nabla^m \rho \nabla \cdot q) \cdot \nabla^m \rho \lesssim \|\nabla^m \rho\|_{L^2}^2 \|\nabla q\|_{L^\infty} \lesssim \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}.$$

Using $\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$ ($s > 0$), we estimate

$$\begin{aligned} I_2 &\lesssim (\|\nabla \rho\|_{L^\infty} \|\nabla q\|_{H^{m-1}} + \|\nabla \rho\|_{H^{m-1}} \|\nabla q\|_{L^\infty}) \|\nabla^m \rho\|_{L^2} \\ &\lesssim \|\nabla \rho\|_{H^{m-1}} \|\nabla q\|_{H^{m-1}} \|\nabla^m \rho\|_{L^2} \\ &\lesssim \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}. \end{aligned}$$

Thus, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla^m \rho\|_{L^2}^2 - \int (\rho \nabla^{m+1} \cdot q) \cdot \nabla^m \rho \lesssim \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}. \quad (2.6)$$

On the other hand, using $\rho > 0$ and (H-KS'), we compute

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \nabla^m q\|_{L^2}^2 = II_1 + II_2$$

with

$$II_1 = \frac{1}{2} \int \nabla \cdot (\rho q) (\nabla^m q)^2,$$

and

$$II_2 = \int \rho \nabla^m q \cdot \nabla^{m+1} \rho.$$

For II_1 , we estimate

$$\begin{aligned} II_1 &\lesssim \|\nabla(\rho q)\|_{L^\infty} \|\nabla^m q\|_{L^2}^2 \\ &\lesssim (\|\nabla \rho\|_{L^\infty} \|q\|_{L^\infty} + \|\rho\|_{L^\infty} \|\nabla q\|_{L^\infty}) \|q\|_{H^m}^2 \\ &\lesssim (\|\rho\|_{L^\infty} + \|\nabla \rho\|_{H^{m-1}}) \|q\|_{H^m}^3. \end{aligned}$$

Using the integration by parts, we compute

$$II_2 = - \int (\nabla \rho \cdot \nabla^m q) \cdot \nabla^m \rho - \int (\rho \nabla^{m+1} \cdot q) \cdot \nabla^m \rho,$$

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so that we have

$$\Pi_2 + \int (\rho \nabla^{m+1} \cdot q) \cdot \nabla^m \rho \lesssim \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}.$$

Thus, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \nabla^m q\|_{L^2}^2 + \int (\rho \nabla^{m+1} \cdot q) \cdot \nabla^m \rho \\ \lesssim (\|\rho\|_{L^\infty} + \|\nabla \rho\|_{H^{m-1}}) \|q\|_{H^m}^3 + \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}. \end{aligned} \quad (2.7)$$

Adding (2.6) to (2.7), we have

$$\frac{d}{dt} \left(\|\nabla^m \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^m q\|_{L^2}^2 \right) \lesssim (\|\rho\|_{L^\infty} + \|\nabla \rho\|_{H^{m-1}}) \|q\|_{H^m}^3 + \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}.$$

With the same argument, we can obtain

$$\begin{aligned} \frac{d}{dt} \sum_{k=1}^m \left(\|\nabla^k \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^k q\|_{L^2}^2 \right) \\ \lesssim (\|\rho\|_{L^\infty} + \|\nabla \rho\|_{H^{m-1}}) \|q\|_{H^m}^3 + \|\nabla \rho\|_{H^{m-1}}^2 \|q\|_{H^m}. \end{aligned} \quad (2.8)$$

We now define

$$X_m := 1 + \|\rho\|_{L^\infty} + \|c\|_{L^\infty} + \|\rho^{-1}\|_{L^\infty} + \|c^{-1}\|_{L^\infty} + \|q\|_{L^2} + \sum_{k=1}^m \left(\|\nabla^k \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^k q\|_{L^2}^2 \right).$$

Then, since

$$\|q\|_{H^m} \approx \|q\|_{L^2} + \|(\sqrt{\rho})^{-1} \sqrt{\rho} \nabla^m q\|_{L^2} \lesssim \|q\|_{L^2} + \sqrt{\|\rho^{-1}\|_{L^\infty}} \|\sqrt{\rho} \nabla^m q\|_{L^2} \lesssim X_m,$$

(2.1)-(2.5), and (2.8) imply that

$$\frac{d}{dt} X_m \lesssim X_m^4.$$

This shows that formally, there exists a time interval $[0, T]$ in which $\|\rho\|_{L^\infty}$, $\|c\|_{L^\infty}$, $\|\rho^{-1}\|_{L^\infty}$, $\|c^{-1}\|_{L^\infty}$, $\|\nabla \rho\|_{H^{m-1}}$, and $\|q\|_{H^m}$ remain finite. Furthermore, we can check that $\|\nabla c\|_{H^m}$ also remains finite on the same time interval, noticing that the definition of q ensures the existence of a constant $C = C(\|c\|_{L^\infty}, \|c^{-1}\|_{L^\infty}, \|q\|_{H^m}) > 0$ such that $\|\nabla c\|_{H^m} \leq C$.

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In order to show that $(\nabla \rho_0, \nabla c_0) \in H^\infty \times H^\infty$ implies $(\nabla \rho(t, \cdot), \nabla c(t, \cdot)) \in H^\infty \times H^\infty$ on the same time interval, we use the induction argument. Assume that $\|\rho\|_{L^\infty}$, $\|c\|_{L^\infty}$, $\|\rho^{-1}\|_{L^\infty}$, $\|c^{-1}\|_{L^\infty}$, $\|\nabla \rho\|_{H^{m-1}}$, $\|q\|_{H^m}$, and $\|\nabla c\|_{H^m}$ are finite on $[0, T]$. Then, using the cancellation as we did above, we can obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla^{m+1} \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2 \right) = \text{III}_1 + \text{III}_2 + \text{IV}_1 + \text{IV}_2$$

with

$$\begin{aligned} \text{III}_1 &= \int (\nabla^{m+2} \rho \cdot q) \cdot \nabla^{m+1} \rho, \\ \text{III}_2 &\approx \int \left(\sum_{1 \leq |\alpha| \leq m+1} \sum_{|\alpha|+|\beta|=m+2} \nabla^\alpha \rho \cdot \nabla^\beta q \right) \cdot \nabla^{m+1} \rho, \\ \text{IV}_1 &= \frac{1}{2} \int (\nabla \cdot (\rho q)) (\nabla^{m+1} q)^2, \end{aligned}$$

and

$$\text{IV}_2 = \int (\nabla \rho \cdot \nabla^{m+1} q) \cdot \nabla^{m+1} \rho.$$

For III_1 , using the integration by parts and the assumption that $\|\nabla q\|_{L^\infty} \lesssim \|q\|_{H^m} \leq C$, we have

$$\text{III}_1 = -\frac{1}{2} \int (\nabla^{m+1} \rho \nabla \cdot q) \cdot \nabla^{m+1} \rho \leq C \|\nabla^{m+1} \rho\|_{L^2}^2.$$

We decompose III_2 into

$$\text{III}_2 = \text{III}_{21} + \text{III}_{22}$$

with

$$\text{III}_{21} \approx \int \left(\sum_{|\alpha|=1} \sum_{|\beta|=m+1} \nabla^\alpha \rho \cdot \nabla^\beta q \right) \cdot \nabla^{m+1} \rho + \int \left(\sum_{|\alpha|=m+1} \sum_{|\beta|=1} \nabla^\alpha \rho \cdot \nabla^\beta q \right) \cdot \nabla^{m+1} \rho,$$

and

$$\text{III}_{22} \approx \int \left(\sum_{2 \leq |\alpha| \leq m} \sum_{|\alpha|+|\beta|=m+2} \nabla^\alpha \rho \cdot \nabla^\beta q \right) \cdot \nabla^{m+1} \rho.$$

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Noticing the assumption that $\|\nabla \rho\|_{L^\infty} \lesssim \|\rho\|_{H^m} \leq C$ and $\|\nabla q\|_{L^\infty} \lesssim \|q\|_{H^m} \leq C$, we estimate

$$\text{III}_{21} \lesssim \|\nabla^{m+1} \rho\|_{L^2} \|\nabla^{m+1} q\|_{L^2} \lesssim \|\nabla^{m+1} \rho\|_{L^2}^2 + \|\nabla^{m+1} q\|_{L^2}^2.$$

Since $\|\nabla^\alpha \rho\|_{L^2} \leq \|\nabla^{m+1} \rho\|_{L^2}$ for $2 \leq |\alpha| \leq m$, $\|q\|_{H^m} \leq C$ again gives

$$\text{III}_{22} \lesssim \|\nabla^{m+1} \rho\|_{L^2}^2.$$

Considering $\|\nabla \cdot (\rho q)\|_{L^\infty} \lesssim \|\rho\|_{H^m} \|q\|_{H^m} \leq C$, we have

$$\text{IV}_1 \lesssim \|\nabla^{m+1} q\|_{L^2}^2.$$

Using $\|\nabla \rho\|_{L^\infty} \lesssim \|\rho\|_{H^m} \leq C$ again, we estimate

$$\text{IV}_2 \lesssim \|\nabla^{m+1} \rho\|_{L^2} \|\nabla^{m+1} q\|_{L^2} \lesssim \|\nabla^{m+1} \rho\|_{L^2}^2 + \|\nabla^{m+1} q\|_{L^2}^2.$$

Combining all, and noticing

$$\|\nabla^{m+1} q\|_{L^2}^2 = \|(\sqrt{\rho})^{-1} \sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2 \leq \sqrt{\|\rho^{-1}\|_{L^\infty}} \|\sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2 \lesssim \|\sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2,$$

we finally have arrived at

$$\frac{d}{dt} \left(\|\nabla^{m+1} \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2 \right) \lesssim \|\nabla^{m+1} \rho\|_{L^2}^2 + \|\sqrt{\rho} \nabla^{m+1} q\|_{L^2}^2.$$

Hence, the Grönwall's inequality implies that $\|\nabla^{m+1} \rho\|_{L^2}$ and $\|\nabla^{m+1} q\|_{L^2}$ remain finite on $[0, T]$. \square

Remark 2.1. *Observing our priori estimate, we can see that*

$$\sup_{t \in [0, T)} \|\rho(t)\|_{W^{1, \infty}(\mathbb{R}^d)} + \|c(t)\|_{W^{2, \infty}(\mathbb{R}^d)}$$

controls blow-up. In other words, for (ρ_0, c_0) satisfying (1.1) and (1.2), we have corresponding solution (ρ, c) satisfying (1.3) on $[0, T]$ as long as

$$\|\rho(t)\|_{W^{1, \infty}(\mathbb{R}^d)} + \|c(t)\|_{W^{2, \infty}(\mathbb{R}^d)} < \infty \quad \text{on } [0, T]. \quad (2.9)$$

We can also ensure $(\nabla \rho, \nabla c) \in L^\infty([0, T]; (H^\infty \times H^\infty)(\mathbb{R}^d))$ if (ρ_0, c_0) further satisfies $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R}^d)$ and (2.9) holds.

2.2 Existence and Uniqueness

To begin with, the proof of existence can be done using the standard argument, such as viscous approximation (refer to [9] for instance).

To prove uniqueness, we assume that there exist two solutions (ρ_i, c_i) ($i = 1, 2$) on $[0, T]$ to (H-KS) satisfying (1.3) and $(\rho_1(t = 0), c_1(t = 0)) = (\rho_2(t = 0), c_2(t = 0))$. We denote

$$q_i = -\frac{\nabla c_i}{c_i} (i = 1, 2), \quad \tilde{\rho} := \rho_1 - \rho_2, \quad \tilde{c} := c_1 - c_2, \quad \text{and} \quad \tilde{q} := q_1 - q_2.$$

From the equation of $\tilde{\rho}$:

$$\partial_t \tilde{\rho} = \nabla \tilde{\rho} \cdot q_1 + \tilde{\rho} \nabla \cdot q_1 + \nabla \rho_2 \cdot \tilde{q} + \rho_2 \nabla \cdot \tilde{q}, \quad (2.1)$$

and the equation of \tilde{q} :

$$\partial_t \tilde{q} = \nabla \tilde{\rho}, \quad (2.2)$$

we have

$$\frac{1}{2} \frac{d}{dt} \left(\|\tilde{\rho}\|_{L^2}^2 + \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2 \right) = \text{I} + \text{II} + \text{III} + \text{IV}.$$

with

$$\begin{aligned} \text{I} &= \int \tilde{\rho} \nabla \tilde{\rho} \cdot q_1 + \int \tilde{\rho}^2 \nabla \cdot q_1, \\ \text{II} &= \int \tilde{\rho} \nabla \rho_2 \cdot \tilde{q} + \int \tilde{\rho} \rho_2 \nabla \cdot \tilde{q}, \\ \text{III} &= \frac{1}{2} \int |\tilde{q}|^2 \nabla \cdot (\rho_2 q_2), \end{aligned}$$

and

$$\text{IV} = \int \rho_2 \tilde{q} \cdot \nabla \tilde{\rho}.$$

Using the integration by parts on the first term of I, we estimate

$$\begin{aligned} \text{I} &= \frac{1}{2} \int \nabla (\tilde{\rho}^2) \cdot q_1 + \int \tilde{\rho}^2 \nabla \cdot q_1 \\ &= -\frac{1}{2} \int \tilde{\rho}^2 \nabla \cdot q_1 + \int \tilde{\rho}^2 \nabla \cdot q_1 \\ &= \frac{1}{2} \int \tilde{\rho}^2 \nabla \cdot q_1 \\ &\leq \|q_1\|_{W^{1,\infty}} \|\tilde{\rho}\|_{L^2}^2. \end{aligned}$$

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The integration by parts also gives the cancellation of II + IV:

$$\text{II} + \text{IV} = 0.$$

Also, we have

$$\text{III} \leq \|\rho_2^{-1}\|_{L^\infty} \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2 \|\rho_2\|_{W^{1,\infty}} \|q_2\|_{W^{1,\infty}}.$$

Note that $\|q_1\|_{W^{1,\infty}}$, $\|\rho_2\|_{W^{1,\infty}}$, $\|q_2\|_{W^{1,\infty}}$, and $\|\rho_2^{-1}\|_{L^\infty}$ are all bounded, and therefore,

$$\frac{1}{2} \frac{d}{dt} \left(\|\tilde{\rho}\|_{L^2}^2 + \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2 \right) \lesssim \|\tilde{\rho}\|_{L^2}^2 + \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2.$$

Since $\left(\|\tilde{\rho}\|_{L^2}^2 + \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2 \right) (0) = 0$, we conclude that $\left(\|\tilde{\rho}\|_{L^2}^2 + \|\sqrt{\rho_2} \tilde{q}\|_{L^2}^2 \right) (t) = 0$ for $t \in [0, T]$. To show $\tilde{c}(t) = 0$ for $t \in [0, T]$, we consider the equation of \tilde{c} :

$$\partial_t \tilde{c} = -\mu (\tilde{c} \rho_1 + c_2 \tilde{\rho}) = -\mu \tilde{c} \rho_1.$$

The second equality follows from $\tilde{\rho} = 0$ on $[0, T]$. Thus using the Grönwall's inequality, $\tilde{c}(t) = 0$ on $[0, T]$. \square

Remark 2.2. *As we can see from the above proof, we are able to prove that the solution is unique on $[0, T]$ as long as the solution satisfies*

$$\|\rho(t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c(t)\|_{W^{2,\infty}(\mathbb{R}^d)} < \infty \quad \text{on } [0, T].$$

Chapter 3

Sufficient conditions of data for finite-time blow-up in \mathbb{R}

In this chapter, we prove Theorem 2 with the aid of Riemann invariants. Henceforth, we always assume that initial data (ρ_0, c_0) satisfies (1.1), (1.4a)-(1.4c).

We firstly show the existence of finite time $T^* > 0$ such that

$$\lim_{t \rightarrow T^*} \left(\|\rho(t)\|_{W^{1,\infty}(\mathbb{R})} + \|c(t)\|_{W^{2,\infty}(\mathbb{R})} \right) = \infty. \quad (3.1)$$

Then we prove that the solution actually satisfies (1.5) for $t \in [0, T^*]$.

Now we begin the proof. Suppose, on the contrary, that for any finite time $T > 0$, the unique solution corresponding to these initial data satisfies

$$\|\rho(t)\|_{W^{1,\infty}(\mathbb{R})} + \|c(t)\|_{W^{2,\infty}(\mathbb{R})} < \infty \quad \text{on } [0, T]. \quad (3.2)$$

Recall that (3.2) is sufficient to guarantee that (ρ, c) is smooth and unique, and has the positive lower bounds on $[0, T]$ by Remark 2.1 and Remark 2.2. Abusing the notation only in this chapter, we denote $q := \frac{\partial_x c}{c}$ (not $q = -\frac{\partial_x c}{c}$), and change (H-KS') into the matrix form:

$$\partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \begin{pmatrix} q & \rho \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that $\begin{pmatrix} q & \rho \\ 1 & 0 \end{pmatrix}$ has eigenvalues: $\lambda_1(\rho, q) = \frac{q - \sqrt{q^2 + 4\rho}}{2}$, $\lambda_2(\rho, q) = \frac{q + \sqrt{q^2 + 4\rho}}{2}$,

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and corresponding eigenvectors: $r_1(\rho, q) = \left(\frac{q - \sqrt{q^2 + 4\rho}}{2}, 1 \right)$, $r_2(\rho, q) = \left(\frac{q + \sqrt{q^2 + 4\rho}}{2}, 1 \right)$.

We can regard λ_i and r_i ($i = 1, 2$) as functions on $\Omega := \{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0\}$.

We now construct a new coordinate system w on Ω , called Riemann invariants.

Lemma 3.1. *There exists a global diffeomorphism $w : \Omega \rightarrow w(\Omega) \subset \mathbb{R}^2$ such that on Ω ,*

$$\bullet \nabla w_i \cdot r_i = 0 \quad (i = 1, 2), \quad (3.3a)$$

$$\bullet \frac{\partial w_1}{\partial z_1} > 0, \quad \frac{\partial w_2}{\partial z_1} < 0. \quad (3.3b)$$

Proof. We firstly solve the following PDE on Ω :

$$\begin{cases} \partial_{z_2} f_1(z_1, z_2) + \frac{z_2 - \sqrt{z_2^2 + 4z_1}}{2} \partial_{z_1} f_1(z_1, z_2) = \frac{1}{\sqrt{z_2^2 + 4z_1}} f_1(z_1, z_2), \\ f_1(z_1, -1) = e^{z_1}. \end{cases} \quad (3.4)$$

Evaluating along the characteristic $\phi_1(z_1, z_2)$ defined by

$$\begin{cases} \partial_{z_2} \phi_1(z_1, z_2) = \frac{z_2 - \sqrt{z_2^2 + 4\phi_1(z_1, z_2)}}{2}, \\ \phi_1(z_1, -1) = z_1, \end{cases}$$

we can check that f_1 is smooth and $f_1 > 0$ on Ω . Furthermore, since (3.4) implies that

$$\partial_{z_2} f_1(z_1, z_2) = \partial_{z_1} \left(\frac{-z_2 + \sqrt{z_2^2 + 4z_1}}{2} f_1(z_1, z_2) \right),$$

the Poincare lemma ensures the existence of a smooth function $w_1 : \Omega \rightarrow \mathbb{R}$ satisfying

$$\frac{\partial w_1}{\partial z_1} = f_1(z_1, z_2), \quad \frac{\partial w_1}{\partial z_2} = \frac{-z_2 + \sqrt{z_2^2 + 4z_1}}{2} f_1(z_1, z_2). \quad (3.5)$$

This gives us that $\nabla w_1 \cdot r_1 = 0$.

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Similarly, with the aid of the characteristic $\phi_2(z_1, z_2)$ defined by

$$\begin{cases} \partial_{z_2} \phi_2(z_1, z_2) = \frac{z_2 + \sqrt{z_2^2 + 4\phi_1(z_1, z_2)}}{2}, \\ \phi(z_1, -1) = z_1, \end{cases}$$

we can obtain a negative smooth function f_2 defined on Ω solving

$$\begin{cases} \partial_{z_2} f_2(z_1, z_2) + \frac{z_2 + \sqrt{z_2^2 + 4z_1}}{2} \partial_{z_1} f_1(z_1, z_2) = -\frac{1}{\sqrt{z_2^2 + 4z_1}} f_2(z_1, z_2), \\ f_1(z_1, 1) = -e^{z_1}. \end{cases} \quad (3.6)$$

Since (3.6) implies that

$$\partial_{z_2} f_2(z_1, z_2) = \partial_{z_1} \left(-\frac{z_2 + \sqrt{z_2^2 + 4z_1}}{2} f_2(z_1, z_2) \right),$$

the Poincare lemma again implies that there exists a smooth function $w_2 : \Omega \rightarrow \mathbb{R}$ satisfying

$$\frac{\partial w_2}{\partial z_1} = f_2(z_1, z_2), \quad \frac{\partial w_2}{\partial z_2} = -\frac{z_2 + \sqrt{z_2^2 + 4z_1}}{2} f_2(z_1, z_2),$$

which yields $\nabla w_2 \cdot r_2 = 0$.

Now we show that $w = (w_1, w_2) : \Omega \rightarrow w(\Omega) \subset \mathbb{R}^2$ is a global diffeomorphism. Since $f_1 > 0$, $f_2 < 0$, and

$$\nabla w = \begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_1}{\partial z_2} \\ \frac{\partial w_2}{\partial z_1} & \frac{\partial w_2}{\partial z_2} \end{pmatrix} = \begin{pmatrix} f_1(z_1, z_2) & f_1(z_1, z_2) \frac{-z_2 + \sqrt{z_2^2 + 4z_1}}{2} \\ f_2(z_1, z_2) & -f_2(z_1, z_2) \frac{z_2 + \sqrt{z_2^2 + 4z_1}}{2} \end{pmatrix}, \quad (3.7)$$

we have

$$\det(\nabla w) = -f_1(z_1, z_2) f_2(z_1, z_2) \sqrt{z_2^2 + 4z_1} > 0 \quad \text{on } \Omega,$$

so that the inverse function theorem implies that w is a local diffeomorphism. Thus, it suffices to prove that w is (globally) one-to-one. Suppose, on the contrary, that $w(\bar{z}_1, \bar{z}_2) = w(\tilde{z}_1, \tilde{z}_2)$ for some $(\bar{z}_1, \bar{z}_2) \neq (\tilde{z}_1, \tilde{z}_2)$. Then by the

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mean value theorem, there exist two points (z_1^*, z_2^*) and (z_1^{**}, z_2^{**}) lying on the line segment with endpoints (\bar{z}_1, \bar{z}_2) and $(\tilde{z}_1, \tilde{z}_2)$ such that

$$\begin{aligned} 0 &= w_1(\bar{z}_1, \bar{z}_2) - w_1(\tilde{z}_1, \tilde{z}_2) \\ &= \nabla w_1|_{(z_1^*, z_2^*)} \cdot (\bar{z}_1 - \tilde{z}_1, \bar{z}_2 - \tilde{z}_2) \\ &= f_1(z_1^*, z_2^*)(\bar{z}_1 - \tilde{z}_1) + f_1(z_1^*, z_2^*) \frac{-z_2^* + \sqrt{z_2^{*2} + 4z_1^*}}{2} (\bar{z}_2 - \tilde{z}_2), \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} 0 &= w_2(\bar{z}_1, \bar{z}_2) - w_2(\tilde{z}_1, \tilde{z}_2) \\ &= \nabla w_2|_{(z_1^{**}, z_2^{**})} \cdot (\bar{z}_1 - \tilde{z}_1, \bar{z}_2 - \tilde{z}_2) \\ &= f_2(z_1^{**}, z_2^{**})(\bar{z}_1 - \tilde{z}_1) - f_2(z_1^{**}, z_2^{**}) \frac{z_2^{**} + \sqrt{z_2^{**2} + 4z_1^{**}}}{2} (\bar{z}_2 - \tilde{z}_2). \end{aligned} \quad (3.9)$$

If $\bar{z}_1 < \tilde{z}_1$ and $\bar{z}_2 \leq \tilde{z}_2$, then (3.8) is less than 0, which is a contradiction. If $\bar{z}_1 < \tilde{z}_1$ and $\bar{z}_2 > \tilde{z}_2$, then (3.9) is less than 0, which is also a contradiction. We can similarly derive contradictions in the other cases, which implies that $(\bar{z}_1, \bar{z}_2) = (\tilde{z}_1, \tilde{z}_2)$. \square

Remark 3.1. *Lemma 3.1 enables us to write that for $(z_1, z_2) \in \Omega$,*

$$\lambda_i(z_1, z_2) = (\lambda_i \circ w^{-1})(w_1(z_1, z_2), w_2(z_1, z_2)), \quad (i = 1, 2),$$

and ensures that each λ_i has the same regularity with $\lambda_i \circ w^{-1}$. Henceforth, we identify λ_i defined on Ω with $\lambda_i \circ w^{-1}$ defined on $w(\Omega)$, and regard λ_i as a function of $(w_1, w_2) \in w(\Omega)$ as well as $(z_1, z_2) \in \Omega$.

Defining a pair of functions $(P, Q) : \mathbb{R} \times [0, \infty) \rightarrow w(\Omega)$ by

$$P(x, t) = w_1(\rho(x, t), q(x, t)), \quad Q(x, t) = w_2(\rho(x, t), q(x, t)),$$

we can check that (P, Q) solves

$$\begin{cases} \partial_t P + \lambda_2(\rho, q) \partial_x P = 0, \\ \partial_t Q + \lambda_1(\rho, q) \partial_x Q = 0, \end{cases} \quad (3.10)$$

with the aid of (3.3a). Since this is Theorem 1 in Section 11.3 of [6], we omit the details.

For the proof of Theorem 2, we need two key ingredients.

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Lemma 3.2. *There exists a universal constant $\delta_0 > 0$ such that*

$$\frac{\partial \lambda_2}{\partial w_1}(P(x, t), Q(x, t)) \geq \delta_0 \quad \text{in } \mathbb{R} \times [0, \infty).$$

Proof. To begin with, we claim that

$$\frac{\partial \lambda_2}{\partial w_1} > 0 \quad \text{on } w(\Omega). \quad (3.11)$$

Indeed, since (3.7) provides us with

$$\begin{pmatrix} \frac{\partial z_1}{\partial w_1} & \frac{\partial z_1}{\partial w_2} \\ \frac{\partial z_2}{\partial w_1} & \frac{\partial z_2}{\partial w_2} \end{pmatrix} = (\nabla w)^{-1} = \begin{pmatrix} \frac{z_2 + \sqrt{z_2^2 + 4z_1}}{2f_1(z_1, z_2)\sqrt{z_2^2 + 4z_1}} & \frac{-z_2 + \sqrt{z_2^2 + 4z_1}}{2f_2(z_1, z_2)\sqrt{z_2^2 + 4z_1}} \\ 1 & -1 \\ \frac{1}{f_1(z_1, z_2)\sqrt{z_2^2 + 4z_1}} & \frac{-1}{f_2(z_1, z_2)\sqrt{z_2^2 + 4z_1}} \end{pmatrix},$$

we have

$$\frac{\partial \lambda_2}{\partial w_1} = \frac{\partial \lambda_2}{\partial z_1} \frac{\partial z_1}{\partial w_1} + \frac{\partial \lambda_2}{\partial z_2} \frac{\partial z_2}{\partial w_1} = \frac{z_2 + \sqrt{z_2^2 + 4z_1}}{f_1(z_1, z_2)(z_2^2 + 4z_1)} > 0.$$

Evaluating along the characteristics defined by each $\lambda_i(\rho, q)$ ($i = 1, 2$), we obtain from (3.10) that

$$P(\mathbb{R} \times [0, \infty)) = P_0(\mathbb{R}) = w_1(\rho_0(\mathbb{R}), q_0(\mathbb{R})) = w_1\left(\rho_0(\mathbb{R}), \frac{\partial_x c_0}{c_0}(\mathbb{R})\right),$$

and

$$Q(\mathbb{R} \times [0, \infty)) = Q_0(\mathbb{R}) = w_2(\rho_0(\mathbb{R}), q_0(\mathbb{R})) = w_2\left(\rho_0(\mathbb{R}), \frac{\partial_x c_0}{c_0}(\mathbb{R})\right),$$

where $P_0(x) := P(x, 0)$ and $Q_0(x) := Q(x, 0)$. Since each w_i is smooth and both $\rho_0(\mathbb{R})$ and $\frac{\partial_x c_0}{c_0}(\mathbb{R})$ are compact by the assumptions (1.1), (1.4a), and (1.4b), $P(\mathbb{R} \times [0, \infty))$ and $Q(\mathbb{R} \times [0, \infty))$ are also compact. Thus, (3.11) and the smoothness of $\frac{\partial \lambda_2}{\partial w_1}$ ensure the existence of the desired $\delta_0 > 0$. \square

Lemma 3.3. *Let x_0 be a point satisfying (1.4c). Then*

$$\partial_x P_0(x_0) < 0.$$

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Proof. By (3.5), (3.3b), (1.1), and (1.4c), we compute at x_0

$$\begin{aligned}\partial_x P_0 &= \frac{\partial w_1}{\partial z_1} \partial_x \rho_0 + \frac{\partial w_1}{\partial z_2} \partial_x q_0 \\ &= f_1(\rho_0, q_0) \partial_x \rho_0 + f_1(\rho_0, q_0) \frac{-q_0 + \sqrt{q_0^2 + 4\rho_0}}{2} \frac{c_0 \partial_{xx} c_0 - (\partial_x c_0)^2}{c_0^2} < 0.\end{aligned}$$

□

We are now in the position to prove Theorem 2. We use a similar argument to the proof of Theorem 2 in section 11.3 of [6]. Let x_0 be the point from (1.4c), and let $x_{\lambda_2}(t)$ be the characteristic curve defined by $\lambda_2(\rho, q)$ with initial value $x_{\lambda_2}(0) = x_0$. Note that $P(x_{\lambda_2}(t), t) = P_0(x_0)$ for $t \geq 0$. For the simplicity, we write $\tilde{P} := \partial_x P$, $\tilde{Q} := \partial_x Q$, and $\tilde{P}_{\lambda_2}(t) := \partial_x P(x_{\lambda_2}(t), t)$. Then (3.10) implies that

$$\partial_t \tilde{P} + \lambda_2 \partial_x \tilde{P} + \frac{\partial \lambda_2}{\partial w_1} \tilde{P}^2 + \frac{\partial \lambda_2}{\partial w_2} \tilde{P} \tilde{Q} = 0,$$

and

$$\partial_t Q + \lambda_2 \partial_x Q = (\lambda_2 - \lambda_1) \tilde{Q}.$$

Combining these, we obtain

$$\partial_t \tilde{P} + \lambda_2 \partial_x \tilde{P} + \frac{\partial \lambda_2}{\partial w_1} \tilde{P}^2 + \left[\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (\partial_t Q + \lambda_2 \partial_x Q) \right] \tilde{P} = 0.$$

Defining a function $\Phi(t)$ by

$$\Phi(t) := \exp \left\{ \left(\int_0^t \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (\partial_t Q + \lambda_2 \partial_x Q)(x_{\lambda_2}(s), s) ds \right) \right\},$$

we have

$$\Phi(t) = \exp \left\{ \left(\int_0^t \frac{d}{ds} \Psi(s) ds \right) \right\} = \exp \{ (\Psi(t) - \Psi(0)) \},$$

where

$$\Psi(s) := \int_0^{Q(x_{\lambda_2}(s), s)} \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (P_0(x_0), v) dv.$$

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Since $\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2}$ is smooth and $Q(\mathbb{R} \times [0, \infty))$ is compact as we saw in the proof of Lemma 3.2, there exist two constants m_Φ, M_Φ such that

$$0 < m_\Phi \leq \Phi(t) \leq M_\Phi \quad \text{for all } t \geq 0. \quad (3.12)$$

Thus, multiplying \tilde{P}_{λ_2} by Φ and differentiating it with respect to time, we obtain the following ODE:

$$\frac{d}{dt} \left(\tilde{P}_{\lambda_2} \Phi \right) = - \frac{\partial \lambda_2}{\partial w_1} \tilde{P}_{\lambda_2}^2 \Phi,$$

so that Lemma 3.2 and (3.12) yield

$$\frac{d}{dt} \left(\tilde{P}_{\lambda_2} \Phi \right) \leq - \frac{\delta_0}{M_\Phi} \left(\tilde{P}_{\lambda_2} \Phi \right)^2.$$

Since $\tilde{P}_{\lambda_2}(0) = \partial_x P_0(x_0) < 0$ by Lemma 3.3, there exists a finite time $T^* > 0$ such that

$$\tilde{P}_{\lambda_2}(t) \leq \frac{M_\Phi \tilde{P}_{\lambda_2}(0)}{M_\Phi + \tilde{P}_{\lambda_2}(0) \delta_0 t} \rightarrow -\infty \quad \text{as } t \rightarrow T^*. \quad (3.13)$$

On the other hand, we claim that there exist two constants m_{f_1}, M_{f_1} such that

$$0 < m_{f_1} \leq f_1(\rho(x, t), q(x, t)) \leq M_{f_1} \quad \text{for } (x, t) \in \mathbb{R} \times [0, \infty). \quad (3.14)$$

Indeed, $f_1(\rho(x, t), q(x, t)) = (f_1 \circ w^{-1})(P(x, t), Q(x, t))$, so that (3.14) follows from the smoothness of f_1 and the compactness of $P(\mathbb{R} \times [0, \infty))$ and $Q(\mathbb{R} \times [0, \infty))$ as in the proof of Lemma 3.2. Hence (3.2) and (3.14) give

$$\begin{aligned} \left\| \tilde{P}(T^*) \right\|_{L^\infty} &= \left\| \partial_x P(T^*) \right\|_{L^\infty} \\ &= \left\| f_1(\rho(T^*), q(T^*)) \left(\partial_x \rho(T^*) + \partial_x q(T^*) \frac{-q(T^*) + \sqrt{q^2(T^*) + 4\rho(T^*)}}{2} \right) \right\|_{L^\infty} \\ &< \infty, \end{aligned}$$

which is a contradiction to (3.13). This shows (3.1).

Now we prove the solution (ρ, c) satisfies (1.5). Note that we have

$$\|\rho(t)\|_{W^{1,\infty}(\mathbb{R})} + \|c(t)\|_{W^{2,\infty}(\mathbb{R})} < \infty$$

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on $[0, T^*)$. This ensures (ρ, c) is smooth and unique, and has the positive lower bounds on $[0, T^*)$ by Remark 2.1 and Remark 2.2. Then we again define $(P, Q) : \mathbb{R} \times [0, T^*) \rightarrow w(\Omega)$ by

$$P(x, t) = w_1(\rho(x, t), q(x, t)), \quad Q(x, t) = w_2(\rho(x, t), q(x, t)).$$

(In this time, we defined (P, Q) on time interval $[0, T^*)$.) Using the characteristic defined by each $\lambda_i(\rho, q)$ ($i = 1, 2$), we can check from (3.10) that

$$P(\mathbb{R} \times [0, T^*)) = P_0(\mathbb{R}) = w_1(\rho_0(\mathbb{R}), q_0(\mathbb{R})) = w_1\left(\rho_0(\mathbb{R}), \frac{\partial_x c_0}{c_0}(\mathbb{R})\right),$$

and

$$Q(\mathbb{R} \times [0, T^*)) = Q_0(\mathbb{R}) = w_2(\rho_0(\mathbb{R}), q_0(\mathbb{R})) = w_2\left(\rho_0(\mathbb{R}), \frac{\partial_x c_0}{c_0}(\mathbb{R})\right),$$

so that $P(\mathbb{R} \times [0, T^*))$ and $Q(\mathbb{R} \times [0, T^*))$ are compact. Hence, noticing $(\rho, q) = w^{-1} \circ (P, Q)$ and $w : \Omega \rightarrow w(\Omega)$ is a global diffeomorphism, $\rho(\mathbb{R} \times [0, T^*)) \times q(\mathbb{R} \times [0, T^*))$ is also compact. This implies that

$$\sup_{t \in [0, T^*)} \left(\|\rho(t)\|_{L^\infty(\mathbb{R})} + \|q(t)\|_{L^\infty(\mathbb{R})} \right) < \infty.$$

From

$$\partial_t c = -\mu c \rho,$$

we have

$$\sup_{t \in [0, T^*)} \|c(t)\|_{L^\infty(\mathbb{R})} < \infty.$$

Moreover, recalling $q = \frac{\partial_x c}{c}$, we obtain

$$\sup_{t \in [0, T^*)} \|\partial_x c(t)\|_{L^\infty(\mathbb{R})} < \infty.$$

This completes the proof of Theorem 2. \square

Chapter 4

Finite-time blow-up in \mathbb{R}^d

In this section, we prove Theorem 3 and Corollary 3.1. We firstly derive the finite propagation speed of (H-KS).

Lemma 4.1 (Finite propagation speed). *Let (ρ_i, c_i) ($i = 1, 2$) be two solutions of (H-KS) corresponding to initial data $(\rho_{i,0}, c_{i,0})$ satisfying (1.1) such that*

$$A := \sup_{t \in [0, T^*)} \left(\|\rho_2(t)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \log c_1(t)\|_{L^\infty(\mathbb{R}^d)} \right) < \infty$$

and

$$\|\rho_i(t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c_i(t)\|_{W^{2,\infty}(\mathbb{R}^d)} < \infty \quad (i = 1, 2)$$

for $t \in [0, T^*)$. Suppose that $(\rho_{1,0}, c_{1,0}) = (\rho_{2,0}, c_{2,0})$ in a ball $B_{A,T^*}(\bar{x})$, where

$$B_{A,T^*}(\bar{x}) := \{x : |x - \bar{x}| \leq 6AdT^*\}.$$

Then $(\rho_1, c_1) = (\rho_2, c_2)$ within a cone $K_{A,T^*}(\bar{x})$, where

$$K_{A,T^*}(\bar{x}) := \{(x, t) : 0 \leq t \leq T^*, |x - \bar{x}| \leq 6Ad(T^* - t)\}.$$

Proof. Note that Remark 2.1 and Remark 2.2 imply our assumptions are sufficient to guarantee that the solution is smooth, unique, and has positive lower bounds on $[0, T^*)$. We denote

$$q_i = -\frac{\nabla c_i}{c_i} \quad (i = 1, 2), \quad \tilde{\rho} := \rho_1 - \rho_2, \quad \tilde{c} := c_1 - c_2, \quad \text{and} \quad \tilde{q} := q_1 - q_2.$$

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Define a local energy

$$E(t) := \frac{1}{2} \left(\|\tilde{\rho}(t)\|_{L^2(B_{A,T^*-t}(\bar{x}))}^2 + \|\sqrt{\rho_2}\tilde{q}(t)\|_{L^2(B_{A,T^*-t}(\bar{x}))}^2 \right),$$

where

$$B_{A,T^*-t}(\bar{x}) := \{x : |x - \bar{x}| \leq 6Ad(T^* - t)\}.$$

Recalling (2.1) and (2.2), we compute

$$\frac{d}{dt}E(t) = \text{I} + \text{II} + \text{III} + \text{IV} - 3Ad \left(\int_{\partial B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 + \int_{\partial B_{A,T^*-t}(\bar{x})} \rho_2 |\tilde{q}|^2 \right)$$

with

$$\begin{aligned} \text{I} &= \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho} \nabla \tilde{\rho} \cdot q_1 + \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 \nabla \cdot q_1, \\ \text{II} &= \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho} \nabla \rho_2 \cdot \tilde{q} + \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho} \rho_2 \nabla \cdot \tilde{q}, \\ \text{III} &= \frac{1}{2} \int_{B_{A,T^*-t}(\bar{x})} |\tilde{q}|^2 \nabla \cdot (\rho_2 q_2), \end{aligned}$$

and

$$\text{IV} = \int_{B_{A,T^*-t}(\bar{x})} \rho_2 \tilde{q} \cdot \nabla \tilde{\rho}.$$

Using the integration by parts on the first term of I, we have

$$\begin{aligned} \text{I} &= \frac{1}{2} \int_{B_{A,T^*-t}(\bar{x})} \nabla (\tilde{\rho}^2) \cdot q_1 + \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 \nabla \cdot q_1 \\ &= -\frac{1}{2} \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 \nabla \cdot q_1 + \frac{1}{2} \int_{\partial B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 \left(\sum_{k=1}^d (q_1)_k \right) + \int_{B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 \nabla \cdot q_1 \\ &\leq \frac{Ad}{2} \int_{\partial B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 + \|q_1\|_{W^{1,\infty}} \|\tilde{\rho}\|_{L^2(B_{A,T^*-t}(\bar{x}))}^2, \end{aligned}$$

where in the second line, we used the notation $q_1 = ((q_1)_1, \dots, (q_1)_d)$. After

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integrating by parts, we employ the Cauchy inequality to obtain

$$\begin{aligned} \text{II} + \text{IV} &= \int_{\partial B_{A,T^*-t}(\bar{x})} \rho_2 \left(\sum_{i=1}^d (\tilde{q})_i \right) \tilde{\rho} \\ &\leq \int_{\partial B_{A,T^*-t}(\bar{x})} \rho_2 \tilde{\rho}^2 + \rho_2 \left(\sum_{k=1}^d (\tilde{q})_k \right)^2 \\ &\leq 2Ad \int_{\partial B_{A,T^*-t}(\bar{x})} \tilde{\rho}^2 + \rho_2 |\tilde{q}|^2. \end{aligned}$$

For III, we have

$$\text{III} \leq \|\rho_2^{-1}\|_{L^\infty} \|\sqrt{\rho_2} \tilde{q}\|_{L^2(B_{A,T^*-t}(\bar{x}))}^2 \|\rho_2\|_{W^{1,\infty}} \|q_2\|_{W^{1,\infty}}.$$

Combining all, we arrive at

$$\frac{d}{dt} E(t) \leq (\|q_1\|_{W^{1,\infty}} + \|\rho_2^{-1}\|_{L^\infty} \|\rho_2\|_{W^{1,\infty}} \|q_2\|_{W^{1,\infty}}) E(t)$$

for $t \in [0, T^*)$. But since $E(0) = 0$, $E(t) = 0$ for $t \in [0, T^*)$. To show $\tilde{c} = 0$ in $K_{A,T^*}(\bar{x})$, we consider the equation of \tilde{c} :

$$\partial_t \tilde{c} = -\mu (\tilde{c} \rho_1 + c_2 \tilde{\rho}) = -\mu \tilde{c} \rho_1,$$

where the second equality follows from $\tilde{\rho} = 0$ in $K_{A,T^*}(\bar{x})$. With the same argument as above, we can show that $\tilde{c}(t) = 0$ for $t \in [0, T^*)$. \square

Proof of Theorem 3. We divide our proof into \mathbb{R} and \mathbb{R}^d ($d \geq 2$) cases.

Case 1. \mathbb{R} :

We consider initial data (ρ_0^1, c_0^1) defined by

$$\rho_0^1 := \bar{\rho} + \psi \quad \text{and} \quad c_0^1 := \bar{c} + \psi, \tag{4.1}$$

where $\bar{\rho}, \bar{c} > 0$ are constants to be determined later, and $\psi : \mathbb{R} \rightarrow [0, 1]$ is a smooth bump function such that

$$\psi(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2). \end{cases}$$

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Then as we have seen in Remark 1.1, (ρ_0, c_0) satisfies all of the conditions given in Theorem 2, so that the corresponding solution (ρ^1, c^1) blows up at some time T^* . The proof consists of two steps. In Step 1, we prove that the blow-up actually occurs on a bounded interval in \mathbb{R} . Then in Step 2, we rescale the solution to make it blow up within any given interval.

Step 1. To get information for the region where the blow-up occurs, we make use of Lemma 4.1. Noticing $(\rho_0^1, c_0^1) = (\bar{\rho}, \bar{c})$ on $\mathbb{R} \setminus (-2, 2)$ and the corresponding solution to $(\bar{\rho}, \bar{c})$ is $(\bar{\rho}, \bar{c}e^{-\mu\bar{\rho}t})$, Lemma 4.1 implies $(\rho^1, c^1) = (\bar{\rho}, \bar{c}e^{-\mu\bar{\rho}t})$ in

$$\{(x, t) \in (\mathbb{R} \setminus (-2, 2)) \times [0, T^*] : x \leq -2 - 6A^1t \text{ or } x \geq 2 + 6A^1t\},$$

where

$$A^1 = \bar{\rho} + \sup_{t \in [0, T^*]} \|\nabla \log c^1(t)\|_{L^\infty(\mathbb{R})}. \quad (4.2)$$

(Note that $\sup_{t \in [0, T^*]} \|\nabla \log c^1(t)\|_{L^\infty(\mathbb{R})}$ is clearly bounded due to (1.5) in Theorem 2.) Hence, we can conclude that our blow-up of (ρ^1, c^1) occurs on $(-2 - 6A^1T^*, 2 + 6A^1T^*) \times \{t = T^*\}$.

Remark 4.1. *In this Step 1, we have not given any restriction to $\bar{\rho}$ and \bar{c} . In other words, for any constants $\bar{\rho}, \bar{c} > 0$, we can show the blow-up region is bounded. This fact will be used in the proof of Corollary 3.1.*

Step 2. It suffices to consider the case when the center of the interval is the origin since (H-KS) is translation invariant. Let $(-r, r)$ be any interval. Our aim is to rescale (ρ^1, c^1) so that the blowup occurs within $(-r, r)$. Since (ρ^1, c^1) is a solution of (H-KS), we can check that (ρ_a^1, c_a^1) is also a solution of (H-KS), where

$$\rho_a^1(x, t) := a^2 \rho^1(ax, a^2t), \quad c_a^1(x, t) := c^1(ax, a^2t) \quad (a > 0).$$

Then since

$$\rho_a^1(x, 0) = a^2(\bar{\rho} + \psi(ax)), \quad c_a^1(x, 0) = \bar{c} + \psi(ax),$$

and $(\rho_a^1(x, 0), c_a^1(x, 0)) = (a^2\bar{\rho}, \bar{c})$ on $\mathbb{R} \setminus (-\frac{2}{a}, \frac{2}{a})$, Lemma 4.1 implies $(\rho_a^1, c_a^1) = (a^2\bar{\rho}, \bar{c}e^{-\mu a^2\bar{\rho}t})$ in

$$\left\{ (x, t) \in \left(\mathbb{R} \setminus \left(-\frac{2}{a}, \frac{2}{a} \right) \right) \times \left[0, \frac{T^*}{a^2} \right] : x \leq -\frac{2}{a} - 6A_a^1t \text{ or } x \geq \frac{2}{a} + 6A_a^1t \right\},$$

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where

$$A_a^1 := a^2 \bar{\rho} + \sup_{t \in [0, \frac{T^*}{a^2})} \|\nabla \log c_a^1(t)\|_{L^\infty(\mathbb{R})} = a^2 \bar{\rho} + a \sup_{t \in [0, T^*)} \|\nabla \log c^1(t)\|_{L^\infty(\mathbb{R})}.$$

Hence, (ρ_a^1, c_a^1) blows up on $\left(-\frac{2}{a} - \frac{6A_a^1 T^*}{a^2}, \frac{2}{a} + \frac{6A_a^1 T^*}{a^2}\right) \times \{t = \frac{T^*}{a^2}\}$. Note that

$$\frac{2}{a} + \frac{6A_a^1 T^*}{a^2} = \frac{2}{a} + \bar{\rho} + \frac{6 \sup_{t \in [0, T^*)} \|\nabla \log c^1(t)\|_{L^\infty(\mathbb{R})} T^*}{a}.$$

Thus, if we choose sufficiently small $\bar{\rho}$ and large a , we can make the blow-up occur within $(-r, r) \in \mathbb{R}$.

Case 2. \mathbb{R}^d ($d \geq 2$) :

We shall use the same notation as the above Case 1. We choose initial data (ρ_0, c_0) defined by

$$\rho_0(x) := \psi(x_1) \prod_{k=2}^d \psi(\delta x_k) + \bar{\rho}, \quad c_0(x) := \psi(x_1) \prod_{k=2}^d \psi(\delta x_k) + \bar{c}, \quad (4.3)$$

where $\bar{\rho}, \bar{c} > 0$ are any constants and $\delta > 0$ is a small constant to be determined later. Note that (ρ_0, c_0) satisfies (1.1) and $(\nabla \rho_0, \nabla c_0) \in (H^\infty \times H^\infty)(\mathbb{R}^d)$. We aim to show that the corresponding solution (ρ, c) undergoes

$$\lim_{t \rightarrow T^*} \left(\|\rho(t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c(t)\|_{W^{2,\infty}(\mathbb{R}^d)} \right) = \infty$$

for some $T^* > 0$. Suppose, toward the contradiction, that (ρ, c) satisfies

$$\|\rho(t)\|_{W^{1,\infty}(\mathbb{R}^d)} + \|c(t)\|_{W^{2,\infty}(\mathbb{R}^d)} < \infty \quad \text{on } [0, T] \quad (4.4)$$

for any $T > 0$. Recall that (4.4) is sufficient to guarantee that (ρ, c) has the positive lower bounds and is unique and smooth on $[0, T]$ by Remark 2.1 and Remark 2.2.

On the other hand, we consider another initial data $(\rho_0^*, c_0^*) : \mathbb{R}^d \rightarrow \Omega$ defined by

$$\rho_0^*(x) := \rho_0^1(x_1), \quad c_0^*(x) := c_0^1(x_1).$$

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Recalling (4.1), we can notice $(\rho_0, c_0) = (\rho_0^*, c_0^*)$ in $\{x : |x_k| \leq \delta^{-1}, k = 2, \dots, d\}$. Moreover, we observe that the unique solution $(\rho^*, c^*) : \mathbb{R}^d \rightarrow \Omega$ corresponding to (ρ_0^*, c_0^*) is

$$(\rho^*(x, t), c^*(x, t)) = (\rho^1(x_1, t), c^1(x_1, t)).$$

Indeed, we can check that (ρ^*, c^*) solves (H-KS) in \mathbb{R}^d and the uniqueness is ensured by Remark 2.2. Hence, recalling Step 1 in the proof of Case 1, (ρ^*, c^*) blows up in $\{|x| \leq 2 + 6A^1T^*\} \times \{t = T^*\}$. Our strategy is to compare (ρ, c) with (ρ^*, c^*) by employing Lemma 4.1. Define A by

$$A := \sup_{t \in [0, T^*)} \left(\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \log c^*(t)\|_{L^\infty(\mathbb{R}^d)} \right).$$

Note that $\sup_{t \in [0, T^*)} \|\rho(t)\|_{L^\infty(\mathbb{R}^d)} < \infty$ by our assumption (4.4), and $\sup_{t \in [0, T^*)} \|\nabla \log c^*(t)\|_{L^\infty(\mathbb{R}^d)} < \infty$ by (1.5). Then Lemma 4.1 asserts that if $(\rho_0, c_0) = (\rho_0^*, c_0^*)$ in $\{x : |x| \leq 2 + 6A^1T^* + 6AdT^*\}$, then $(\rho, c) = (\rho^*, c^*)$ within $\{(x, t) : 0 \leq t \leq T^*, |x| \leq 2 + 6A^1T^* + 6Ad(T^* - t)\}$, where A^1 is the constant defined in (4.2).

Since $(\rho_0, c_0) = (\rho_0^*, c_0^*)$ in $\{x : |x_k| \leq \delta^{-1}, k = 2, \dots, d\}$, if we choose sufficiently small δ satisfying

$$2 + 6A^1T^* + 6AdT^* \leq \delta^{-1},$$

Lemma 4.1 guarantees $(\rho, c) = (\rho^*, c^*)$ within $\{(x, t) : 0 \leq t \leq T^*, |x| \leq 2 + 6A^1T^* + 6Ad(T^* - t)\}$. In particular, $(\rho, c) = (\rho^*, c^*)$ in $\{|x| \leq 2 + 6A^1T^*\} \times \{t = T^*\}$. But since (ρ^*, c^*) blows up in $\{|x| \leq 2 + 6A^1T^*\} \times \{t = T^*\}$, this is a contradiction to (4.4). This completes the proof of Theorem 3. \square

Proof of Corollary 3.1. We construct the desired initial data by slightly changing (4.3): we redefine (ρ_0, c_0) as follows:

$$\rho_0(x) := \delta^N \psi(x_1) \prod_{k=2}^d \psi(\delta x_k) + \bar{\rho}, \quad c_0(x) := \delta^N \psi(x_1) \prod_{k=2}^d \psi(\delta x_k) + \bar{c}.$$

for large N . Here $\bar{\rho}, \bar{c} > 0$ are any constants. Recalling our proof of the above Case 2, we needed the smallness of δ . Thus we can control δ, N so that our

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new (ρ_0, c_0) satisfies (1.8), and then proceed as we did in the above proof of Case 2 to obtain the finite blow-up result. We omit the details. \square

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국문초록

이 논문은 \mathbb{R}^d ($d \geq 1$)에서 다음과 같은 로그 민감도를 갖는 쌍곡 Keller-Segel 소비 유형 시스템에 대해 다룬다.

$$\partial_t \rho = -\chi \nabla \cdot (\rho \nabla \log c), \quad \partial_t c = -\mu c \rho \quad (\chi, \mu > 0)$$

우리는 먼저 영이 아닌 매끄러운 초기 데이터에 해당하는 매끄러운 해의 국소적 존재성을 보인다. 다음으로 리만 불변량을 통해 $d = 1$ 일 때 유한 시간 특이점 형성을 위한 초기 데이터의 몇 가지 조건을 제시한다. 그런 다음 모든 $d \geq 1$ 에 대해 일부 영이 아닌 C^∞ 데이터가 유한한 시간 내에 폭발할 수 있음을 증명한다. 따름정리로, 우리는 모든 $d \geq 1$ 에 대해 유한한 시간 내에 폭발하는 모든 상수 평형 상태 근처에 초기 데이터를 건설한다.

주요어휘: Keller-Segel, 소비 유형, 유한시간 폭발, 리만 불변량
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