이학박사 학위논문

Homological Mirror Symmetry and Geometry of Degenerate Cusp Singularities

(퇴화첨단특이점의 호몰로지거울대칭과 기하학)

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이 논문을 이학박사 학위논문으로 제출함

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Abstract

Homological Mirror Symmetry and Geometry of Degenerate Cusp Singularities

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Under homological mirror symmetry, we establish an explicit correspondence of Lagrangians in pair-of-pants surface (A-model) and Cohen-Macaulay modules over the degenerate cusp singularity defined by xyz=0 (B-model).

In A-model, we show that Cho-Hong-Lau's localized mirror functor is stable under homotopy of Lagrangians. It motivates us to introduce loop/arc data to parameterize Lagrangians equipped with a holonomy. Their mirror images under the mirror functor provide the canonical form of matrix factorizations of xyz.

In B-model, we review Burban-Drozd's representation of Cohen-Macaulay modules, whose indecomposable isomorphism classes are classified and parameterized by band/string data. We introduce a combinatorial method to compute their corresponding matrix factorizations of xyz.

It turns out that there is a conversion formula between loop/arc data and band/string data, which translates Burban-Drozd's classification result of Cohen-Macaulay modules into the language of matrix factorizations, presenting their explicit canonical form. It realizes a one-to-one correspondence between indecomposable isomorphism classes of Lagrangians and Cohen-Macaulay modules. As a consequence, we find their relation with geodesics in hyperbolic pair-of-pants.

For a geometric understanding of Cohen-Macaulay modules over degenerate cusp singularities, we develop a new geometric notion called degenerate vector bundles over those singularities. It naturally induces Cohen-Macaulay modules from Burban-Drozd's representation, providing the concrete geometry underlying our computation.

keywords : homological mirror symmetry, degenerate cusp singularity, Lagrangian submanifold, matrix factorization, Cohen-Macaulay module, degenerate vector bundle
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국문초록

이 학위논문에서는 호몰로지 거울대칭 하에서 바지 곡면(pair-of-pants surface)의 Lagrangian(A-모델)과, xyz=0으로 정의되는 퇴화첨단특이점(degenerate cusp singularity) 위의 Cohen-Macaulay 모듈(B-모델)의 대응 관계를 밝힌다.

먼저 A-모델에서 Cho-Hong-Lau의 지역화된 거울함자(localized mirror functor)가 Lagrangian의 호모토피 하에서 안정적(stable)이라는 것을 보인다. 이로부터 루프/아크 데이터를 정의하여 홀로노미가 부여된 Lagrangian들을 매개화한다. 지역화된 거울함자 하에서 이들의 거울대칭 상은 xyz의 행렬 인수분해의 표준형을 제공한다.

한편 B-모델에서는 Cohen-Macaulay 모듈의 표현에 대한 Burban-Drozd의 이론을 복습한다. 여기서 이들의 분해불가능한 동형 클래스들은 밴드/스트링 데이터로 분류 및 매개화된다. 우리는 조합론적인 방법론을 도입하여 이들에 대응하는 xyz의 행렬 인수분해를 계산한다.

루프/아크 데이터와 밴드/스트링 사이에는 변환 공식이 있음이 밝혀지는데, 이는 Cohen-Macaulay 모듈에 대한 Burban-Drozd의 분류 결과를 행렬 인수분해의 언어로 바꾸며 명시적인 표준형을 제시한다. 이것은 또한 Lagrangian과 Cohen-Macaulay 모듈의 분해불가능한 동형 클래스들 간의 일대일 대응을 구체화한다. 결과적으로 이들이 하이퍼볼릭 바지 곡면의 측지선들과 연관되어있음을 보인다.

퇴화첨단특이점 위의 Cohen-Macaulay 모듈의 기하학적인 이해를 위해, 특이점 위의 퇴화벡터다발(degenerate vector bundle)이라는 새로운 기하학적 개념을 도입한다. 이것은 Burban-Drozd의 표현으로부터 Cohen-Macaulay 모듈을 자연스럽게 유도하며, 우리의 계산의 기저에 있는 기하학을 제시한다.

주요어 : 호몰로지 거울대칭, 퇴화첨단특이점, 라그랑지안 부분다양체, 행렬 인수분해, 코헨-맥컬리 모듈, 퇴화벡터다발
학 번 : 2014-21204

Contents

Chapter 1

Introduction

Homological mirror symmetry (HMS) conjecture comes from Kontsevich's original formulation

$$D^{\pi}\left(\mathrm{Fuk}\left(X\right)\right) \simeq D^{b}\,\mathrm{Coh}\left(\check{X}\right),$$

saying that the derived Fukaya category of some compact symplectic manifold X (A-model) and the bounded derived category of coherent sheaves on a mirror dual compact complex variety $\check{X}$ (B-model) are equivalent. Allowing X to be non-compact and $\check{X}$ to be singular, it has been generalized into the equivalence

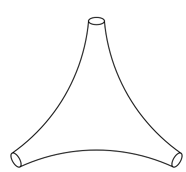
$$D^{\pi}\left(W\,\mathrm{Fuk}\left(X\right)\right) \simeq D_{\mathrm{sing}}\left(W^{-1}\left(0\right)\right)$$

of the derived wrapped Fukaya category of the symplectic manifold X (A-model) and the singularity category of its mirror Landau-Ginzburg (LG) model $\left(Y, W : Y \to \Bbbk\right)$ (B-model).

HMS between punctured Riemann spheres and the corresponding LG models was first established in [AAE+13], where the authors constructed a mirror LG model $\left(Y\left(n\right), W : Y\left(n\right) \to \Bbbk\right)$ corresponding to the n-punctured sphere $X = S^2 \setminus \left\{n \text{ points}\right\}$ for each $n \geq 3$ and proved the above equivalence of categories.

HMS of 3-punctured sphere and singularity $xyz = 0$**.** In this thesis, we focus on HMS between the 3-punctured sphere and the corresponding LG model $\left(\Bbbk^3, xyz\right)$.

A-model



$\mathscr{P} = S^2 \setminus \left\{3 \text{ points}\right\}$

pair-of-pants surface

symplectic geometry

$D^{\pi}\left(W\,\mathrm{Fuk}(\mathscr{P})\right)$

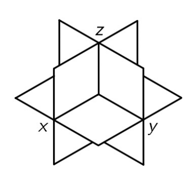
derived wrapped Fukaya category

mirror pair

$\simeq$

(AAEKO '13)

B-model



$X = \left\{xyz = 0\right\} \subset \Bbbk^3$

normal crossing singularity

algebraic geometry

$D_{\mathrm{sing}}\left(X\right)$

category of singularities

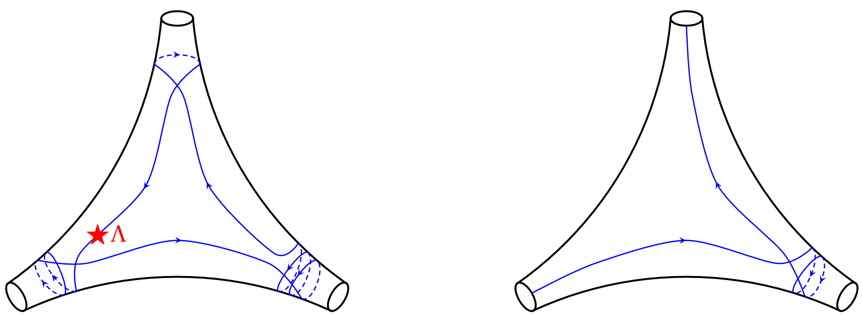
In A-model, we consider a 3-punctured sphere $\mathscr{P} := S^2 \setminus \{3 \text{ points}\}$. Together with any area form ω, it becomes a symplectic manifold called a **pair-of-pants surface**. It gives rise to an invariant $D^\pi(\mathcal{W}\operatorname{Fuk}(\mathscr{P}))$, the **derived wrapped Fukaya category**.

In B-model, we consider the algebraic variety X in $\Bbbk^3$ defined by a single equation $xyz = 0$. More precisely, we define $X := \operatorname{Spec}(A)$ as the spectrum of a complete local ring $A := \Bbbk[[x,y,z]]/(xyz)$. It has a non-isolated singularity at the origin called a **normal crossing singularity**. It gives rise to an invariant $D_{\text{sing}}(X)$, the **category of singularities**.

Two categories seem very different at the first glance, but are proven to be equivalent in [AAE+13]. The first part of this thesis is dedicated to investigate the classification and explicit correspondence of indecomposable objects in both categories.

Objects in A-model. We consider objects in the Fukaya category $D^\pi(\mathcal{W}\operatorname{Fuk}(\mathscr{P}))$ given by

- closed immersed oriented loops $L : S^1 \to \mathscr{P}$ equipped with a rank μ local system (E, ∇), and
- open immersed oriented arcs $L : \mathbb{R} \to \mathscr{P}$ which start and end at punctures.



(a) A closed loop with holonomy $\Lambda \in \mathrm{GL}_\mu(\Bbbk)$ (b) An open arc

Figure 1.1: Objects in $D^\pi(\mathcal{W}\operatorname{Fuk}(\mathscr{P}))$

A rank μ local system on the loop L is given by a $\Bbbk$-vector bundle E of rank μ along L together with a (flat) connection ∇ on it. It turns out that its total holonomy $\Lambda \in \mathrm{GL}_\mu(\Bbbk)$ up to basis change is the only invariant under the quasi-isomorphism in the Fukaya category. We mark it on some point of the loop as a star mark(★). From now on, a loop with local system (L, E, ∇) will be simply shortened to a loop L with (total) holonomy Λ.

Objects in B-model. The category of singularities is defined as a quotient

$$D_{\text{sing}}(X) := D^b\operatorname{Coh}(X)/\operatorname{Perf}(X)$$

of the bounded derived category of coherent sheaves on X by its full subcategory of perfect complexes. This was introduced and studied in [Buc21, Orl09]. In the current paper, however, we will work with another well-known equivalent categories:

$$\underline{\mathrm{MF}}(xyz) \overset{\text{Eisenbud}}{\overset{\text{coker}}{\xrightarrow{\simeq}}} \underline{\mathrm{CM}}(A) \overset{\text{Buchweitz}}{\xrightarrow{\simeq}} D_{\text{sing}}(X)$$

Here, $\underline{\mathrm{MF}}(xyz)$ is the stable category of **matrix factorizations** of xyz, and $\underline{\mathrm{CM}}(A)$ is the stable category of (maximal) **Cohen-Macaulay modules** over A. Hereafter we will omit the word *maximal* because it doesn't change the definition in this case.

The objects in $\underline{\mathrm{MF}}(xyz)$ are especially easy to describe in a concrete way, as shown in the following example:

$$\underbrace{\begin{pmatrix} z & 0 & 0 \\ -y^2 & x & -z \\ -\lambda x^2 & 0 & y \end{pmatrix}}_{\varphi}\underbrace{\begin{pmatrix} xy & 0 & 0 \\ y^3+\lambda z x^2 & yz & z^2 \\ \lambda x^3 & 0 & zx \end{pmatrix}}_{\psi} = xyz \cdot I_3.$$

The above pair (φ, ψ) of 3×3 matrices form a matrix factorization of xyz for any $\lambda \in \Bbbk$, providing a one-parameter family of objects in $\underline{\mathrm{MF}}(xyz)$. In general, a pair (φ,ψ) of $n \times n$ matrices is called a *matrix factorization* of xyz if it satisfies $\varphi\psi = \psi\varphi = xyzI_n$. Because either φ or ψ is automatically determined by the other factor, we often write only the first factor φ to indicate the whole pair (φ,ψ).

The matrix φ induces a map $\underline{\varphi} : A^n \to A^n$. Taking cokernel of $\underline{\varphi}$ yields a Cohen-Macaulay module $\operatorname{coker}\underline{\varphi}$ over A. This induces a functor from $\underline{\mathrm{MF}}(xyz)$ to $\underline{\mathrm{CM}}(A)$, which is proven to be an equivalence by Eisenbud.

Localized mirror functor. Putting the above discussion together, we know that all categories $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$, $D_{\mathrm{sing}}(X)$, $\underline{\mathrm{CM}}(A)$, and $\underline{\mathrm{MF}}(xyz)$ are equivalent. But it is still not easy to draw an explicit correspondence between objects in the Fukaya category and objects in the other categories.

In this situation, Cho-Hong-Lau's *localized mirror functor* [CHL17]

$$\mathscr{F}^{\mathbb{L}} : D^\pi(W\,\mathrm{Fuk}(\mathscr{P})) \xrightarrow{\simeq} \underline{\mathrm{MF}}(xyz)$$

becomes a powerful tool to convert objects in the Fukaya category into matrix factorizations of xyz. It uses the Seidel Lagrangian $\mathbb{L}$ as a reference, which is a weakly unobstructed object in the Fukaya category with potential xyz.

Given any loop or arc L in $\mathscr{P}$, we can compute the corresponding matrix factorization $\mathscr{F}^{\mathbb{L}}(L) = (\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L))$ using Lagrangian intersection Floer theory developed in [FOOO09]. More specifically, the size of matrices is determined by the number of intersections of L and $\mathbb{L}$, and the entries are given by counting immersed polygons bounded by L and $\mathbb{L}$. In Section 2.1.1, we will explain this in more details with examples.

Homotopy invariance. We prove that the localized mirror functor $\mathscr{F}^{\mathbb{L}}$ is stably invariant under free homotopy of immersed loops and conjugation of total holonomies. Similarly, $\mathscr{F}^{\mathbb{L}}$ is also invariant under homotopy of immersed arcs, keeping the end points in the same boundaries.

Proposition 1.0.1. *(1) If two loops L and L' in $\mathscr{P}$ are freely homotopic to each other and have the same total holonomies Λ and Λ' up to basis change, then their mirror matrix factorizations $\mathscr{F}^{\mathbb{L}}(L)$ and $\mathscr{F}^{\mathbb{L}}(L')$ are isomorphic in the stable category $\underline{\mathrm{MF}}(xyz)$.*

(2) If two arcs L and L' in $\mathscr{P}$ are homotopic to each other under a homotopy keeping the end points in the same boundaries, then their mirrors $\mathscr{F}^{\mathbb{L}}(L)$ and $\mathscr{F}^{\mathbb{L}}(L')$ are isomorphic in $\underline{\mathrm{MF}}(xyz)$.

Loop/Arc word. We say that a loop L is *non-essential* if it winds around only one of three holes. A loop/arc which is not freely homotopic to any non-essential loop is called *essential*. A non-essential loop L can always be deformed so that it doesn't meet the reference $\mathbb{L}$ at all, which means that $\mathscr{F}^{\mathbb{L}}(L)$ is stably trivial. Therefore, non-essential loops will be excluded from our consideration.

The homotopy invariance of $\mathscr{F}^{\mathbb{L}}$ suggests that it is enough for us to consider only one loop/arc in each essential (free) homotopy class.

For the loop case, we take the following specific representatives: Given a **loop word**

$$w' = \left(l'_1, m'_1, n'_1, l'_2, m'_2, n'_2, \dots, l'_\tau, m'_\tau, n'_\tau\right) \in \mathbb{Z}^{3\tau}$$

($\tau \in \mathbb{Z}_{\geq 1}$), consider the loop described in Figure 1.2. It visits 3 holes A, B, and C in turn, winding them around the number of times specified in w'. Namely, it winds hole A l'_1-times, hole B m'_1-times, hole C n'_1-times, hole A l'_2-times, hole B m'_2-times, and so on. After finally it winds hole C n'_τ-times, it returns to the starting point to form a closed loop. The loop thus constructed is denoted as $L(w')$, and its free homotopy class is denoted as $\left[L(w')\right]$.

In the case of an arc, we should decide which hole to starts from and which hole to ends at. So there are $3 \times 3 = 9$ cases. For example, we use an **arc word**

$$w' = \left(\mathrm{A}, m'_1, n'_1, l'_2, m'_2, n'_2, \dots, l'_\tau, \mathrm{B}\right) \in \{\mathrm{A}\} \times \mathbb{Z}^{3\tau-3} \times \{\mathrm{B}\}$$

($\tau \in \mathbb{Z}_{\geq 1}$) to denote an arc which starts from hole A, winds hole B m'_1-times, winds hole C n'_1-times, and so on, and finally ends at hole B. Similarly, there are 9 types of arc words, each belonging to

$$\begin{array}{lll} \{A\} \times \mathbb{Z}^{3\tau-1} \times \{A\}, & \{A\} \times \mathbb{Z}^{3\tau-3} \times \{B\}, & \{A\} \times \mathbb{Z}^{3\tau-2} \times \{C\}, \\ \{B\} \times \mathbb{Z}^{3\tau-2} \times \{A\}, & \{B\} \times \mathbb{Z}^{3\tau-1} \times \{B\}, & \{B\} \times \mathbb{Z}^{3\tau-3} \times \{C\}, \\ \{C\} \times \mathbb{Z}^{3\tau-3} \times \{A\}, & \{C\} \times \mathbb{Z}^{3\tau-2} \times \{B\}, & \{C\} \times \mathbb{Z}^{3\tau-1} \times \{C\} \end{array}$$

for some $\tau \in \mathbb{Z}_{\geq 1}$. The arc constructed according to the arc word w' is denoted as $L(w')$, and its homotopy class is denoted as $\left[L(w')\right]$.

Example 1.0.2. *The loop drawn in Figure 1.1a is* $L((3, -2, 2))$*, and the arc in Figure 1.1b is* $L((A, 3, C))$.

Loops/arcs constructed in this way exhaust all (free) homotopy classes, but some of them may represent the same class. For example, if a loop word w' is a *shift* of another loop word w'_*, they give the same free homotopy class $\left[L(w')\right] = \left[L(w'_*)\right]$. This can also happen even when w' is not a shift of w'_*. We define the *normality condition* of loop/arc words so that *normal* words (up to shifting) produce only one loop/arc in each essential (free) homotopy class. Especially, they automatically exclude the non-essential loop classes.

Normal loop/arc words and geodesics. Now we give $\mathscr{P}$ a hyperbolic metric with three cusps. From an elementary fact in hyperbolic geometry, we know that there is at most one geodesic in each (free) homotopy class of loops/arcs in $\mathscr{P}$. To be specific, there is no geodesic in non-essential class, and there is exactly one for the other classes. So we have another nice description of normal loop/arc words.

Proposition 1.0.3. *We give $\mathscr{P}$ a hyperbolic metric with three cusps. Then there are one-to-one correspondences*

$$\begin{aligned} \{\textit{normal loop words}\}/\sim_{\textit{shifting}} &\overset{1:1}{\leftrightarrow} \{\textit{closed geodesics in } \mathscr{P}\} \\ \& \ \{\textit{normal arc words}\} &\overset{1:1}{\leftrightarrow} \{\textit{open geodesics in } \mathscr{P}\}. \end{aligned}$$

Loop/Arc data. As total holonomies on a loop $L(w')$, we also consider only one in each conjugacy class, namely, the $\mu \times \mu$ Jordan block $J_\mu(\lambda') \in \mathrm{GL}_\mu(\Bbbk)$ with eigenvalue $\lambda' \in \Bbbk^\times$. We denote by $L(w', \lambda', \mu)$ the loop $L(w')$ equipped with a rank μ local system whose total holonomy is $J_\mu(\lambda')$.

Definition 1.0.4. *(1) A* ***loop datum*** (w', λ', μ) *consists of a* ***normal loop word*** $w' \in \mathbb{Z}^{3\tau}$ *($\tau \in \mathbb{Z}_{\geq 1}$), the* ***holonomy parameter*** $\lambda' \in \Bbbk^\times$*, and the* ***multiplicity*** $\mu \in \mathbb{Z}_{\geq 1}$*. We refer to the corresponding loop with holonomy*

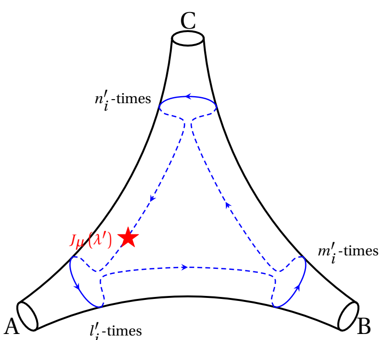
$$L(w', \lambda', \mu)$$

as the ***canonical form of*** **loop-type** ***objects*** *in $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$. Note that they consist countably many one-parameter families.*

(2) An ***arc datum*** w' *consists of only a* ***normal arc word*** $w' \in \{A, B, C\} \times \mathbb{Z}^{3\tau - *} \times \{A, B, C\}$*. We refer to the corresponding arc*

$$L(w')$$

as the ***canonical form of*** **arc-type** ***objects*** *in $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$. There are only countably many of them.*



$L\left(w' = (l'_i, m'_i, n'_i)_{i=1}^{\tau}, \lambda', \mu\right)$ $\xrightarrow[\cong]{\Phi^{\mathbb{L}}}$

$$\begin{pmatrix} zI_\mu & -y^{m'_1-1}I_\mu & 0 & 0 & \cdots & 0 & -x^{-l'_1}J_\mu(\lambda)^{-1} \\ -y^{-m'_1}I_\mu & xI_\mu & -z^{n'_1-1}I_\mu & 0 & \cdots & 0 & 0 \\ 0 & -z^{-n'_1}I_\mu & yI_\mu & -x^{l'_2-1}I_\mu & \cdots & 0 & 0 \\ 0 & 0 & -x^{-l'_2}I_\mu & zI_\mu & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -y^{m'_\tau-1}I_\mu & 0 \\ 0 & 0 & 0 & \cdots & -y^{-m'_\tau}I_\mu & xI_\mu & -z^{n'_\tau-1}I_\mu \\ -x^{l'_1-1}J_\mu(\lambda) & 0 & 0 & \cdots & 0 & -z^{-n'_\tau}I_\mu & yI_\mu \end{pmatrix}_{3\tau\mu \times 3\tau\mu}$$

where x^a, y^a, z^a are regarded as 0 if $a < 0$

$$\varphi\left(w' = (l'_i, m'_i, n'_i)_{i=1}^{\tau}, \lambda, \mu\right)$$

Figure 1.2: Canonical form of loop-type objects in $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$ and $\underline{\mathrm{MF}}(xyz)$

Matrix factorizations from Lagrangians. Localized mirror functor $\mathscr{F}^{\mathbb{L}} : D^\pi(W\,\mathrm{Fuk}(\mathscr{P})) \to \underline{\mathrm{MF}}(xyz)$ converts each loop with holonomy $L(w', \lambda', \mu)$ and each arc $L(w')$ into matrix factorizations $\mathscr{F}^{\mathbb{L}}(L(w', \lambda', \mu))$ and $\mathscr{F}^{\mathbb{L}}(L(w'))$, respectively. We compute them and show that they are isomorphic to a very nice form.

For a normal loop word $w' \in \mathbb{Z}^{3\tau}$ ($\tau \in \mathbb{Z}_{\geq 1}$), a nonzero scalar $\lambda \in \Bbbk^\times$, and a positive integer $\mu \in \mathbb{Z}_{\geq 1}$, we define the corresponding matrix factorization $(\varphi(w', \lambda, \mu), \psi(w', \lambda, \mu))$, whose first component is shown in Figure 1.2. But in *degenerate* cases where $w' = (2,2,2)^{\#\tau}$ and $\lambda = 1$, the second factor $\psi(w', \lambda, \mu)$ is not defined, so we use an alternative form $(\varphi_{\deg}((2,2,2)^{\#\tau}, 1, \mu), \psi_{\deg}((2,2,2)^{\#\tau}, 1, \mu))$.

For a normal arc word w', we similarly define the corresponding matrix factorization $(\varphi(w'), \psi(w'))$.

Theorem 1.0.5. *(1) For a non-degenerate loop datum (w', λ', μ), there is an isomorphism*

$$\mathscr{F}^{\mathbb{L}}(L(w', \lambda', \mu)) \;\cong\; (\varphi(w', \lambda, \mu), \psi(w', \lambda, \mu))$$

in $\underline{\mathrm{MF}}(xyz)$, where λ is either λ' or $-\lambda'$ depending on w'.

In degenerate cases ($w' = (2,2,2)^{\#\tau}, \lambda' = -1$), we have

$$\mathscr{F}^{\mathbb{L}}\left(L\left((2,2,2)^{\#\tau}, -1, \mu\right)\right) \;\cong\; \left(\varphi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right), \psi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right)\right).$$

(2) For an arc datum w', there is an isomorphism

$$\mathscr{F}^{\mathbb{L}}\left(L\left(w'\right)\right) \;\cong\; \left(\varphi\left(w'\right), \psi\left(w'\right)\right).$$

Inspired by this observation, we propose the **canonical form of matrix factorizations of xyz** in the following way.

Definition 1.0.6. *(1) For a normal loop word $w' \in \mathbb{Z}^{3\tau}$ ($\tau \in \mathbb{Z}_{\geq 1}$), a nonzero scalar $\lambda \in \Bbbk^{\times}$, and a positive integer $\mu \in \mathbb{Z}_{\geq 1}$, we refer to the corresponding matrix factorization*

$$\left(\varphi\left(w', \lambda, \mu\right), \psi\left(w', \lambda, \mu\right)\right) \quad \textit{or} \quad \left(\varphi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right), \psi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right)\right)$$

as the ***canonical form of*** **loop-type** ***objects*** *in $\underline{\mathrm{MF}}\left(xyz\right)$. The latter is chosen only in degenerate cases ($w' = (2,2,2)^{\#\tau}$, $\lambda = 1$).*

(2) For a normal arc word $w' \in \{A, B, C\} \times \mathbb{Z}^{3\tau-*} \times \{A, B, C\}$, we refer to the corresponding matrix factorization

$$\left(\varphi\left(w'\right), \psi\left(w'\right)\right)$$

as the ***canonical form of*** **arc-type** ***objects*** *in $\underline{\mathrm{MF}}\left(xyz\right)$.*

It is in general a non-trivial problem to determine whether a given object is indecomposable or not in the Fukaya category $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$ or the stable category $\underline{\mathrm{MF}}(xyz)$.

Question 1.0.7. *When is the above canonical form indecomposable?*

Moreover, we do not know yet if they are all different and exhaust all indecomposable isomorphism classes.

Question 1.0.8. *Does the canonical form uniquely represents every isomorphism class of indecomposable objects in $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$ or $\underline{\mathrm{MF}}(xyz)$?*

We need another algebraic framework to answer the above questions.

Representation of Cohen-Macaulay modules. In a recent work [BD17] of Burban-Drozd, they developed a new representation-theoretic method to deal with Cohen-Macaulay modules over non-isolated surface singularities such as $A = \Bbbk[[x, y, z]]/(xyz)$. As a consequence, they classified all indecomposable Cohen-Macaulay modules, proving that those non-isolated surface singularities have *tame* Cohen-Macaulay representation type.

More precisely, they introduced new categories $\mathrm{Tri}(A)$ (*category of triples*) and $\mathrm{Rep}(\mathfrak{X}_A)$ (*category of elements* of a certain *bimodule* $\mathfrak{X}_A$) and constructed functors

$$\mathrm{CM}(A) \xrightarrow[\simeq]{\mathbb{F}} \mathrm{Tri}(A) \xrightarrow{\mathbb{H}} \mathrm{Rep}(\mathfrak{X}_A)$$

such that $\mathbb{F}$ is an equivalence of categories and $\mathbb{H}$ preserves indecomposability and isomorphism classes of objects. Objects in $\operatorname{Rep}(\mathfrak{X}_A)$ can be represented as

$$\begin{tikzcd}[column sep=small]
& \Bbbk((t))^{l_z} \arrow[dl, "\theta^{z}_{zx}"'] \arrow[dr, "\theta^{z}_{yz}"] & \\
\Bbbk((t))^{d_{zx}} & & \Bbbk((t))^{d_{yz}} \\
\Bbbk((t))^{l_x} \arrow[u, "\theta^{x}_{zx}"] \arrow[dr, "\theta^{x}_{xy}"'] & & \Bbbk((t))^{l_y} \arrow[u, "\theta^{y}_{yz}"'] \arrow[dl, "\theta^{y}_{xy}"] \\
& \Bbbk((t))^{d_{xy}} &
\end{tikzcd} \tag{1.0.1}$$

for six $\Bbbk((t))$-vector spaces and six $\Bbbk((t))$-linear maps between them. Consequently, the classification of indecomposable objects in $\operatorname{CM}(A)$ is equivalent to the classification in $\operatorname{Rep}(\mathfrak{X}_A)$, which reduces to a certain problem of linear algebra called the *matrix problem*.

A slight modification of morphisms in $\operatorname{Rep}(\mathfrak{X}_A)$ allows the functor $\mathbb{H}$ to be an equivalence. We denote the underlying quiver as $Q_A := \begin{tikzcd}[sep=tiny] & \bullet \arrow[dl] \arrow[dr] & \\ \circ & & \circ \\ \bullet \arrow[u] \arrow[dr] & & \bullet \arrow[u] \arrow[dl] \\ & \circ & \end{tikzcd}$ and the modified category as $\operatorname{Rep}(Q_A)$ (**category of decorated representations** on Q_A). Then we have an equivalence

$$\mathbb{F}_{\mathrm{BD}} : \operatorname{CM}(A) \xrightarrow[\simeq]{\mathbb{F}} \operatorname{Tri}(A) \xrightarrow[\simeq]{\mathbb{H}'} \operatorname{Rep}(Q_A), \tag{1.0.2}$$

as will be shown in Chapter 5. Because the natural functor $\operatorname{Rep}(Q_A) \to \operatorname{Rep}(\mathfrak{X}_A)$ preserves indecomposability and isomorphism classes of objects, the classification of indecomposable objects agrees in both categories.

Band/String data. Burban-Drozd provided a complete list of indecomposable objects in $\operatorname{Rep}(\mathfrak{X}_A)$ (hence in $\operatorname{Rep}(Q_A)$), which fall into two types: *band-type* and *string-type*. The **canonical form** $\Theta(w, \lambda, \mu)$ **of *band-type* objects** is given in Figure 1.3, which is parameterized by the **band word**

$$w = (l_1, m_1, n_1, l_2, m_2, n_2, \ldots, l_\tau, m_\tau, n_\tau) \in \mathbb{Z}^{3\tau}$$

($\tau \in \mathbb{Z}_{\geq 1}$), the **eigenvalue** $\lambda \in \Bbbk^\times$, and the **multiplicity** $\mu \in \mathbb{Z}_{\geq 1}$, which together form the **band datum** (w, λ, μ). It is *decomposable* if and only if the band word w is *periodic*, i.e., $w = \tilde{w}^{\# N}$ for some band word $\tilde{w}$ and $N \in \mathbb{Z}_{\geq 2}$. The band-type canonical form consists countably many one-parameter families.

$$
\begin{tikzcd}[column sep=small]
 & \Bbbk((t))^{\tau\mu} \arrow[dl, "{\begin{pmatrix} t^{n_1^-+1}I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{n_\tau^-+1}I_\mu \end{pmatrix}}"'] \arrow[dr, "{\begin{pmatrix} t^{n_1^++1}I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{n_\tau^++1}I_\mu \end{pmatrix}}"] & \\
\Bbbk((t))^{\tau\mu} & & \Bbbk((t))^{\tau\mu} \\
\Bbbk((t))^{\tau\mu} \arrow[u, "{\begin{pmatrix} 0 & t^{l_2^++1}I_\mu & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & t^{l_\tau^++1}I_\mu \\ t^{l_1^++1}J_\mu(\lambda) & 0 & \cdots & 0 \end{pmatrix}}"] \arrow[dr, "{\begin{pmatrix} t^{l_1^-+1}I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{l_\tau^-+1}I_\mu \end{pmatrix}}"'] & & \Bbbk((t))^{\tau\mu} \arrow[u, "{\begin{pmatrix} t^{m_1^-+1}I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{m_\tau^-+1}I_\mu \end{pmatrix}}"'] \arrow[dl, "{\begin{pmatrix} t^{m_1^++1}I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{m_\tau^++1}I_\mu \end{pmatrix}}"] \\
 & \Bbbk((t))^{\tau\mu} &
\end{tikzcd}
$$

where $a^+ := \max\{0, a\}$ and $a^- := \max\{0, -a\}$ for $a \in \mathbb{Z}$

$$\Theta\left(w = (l_i, m_i, n_i)_{i=1}^{\tau}, \lambda, \mu\right)$$

Figure 1.3: Canonical form of band-type indecomposable objects in Rep $(\mathfrak{X}_A)$

The **canonical form** $\Theta(w)$ **of *string-type* objects** is parameterized the following 9 sets:

$$
\begin{array}{lll}
\{x\} \times \mathbb{Z}^{3\tau-4} \times \{x\}, & \{x\} \times \mathbb{Z}^{3\tau-3} \times \{y\}, & \{x\} \times \mathbb{Z}^{3\tau-2} \times \{z\}, \\
\{y\} \times \mathbb{Z}^{3\tau-5} \times \{x\}, & \{y\} \times \mathbb{Z}^{3\tau-4} \times \{y\}, & \{y\} \times \mathbb{Z}^{3\tau-3} \times \{z\}, \\
\{z\} \times \mathbb{Z}^{3\tau-6} \times \{x\}, & \{z\} \times \mathbb{Z}^{3\tau-5} \times \{y\}, & \{z\} \times \mathbb{Z}^{3\tau-4} \times \{z\}.
\end{array}
$$

Each element w is called the **string word** (or the **string datum**). They are all *indecomposable*. There are countably many of them.

The corresponding **canonical form of *band-type* and *string-type* objects** in $\mathrm{CM}(A)$ are denoted as $M(w, \lambda, \mu)$ and $M(w)$, respectively. They were obtained in [BD17] by an algebraic way called the *reconstruction procedure*.

Theorem 1.0.9 (Burban-Drozd)**.** *Any indecomposable Cohen-Macaulay module over $A = \Bbbk[[x, y, z]]/(xyz)$ is isomorphic to either $M(w, \lambda, \mu)$ for some non-periodic band datum (w, λ, μ) or $M(w)$ for some string datum w. They are not isomorphic to each other, except that $M(w, \lambda, \mu) \cong M(w_*, \lambda, \mu)$ if a band word w_* is a* shift *of another band word w.*

This reveals that the singularity A has **tame** Cohen-Macaulay representation type. Namely, indecomposable isomorphism classes in $\mathrm{CM}(A)$ is neither finite nor countable, but still they consist countably many one-parameter families.

This makes A a very important object from a representational point of view.

The same method was also applied to every degenerate cusp singularities to show that they are all Cohen-Macaulay tame.

The stable category $\underline{\mathrm{CM}}(A) = \mathrm{CM}(A)/\{A\}$ shares all isomorphism classes of indecomposable objects with the original category $\mathrm{CM}(A)$, but loses just one class containing the free module A. This is because the free module A is regarded as a zero object in $\underline{\mathrm{CM}}(A)$. It is written in the canonical form as $A = M((0,0,0), 1, 1)$ in the original category $\mathrm{CM}(A)$.

Under Eisenbud's equivalence $\underline{\mathrm{MF}}(xyz) \simeq \underline{\mathrm{CM}}(A)$, Burban-Drozd obtained matrix factorizations corresponding to indecomposable Cohen-Macaulay modules $M(w,\lambda,\mu)$ in the case of *rank* 1 band data.

Question 1.0.10. *For general band/string data, what matrix factorization corresponds to the Burban-Drozd's canonical form of indecomposable Cohen-Macaulay modules?*

Then they raised a question of what their symplectic images would be in the Fukaya category $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$ under homological mirror symmetry (Remark 9.8 in [BD17]). This naturally leads to the more general next question.

Question 1.0.11. *What are symplectic images of indecomposable Cohen-Macaulay modules in the Fukaya category, and are they* geometric*?*

Question 1.0.10 is purely algebraic, but we will use homological mirror symmetry to answer it. We already have candidates for the corresponding matrix factorizations, that is, the canonical form of loop/arc-type matrix factorizations. If they turn out to be suitable, we can also answer Question 1.0.11. Recall that they came from corresponding canonical form of loops/arcs with holonomies, which are geometric objects in the Fukaya category.

Mirror symmetry correspondence. Summing up, we have equivalences and correspondences as follows.

$$\begin{array}{ccccccc}
 & \substack{\text{localized}\\ \text{mirror functor}} & & \text{Eisenbud} & & \substack{\text{Burban}\\ \text{-Drozd}} & \\
D^\pi(W\,\mathrm{Fuk}(\mathscr{P})) & \xrightarrow[\simeq]{\Phi^{\mathbb{L}}} & \underline{\mathrm{MF}}(xyz) & \xrightarrow[\simeq]{\mathrm{coker}} & \underline{\mathrm{CM}}(A) & \xrightarrow[\simeq]{\mathbb{F}_{\mathrm{BD}}} & \underline{\mathrm{Rep}}(Q_A) \\
L(w',\lambda',\mu) & \overset{\cong}{\mapsto} & \varphi(w',\lambda,\mu) & \overset{?}{\longleftrightarrow} & M(w,\lambda,\mu) & \mapsto & \Theta(w,\lambda,\mu) \\
L(w') & \overset{\cong}{\mapsto} & \varphi(w') & \overset{?}{\longleftrightarrow} & M(w) & \mapsto & \Theta(w)
\end{array}$$

Note that objects in each category are parameterized by loop/arc data or band/string data. The main task of the current thesis is to establish the correspondence between the canonical form in $\underline{\mathrm{MF}}(xyz)$ which are parameterized by loop/arc data and the canonical form in $\underline{\mathrm{CM}}(A)$ parameterized by band/string data. But we don't yet have enough language to interpret the module $M(w,\lambda,\mu)$ (or $M(w)$) and compute an explicit form of the corresponding matrix factorization.

The conceptual understanding of the Cohen-Macaulay module corresponding to the decorated representation will be postponed to Chapter 5, where we will discuss a geometric construction of a quasi-inverse functor of $\mathbb{F}_{\mathrm{BD}}$.

Matrix factorizations from Cohen-Macaulay modules. We deal with the computational aspect first. Following the reconstruction procedure in [BD17], what we are explicitly given first are generators of an A-module $\tilde{M}(w,\lambda,\mu)$, which is not Cohen-Macaulay in general. Then a process called *Macaulayfication* extends it to the actual Cohen-Macaulay module $M(w,\lambda,\mu)$ by adding *Macaulayfying elements*. Next, we need to find a matrix factorization (φ,ψ) such that $\mathrm{coker}\,\underline{\varphi}$ is isomorphic to $M(w,\lambda,\mu)$, or equivalently, find a free resolution of $M(w,\lambda,\mu)$. However, because the Macaulayfying elements appear irregularly depending on the band/string data, it is not easy to establish free resolutions involving them all at once.

To address this difficulty, we introduce a combinatorial tool called **generator diagram** which allows us to perform a systematic computation in the following steps.

(1) Represent generators of $\tilde{M}(w,\lambda,1)$ on the generator diagram.

(2) Add Macaulayfying elements of $\tilde{M}(w,\lambda,1)$ to Macaulayfy it to $M(w,\lambda,1)$.

(3) Find a free resolution of $M(w,\lambda,1)$, which yields the corresponding matrix factor $\varphi\left(w',\lambda,1\right)$.

(4) Substitute $J_\mu(\lambda)$ for λ in $\varphi\left(w',\lambda,1\right)$ to obtain the matrix factor $\varphi\left(w',\lambda,\mu\right)$ corresponding to $M\left(w,\lambda,\mu\right)$.

The situation is similar for the string-type $M(w)$, and we apply steps (1) to (3) above.

Conversion formula. Interestingly, it turns out that there is a *conversion formula* between normal loop/arc words w' and band/string words w, which relates the canonical form $\varphi\left(w',\lambda,\mu\right)$ or $\varphi\left(w'\right)$ of matrix factorizations with the canonical form $M\left(w,\lambda,\mu\right)$ or $M(w)$ of Cohen-Macaulay modules.

Theorem 1.0.12 (Main Theorem I)**.** *(1) We have a conversion formula*

$$\begin{array}{ccc} \{\textit{normal loop words}\} & \overset{1:1}{\leftrightarrow} & \{\textit{band words}\} \\ w' & \leftrightarrow & w \end{array}$$

under which there is an isomorphism

$$\operatorname{coker}\underline{\varphi}\left(w',\lambda,\mu\right) \;\cong\; M\left(w,\lambda,\mu\right)$$

in $\underline{\mathrm{CM}}(A)$*, except for the degenerate cases where* $w' = (2,2,2)^{\#\tau} \leftrightarrow w = (0,0,0)^{\#\tau}$ *and* $\lambda = 1$*.*

In a non-periodic degenerate case where $w' = (2,2,2) \leftrightarrow w = (0,0,0)$*,* $\lambda = 1$ *and* $\mu \in \mathbb{Z}_{\geq 2}$*, we have*

$$\operatorname{coker}\underline{\varphi_{\deg}}\left((2,2,2),1,\mu-1\right) \;\cong\; M\left((0,0,0),1,\mu\right),$$

in $\underline{\mathrm{CM}}(A)$*. Note that* $\mu = 1$ *case corresponds to the zero object, which is not counted as an indecomposable object by definition.*

(2) We have a conversion formula

$$\begin{array}{ccc} \{\textit{normal arc words}\} & \overset{1:1}{\leftrightarrow} & \{\textit{string words}\} \\ w' & \leftrightarrow & w \end{array}$$

under which there is an isomorphism in $\underline{\mathrm{CM}}(A)$

$$\operatorname{coker}\underline{\varphi}\left(w'\right) \;\cong\; M(w).$$

The conversion formula between loop/arc data and band/string data realizes one-to-one correspondences between indecomposable objects (up to isomorphism) in each category as

$$
\begin{aligned}
&\left\{\textbf{loop-type} \text{ indecomposable objects in } D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P})) \text{ or } \underline{\mathrm{MF}}(xyz)\right\}\big/\sim_{\text{isomorphism}}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \{\text{non-periodic loop data}\}\big/\sim_{\text{shifting}}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \left(\{\text{non-periodic band data}\}\setminus\{((0,0,0),1,1)\}\right)\big/\sim_{\text{shifting}}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \left\{\textbf{band-type} \text{ indecomposable objects in } \underline{\mathrm{CM}}(A) \text{ or } \underline{\mathrm{Rep}}(Q_A)\right\}\Big/\sim_{\text{isomorphism}}\\
\&\ &\left\{\textbf{arc-type} \text{ indecomposable objects in } D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P})) \text{ or } \underline{\mathrm{MF}}(xyz)\right\}\big/\sim_{\text{isomorphism}}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \{\text{arc data}\}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \{\text{string data}\}\\
&\quad\overset{1:1}{\leftrightarrow}\quad \left\{\textbf{string-type} \text{ indecomposable objects in } \underline{\mathrm{CM}}(A) \text{ or } \underline{\mathrm{Rep}}(Q_A)\right\}\Big/\sim_{\text{isomorphism}}.
\end{aligned}
$$

This answers to all questions raised so far. Regarding Question 1.0.7, the arc-type canonical form is always indecomposable, while the loop-type canonical form is indecomposable if and only if the loop word is non-periodic. This implies that a periodic loop is quasi-isomorphic to the direct sum of non-periodic ones in the Fukaya category, which is not obvious at all.

The above shown correspondences also provide a positive answer to Question 1.0.8: The canonical form of loop/arc-type indecomposable objects represents every isomorphism class of indecomposable objects in $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$ or $\underline{\mathrm{MF}}(xyz)$.

Those also translates Burban-Drozd's classification result of Cohen-Macaulay modules into the language of matrix factorizations, providing the explicit canonical form which would have hardly been found without the aid of mirror symmetry. This gives a complete answer to Question 1.0.10.

Because the canonical form $L(w',\lambda',\mu)$ or $L(w')$ is *geometric* (not need to be expressed as a twisted complex of another geometric objects) in the derived Fukaya category $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$, we have a complete and positive answer to Question 1.0.11.

Correspondence of modules and geodesics. Combining Proposition 1.0.3, Theorem 2.4.2 and Theorem 1.0.12, we get the following neat and intuitive conclusion.

Corollary 1.0.13. *Under homological mirror symmetry, there is a one-to-one correspondence*

$$
\begin{aligned}
&\left\{\textit{isomorphism classes of } \textbf{\textit{indecomposable Cohen-Macaulay modules}} \textit{ over } A\right\}\setminus\{[A]\}\\
\overset{1:1}{\leftrightarrow}\quad &\left\{\textbf{\textit{closed geodesics}} \textit{ in } \mathscr{P} \textit{ with holonomy } J_{\mu}(\lambda')\in \mathrm{GL}_{\mu}(\Bbbk)\right\}\cup\left\{\textbf{\textit{open geodesics}} \textit{ in } \mathscr{P}\right\}
\end{aligned}
$$

where $\mathscr{P}$ is given a hyperbolic metric with three cusps.

Geometric interpretation of tameness. In a representation-theoretic point of view, the above symplectic-geometric interpretation explains very naturally the tameness of Cohen-Macaulay modules over A. As already pointed out in Definition 1.0.4, it is very obvious that loop words (or free homotopy classes of loops or closed geodesics) and arc words (or homotopy classes relative to bounded end points or open geodesics) in $\mathscr{P}$ are only countably many. Then an indecomposable local systems lying on a fixed loop and having a fixed rank μ are only parameterized by the holonomy parameter $\lambda' \in \Bbbk^\times$ up to gauge equivalence. It counts for the reason why loop data consist countably many one-parameter families.

Applications. The above parameterization and correspondence interchange plenty of algebraic structure and geometric symmetry lying naturally on each category. In particular, we can gain insights or perform certain calculations on one side that are difficult on the other side. Here we present only three of those kinds of application, while expecting that there would be further interesting translation between two different languages.

First, categories involved in HMS have a natural **triangulated structure**. Along with the parameterization by band or loop data, an object with higher multiplicity is given by some *mapping cone* or *twisted complex* consisting of objects with lower multiplicities, which can be also understood as a higher rank local system attached to the Lagrangian. This reveals that the twisted complexes can be well expressed using band or loop data and makes the computations regarding them much easier.

Second, we show that taking **dual** of modules in $\underline{\mathrm{CM}}(A)$ corresponds to an **inversion** of curves in $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$ that flips the punctured sphere back and forth. The operations can be done by multiplying -1 to band/string or loop/arc words. See Proposition 4.2.1.

Finally, the **Auslander-Reiten translation** in $\underline{\mathrm{CM}}(A)$ is in general not easy to compute in terms of the band data. But it is equivalent to **reversing the orientation** of underlying curves in $D^\pi(W\,\mathrm{Fuk}(\mathscr{P}))$, which we can understand more intuitively. It also have a neat description in loop data, which can then be converted into band data. See Proposition 4.2.2.

Introduction to Chapter 5. The discussion so far shows that Cohen-Macaulay modules over degenerate cusp singularities have a quite complicated nature. Representation theory of Cohen-Macaulay modules in [BD17] is highly non-trivial and the reason why the hexagonal quiver is involved is also not obvious at all. We already introduced the generator diagram as a visual tool to understand the combinatorics of generators of Cohen-Macaulay modules, however, it doesn't still account for the meaning of those generators and their relations.

Here we study the underlying geometry of degenerate cusp singularities as well as Cohen-Macaulay modules over them, which would provide a more intuitive and direct relation between Cohen-Macaulay modules and decorated quiver representations. It will also reveal the underlying geometry of the combinatorics over the generator diagram.

As another consequence, it provides an explicit quasi-inverse of the Burban-Drozd functor

$$\mathbb{F}_{\mathrm{BD}} : \mathrm{CM}\,(A) \longrightarrow \mathrm{Tri}\,(A) \longrightarrow \mathrm{Rep}\,(Q_A)\,.$$

discussed in Equation 1.0.2. It has also appeared in [BZ20] in a different form, but our construction will still add it a geometric flavor.

The construction below is also applicable to another degenerate cusp singularities whose normalization is regular, but in this thesis we focus only on A for simplicity.

Geometry of $X = \mathrm{Spec}(A)$. The affine scheme $X = \mathrm{Spec}(A)$ is defined by a single equation $xyz = 0$ in an infinitesimal neighborhood of the origin. In an elementary-geometric language, it is nothing but a union of three planes which perpendicularly meet at the origin. Again in the language of schemes, X is still a gluing of three '*infinitesimal planes*' $\mathrm{Spec}\Bbbk[[x,y]]$, $\mathrm{Spec}\Bbbk[[y,z]]$, $\mathrm{Spec}\Bbbk[[z,x]]$ along three '*infinitesimal axes*' $\mathrm{Spec}\Bbbk((x))$, $\mathrm{Spec}\Bbbk((y))$, $\mathrm{Spec}\Bbbk((z))$ in the following sense. Together with six natural '*inclusion maps*' from axes to planes, they form a hexagon diagram as in Figure 1.4.

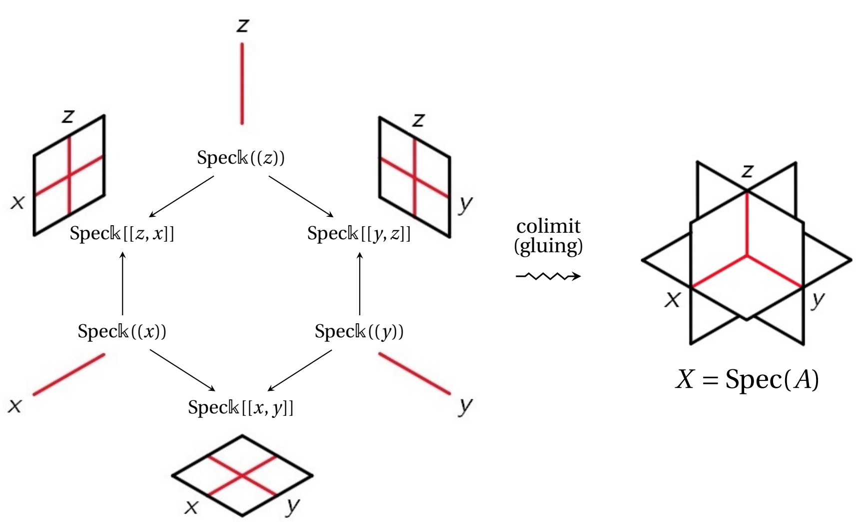


Figure 1.4: Geometric Construction of $X = \mathrm{Spec}(A)$

Proposition 1.0.14. *The affine scheme $X = \mathrm{Spec}(A)$ is the (categorical) colimit of the hexagon diagram of* *affine schemes in in Figure 1.4.*

Degenerate Vector Bundles. The above observation on the geometry of X gives us great insight into motivating the following definition.

Definition 1.0.15. *Given a decorated representation* Θ *on* Q_A *as in Diagram 1.0.1, we construct a hexagon diagram* $\hat{\Theta}$ *of affine* X*-schemes as shown in Figure 1.5. We attach (trivial) vector bundles* $\operatorname{Spec}\Bbbk[[x,y]] \times \mathbb{A}^{d_{\mathrm{xy}}}$, $\operatorname{Spec}\Bbbk[[y,z]] \times \mathbb{A}^{d_{\mathrm{yz}}}$, *and* $\operatorname{Spec}\Bbbk[[z,x]] \times \mathbb{A}^{d_{\mathrm{zx}}}$ *over infinitesimal planes at three target vertices, and (trivial) vector bundles* $\operatorname{Spec}\Bbbk((x)) \times \mathbb{A}^{l_{\mathrm{x}}}$, $\operatorname{Spec}\Bbbk((y)) \times \mathbb{A}^{l_{\mathrm{y}}}$, *and* $\operatorname{Spec}\Bbbk((z)) \times \mathbb{A}^{l_{\mathrm{z}}}$ *over infinitesimal axes at three source vertices. The ranks of them are determined by the dimension of* $\Bbbk((t))$*-vector spaces in* Θ*. At six arrows we attach appropriate vector bundle maps* $\widehat{\theta^{\blacktriangle}_{\bullet\blacksquare}}$ *which come from six* $\Bbbk((t))$*-linear maps* $\theta^{\blacktriangle}_{\bullet\blacksquare}$ *in* Θ*. Then we define the* ***degenerate vector bundle*** $\mathscr{E}(\Theta)$ *over* X *associated to* Θ *as the colimit*

$$\mathscr{E}(\Theta) := \operatorname{colim} \widehat{\Theta}$$

of the diagram $\widehat{\Theta}$ *in the category of affine* X*-schemes.*

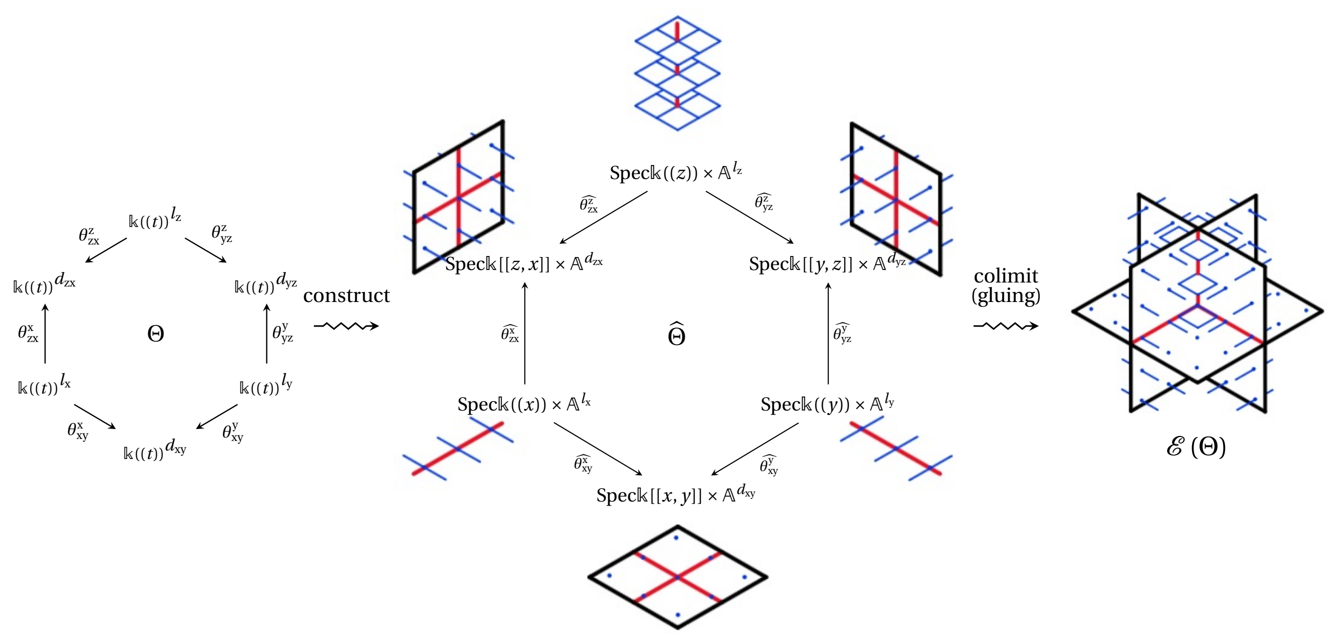


Figure 1.5: Degenerate Vector Bundle $\mathscr{E}(\Theta)$ associated to the Representation Θ

The base spaces ($\operatorname{Spec}\Bbbk[[\bullet,\blacksquare]]$ and $\operatorname{Spec}\Bbbk((\bullet))$) are glued under the natural inclusion maps and the fibers ($\mathbb{A}^{d_{\bullet\blacksquare}}$ and $\mathbb{A}^{l_\bullet}$) are glued according to maps $\theta^{\blacktriangle}_{\bullet\blacksquare}$ in Θ. Then the universal property of the colimit, or intuitively just by forgetting fibers, we get the *projection* map $\widehat{\pi} : \mathscr{E}(\Theta) \to X$, which makes $\mathscr{E}(\Theta)$ an X-scheme.

Because the rank of individual trivial bundles are different over each component, we cannot call $\mathscr{E}(\Theta)$ as a vector bundle in a traditional sense. But together with the projection map we can still enjoy similar concepts and properties of it.

Degenerate vector bundles over X form a subcategory $\mathrm{DVB}(X)$ of category of X-schemes. Then the above construction gives rise to a functor $\mathscr{E} : \mathrm{Rep}\,(Q_A) \to \mathrm{DVB}(X)$, which is shown to be an equivalence of categories.

Global sections. A *(global) section* $\hat{s} : X \to \mathscr{E}\,(\Theta)$ is by definition an X-morphism, that is, a morphism of schemes satisfying $\hat{\pi} \circ \hat{s} = \mathrm{id}_X$. It is made from 6 individual sections

$$\begin{aligned}
&s_{\mathrm{xy}} \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}} : \mathrm{Spec}\Bbbk[[x,y]] \to \mathrm{Spec}\Bbbk[[x,y]] \times \mathbb{A}^{d_1}, \quad && r_{\mathrm{x}} \in \Bbbk((x))^{l_{\mathrm{x}}} : \mathrm{Spec}\Bbbk((x)) \to \mathrm{Spec}\Bbbk((x)) \times \mathbb{A}^{l_x},\\
&s_{\mathrm{yz}} \in \Bbbk[[y,z]]^{d_{\mathrm{yz}}} : \mathrm{Spec}\Bbbk[[x,y]] \to \mathrm{Spec}\Bbbk[[x,y]] \times \mathbb{A}^{d_2}, \quad && r_{\mathrm{y}} \in \Bbbk((y))^{l_{\mathrm{y}}} : \mathrm{Spec}\Bbbk((y)) \to \mathrm{Spec}\Bbbk((y)) \times \mathbb{A}^{l_y},\\
&s_{\mathrm{zx}} \in \Bbbk[[z,x]]^{d_{\mathrm{zx}}} : \mathrm{Spec}\Bbbk[[x,y]] \to \mathrm{Spec}\Bbbk[[x,y]] \times \mathbb{A}^{d_3}, \quad && r_{\mathrm{z}} \in \Bbbk((z))^{l_{\mathrm{z}}} : \mathrm{Spec}\Bbbk((z)) \to \mathrm{Spec}\Bbbk((z)) \times \mathbb{A}^{l_z},
\end{aligned}$$

of trivial vector bundles satisfying 6 gluing conditions

$$\begin{aligned}
s_{\mathrm{zx}}(0,x) &= \theta^{\mathrm{zx}}_{\mathrm{x}}(x)\, r_{\mathrm{x}}(x), \quad & s_{\mathrm{xy}}(x,0) &= \theta^{\mathrm{xy}}_{\mathrm{x}}(x)\, r_{\mathrm{x}}(x)\,,\\
s_{\mathrm{xy}}\left(0,y\right) &= \theta^{\mathrm{xy}}_{\mathrm{y}}\left(y\right) r_{\mathrm{y}}\left(y\right), \quad & s_{\mathrm{yz}}\left(y,0\right) &= \theta^{\mathrm{yz}}_{\mathrm{y}}\left(y\right) r_{\mathrm{y}}\left(y\right),\\
s_{\mathrm{yz}}(0,z) &= \theta^{\mathrm{yz}}_{\mathrm{z}}(z)\, r_{\mathrm{z}}(z), \quad & s_{\mathrm{zx}}(z,0) &= \theta^{\mathrm{zx}}_{\mathrm{z}}(z)\, r_{\mathrm{z}}(z)\,.
\end{aligned}$$

Namely, they must be compatible with the gluing of the underlying degenerate vector bundle to be merged together. If then, the universal property of colimit ensures a unified section $\hat{s}$.

Proposition 1.0.16. *Any section of the degenerate vector bundle* $\mathscr{E}\,(\Theta)$ *is induced in the above way.*

Whenever a tuple $\left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}, r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right)$ induces a section $\hat{s}$, it turns out that $\left(r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right)$ is uniquely determined by $\left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right)$. Therefore the section $\hat{s}$ can be identified with an element $s := \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right)$. In this respect, we can identify the **set of all sections** $\Gamma\,(\mathscr{E}\,(\Theta))$ with a subset of $\Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$ satisfying the gluing conditions for some $\left(r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right)$. Moreover, it has a natural A-module structure because A is the function ring of the base scheme X.

Lemma 1.0.17. *The set* $\Gamma\,(\mathscr{E}\,(\Theta))$ *of all sections is a Cohen-Macaulay* A*-module.*

Taking global sections gives a functor $\Gamma : \mathrm{DVB}(X) \to \mathrm{CM}(A)$.

Equivalence of categories. So far we constructed the category $\mathrm{DVB}(X)$ and two functors $\mathscr{E} : \mathrm{Rep}\,(Q_A) \to \mathrm{DVB}(X)$ and $\Gamma : \mathrm{DVB}(X) \to \mathrm{CM}(A)$. Now we can state our main theorem.

Theorem 1.0.18 (Main Theorem II)**.**

$$\begin{tikzcd}
& \mathrm{DVB}\,(X) \arrow[dl, "\Gamma"', "\substack{\textit{global}\\ \textit{section}}"] & \\
\mathrm{CM}(A) \arrow[rr, "\mathbb{F}_{\mathrm{BD}}"] & & \mathrm{Rep}\,(Q_A) \arrow[ul, "\mathscr{E}\ \textit{gluing functor}"']
\end{tikzcd}$$

(1) $(\Gamma \circ \mathscr{E}) \circ \mathbb{F}_{\mathrm{BD}} \simeq \mathrm{id}_{\mathrm{CM}(A)}$

(2) $\mathbb{F}_{\mathrm{BD}} \circ (\Gamma \circ \mathscr{E}) \simeq \mathrm{id}_{\mathrm{Rep}(Q_A)}$

Therefore, categories $\mathrm{CM}\,(A)$ *and* $\mathrm{Rep}\,(Q_A)$ *are equivalent. As the gluing functor* $\mathscr{E}$ *is also an equivalence, the category* $\mathrm{DVB}\,(X)$ *is also equivalent to them.*

Therefore we can identify Cohen-Macaulay modules and decorated quiver representations with degenerate vector bundles, adding geometry to previous concepts. We hope that this new geometric perspective would enhance our understanding of algebraic operations such as dual, tensor product, and triangulated structures on Cohen-Macaulay modules.

Chapter 2

Wrapped Fukaya Category of Pair-of-Pants Surface

2.1 Fukaya Category and Localized Mirror Functor

Let us recall our geometric setup of Fukaya category and localized mirror functor. We refer readers to Fukaya-Oh-Ohta-Ono [FOOO09], Seidel [Sei11], Akaho-Joyce [AJ10] for general definitions and properties of Fukaya category, and [CHL17] for localized mirror functor formalism.

In this paper, a symplectic manifold (M,ω) is given by a punctured Riemann surface Σ with an area form ω on it. In particular, many operations on the Fukaya category can be explained combinatorially as counts of suitable (immersed) polygons (instead of counting solutions of J-holomorphic curve equation).

An object of Fukaya category of Σ will be given by oriented immersed curves $\iota : L \to M$, which automatically satisfies the Lagrangian condition ($\iota^*(\omega) = 0, \dim(L) = \frac{1}{2}\dim(M)$). We will call L an *immersed Lagrangian*. Our Lagrangian is always oriented, and hence we will omit it from now on. (In fact, we will only consider regular immersed curves; see Definition 2.2.2). We allow non-compact Lagrangians that start and end at punctures (morphisms between them will be defined as in wrapped Fukaya category). Our Fukaya category is defined over the Novikov field Λ where

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{C}, \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

This was introduced to handle infinite sums whose energy (exponent of T) of summands approach infinity. Note that M is an exact symplectic manifold $\omega = d\theta$, and if we consider exact Lagrangians $\iota : L \to M$ only (with $\iota^*\theta = df_L$ for some function $f_L : L \to \mathbb{R}$), we can work with $\mathbb{C}$-coefficients. With exact Lagrangians, a J-holomorphic curve with prescribed inputs and an output has a fixed energy, and hence its count is finite from the Gromov-Compactness theorem.

But compact immersed Lagrangians that we are interested in are *not* exact, hence we need to work with Λ a priori. The energy filtration of Λ is used to run Maurer-Cartan formalism as well as the localized mirror functor.

To compare with the matrix factorizations over $\mathbb{C}$, we will make an evaluation $T = 1$ later. In general, an evaluation $T = 1$ for an element of Λ does not make sense. But for regular immersed curves, even though they are not exact, we will be able to make the evaluation $T = 1$ (see Appendix 2.2).

Let us recall the definition (and convention) of an A_∞-category over the field Λ.

Definition 2.1.1. *An A_∞-category $\mathscr{C}$ over Λ consists of a collection of objects $Ob(\mathscr{C})$, a (graded) Λ-module* $\mathrm{Hom}(A_1, A_2)$ *for $A_1, A_2 \in Ob(\mathscr{C})$, and a set of A_∞-operations $\{\mathfrak{m}_k\}_{k\geq 1}$ where*

$$\mathfrak{m}_k : \mathrm{Hom}(A_1, A_2) \otimes \cdots \otimes \mathrm{Hom}(A_k, A_{k+1}) \to \mathrm{Hom}(A_1, A_{k+1}).$$

They satisfy A_∞-relations

$$\sum_{p,q} (-1)^{\dagger} \mathfrak{m}_{n-q+1} \left(f_1, \ldots, f_p, \mathfrak{m}_q \left(f_{p+1}, \ldots, f_{p+q}\right), f_{p+q+1}, \ldots, f_n\right) = 0$$

for any fixed $k \geq 1$ and possible $p, q \geq 1$. Here, $\dagger = |a_1| + \cdots + |a_p| - p$ is related to the grading of inputs.

An A_∞-functor $\mathscr{F}$ between two A_∞-categories $\mathscr{A}$, $\mathscr{B}$ consists of maps $\{\mathscr{F}_k\}_{k\geq 0}$ where

$$\mathscr{F}_0 : Ob(\mathscr{A}) \to Ob(\mathscr{B})$$

and for $k \geq 1$,

$$\mathscr{F}_k : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{A}}(A_k, A_{k+1}) \to \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}_0(A_1), \mathscr{F}_0(A_{k+1})).$$

They satisfy similar A_∞-relations

$$\begin{aligned} &\sum_{t, i_1+\cdots+i_t = n} \mathfrak{m}_t \left(\mathscr{F}_{i_1}\left(a_1, \ldots, a_{i_1}\right), \ldots, \mathscr{F}_{i_t}\left(a_{i_{t-1}+1}, \ldots, a_n\right)\right) \\ &= \sum_{p,q} (-1)^{\dagger} \mathscr{F}_{n-q+1}\left(a_1, \ldots, a_p, \mathfrak{m}_q\left(a_{p+1}, \ldots, a_{p+q}\right), \ldots, a_n\right). \end{aligned}$$

We will only consider a countable family of regular immersed Lagrangians on Σ as objects of Fukaya category. Without loss of generality, we may assume that these curves intersect transversely away from self intersection points and there are no triple (or higher) intersections.

Since M is non-compact and has cylindrical ends toward the punctures, we choose a Hamiltonian function H on M which is quadratic at infinity (to define wrapped Fukaya category). Let ϕ_H be a time one map of its Hamiltonian vector field X_H. Often we will simply write L instead of writing the immersion $\iota : L \to M$ for convenience.

Definition 2.1.2. *For two immersed oriented Lagrangians L_1, L_2 that intersect transversely, $CF^*(L_1, L_2)$ is a $\mathbb{Z}/2$-graded vector space over Λ generated by $\phi_H(L_1) \cap L_2$. Here an intersection $p \in CF^*(L_1, L_2)$ is odd if the orientation of $T_pL_1 \oplus T_pL_2$ agrees with that of T_pM and it is even otherwise. The self Hom space $CF^*(L, L)$ is a $\mathbb{Z}/2$-graded vector space over Λ generated by $\phi_H(L) \cap L$.*

It is well-known how to define a differential and A_∞-operations in general. We will add another object, compact immersed Lagrangian in the pair of pants, called Seidel Lagrangian $\mathbb{L}$. To run the Maurer-Cartan theory, we can work with the following Morse complex version of self-Hom space instead of Hamiltonian perturbation. We refer readers to Seidel [Sei11] for more details, and in particular the sign convention.

Let $\iota : L \to M$ be a regular *compact* Lagrangian immersion. We fix a Morse function on L so that its critical points are away from immersed points, and intersection points with other Lagrangians, and denote by $C^*_{Morse}(L)$ the resulting Morse complex of L over Λ. Let us explain the association of two immersed generators for each self intersection point $p \in M$ of ι. For two branches $\tilde{L}_1, \tilde{L}_2$ of $\iota(L)$ in the neighborhood of p, we can associate two local Floer generators $p \in CF(\tilde{L}_1, \tilde{L}_2)$, $\overline{p} \in CF(\tilde{L}_2, \tilde{L}_1)$. We may choose branches so that p is odd, and $\overline{p}$ is even. For the Seidel Lagrangian $\mathbb{L}$, we define

$$CF^*(\mathbb{L},\mathbb{L}) := C^*_{Morse}(\mathbb{L}) \bigoplus_p (\Lambda\langle p\rangle \oplus \Lambda\langle \overline{p}\rangle)$$

where the sum is over the self intersection points of $\mathbb{L}$.

If $L_0, \ldots, L_k$ are immersed Lagrangians and for $w_i \in CF^*(L_{i-1}, L_i)$ for $i = 1, \cdots k$ given by transverse intersections or immersed generators, an A_∞-operation $m_k(w_1, \ldots, w_k)$ is defined by counting immersed rigid (holomorphic) polygons with convex corners at $w_1, \cdots, w_k, w_0$ with $w_0 \in CF^{|w_0|}(L_k, L_0)$, contributing a term $\pm T^{\omega(u)} \cdot \overline{w}_0 \in CF^{1-|w_0|}(L_0, L_k)$. We take the sum over all such u and w_0. If a polygon has a non-convex corner, it is not difficult to show that such a polygon comes with at least one parameter family of holomorphic polygons and thus it is not rigid.

Let $\mathscr{P}$ be a pair of pants which is a three-punctured sphere $\mathbb{P}^1 \setminus \{a,b,c\}$ and $\mathbb{L}$ be the Seidel Lagrangian in $\mathscr{P}$ as in Figure **??**. Since $\mathbb{L}$ has 3 self-intersections, a Floer complex of $\mathbb{L}$ has 8 generators

$$CF^*(\mathbb{L},\mathbb{L}) = \langle e, X, Y, Z, \overline{X}, \overline{Y}, \overline{Z}, p\rangle$$

where e and p are a minimum and a maximum of the chosen Morse function on $\mathbb{L}$.

Theorem 2.1.3. *[CHL17] Assume that the areas of two triangles bounded by $\mathbb{L}$ in $\mathscr{P}$ are the same. A linear combination $b = xX + yY + zZ \in CF^*(\mathbb{L},\mathbb{L}) \otimes_\Lambda \Lambda\langle x, y, z\rangle$ is a weak bounding cochain. Namely, we have*

$$\sum_{i=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = xyz \cdot e. \tag{2.1.1}$$

The only nontrivial A_∞-operation is $\mathfrak{m}_k(X, Y, Z)$ which counts the front triangle bounded by $\mathbb{L}$ passing through e, and we have the mirror potential $W = xyz$ from the pair of pants $\mathscr{P}$.

For weakly unobstructed Lagrangian $\mathbb{L}$ in symplectic manifold M, localized mirror functor formalism [CHL17] gives an A_∞-functor from Fukaya category of M to the matrix factorization category of $W^{\mathbb{L}}$ (here we follow the convention in [CHL15]).

Theorem 2.1.4. *[CHL17] Let $W^{\mathbb{L}}$ be the disc potential of $\mathbb{L}$. The localized mirror functor $\mathscr{F}^{\mathbb{L}} : \mathscr{WF}(X) \to \mathscr{MF}(W^{\mathbb{L}})$ is defined as follows.*

- *For a given Lagrangian L, mirror object $\mathscr{F}^{\mathbb{L}}(L)$ is given by the following matrix factorization M_L*

$$\left(CF(L,\mathbb{L}), -\mathfrak{m}_1^{0,b}\right), \textit{where } \mathfrak{m}_1^{0,b}(x) = \sum_{l=0}^{\infty} \mathfrak{m}_k(x, \underbrace{b, \ldots, b}_{l}).$$

- *Higher component of the A_∞-functor*

$$\mathscr{F}_k^{\mathbb{L}} : CF(L_1, L_2) \otimes \cdots \otimes CF(L_k, L_{k+1}) \to \mathscr{MF}(M_{L_1}, M_{L_{k+1}})$$

is given by

$$\mathscr{F}_k^{\mathbb{L}}(f_1, \ldots, f_k) := \mathfrak{m}_{k+1}^{0,\ldots,0,b}(f_1, \ldots, f_k, \bullet) = \sum_{l=0}^{\infty} \mathfrak{m}_{k+1+l}(f_1, \ldots, f_k, \bullet, \underbrace{b, \ldots, b}_{l}).$$

Here the input $\bullet$ is an element of $M_{L_{k+1}} = CF(L_{k+1}, \mathbb{L})$.

Then, $\mathscr{F}^{\mathbb{L}}$ defines an A_∞-functor, which is cohomologically injective on $\mathbb{L}$.

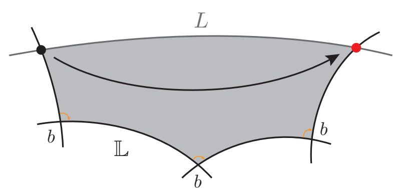


Figure 2.1: Reading off entries of matrix factorization from decorated strips

The above theorem prescribes how to find the mirror matrix factorization from geometry. Given a curve L, take the $\mathbb{Z}/2$-graded vector space generated by the intersection $L \cap \mathbb{L}$. Count decorated strips bounded by L and $\mathbb{L}$ as in Figure 2.1. Here decorated means that we allow strips to have arbitrary many corners (of X, Y, Z), and the resulting count records the labels of them (by x, y, z). By index reasons, each strip should map an even (resp. odd) intersection to an odd (resp. even) intersection. If we record all these data in two matrices, they become the matrix factorization of the disc potential function $W_{\mathbb{L}}$ corresponding to L.

In our case, a disc potential of Seidel Lagrangian $\mathbb{L}$ is $W^{\mathbb{L}} = xyz$, and three simplest non-compact Lagrangians connecting different punctures are mapped to the three factorizations $x \cdot yz, y \cdot xz, z \cdot zy$. From this, $\mathscr{F}^{\mathbb{L}}$ becomes an quasi-equivalence recovering the results of [AAE+13] for $\mathscr{P}$.

Theorem 2.1.5. *$\mathscr{WF}(\mathscr{P})$ is derived equivalent to* $\mathrm{MF}(xyz)$.

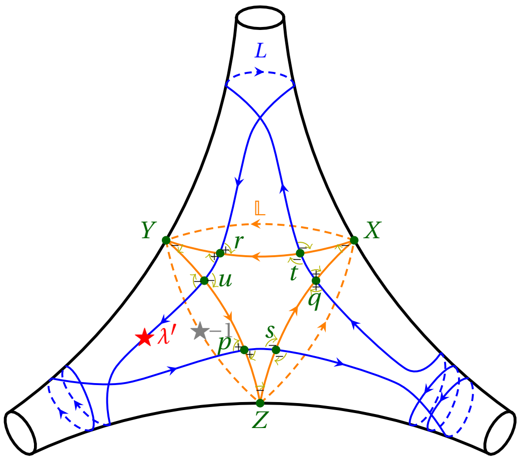
2.1.1 Computation of localized mirror functor

In this section, we illustrate the computation of the localized mirror functor with a relatively simple example. Namely, we will find the mirror matrix factorization of the loop with holonomy $L := L\big((3, -2, 2), \lambda', 1\big)$.

Note that L and $\mathbb{L}$ have 6 intersections, say p, q, r, and s, t, u as shown in Figure 2.2. Here we will consider them elements of $\mathrm{Hom}(L, \mathbb{L})$. It is convenient to view them as clockwise angles from L to $\mathbb{L}$. Their complement clockwise angles from $\mathbb{L}$ to L consist $\mathrm{Hom}(\mathbb{L}, L)$ and are denoted as $\overline{p}$, $\overline{q}$, $\overline{r}$, and $\overline{s}$, $\overline{t}$, $\overline{u}$, respectively. We are using a $\mathbb{Z}/2$-grading on $\mathrm{Hom}(L, \mathbb{L})$, where the degree of each angle is determined by orientations of two curves L and $\mathbb{L}$ at the angle. Namely, the orientations of L and $\mathbb{L}$ are the same at *even*(+ or 0) -degree angles p, q, and r, and different at *odd*(− or 1) -degree angles s, t, and u. So we have a decomposition as

$$\mathrm{Hom}(L, \mathbb{L}) = \mathrm{Hom}^0(L, \mathbb{L}) \oplus \mathrm{Hom}^1(L, \mathbb{L}) = \Bbbk\langle p, q, r\rangle \oplus \Bbbk\langle s, t, u\rangle.$$

Then two restricted operators $\mathfrak{m}_1^{0,b} : \mathrm{Hom}^0(L,\mathbb{L}) \to \mathrm{Hom}^1(L,\mathbb{L})$ and $\mathfrak{m}_1^{0,b} : \mathrm{Hom}^1(L,\mathbb{L}) \to \mathrm{Hom}^0(L,\mathbb{L})$ with respect to ordered bases $\{p,q,r\}$ and $\{s,t,u\}$ yield two 3×3 matrices $\Phi^{\mathbb{L}}(L)$ and $\Psi^{\mathbb{L}}(L)$, respectively.



$$\xrightarrow{\mathscr{F}^{\mathbb{L}}} \quad \underbrace{\begin{array}{c} \\ s \\ t \\ u \end{array}\begin{array}{c} \begin{array}{ccc} p & q & r \end{array} \\ \begin{pmatrix} z & 0 & 0 \\ y^2 & x & -z \\ -\lambda' x^2 & 0 & y \end{pmatrix}\end{array}}_{\Phi^{\mathbb{L}}(L)} \quad \underbrace{\begin{array}{c} \\ p \\ q \\ r \end{array}\begin{array}{c} \begin{array}{ccc} s & t & u \end{array} \\ \begin{pmatrix} xy & 0 & 0 \\ -y^3 + \lambda' z x^2 & yz & z^2 \\ \lambda' x^3 & 0 & zx \end{pmatrix}\end{array}}_{\Psi^{\mathbb{L}}(L)}$$

Figure 2.2: Matrix factorization of xyz corresponding to $L = L\big((3,-2,2),\lambda',1\big)$

Entries can be computed by reading off the coefficients of $\mathfrak{m}_1^{0,b}$ evaluated at the corresponding inputs. Namely, the $(\blacksquare,\bullet)$-entry of $\Phi^{\mathbb{L}}(L)$ is the coefficient of $\blacksquare$ in $\mathfrak{m}_1^{0,b}(\bullet)$ for $\bullet \in \{p,q,r\}$ and $\blacksquare \in \{s,t,u\}$. It is by definition the weighted signed counting of immersed polygons (or *deformed strip*) bounded by L and $\mathbb{L}$, whose vertices consist of $\bullet$, elements in $\{X,Y,Z\}$ as many as needed, and then $\bar{\blacksquare}$, in a counterclockwise direction. Whenever there is a such polygon, a monomial of x, y, and z is added to the $\blacksquare$-coefficient in $\mathfrak{m}_1^{0,b}(\bullet)$, whose multiplicities are equal to the number of times X, Y, and Z are passed in the polygon.

We follow the sign rule established in Section 7 of [Sei11]: Consider the boundary orientation of the polygon as usual, that is, it is given in such a way that the polygon lies on the left along the orientation of the boundary. The orientation of L is irrelevant. Let the pieces of $\mathbb{L}$ constituting the polygon be $\mathbb{L}_1,\ldots,\mathbb{L}_d$ in a counterclockwise direction. If the orientation of $\mathbb{L}_1$ coincides with the boundary orientation of the polygon, we do nothing. If they don't coincide, we multiply $(-1)^{|\bullet|}$ to the monomial, which changes the sign only when the angle $\bullet$ from L to $\mathbb{L}_1$ has odd-degree. For $1<k<d$, whenever the orientation of $\mathbb{L}_k$ differs from the boundary orientation, we change the sign, because angles X, Y, or Z from $\mathbb{L}_{k-1}$ to $\mathbb{L}_k$ all have odd-degree. Finally, if the orientation of $\mathbb{L}_d$ does not match the boundary orientation, we change the sign by $(-1)\times(-1)^{|\blacksquare|}$, where the first one comes from one of X, Y, or Z from $\mathbb{L}_{d-1}$ to $\mathbb{L}_d$ and the second one from the degree of the output angle $\blacksquare$ from L to $\mathbb{L}_d$.

Note that we are considering loops L and $\mathbb{L}$ equipped with a rank 1 local system, which has a scalar holonomy λ' or $-1 \in \Bbbk^\times$, respectively. (We will explain later how we deal with higher rank local systems, which has a matrix-valued holonomy $\Lambda \in \mathrm{GL}_\mu(\Bbbk)$). It is marked as a star mark(★) on some point (say *holonomy point*) on loops. Whenever the boundary of the polygon passes through a holonomy point, the holonomy contributes to the corresponding monomial. Here we distinguish whether the holonomy is written inside or outside the polygon at the point. If the polygon contains the holonomy λ' inside, it is just multiplied to the monomial. Otherwise if it is placed outside the polygon, we multiply λ'^{-1} instead.

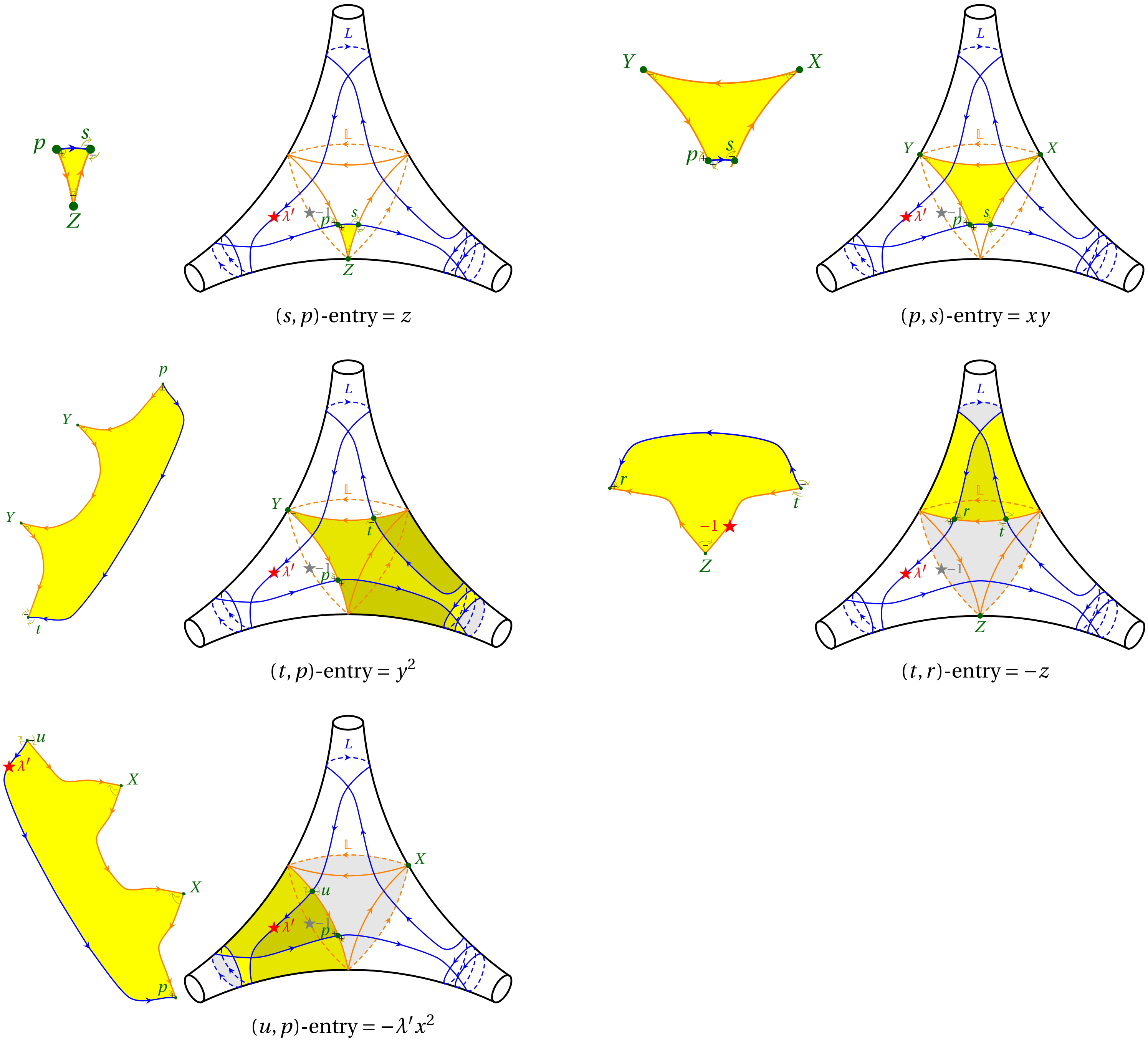


Figure 2.3: Some entries of $\Phi^{\mathbb{L}}(L)$ and $\Psi^{\mathbb{L}}(L)$

2.2 Finiteness and Homotopy Invariance

In this section, we prove two Propositions 2.2.3 and 2.2.5 about the localized mirror functor.

The first proposition is about the '**finiteness**' of the mirror matrix factorizations. Let us first remark that for exact Lagrangians in exact symplectic manifolds, their Fukaya category can be defined over $\Bbbk$. Punctured Riemann surfaces are exact symplectic manifolds, but the immersed loops/arcs L that we consider are not exact, hence a priori, we need to work on the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \Bbbk, \lim_{i\to\infty} \lambda_i = \infty \right\}$$

to define their localized mirror functor images $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$. This means, for an arbitrary unobstructed loop/arc L, each entry of $\Phi^{\mathbb{L}}(L)$ or $\Psi^{\mathbb{L}}(L)$ is an element of $\Lambda[[x,y,z]]$ and each coefficient of a monomial (consisting of x, y and z) in it is an element of Λ. In this case, the evaluation $T=1$ may not make sense for general infinite sums in Λ.

Definition 2.2.1. *We say that* $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ *is* finite *if each coefficient (in Λ) of a monomial in any entry of $\Phi^{\mathbb{L}}(L)$ and $\Psi^{\mathbb{L}}(L)$ is a finite sum. In this case, we can make $T=1$ substitution in $\mathscr{F}^{\mathbb{L}}(L)$ to get a matrix factorization of xyz over $\Bbbk$, or an object in* $\mathrm{MF}(xyz)$.

We need the concept of '*regularity*' of immersed loops/arcs to ensure the finiteness of its mirror image.

Definition 2.2.2. *Consider an immersed loop $L : S^1 \to \mathscr{P}$ or an immersed arc $L : \mathbb{R} \to \mathscr{P}$. We also regard the Seidel Lagrangian $\mathbb{L}$ as an immersed loop $\mathbb{L} : S^1 \to \mathscr{P}$.*

(1) L is said to bound an immersed 'fish-tale' *if there is an immersion $i : D^2 \to \mathscr{P}$ which satisfies $i\left(e^{2\pi i t}\right) = L(\imath(t))$ for some immersion $\imath : [0,1] \to S^1$ or $\mathbb{R}$.*

(2) Loops L and $\mathbb{L}$ are said to bound an immersed 'cylinder' *if there is a continuous map $j : S^1 \times [0,1] \to \mathscr{P}$ which is an immersion on $S^1 \times (0,1)$ and satisfies $j\left(\left(e^{2\pi i t}, 0\right)\right) = L(\imath(t))$ and $j\left(\left(e^{2\pi i t}, 1\right)\right) = \mathbb{L}(\jmath(t))$ for some immersions $\imath$, $\jmath : S^1 \to S^1$.*

(3) L is called unobstructed *if it satisfies the following conditions:*

- *L and $\mathbb{L}$ meet transversally,*
- *$L \cup \mathbb{L}$ does not have a triple intersection, and*
- *L does not bound any immersed disks or fish-tales.*

(4) L is called regular *if it is unobstructed and satisfies the following additional condition:*

- *L and $\mathbb{L}$ do not bound any immersed cylinders.*

Proposition 2.2.3. *For a regular immersed loop/arc L, its mirror matrix factorization $\mathscr{F}^{\mathbb{L}}(L)$ is finite.*

Remark 2.2.4. *It is not hard to show the converse statement that an unobstructed Lagrangian L with $\mathscr{F}^{\mathbb{L}}(L)$ finite can be shown to be a regular immersed loop (up to Hamiltonian isotopy).*

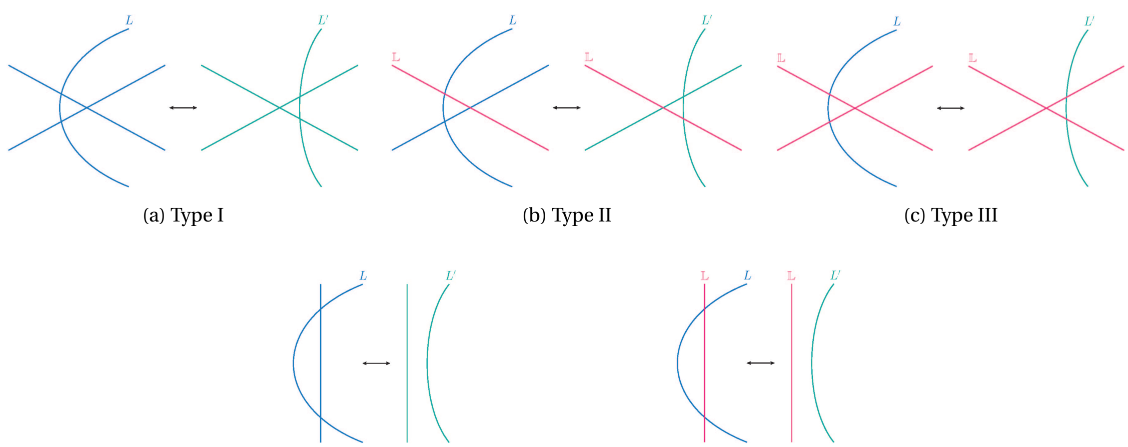
The second proposition is about the '**homotopy invariance**' of the localized mirror functor on regular loops. It was already shown in [CHL19] Proposition 5.4 that the localized mirror functor takes Hamiltonian equivalence of the Fukaya category to the homotopy equivalence of the DG-category matrix factorization (over Λ). But a homotopy between immersed loops that we consider here, are more general than a Hamiltonian isotopy, and also we want to establish such homotopy equivalence over $\Bbbk$.

Proposition 2.2.5. *(1) If two regular immersed loops L and L' are freely homotopic to each other and have the same total holonomy up to basis change, then their mirror matrix factorizations $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ and $\mathscr{F}^{\mathbb{L}}(L') = \left(\Phi^{\mathbb{L}}(L'), \Psi^{\mathbb{L}}(L')\right)$ over $\Bbbk$ are homotopically equivalent in* $\mathrm{MF}(xyz)$.

(2) If two immersed arc L and L' are homotopic to each other under a homotopy keeping the end points in the same boundaries, then their mirrors $\mathscr{F}^{\mathbb{L}}(L)$ and $\mathscr{F}^{\mathbb{L}}(L')$ over $\Bbbk$ are homotopically equivalent in $\mathrm{MF}(xyz)$.

An overview of the proof of Propositions 2.2.3 and 2.2.5 is as follows. As we set $T = 1$, we only concern about the signed count of holomorphic polygons with holonomy contributions, not about the area of such polygons. Therefore, any homotopy of a Lagrangian L, which does not change the intersection and holonomy pattern with $\mathbb{L}$, provides exactly the same matrix factorization over $\Bbbk$.

Meanwhile, any change of intersection pattern with fixed $\mathbb{L}$ in the same free homotopy class of L can be made into a composition of five types of homotopy moves described in Figure 2.4 (Lemma 2.2.6), with some holonomy movements. But here we focus only on homotopy moves, as holonomy moves are easier to deal with. (Actually, it was already shown in [CHL19] Lemma 5.3 that the gauge equivalence of Lagrangians yields an isomorphism in their mirror matrix factorizations.)



(a) Type I (b) Type II (c) Type III

(d) Type IV (e) Type V

Figure 2.4: 5 types of homotopy moves: L is green and $\mathbb{L}$ is red

The type I, II or III is when L slides across a self-intersection of L, an intersection of L and $\mathbb{L}$ or a self-intersection of $\mathbb{L}$, respectively. Later we will further divide type III into subcases III-i to III-iv as illustrated in Figure 2.5, according to the orientation patterns given to the curves, because they give different results. The type IV or V is when L unpoke out or poke in a part of L or $\mathbb{L}$, respectively. We sometimes denote by $\mathrm{IV}^+[\mathrm{IV}^-]$ and $\mathrm{V}^+[\mathrm{V}^-]$ for moves from the right[left] to the left[right] for convenience.

Then we look at how matrix factorizations are related under those moves. Type I, II, III-i, III-ii and IV moves don't change the resulting matrix factorizations at all, regardless of orientations (unless specified) on curves. In type III-iii and III-iv moves, however, they are no longer the same, but are still isomorphic to each other by some row or column operations. Especially, the finiteness of the matrix factorizations is preserved under type I, II, III and IV moves (Lemma 2.2.8).

In type V moves, the resulting matrix factorizations have different sizes so they can no longer be isomorphic in $\mathrm{MF}(xyz)$. However, we show that if L' is regular and $\mathscr{F}^{\mathbb{L}}(L)$ is finite, then $\mathscr{F}^{\mathbb{L}}(L')$ is also finite and completely determined by $\mathscr{F}^{\mathbb{L}}(L)$, and they are homotopic to each other in $\mathrm{MF}(xyz)$ (Lemma 2.2.9).

Finally, to show Proposition 2.2.3, we adopt the notion of '*admissibility*' introduced in the paper of Azam and Blanchet [AB22] and show that $\mathscr{F}^{\mathbb{L}}(L)$ is finite for any admissible loop L (Definition 2.2.12 and Corollary 2.2.19). Next we show that any regular loop L' can be deformed to an admissible loop L using finitely many homotopy (poking) moves of type V (Lemma 2.2.21). Then we deduce the finiteness of $\mathscr{F}^{\mathbb{L}}(L')$ from the finiteness of $\mathscr{F}^{\mathbb{L}}(L)$ using Lemmas 2.2.8 and 2.2.9 inductively.

After the finiteness is insured for regular loops, the proof of Proposition 2.2.5 follows automatically from Lemmas 2.2.6, 2.2.8 and 2.2.9.

2.2.1 Matrix Transformations under Homotopy Moves

In this subsection, we find the relations between matrix factorizations before and after the homotopy moves. In the process, we will have '*row/column operation*' according to a sliding of a curve in Lemma 2.2.8.(2) and the '*matrix reduction*' according to removing a bigon in Lemma 2.2.9. In each type, we derive explicit isomorphisms or homotopies. They are very useful for practical calculations as well as establishing finiteness or homotopy equivalence over $\mathbb{C}$.

Lemma 2.2.6. *If two unobstructed loops L and L' are freely homotopic to each other, the intersection pattern of L with $\mathbb{L}$ can be transformed to that of L' with $\mathbb{L}$ (and vice versa) by a finite composition of type I to V homotopy moves. Moreover, if L and L' are regular, we can also choose all loops that appear in the process as regular loops.*

Remark 2.2.7. *(1) Except the regularity condition, the assertion is just a classical Reidemeister move type proposition. So we only explain how to construct a homotopy sequence which preserves regularity.*

(2) To prove the lemma, we need the notion of 'strong admissibility' which will be introduced in Section 2.2.2. So we will give the proof after proving Lemma 2.2.21.

Next we seek how the matrix factorizations transform under each type of homotopy moves. We take the strategy to minimize direct disk countings and find non-trivial ones by A_∞-relations. It will be accomplished by finding isomorphisms in the Fukaya category between loops before and after the move.

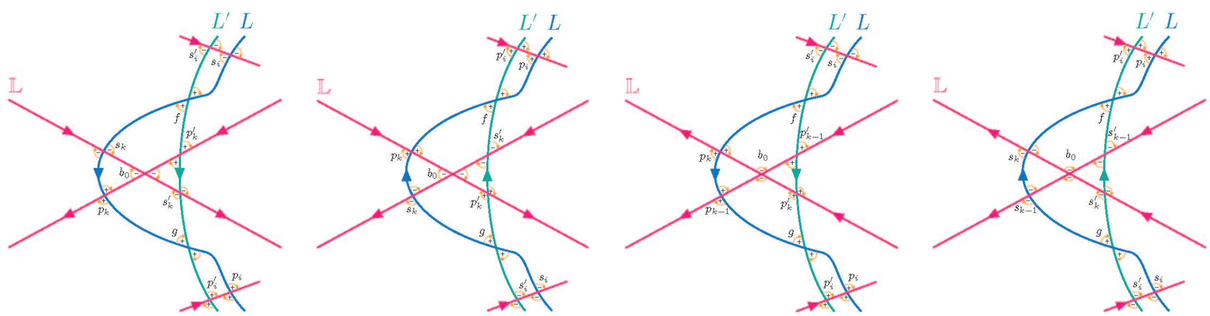
Lemma 2.2.8. *Let two unobstructed loops L and L' can be made into each other by one of type I to IV moves. Then their mirror matrix factorizations $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ and $\mathscr{F}^{\mathbb{L}}(L') = \left(\Phi^{\mathbb{L}}(L'), \Psi^{\mathbb{L}}(L')\right)$ are determined by each other by the following rules. Especially, if one is finite, so is the other.*

(1) In type I, II, III-i, III-ii and IV cases, $\mathscr{F}^{\mathbb{L}}(L)$ and $\mathscr{F}^{\mathbb{L}}(L')$ are exactly the same.

(2) In a type III-iii case, let χ be one of x, y or z according to whether b_0 is X, Y or Z (in Figure 2.5c). Then $\Phi^{\mathbb{L}}\left(L'\right)$ can be obtained from $\Phi^{\mathbb{L}}(L)$ by adding χ times the column corresponding to p_{k-1} to the column corresponding to p_k, and $\Psi^{\mathbb{L}}\left(L'\right)$ can be obtained from $\Psi^{\mathbb{L}}(L)$ by subtracting χ times the row corresponding to p_k from the row corresponding to p_{k-1}. In a type III-iv case, exactly the same holds if we exchange $\Phi^{\mathbb{L}} \leftrightarrow \Psi^{\mathbb{L}}$ and $p_i \leftrightarrow s_i$.

Proof. We restrict to a type III-iii case, but most part of the proof works also for the other cases.

We can assume that L' is a C^0-small perturbation of L as shown in Figure 2.5c. Namely, they have two intersections near the self-intersection $b_0 \in \{X, Y, Z\} \subset \operatorname{Hom}^1(\mathbb{L},\mathbb{L})$ of $\mathbb{L}$, denoted by $f \in \operatorname{Hom}^0\left(L', L\right)$ and $g \in \operatorname{Hom}^0\left(L, L'\right)$, so that b_0 is bounded by the '*small*' embedded bigon made by a perturbation L' of L between vertices f and g. We further assume that the remaining part of L is perturbed to L' in the opposite direction, which is possible as the underlying surface $\mathscr{P}$ is orientable. Therefore, we have one more '*long*' immersed bigon between f and g, only the ends of which are shown in the figure and the middle part of which might intersect with itself.



(a) Type III-i (b) Type III-ii (c) Type III-iii (d) Type III-iv

Figure 2.5: Isomorphisms in type III homotopy moves

Then the small and long bigons together give the following 4 equations:

$$\mathfrak{m}_1(f) = \overline{g} - \overline{g} = 0, \quad \mathfrak{m}_1(g) = -\overline{f} + \overline{f} = 0, \quad \mathfrak{m}_2(f, g) = \mathrm{id}_{L'} \quad \mathfrak{m}_2(g, f) = \mathrm{id}_L.$$

Here $\overline{g} \in \operatorname{Hom}^1\left(L', L\right)$ in the first equation is the complement of g, where the first and second terms come from the small and long bigons, respectively. These 4 equations show that f and g are isomorphisms between L and L' in the A_∞-category $\operatorname{Fuk}(\mathscr{P})$ (though we need only the first equation to prove this lemma).

Note that each intersection of L and $\mathbb{L}$ is paired with an intersection of L' and $\mathbb{L}$. So we can write

$$\operatorname{Hom}^0(L,\mathbb{L}) = \mathbb{C}\langle p_1, \ldots, p_k\rangle, \quad \operatorname{Hom}^1(L,\mathbb{L}) = \mathbb{C}\langle s_1, \ldots, s_k\rangle,$$
$$\operatorname{Hom}^0\left(L',\mathbb{L}\right) = \mathbb{C}\langle p_1', \ldots, p_k'\rangle, \quad \operatorname{Hom}^1\left(L',\mathbb{L}\right) = \mathbb{C}\langle s_1', \ldots, s_k'\rangle,$$

where p_{k-1}, p_{k-1}', p_k and p_k' denote the intersections on the boundary of the small bigon.

As $\mathscr{F}^{\mathbb{L}}$ is an A_∞-functor, we know from Definition 2.1.1 that

$$\mathfrak{m}_1^{\mathscr{MF}(xyz)}\left(\mathscr{F}_1^{\mathbb{L}}(f)\right) = \mathscr{F}_1^{\mathbb{L}}\left(\mathfrak{m}_1(f)\right)$$

holds. Its left side is calculated as

$$\left(\Phi^{\mathbb{L}}\left(L'\right)\circ\Phi_1^{\mathbb{L}}\left(f\right)-\Psi_1^{\mathbb{L}}\left(f\right)\circ\Phi^{\mathbb{L}}\left(L\right),\quad \Psi^{\mathbb{L}}\left(L'\right)\circ\Psi_1^{\mathbb{L}}\left(f\right)-\Phi_1^{\mathbb{L}}\left(f\right)\circ\Psi^{\mathbb{L}}\left(L\right)\right)\in \mathrm{Hom}^1\left(\mathscr{F}^{\mathbb{L}}\left(L\right),\mathscr{F}^{\mathbb{L}}\left(L'\right)\right),$$

whereas the right side is just $(0,0)$ as $\mathfrak{m}_1\left(f\right)=0$. This means that the following diagram commutes.

$$\begin{tikzcd}
S\langle p_1,\dots,p_k\rangle \arrow[r, "\Phi^{\mathbb{L}}(L)"] \arrow[d, "\Phi_1^{\mathbb{L}}(f)"] & S\langle s_1,\dots,s_k\rangle \arrow[r, "\Psi^{\mathbb{L}}(L)"] \arrow[d, "\Psi_1^{\mathbb{L}}(f)"] & S\langle p_1,\dots,p_k\rangle \arrow[d, "\Phi_1^{\mathbb{L}}(f)"] \\
S\langle p'_1,\dots,p'_k\rangle \arrow[r, "\Phi^{\mathbb{L}}(L')"] & S\langle s'_1,\dots,s'_k\rangle \arrow[r, "\Psi^{\mathbb{L}}(L')"] & S\langle p'_1,\dots,p'_k\rangle
\end{tikzcd}$$

Now we perform some polygon counting containing f to get

$$\Phi_1^{\mathbb{L}}\left(f\right)\left(p_i\right)=\mathfrak{m}_2^{0,0,b}\left(f,p_i\right)=\begin{cases} p'_i & \text{if } i=1,\dots,k-1,\\ -xp'_{k-1}+p'_k & \text{if } i=k\end{cases}\quad\text{and}\quad \Psi_1^{\mathbb{L}}\left(f\right)\left(s_i\right)=\mathfrak{m}_2^{0,0,b}\left(f,s_i\right)=s'_i\quad\text{if } i=1,\dots,k,$$

which are obvious from Figure 2.5c. These yield

$$\Phi_1^{\mathbb{L}}\left(f\right)=\begin{pmatrix} I_{k-1} & -x\mathbf{e}_{k-1}\\ 0 & 1\end{pmatrix}\quad\text{and}\quad \Psi_1^{\mathbb{L}}\left(f\right)=I_k$$

as matrices of maps $S^k\to S^k$, where $\mathbf{e}_{k-1}$ is the column vector in S^{k-1} whose $(k-1)$-th entry is 1 and the rest are 0. This proves the lemma in type III-iii case. In type I, II, III-i, III-ii and IV cases, we have $\Phi_1^{\mathbb{L}}\left(f\right)=\Psi_1^{\mathbb{L}}\left(f\right)=I_k$, and in type III-iv case, $\Phi_1^{\mathbb{L}}\left(f\right)$ and $\Psi_1^{\mathbb{L}}\left(f\right)$ are just swapped, which prove the lemma. □

We next illustrate the '*matrix reduction*' process according to a type V move, where L' is obtained from L by removing a bigon made by L and $\mathbb{L}$. As in Figure 2.6, we divided type V moves into 4 sub-cases according to the orientation patterns. Namely, in type V-i and V-ii cases (resp. V-iii and V-iv cases) the vertices of the bigon reverse (resp. preserve) the orientation of curves.

Lemma 2.2.9. *Let an unobstructed loop L' can be obtained from an unobstructed loop L by a type V^- homotopy move, that is, by removing a bigon made by L and $\mathbb{L}$. Suppose that L' is regular and $\mathscr{F}^{\mathbb{L}}(L)$ is finite so that written as*

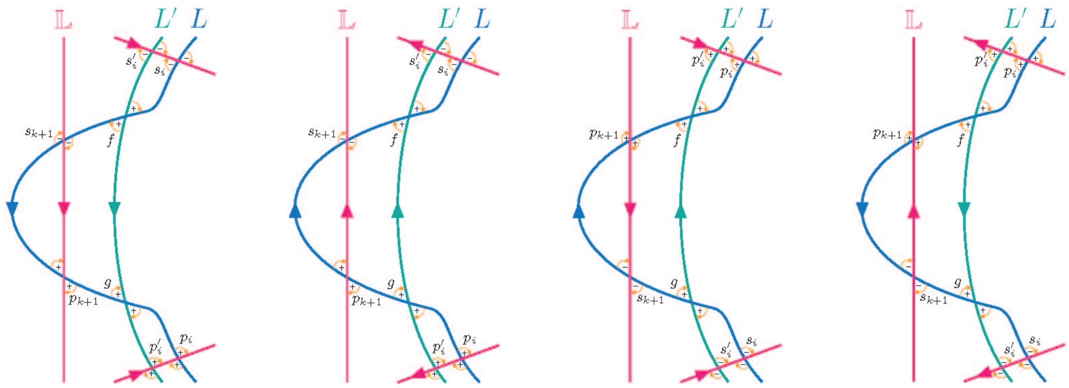
$$\mathscr{F}^{\mathbb{L}}\left(L\right)=\left(\Phi^{\mathbb{L}}\left(L\right),\Psi^{\mathbb{L}}\left(L\right):S^k\oplus S\to S^k\oplus S\right)=\left(\begin{pmatrix} C & D\\ E^T & u\end{pmatrix},\begin{pmatrix} F & G\\ H^T & v\end{pmatrix}\right),$$

for some matrices $C,F\in S^{k\times k}$, $D,E,G,H\in S^{k\times 1}$ and $u,v\in S$, where the last column and row of each matrix correspond to intersections p_{k+1} or s_{k+1} on the bigon (as in Figure 2.6). Then u (resp. v) is a unit in S in type V-i or V-ii cases (resp. V-iii or V-iv cases), and $\mathscr{F}^{\mathbb{L}}(L)$ reduces to the matrix factorization of L' as

$$\mathscr{F}^{\mathbb{L}}\left(L'\right)=\left(\Phi^{\mathbb{L}}\left(L'\right),\Psi^{\mathbb{L}}\left(L'\right):S^k\to S^k\right)=\begin{cases}\left(C-Du^{-1}E^T,\ F\right) & \textit{in type V-i or V-ii moves, or}\\ \left(C,\ F-Gv^{-1}H^T\right) & \textit{in type V-iii or V-iv moves.}\end{cases}$$

In particular, $\mathscr{F}^{\mathbb{L}}\left(L'\right)$ is also finite and completely determined by $\mathscr{F}^{\mathbb{L}}(L)$. Moreover, they are homotopically equivalent (over $\mathbb{C}$) to each other in $\mathrm{MF}\left(xyz\right)$.

Proof. We proceed in a similar way as in the proof of Lemma 2.2.8. Here we only consider type V-i case. For the V-ii type, the same can be done with a slight sign considerations. For the other cases, we can do the same by just switching the role of $\Phi^{\mathbb{L}}$ and $\Psi^{\mathbb{L}}$.



(a) Type V-i (b) Type V-ii (c) Type V-iii (d) Type V-iv

Figure 2.6: Isomorphisms in type V homotopy moves

As in the previous lemma, f and g are isomorphisms between L and L' in $\mathrm{Fuk}(\mathscr{P})$. But note in this case that L has 2 more intersections with $\mathbb{L}$ than L' does. So we can write

$$\mathrm{Hom}^0(L,\mathbb{L}) = \mathbb{C}\langle p_1,\ldots,p_k,p_{k+1}\rangle, \quad \mathrm{Hom}^1(L,\mathbb{L}) = \mathbb{C}\langle s_1,\ldots,s_k,s_{k+1}\rangle,$$
$$\mathrm{Hom}^0(L',\mathbb{L}) = \mathbb{C}\langle p'_1,\ldots,p'_k\rangle, \quad \mathrm{Hom}^1(L',\mathbb{L}) = \mathbb{C}\langle s'_1,\ldots,s'_k\rangle,$$

where p_{k+1} and s_{k+1} denote the intersections of L and $\mathbb{L}$ on the boundary of the small bigon.

Denote $\Phi^{\mathbb{L}}(L) =: \begin{pmatrix} C & D \\ E^T & u \end{pmatrix}$ and $\Psi^{\mathbb{L}}(L) =: \begin{pmatrix} F & G \\ H^T & v \end{pmatrix}$ as matrices of maps

$$S^k \oplus S \cong S\langle p_1,\ldots,p_k\rangle \oplus S\langle p_{k+1}\rangle \underset{\Psi^{\mathbb{L}}(L)}{\overset{\Phi^{\mathbb{L}}(L)}{\rightleftarrows}} S^k \oplus S \cong S\langle s_1,\ldots,s_k\rangle \oplus S\langle s_{k+1}\rangle.$$

Then we have the following commuting (ignoring gray arrows) diagram from an A_∞-relation on $\mathscr{F}^{\mathbb{L}}$ and the vanishing of $\mathfrak{m}_1(f)$ and $\mathfrak{m}_1(g)$, as in the previous lemma.

$$
\begin{tikzcd}
S\langle p_1,\dots,p_k\rangle \oplus S\langle p_{k+1}\rangle \arrow[r, "\Phi^{\mathbb{L}}(L)=\begin{pmatrix} C & D \\ E^T & u \end{pmatrix}"] \arrow[d, "\Phi_1^{\mathbb{L}}(f)=\begin{pmatrix} I_k & 0 \end{pmatrix}"] & S\langle s_1,\dots,s_k\rangle \oplus S\langle s_{k+1}\rangle \arrow[r, "\Psi^{\mathbb{L}}(L)=\begin{pmatrix} F & G \\ H^T & v \end{pmatrix}"] \arrow[d, "\Psi_1^{\mathbb{L}}(f)=\begin{pmatrix} I_k & -Du^{-1} \end{pmatrix}"] \arrow[ddl, "-\Psi_2^{\mathbb{L}}(g,f)=\begin{pmatrix} 0 & 0 \\ 0 & -u^{-1} \end{pmatrix}"'] & S\langle p_1,\dots,p_k\rangle \oplus S\langle p_{k+1}\rangle \arrow[d, "\Phi_1^{\mathbb{L}}(f)=\begin{pmatrix} I_k & 0 \end{pmatrix}"] \arrow[ddl, "-\Phi_2^{\mathbb{L}}(g,f)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}"'] \\
S\langle p'_1,\dots,p'_k\rangle \arrow[r, "\Phi^{\mathbb{L}}(L')=C-Du^{-1}E^T"] \arrow[d, "\Phi_1^{\mathbb{L}}(g)=\begin{pmatrix} I_k \\ -u^{-1}E^T \end{pmatrix}"] & S\langle s'_1,\dots,s'_k\rangle \arrow[r, "\Psi^{\mathbb{L}}(L')=F"] \arrow[d, "\Psi_1^{\mathbb{L}}(g)=\begin{pmatrix} I_k \\ 0 \end{pmatrix}"] & S\langle p'_1,\dots,p'_k\rangle \arrow[d, "\Phi_1^{\mathbb{L}}(g)=\begin{pmatrix} I_k \\ -u^{-1}E^T \end{pmatrix}"] \\
S\langle p_1,\dots,p_k\rangle \oplus S\langle p_{k+1}\rangle \arrow[r, "\Phi^{\mathbb{L}}(L)=\begin{pmatrix} C & D \\ E^T & u \end{pmatrix}"] & S\langle s_1,\dots,s_k\rangle \oplus S\langle s_{k+1}\rangle \arrow[r, "\Psi^{\mathbb{L}}(L)=\begin{pmatrix} F & G \\ H^T & v \end{pmatrix}"] & S\langle p_1,\dots,p_k\rangle \oplus S\langle p_{k+1}\rangle
\end{tikzcd}
$$

The second A_∞-relation on $\mathscr{F}^{\mathbb{L}}$ from Definition 2.1.1 evaluated at (g,f) is written as

$$
\begin{aligned}
\mathfrak{m}_1^{\mathscr{MF}(xyz)}\left(\mathscr{F}_2^{\mathbb{L}}(g,f)\right) &+ \mathfrak{m}_2^{\mathscr{MF}(xyz)}\left(\mathscr{F}_1^{\mathbb{L}}(g),\mathscr{F}_1^{\mathbb{L}}(f)\right) \\
&= \mathscr{F}_2^{\mathbb{L}}\left(\mathfrak{m}_1(g),f\right) + (-1)^{|g|'}\mathscr{F}_2^{\mathbb{L}}\left(g,\mathfrak{m}_1(f)\right) + \mathscr{F}_1^{\mathbb{L}}\left(\mathfrak{m}_2(g,f)\right).
\end{aligned}
$$

The left hand side is calculated as

$$
\begin{aligned}
\text{(LHS)} = \Big(&\Psi^{\mathbb{L}}(L)\circ\Phi_2^{\mathbb{L}}(g,f) + \Psi_2^{\mathbb{L}}(g,f)\circ\Phi^{\mathbb{L}}(L) + \Phi_1^{\mathbb{L}}(g)\circ\Phi_1^{\mathbb{L}}(f), \\
&\Phi^{\mathbb{L}}(L)\circ\Psi_2^{\mathbb{L}}(g,f) + \Phi_2^{\mathbb{L}}(g,f)\circ\Psi^{\mathbb{L}}(L) + \Psi_1^{\mathbb{L}}(g)\circ\Psi_1^{\mathbb{L}}(f)\Big)
\end{aligned}
$$

and the right hand side is done as

$$
\text{(RHS)} = \mathscr{F}_1^{\mathbb{L}}(\mathrm{id}_L) = \left(\Phi_1^{\mathbb{L}}(\mathrm{id}_L),\Psi_1^{\mathbb{L}}(\mathrm{id}_L)\right) = \left(\mathfrak{m}_2^{0,0,b}(\mathrm{id}_L,\bullet),\mathfrak{m}_2^{0,0,b}(\mathrm{id}_L,\bullet)\right) = (I_{k+1},I_{k+1}),
$$

using $\mathfrak{m}_1(f)=0$, $\mathfrak{m}_1(g)=0$, $\mathfrak{m}_2(g,f)=\mathrm{id}_L$ and the property of the unit element id_L. Thus comparing both sides yields two equations

$$
\begin{cases}
\Phi_1^{\mathbb{L}}(g)\circ\Phi_1^{\mathbb{L}}(f) = I_{k+1} + \Psi^{\mathbb{L}}(L)\circ\left(-\Phi_2^{\mathbb{L}}(g,f)\right) + \left(-\Psi_2^{\mathbb{L}}(g,f)\right)\circ\Phi^{\mathbb{L}}(L) \\
\Psi_1^{\mathbb{L}}(g)\circ\Psi_1^{\mathbb{L}}(f) = I_{k+1} + \Phi^{\mathbb{L}}(L)\circ\left(-\Psi_2^{\mathbb{L}}(g,f)\right) + \left(-\Phi_2^{\mathbb{L}}(g,f)\right)\circ\Psi^{\mathbb{L}}(L)
\end{cases}. \tag{2.2.1}
$$

On the other hand, by some direct polygon counting containing f or g, we get

$$\begin{cases}
\Phi_1^{\mathbb{L}}(f)(p_i) &= \mathfrak{m}_2^{0,0,b}(f,p_i) &= \begin{cases} p_i' & \text{for } i=1,\dots,k, \\ 0 & \text{for } i=k+1 \end{cases} & \Rightarrow & \Phi_1^{\mathbb{L}}(f) &= \begin{pmatrix} I_k & 0 \end{pmatrix} \\
\Psi_1^{\mathbb{L}}(f)(s_i) &= \mathfrak{m}_2^{0,0,b}(f,s_i) &= s_i' \quad \text{for } i=1,\dots,k & \Rightarrow & \Psi_1^{\mathbb{L}}(f) &= \begin{pmatrix} I_k & * \end{pmatrix} \\
\Phi_1^{\mathbb{L}}(g)(p_i') &= \mathfrak{m}_2^{0,0,b}(g,p_i') &= p_i+(*)p_{k+1} \quad \text{for } i=1,\dots,k & \Rightarrow & \Phi_1^{\mathbb{L}}(g) &= \begin{pmatrix} I_k \\ * \end{pmatrix} \\
\Psi_1^{\mathbb{L}}(g)(s_i') &= \mathfrak{m}_2^{0,0,b}(g,s_i') &= s_i \quad \text{for } i=1,\dots,k & \Rightarrow & \Psi_1^{\mathbb{L}}(g) &= \begin{pmatrix} I_k \\ 0 \end{pmatrix} \\
\Phi_2^{\mathbb{L}}(g,f)(p_i) &= \mathfrak{m}_3^{0,0,b}(g,f,p_i) &= 0 \quad \text{for } i=1,\dots,k+1 & \Rightarrow & \Phi_2^{\mathbb{L}}(g,f) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Psi_2^{\mathbb{L}}(g,f)(s_i) &= \mathfrak{m}_3^{0,0,b}(g,f,s_i) &= \begin{cases} 0 & \text{for } i=1,\dots,k, \\ w p_{k+1} & \text{for } i=k+1 \end{cases} & \Rightarrow & \Psi_2^{\mathbb{L}}(g,f) &= \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}
\end{cases}$$

for some $w \in S$. Here, a priori, some coefficient of a monomial in w might consists of an infinite sum so we cannot assume that w is indeed an element of S. But we will demonstrate that this is not the case in the next two paragraphs, under the regularity of L' and the finiteness of $\mathscr{F}^{\mathbb{L}}(L)$.

First, we claim that the constant term (not containing x, y or z) of w is just 1 (after substituting $T=1$). This is equivalent to showing that $\mathfrak{m}_3(g,f,s_{k+1})$ has only one term p_{k+1}, coming from the obvious smallest (embedded) quadrangle $(\square g f s_{k+1} \overline{p_{k+1}})$ in the figure with vertices g, f, s_{k+1} and $\overline{p_{k+1}}$. Indeed, if there is an another (immersed) quadrangle $(\tilde{\square} g f s_{k+1} \overline{p_{k+1}})$ having the same vertices, it should contain $(\square g f s_{k+1} \overline{p_{k+1}})$ at least twice at its ends, and subtracting one of them from one end would give an immersed cylinder bounded by L' and $\mathbb{L}$, which contradicts the regularity of L'.

Next, substituting the above computations into Equations 2.2.1 gives $wu = uw = 1$ (which should hold a priori over Λ). Therefore, we know that the constant term of u is also 1, implying that u is a unit (both in $\Lambda[[x,y,z]]$ and S). Furthermore, by the finiteness of $\mathscr{F}^{\mathbb{L}}(L)$, each coefficient of a monomial in u consists of a finite sum (in Λ before substituting $T=1$) and then so does $w=u^{-1}$, resulting in $w \in S$.

Equations 2.2.1 also fill in all other missing entries so that we have

$$\Psi_1^{\mathbb{L}}(f) = \begin{pmatrix} I_k & -Du^{-1} \end{pmatrix}, \quad \Phi_1^{\mathbb{L}}(g) = \begin{pmatrix} I_k \\ -u^{-1}E^T \end{pmatrix} \quad \text{and} \quad \Psi_2^{\mathbb{L}}(g,f) = \begin{pmatrix} 0 & 0 \\ 0 & u^{-1} \end{pmatrix}$$

and finally, commuting of the diagram determines

$$\Phi^{\mathbb{L}}(L') = C - Du^{-1}E^T \quad \text{and} \quad \Psi^{\mathbb{L}}(L') = F.$$

Note that $\mathscr{F}_1^{\mathbb{L}}(f) \circ \mathscr{F}_1^{\mathbb{L}}(g)$ is exactly the identity morphism of $\mathscr{F}^{\mathbb{L}}(L')$, and $\mathscr{F}_1^{\mathbb{L}}(g) \circ \mathscr{F}_1^{\mathbb{L}}(f)$ is homotopic to the identity morphism of $\mathscr{F}^{\mathbb{L}}(L)$ through an explicit homotopy

$$\left(-\Psi_2^{\mathbb{L}}(g,f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ -\Phi_2^{\mathbb{L}}(g,f) = \begin{pmatrix} 0 & 0 \\ 0 & -u^{-1} \end{pmatrix} \right),$$

as described in Equations 2.2.1. That is, $\mathscr{F}^{\mathbb{L}}(L)$ and $\mathscr{F}^{\mathbb{L}}(L')$ are homotopically equivalent. $\square$

An example of the matrix reduction according to removing a bigon was given in Proposition **??**. We remark here that actually a type III-iii or III-iv move can also be obtained by a composition of type III-i, III-ii and V moves. It would be a good exercise to check that the row/column operation process in Lemma 2.2.8.(2) matches that obtained from Lemma 2.2.8.(1) and the matrix reduction in Lemma 2.2.9.

Remark 2.2.10. *According to Eisenbud's theorem, two homotopically equivalent matrix factorizations of xyz should yield the same Cohen-Macaulay modules over A up to free modules. In the case of Lemma 2.2.9, in particular, we can find an explicit stable isomorphism between them as below.*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
S\langle p_1,\dots,p_k,p_{k+1}\rangle \arrow[r, "\Phi^{\mathbb{L}}(L)=\begin{pmatrix} C & D \\ E^T & u \end{pmatrix}"] \arrow[d, "\wr\shortparallel", "\begin{pmatrix} I_k & 0 \\ u^{-1}E^T & 1 \end{pmatrix}"'] & S\langle s_1,\dots,s_k,s_{k+1}\rangle \arrow[r, "\Psi^{\mathbb{L}}(L)=\begin{pmatrix} F & G \\ H^T & v \end{pmatrix}"] \arrow[d, "\wr\shortparallel"', "\begin{pmatrix} I_k & -Du^{-1} \\ 0 & u^{-1} \end{pmatrix}"] & S\langle p_1,\dots,p_k,p_{k+1}\rangle \arrow[d, "\wr\shortparallel"', "\begin{pmatrix} I_k & 0 \\ u^{-1}E^T & 1 \end{pmatrix}"] \\
S\langle p'_1,\dots,p'_k\rangle\oplus S \arrow[r, "\Phi^{\mathbb{L}}(L') = \begin{pmatrix} C-Du^{-1}E^T & 0 \\ 0 & 1 \end{pmatrix}"] & S\langle s'_1,\dots,s'_k\rangle\oplus S \arrow[r, "\Psi^{\mathbb{L}}(L') = \begin{pmatrix} F & 0 \\ 0 & xyz \end{pmatrix}"] & S\langle p'_1,\dots,p'_k\rangle\oplus S
\end{tikzcd}$$

Commuting of the diagram is immediate from some matrix calculations and the fact that $\left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ is a matrix factorization of xyz. Also, note that the vertical maps are all isomorphisms. From this we get isomorphisms

$$\begin{aligned}
\operatorname{coker} \underline{\Phi^{\mathbb{L}}(L)} &\cong \operatorname{coker} \underline{\begin{pmatrix} \Phi^{\mathbb{L}}(L') & 0 \\ 0 & 1 \end{pmatrix}} \cong \operatorname{coker} \underline{\Phi^{\mathbb{L}}(L')} \quad \textit{and} \\
\operatorname{coker} \underline{\Psi^{\mathbb{L}}(L)} &\cong \operatorname{coker} \underline{\begin{pmatrix} \Psi^{\mathbb{L}}(L') & 0 \\ 0 & 0 \end{pmatrix}} \cong \operatorname{coker} \underline{\Psi^{\mathbb{L}}(L')} \oplus A \quad \textit{in } \mathrm{CM}(A).
\end{aligned}$$

This is also closely related to the algebraic version of matrix reduction process used in Lemma 3.6.8 which corresponds to the removing a redundant generator of a module.

2.2.2 Admissibility

In this subsection, we recall the notion of '*admissibility*' introduced in the paper of Azam and Blanchet [AB22], and prove some related lemmas that we need for our propositions. In particular, we define the notion of '*strong admissibility*', which is a special type of admissibility that is easier to see whether a given loop has the property.

Let L be an unobstructed loop which intersects transversally with the Seidel Lagrangian $\mathbb{L}$ and L itself. Then $\mathscr{P} \setminus (L \cup \mathbb{L})$ and $(L \cup \mathbb{L}) \setminus \{\text{intersection points}\}$ have only finitely many path connected components. Define $C_2 = C_2(\mathscr{P}; \mathbb{L}, L; \mathbb{Z})$ as a free $\mathbb{Z}$-module generated by components which do not contain punctures. Also define $C_1 = C_1(\mathscr{P}; \mathbb{L}, L; \mathbb{Z})$ as a free $\mathbb{Z}$-module generated by components of $(L \cup \mathbb{L}) \setminus \{\text{intersections points}\}$. Then there is a natural boundary operator $\partial : C_2 \to C_1$. The orientations of elements in C_1 and C_2 are inherited from the orientation of $\mathscr{P}, L$ and $\mathbb{L}$. Let us call an element of C_i an *i-chain* and a basis element of C_i an *i-basis* for $i = 1, 2$. Let us denote by $\langle x, \tau \rangle$ the coefficient of τ in x, where τ is an i-basis and x is an i-chain for $i = 1, 2$. Note that for any 2-basis σ and 1-basis τ, $\langle \partial\sigma, \tau \rangle = 1, 0,$

or -1. Also note that for each 1-basis τ, there are at most two 2-basis σ such that $\langle \partial\sigma, \tau\rangle \neq 0$ and if there are two, then they should have opposite sign.

Note that there are exactly three component containing punctures. Let us denote them by A, B, and C. By measuring distance from the puncture, we can give a natural grading called *level* to each 1- and 2-basis as follows. First, boundary components of A, B and C are set to be of level 0. If a 2-basis has a boundary component of level 0, then is is said to be of level 1. For a positive integer n, a 1-basis is said to be of level n if it is not of level k for each $k < n$, and it is a boundary of some level n-component. Also, a 2-basis is said to be level n if it is not of level k for each $k < n$, and it has a boundary component of level $(n-1)$. Since basis is finite, level is well defined.

Lemma 2.2.11. *The boundary map ∂ is injective.*

Proof. Suppose that x is a 2-chain such that $\partial(x) = 0$. Take a 2-basis σ of minimal level k in x. Then, by definition, ∂x has level k boundary 1-basis. To eliminate it, there should be 2-basis of level $(k-1)$ in x, which contradicts minimality of σ. Thus $x = 0$, which implies that ∂ is injective. □

Now recall the definition of the Euler measure $e : C_2 \to \mathbb{Z}$. For a 2-basis σ, define its Euler measure as $e(\sigma) = 1 - \frac{1}{4}\#\{\text{boundary components of } \sigma\}$ and extend linearly to whole of C_2. Let $[\mathbb{L}]$ and $[L]$ be 1-chains obtained from the Lagrangians $\mathbb{L}$ and L, respectively and define a set $H(L)$ consisting of 2-chains with Euler measure 0 whose boundary is a linear combination of $[\mathbb{L}]$ and $[L]$.

Definition 2.2.12. *An unobstructed loop L is said to be* admissible *if any nonzero 2-chain in $H(L)$ has both positive and negative coefficients.*

Remark 2.2.13. *The regular loops turn out to be a wider class than the admissible loops. Indeed, we can show with an obvious argument that an admissible loop is always regular, but the converse is false.*

Example 2.2.14. *Let L be an unobstructed loop. Then there are three components containing puncture, namely A, B, and C. Suppose that the boundary $\partial(A), \partial(B), \partial(C)$ have both positive and negative components of $[L]$. Since the boundary operator ∂ is injective and A, B, and C are not 2-basis, there are not positive or negative 2-chain of which boundary is a linear combination of $[L]$ and $[\mathbb{L}]$. Thus L is admissible. This motivates the following definition.*

Definition 2.2.15. *An unobstructed loop L is said to be* strongly admissible *if the boundary of punctured components A, B, C have both positive and negative components.*

From the definition, the proposition below immediately follows.

Lemma 2.2.16. *If an unobstructed loop is strongly admissible, then it is admissible.*

Let $p, q \in L \cap \mathbb{L}$ and $b_1, \cdots, b_r$ be self intersection points of $\mathbb{L}$, and denote the set of decorated strips whose vertices are $p, b_1, \cdots, b_r, q$ by $\mathcal{M}(p, b_1, \cdots, b_r, q)$. Note that an element of $\mathcal{M}(p, b_1, \cdots, b_r, q)$ gives a 2-chain. By mimicking the proof of Lemma 3.3, Proposition 3.4, and Corollary 3.5 in [AB22], we get the following result.

Lemma 2.2.17. *Let n be a positive integer and $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. Then for any infinite subset $S \subseteq \mathbb{Z}^n_{\geq 0}$, there are distinct elements $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ such that $x_i \leq y_i$ for all i.*

Lemma 2.2.18. *Let L be an admissible loop and $p, q \in L \cap \mathbb{L}$ be intersection points. Let also $b_1, \cdots, b_r \in \{X, Y, Z\}$ be self intersections of $\mathbb{L}$. Then the moduli space $\mathcal{M}(p, b_1, \cdots, b_r, q)$ is finite.*

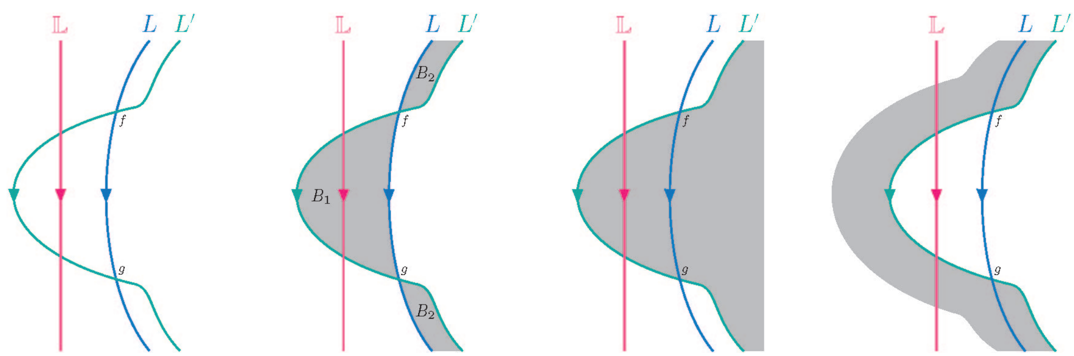
Corollary 2.2.19. *For an admissible loop L, its mirror matrix factorization $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ is finite.*

As pointed out in Remark 2.2.13, there are regular loops that are not admissible. To show that the finiteness still holds for regular loops, by Lemma 2.2.9, it is enough to show that any regular loop can be obtained from an admissible loop by only some type V^- moves (Lemma 2.2.21). But we also require the intermediate loops to be regular, for the proof of Lemma 2.2.6. So we first show the following lemma.

Lemma 2.2.20. *Let L be a regular loop and L' be a loop obtained from L by type I, II, III, IV and V^+ moves. Then L' is also regular.*

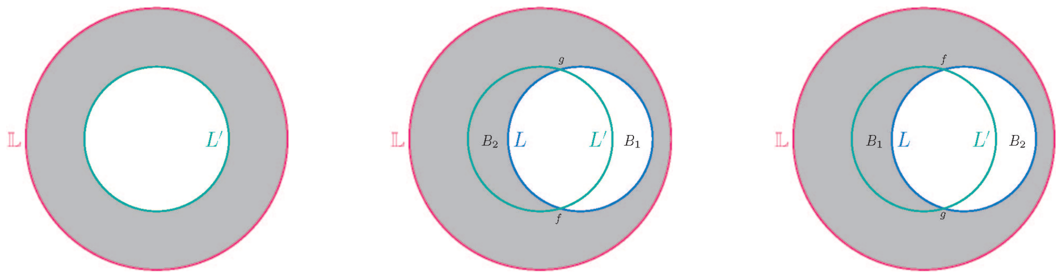
Proof. As already mentioned above, type I, II and IV moves do not change intersection patters of L with $\mathbb{L}$, so they do not affect regularity. We restrict to the type V^+ case, but the proof also works for the type III case. Let us prove that if a perturbed Lagrangian is not regular, then so is the original one.

Let L' be a Type V^+-move of L as in Figure 2.7a, and let us denote B_1, B_2 the bigons with vertices f, g as in Figure 2.7b. Suppose that L' is not regular so that there is an immersed cylinder bounded by L' and $\mathbb{L}$. There are two cases depending on the direction of the cylinder as in Figure 2.7c and 2.7d. Suppose that the cylinder is on the left. Then the bigon B_1 is totally contained in the cylinder. Thus, by cutting out B_1 and attaching B_2 to the cylinder, we get another cylinder bounded by L and $\mathbb{L}$. The old and new cylinders are illustrated in Figure 2.8. Similarly, if the cylinder is on the right, one can obtain a new cylinder by cutting out B_2 and attaching B_1. Hence we prove that if L' is non-regular, so is L.



(a) Perturbation (b) Bigons (c) Cylinder at the left side (d) Cylinder at the right disc

Figure 2.7



(a) Domain of the cylinder (b) Perturbed domain of the left side cylinder (c) Perturbed domain of the right side cylinder

Figure 2.8: Cylinders

□

Lemma 2.2.21. *For any regular loop L', there are regular loops $L' = L^{(0)}, L^{(1)}, \dots, L^{(d)} = L$ satisfying the following:*

- *L is admissible, and*
- *For $i = 1, \dots, d$, the intersection pattern of $L^{(i)}$ with $\mathbb{L}$ is obtained from that of $L^{(i-1)}$ with $\mathbb{L}$ by performing a move of type V^+.*

Proof. We show that any regular loop can be deformed into a strongly admissible one by using only type V^+ moves, which preserve regularity by Lemma 2.2.20. Choose any puncture, say a and take paths from a to ℓ_x and $\tilde{\ell}_x$. We may take the paths so that they do not go through self intersections of L' and meet L' transversally. Then perturb L' along the path as in Figure 2.9, the component containing a has both positive and negative boundary component of the Seidel Lagrangian, which implies that the resulting loop is strongly admissible. This proves the lemma.

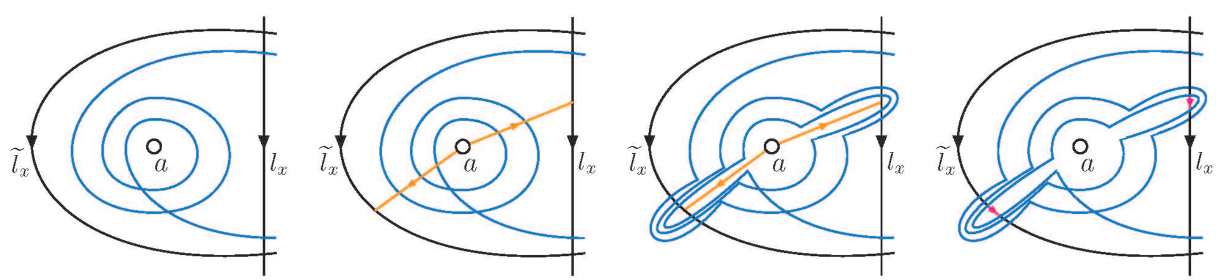


Figure 2.9: Deforming process

□

Now we are ready to give the proof of Lemma 2.2.6.

Proof of Lemma 2.2.6. Suppose that homotopic unobstructed regular loops L_0, L_1 are given. In the proof of Lemma 2.2.21, we may take paths from a puncture to Seidel Lagrangian so that they satisfy intersection condition for L_0 and L_1 simultaneously. Then L_0 and L_1 are deformed to strongly admissible loops $\tilde{L}_0$ and $\tilde{L}_1$, while maintaining the regularity. Note that two loops $\tilde{L}_0$ and $\tilde{L}_1$ are still homotopic to each other (As in Figure 2.10, we may pretend that the puncture became larger, containing the chosen paths). Moreover, loops appearing through the homotopy are still strongly admissible, which implies regularity. Thus we have a sequence of regular loops connecting L_0 and L_1.

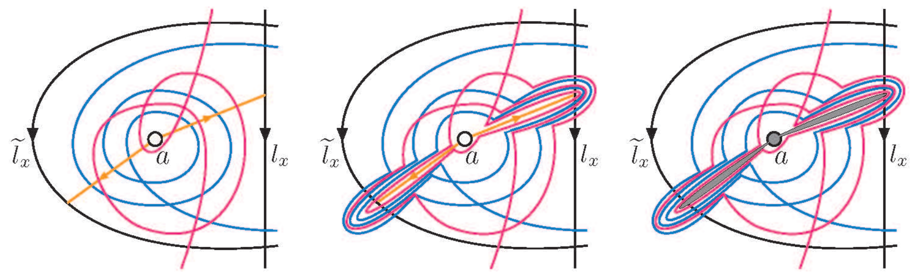


Figure 2.10: Deforming process

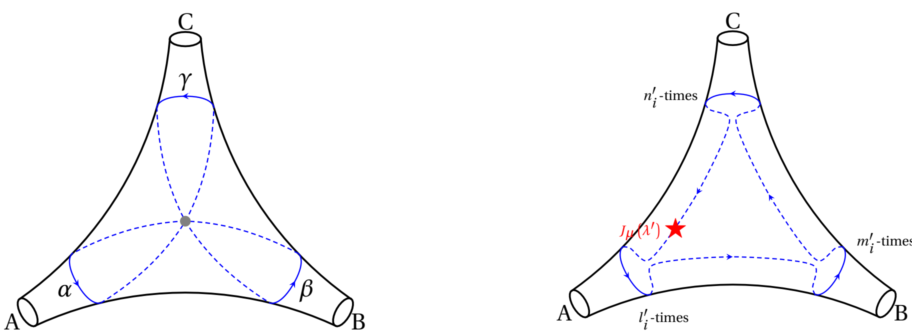
□

2.3 Loop Data and Arc Data

The homotopy invariance of $\mathscr{F}^{\mathbb{L}}$ suggests that it is enough for us to consider only one loop in each essential free homotopy class, and only one arc in each homotopy class keeping the end points in the same boundaries. So we will take some specific representative in each class.

2.3.1 Loop/arc words for immersed Lagrangians

Note that the fundamental group of $\mathscr{P}$ can be presented as $\pi_1(\mathscr{P}) = \langle \alpha, \beta, \gamma \,|\, \alpha\beta\gamma = 1 \rangle$ with the based loops α, β and γ in $\mathscr{P}$ shown in Figure 2.11a. Also recall that there is a one-to-one correspondence between the free homotopy classes and the conjugacy classes in $\pi_1(\mathscr{P})$.



(a) Generators α, β, γ of $\pi_1(\mathscr{P})$

(b) Canonical form $L(w', \lambda', \mu)$ of loops with holonomy

Figure 2.11: Fundamental group and loop data

Definition 2.3.1. *A* ***loop word*** *of* length 3τ *is*

$$w' = (l_1', m_1', n_1', l_2', m_2', n_2', \dots, l_\tau', m_\tau', n_\tau') \in \mathbb{Z}^{3\tau}$$

($\tau \in \mathbb{Z}_{\geq 1}$). The associated loop $L(w')$ is illustrated in Figure 2.11b. It visits 3 holes A, B, and C in turn, winding them around the number of times specified in w'. Namely, it winds hole A l_1'-times, hole B m_1'-times, hole C n_1'-times, hole A l_2'-times, hole B m_2'-times, and so on. After finally it winds hole C n_τ'-times, it returns to the starting point to form a closed loop. Note that its free homotopy class in $[S^1, \mathscr{P}]$ is

$$[L(w')] = \left[\alpha^{l_1'}\beta^{m_1'}\gamma^{n_1'}\alpha^{l_2'}\beta^{m_2'}\gamma^{n_2'}\cdots\alpha^{l_\tau'}\beta^{m_\tau'}\gamma^{n_\tau'}\right].$$

An ***arc word*** *w' is an element in one of the sets*

$$\begin{array}{lll} \{A\} \times \mathbb{Z}^{3\tau-1} \times \{A\}, & \{A\} \times \mathbb{Z}^{3\tau-3} \times \{B\}, & \{A\} \times \mathbb{Z}^{3\tau-2} \times \{C\}, \\ \{B\} \times \mathbb{Z}^{3\tau-2} \times \{A\}, & \{B\} \times \mathbb{Z}^{3\tau-1} \times \{B\}, & \{B\} \times \mathbb{Z}^{3\tau-3} \times \{C\}, \\ \{C\} \times \mathbb{Z}^{3\tau-3} \times \{A\}, & \{C\} \times \mathbb{Z}^{3\tau-2} \times \{B\}, & \{C\} \times \mathbb{Z}^{3\tau-1} \times \{C\} \end{array}$$

($\tau \in \mathbb{Z}_{\geq 1}$). The associated arc $L(w')$ starts from the hole specified in the first entry of w' and ends at the hole specified in the last entry of w'. In between, it sequentially winds holes A, B and C in turn the number of times specified in the remaining entries of w'. Its homotopy class is denoted as $[L(w')]$. See Example 1.0.2.

Two loop/arc words w' and w_*' are regarded as equivalent *if* $[L(w')] = [L(w_*')]$.

We sometimes denote the j-th value of a loop/arc word w' as w'_j so that

$$w' = \left(w'_1, w'_2, w'_3, \ldots, w'_{3\tau-1}, w'_{3\tau}\right) \in \mathbb{Z}^{3\tau} \quad \text{in the loop case, and}$$
$$w' = \left(*, w'_1, w'_2, \ldots, w'_{3\tau-*}, *\right) \in \{\mathrm{A},\mathrm{B},\mathrm{C}\} \times \mathbb{Z}^{3\tau-*} \times \{\mathrm{A},\mathrm{B},\mathrm{C}\} \quad \text{in the arc case.}$$

Then any tuple $(w'_k, w'_{k+1}, \ldots, w'_l)$ for some distinct $k, l \in \mathbb{Z}_{3\tau}$ is called a *subword* in w'.

In the loop case, we regard the index i of l'_i, m'_i and n'_i to be in $\mathbb{Z}_\tau$ (hence $3i \in \mathbb{Z}_{3\tau}$) and the index j of w'_j to be in $\mathbb{Z}_{3\tau}$. Therefore, for example, $(w'_{3\tau-1}, w'_{3\tau}, w'_1)$ is a subword. We define the 1-*shift* of a loop word w' to be

$$w'^{(1)} = \left(l'_2, m'_2, n'_2, \ldots, l'_\tau, m'_\tau, n'_\tau, l'_1, m'_1, n'_1\right) \in \mathbb{Z}^{3\tau}$$

and k-*shift* to be $w'^{(k)}$ which is obtained from w' by applying the 1-shift k-times ($k \in \mathbb{Z}$).

The following lemma is easy to check.

Lemma 2.3.2. *The following operations on a loop/arc word w' do not change the equivalence class of w':*

- *(inserting 0s) insert the subword $(0,0,0)$ somewhere in the middle of w',*
- *(removing 0s) remove a subword $(0,0,0)$ in w' if it exists,*
- *(adding 1s around 0) add $(1,1,1)$ to the subword $(w'_{j-1}, 0, w'_{j+1})$ in w' where $w'_j = 0$, and*
- *(subtracting 1s around 1) subtract $(1,1,1)$ from the subword $(w'_{j-1}, 1, w'_{j+1})$ in w' where $w'_j = 1$,*
- *(shifting) take k-shift of a loop word w' for some $k \in \mathbb{Z}$.*

The converse statement is also true, but its proof involves some non-trivial word problem.

Proposition 2.3.3. *Two loop/arc words w' and w'_* are equivalent if and only if w'_* can be obtained from w' by performing the above operations finitely many times.*

Proof. See [CJKR22]. □

Note that several equivalent loop/arc words can represent the same (free) homotopy class. To find a unique representative in each class, we propose the following **normal form** of loop/arc words. It turns out that this also plays an important role in the conversion formula between loop/arc data and band/string data.

Definition 2.3.4. *A loop/arc word w' is said to be **normal** if it satisfies the following conditions:*

- *any subword of the form $(a,1,b)$ in w' satisfies $a, b \leq 0$,*
- *any subword of the form $(a,0,b)$ in w' satisfies $a \leq -1$, $b \geq 1$ or $a \geq 1$, $b \leq -1$ or $a, b \geq 1$,*
- *w' has no subword of the form $(0,-1,-1,\ldots,-1,0)$, and*
- *in the loop case, w' does not consist only of -1, that is, $w' \neq (-1,-1,\ldots,-1)$.*

We say that a loop L is *non-essential* if it winds around only one of three holes. A loop/arc which is not freely homotopic to any non-essential loop is called *essential.* A non-essential loop L can always be deformed so that it doesn't meet the reference $\mathbb{L}$ at all, which means that $\mathscr{F}^{\mathbb{L}}(L)$ is stably trivial. Therefore, non-essential loops will be excluded from our consideration.

According to Definition 2.3.1, non-essential loops are produced by *non-essential* loop words, which are by definition equivalent to a loop word of the form $\left(l',0,0\right)$, $\left(0,m',0\right)$ or $\left(0,0,n'\right)$ for some $l',m',n' \in \mathbb{Z}$. Loop/arc words other than these are called *essential.* Interestingly, the above normality condition automatically rules out non-essential loop words. Except for those, the normal form up to shifting gives exactly one representative among equivalent essential loop/arc words.

Proposition 2.3.5. *Any normal loop word is essential. Moreover, any essential loop/arc word is equivalent to a unique normal loop/arc word up to shifting.*

Proof. See [CJKR22]. □

For an immersed loop $L : S^1 \to \mathscr{P}$, we define its N*-concatenation* by the immersed loop

$$L^{\#N} : S^1 \to \mathscr{P}, \quad e^{2\pi i t} \mapsto L\left(e^{2N\pi i t}\right).$$

A free homotopy class of loops is called *periodic* if it is represented by an N-concatenation of another loop for some $N \in \mathbb{Z}_{\geq 2}$.

For a normal loop word w', we define its N*-concatenation* $\left(w'\right)^{\#N}$ as the N repetitions of w'. It is called *periodic* if it is N-concatenation of another normal loop word for some $N \in \mathbb{Z}_{\geq 2}$. The notion of concatenation and periodicity for a normal loop word w' and corresponding loop $L\left(w'\right)$ (or $\left[L\left(w'\right)\right]$) is compatible in an obvious way.

2.3.2 Regularity of $L\left(w'\right)$ and perturbation in case $w' = (2,2,2)^{\#\tau}$

According to Proposition 2.2.3, the regularity of loop/arc $L\left(w'\right)$ must be guaranteed to find its mirror image.

Proposition 2.3.6. *For a normal loop/arc word w' other than of the form $(2,2,2)^{\#\tau}$, the corresponding loop/arc $L\left(w'\right)$ is regular.*

In case of $w' = (2,2,2)^{\#\tau}$ for some $\tau \in \mathbb{Z}_{\geq 1}$, the corresponding loop $L\left((2,2,2)^{\#\tau}\right)$ has the same free homotopy type with $\mathbb{L}^{\#(-\tau)}$. Then they have the potential to bound an immersed cylinder, which breaks the regularity of $L\left((2,2,2)^{\#\tau}\right)$. To prevent this, we give a very specific perturbed version for this case. We first define the loop $L((2,2,2))$ as shown in Figure 2.12. Then we define $L\left((2,2,2)^{\#\tau}\right)$ as the τ-concatenation of it.

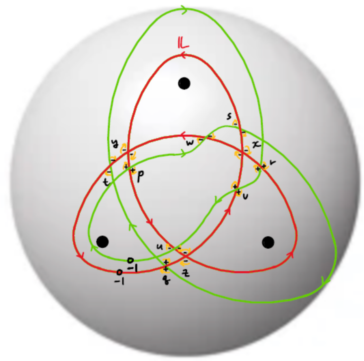


Figure 2.12: Perturbed loop $L((2,2,2))$

2.3.3 Normal loop/arc words and geodesics

Give $\mathscr{P}$ a hyperbolic metric on with three cusps. It can be achieved by considering the Poincaré disc as the universal cover of $\mathscr{P}$ as shown in Figure 2.13.

From an elementary fact in hyperbolic geometry, we know that there is at most one geodesic in each (free) homotopy class of loops/arcs in $\mathscr{P}$. There is no geodesic that winds around only one cusp, and there is exactly one for the other classes. That is, geodesics are in one-to-one correspondence with essential (free) homotopy classes. This provides another nice description of normal loop/arc words.

Proposition 2.3.7. *We give $\mathscr{P}$ a hyperbolic metric with three cusps. Then there are one-to-one correspondences*

$$\{\textit{normal loop words}\}/\sim_{\textit{shifting}} \overset{1:1}{\leftrightarrow} \{\textit{closed geodesics in } \mathscr{P}\}$$
$$\& \ \{\textit{normal arc words}\} \overset{1:1}{\leftrightarrow} \{\textit{open geodesics in } \mathscr{P}\}.$$

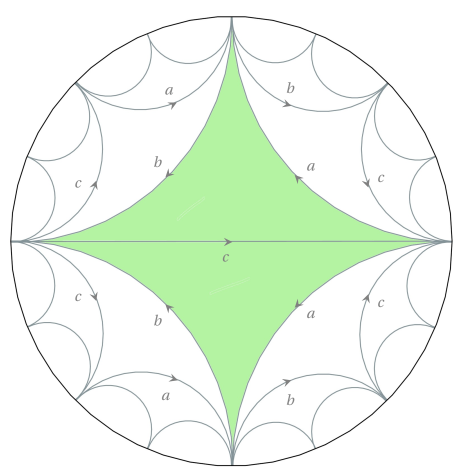


Figure 2.13: Fundamental domain of $\mathscr{P}$ in its universal cover (Poincaré disc)

2.3.4 Holonomies on loops and canonical form in $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$

As total holonomies on a loop $L(w')$, we consider only one in each conjugacy class, namely, the $\mu \times \mu$ Jordan block $J_{\mu}(\lambda') \in \mathrm{GL}_{\mu}(\Bbbk)$ with eigenvalue $\lambda' \in \Bbbk^{\times}$.

Definition 2.3.8. *(1) A* ***loop datum*** (w', λ', μ) *consists of the following:*

- *(normal loop word)* $w' = (l'_1, m'_1, n'_1, l'_2, m'_2, n'_2, \ldots, l'_{\tau}, m'_{\tau}, n'_{\tau}) \in \mathbb{Z}^{3\tau}$ *for some* $\tau \in \mathbb{Z}_{\geq 1}$,
- *(holonomy parameter)* $\lambda' \in \Bbbk^{\times}$,
- *(multiplicity)* $\mu \in \mathbb{Z}_{\geq 1}$.

It represents the loop $L(w')$ *equipped with a rank* μ *local system whose total holonomy is* $J_{\mu}(\lambda)$, *which we denote as*

$$L(w', \lambda', \mu).$$

For a non-periodic normal loop word w', *we refer to it as the* ***canonical form of loop-type indecomposable objects*** *in* $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$.

(2) An ***arc datum*** w' *consists of only a* ***normal arc word*** $w' \in \{A, B, C\} \times \mathbb{Z}^{3\tau - *} \times \{A, B, C\}$. *We refer to the corresponding arc*

$$L(w')$$

as the ***canonical form of arc-type indecomposable objects*** *in* $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$.

2.4 Matrix Factorizations from Lagrangians: Loop-Type and Arc-Type

Now we compute the mirror image of canonical form of loop/arc-type objects in $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$. For a loop datum (w', λ', μ) where

$$w' = \left(l'_1, m'_1, n'_1, l'_2, m'_2, n'_2, \ldots, l'_\tau, m'_\tau, n'_\tau\right),$$

the corresponding loop $L(w')$ has 6τ intersections with $\mathbb{L}$. We name even-degree angles from L to $\mathbb{L}$ by $p_1, q_1, r_1, \ldots, p_\tau, q_\tau, r_\tau \in \mathrm{Hom}^0(L, \mathbb{L})$ and odd-degree angles from L to $\mathbb{L}$ by $s_1, t_1, u_1, \ldots, s_\tau, t_\tau, u_\tau \in \mathrm{Hom}^1(L, \mathbb{L})$ in the order in L.

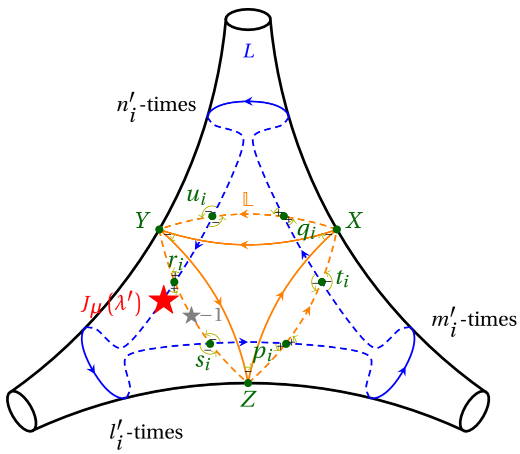


Figure 2.14: Elements in $\mathrm{Hom}(L, \mathbb{L})$

Proposition 2.4.1. *The matrix factor corresponding to $L(w', \lambda', \mu)$ is given by*

$$\Phi^{\mathbb{L}}\left(L\left(w', \lambda', \mu\right)\right) = \begin{pmatrix} zI_\mu & -y^{m'_1-1}I_\mu & 0 & 0 & \cdots & 0 & -(-x)^{-l'_1}J_\mu(\lambda)^{-1} \\ y^{-m'_1}I_\mu & xI_\mu & -z^{n'_1-1}I_\mu & 0 & \cdots & 0 & 0 \\ 0 & z^{-n'_1}I_\mu & yI_\mu & -(-x)^{l'_2-1}I_\mu & \cdots & 0 & 0 \\ 0 & 0 & -(-x)^{-l'_2}I_\mu & zI_\mu & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -y^{m'_\tau-1}I_\mu & 0 \\ 0 & 0 & 0 & \cdots & y^{-m'_\tau}I_\mu & xI_\mu & -z^{n'_\tau-1}I_\mu \\ -(-x)^{l'_1-1}J_\mu(\lambda) & 0 & 0 & \cdots & 0 & z^{-n'_\tau}I_\mu & yI_\mu \end{pmatrix}_{3\tau\mu\times 3\tau\mu}.$$

where x^a, y^a, z^a are regarded as 0 if $a < 0$

Theorem 2.4.2. *(1) For a non-degenerate loop datum (w', λ', μ), there is an isomorphism*

$$\mathscr{F}^{\mathbb{L}}\left(L\left(w', \lambda', \mu\right)\right) \;\cong\; \left(\varphi\left(w', \lambda, \mu\right), \psi\left(w', \lambda, \mu\right)\right)$$

in $\underline{\mathrm{MF}}(xyz)$, where λ is either λ' or $-\lambda'$ depending on w'.

In degenerate cases ($w' = (2,2,2)^{\#\tau}, \lambda' = -1$), we have

$$\mathscr{F}^{\mathbb{L}}\left(L\left((2,2,2)^{\#\tau}, -1, \mu\right)\right) \;\cong\; \left(\varphi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right), \psi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right)\right).$$

(2) For an arc datum w', there is an isomorphism

$$\mathscr{F}^{\mathbb{L}}\left(L\left(w'\right)\right) \;\cong\; \left(\varphi\left(w'\right), \psi\left(w'\right)\right).$$

Inspired by this observation, we propose the **canonical form of matrix factorizations of** xyz in the following way.

Definition 2.4.3. *(1) For a normal loop word $w' \in \mathbb{Z}^{3\tau}$ ($\tau \in \mathbb{Z}_{\geq 1}$), a nonzero scalar $\lambda \in \Bbbk^{\times}$, and a positive integer $\mu \in \mathbb{Z}_{\geq 1}$, we refer to the corresponding matrix factorization*

$$\left(\varphi\left(w', \lambda, \mu\right), \psi\left(w', \lambda, \mu\right)\right) \quad \textit{or} \quad \left(\varphi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right), \psi_{\deg}\left((2,2,2)^{\#\tau}, 1, \mu\right)\right)$$

*as the **canonical form of*** **loop-type** ***objects** in $\underline{\mathrm{MF}}\left(xyz\right)$. The latter is chosen only in degenerate cases ($w' = (2,2,2)^{\#\tau}$, $\lambda = 1$).*

(2) For a normal arc word $w' \in \{A, B, C\} \times \mathbb{Z}^{3\tau - *} \times \{A, B, C\}$, we refer to the corresponding matrix factorization

$$\left(\varphi\left(w'\right), \psi\left(w'\right)\right)$$

*as the **canonical form of*** **arc-type** ***objects** in $\underline{\mathrm{MF}}\left(xyz\right)$.*

Chapter 3

Representation of Cohen-Macaulay Modules over Singularity $xyz = 0$

3.1 Cohen-Macaulay Modules and Eisenbud's Matrix Factorization Theorem

In this section, we give a gentle introduction to the necessary algebraic notions for geometry minded readers.

3.1.1 Algebraic preliminaries

Recall that for a local ring $(A, \mathfrak{m})$, the inverse system $(A/\mathfrak{m}^n)$ gives the *completion* $\hat{A} = \varprojlim(A/\mathfrak{m}^n)$ together with the canonical map $\iota_A : A \to \hat{A}$ induced from the canonical projection $A \to A/\mathfrak{m}^n$ for each $n \in \mathbb{Z}_{>0}$. A local ring $(A, \mathfrak{m})$ is said to be *complete* if the canonical map ι_A is an isomorphism. We are interested in a complete reduced Noetherian local ring with the residue field $\mathbb{C}$. For example, the power series ring $\mathbb{C}[[x_1, \cdots, x_n]]$ and its quotient ring $\mathbb{C}[[x_1, \cdots, x_n]]/I$, where I is a radical ideal, are complete reduced local ring with the maximal ideal $(x_1, \cdots, x_n)$. Note that each of them has the residue field $\mathbb{C}$. Also we recall some notion of dimension.

Definition 3.1.1. *The* Krull dimension $\dim_{Kr}(A)$ *of a local ring* $(A, \mathfrak{m})$ *is defined as the maximal length of chains of prime ideals.*

For example, the Krull dimension of $\mathbb{C}[[x]]$ is 1 since it has only one non-zero prime ideal (x). More generally, the Krull dimension of the power series ring $\mathbb{C}[[x_1, \cdots, x_n]]$ is n. The following example is our main object.

Example 3.1.2. *The Krull dimension of* $A = \mathbb{C}[[x, y, z]]/(xyz)$ *is* 2. *Indeed, suppose that* $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ *be a maximal chain of prime ideals of* A. *Then, since* $xyz = 0$ *in* A, *one of* $x, y,$ *and* z *should be in* $\mathfrak{p}_0$. *Without loss of generality, we may assume* $x \in \mathfrak{p}_0$. *Then the chain can be viewed as a chain of prime ideals of* $A/(x) \cong \mathbb{C}[[y, z]]$. *Hence* $\dim_{Kr}(A) = \dim_{Kr}(\mathbb{C}[[y, z]]) = 2$.

There is also a notion of algebraic dimension, *depth*.

Definition 3.1.3. *Let* $(A, \mathfrak{m})$ *be a local ring. A sequence of elements* $(f_1, \cdots, f_r)$ *is said to be* regular *if the following are satisfied.*

- f_1 *is neither a unit nor a zero divisor in* A.
- *For each* $i \geq 2$, f_i *is neither a unit nor a zero divisor in* $A / \left(f_1, \cdots, f_{i-1}\right)$.

The depth *of* A *is the maximal length of regular sequences of* A *and is denoted by* $\operatorname{depth}(A)$.

There is a ring whose Krull dimension and depth are not equal. If the Krull dimension and depth of a ring are the same, the ring is called a Cohen-Macaulay ring.

Definition 3.1.4. *A local ring* $(A, \mathfrak{m})$ *is said to be* Cohen-Macaulay *if* $\dim_{Kr}(A) = \operatorname{depth}(A)$.

Example 3.1.5. *For example, a sequence* $(x+y+z, y-z)$ *in* $\mathbb{C}[[x, y, z]]/(xyz)$ *is regular. Hence* $\operatorname{depth}(A) \geq 2$. *It is known that* $\operatorname{depth}(A) \leq \dim_{Kr}(A)$. *Hence* $\operatorname{depth}(A) = 2$. *Thus the ring* $\mathbb{C}[[x, y, z]]/(xyz)$ *is Cohen-Macaulay.*

Remark 3.1.6. *It is known that a local ring* $(A, \mathfrak{m})$ *with a residue field* $\mathbb{C}$ *is Cohen-Macaulay if and only if* $\operatorname{Ext}^i_A(\mathbb{C}, A) = 0$ *for* $i < \operatorname{depth}(A)$.

Definition 3.1.7. *Let* A *be a local ring with a residue field* $\mathbb{C}$. *A finitely generated* A*-module* M *is said to be* maximal Cohen-Macaulay *if* $\operatorname{Ext}^i_A(\mathbb{C}, M) = 0$ *for* $i < \operatorname{depth}(A)$.

For a ring A, denote by $\operatorname{Mod}(A)$ the category of A-modules. Denote by $\operatorname{CM}(A)$ the full subcategory of $\operatorname{Mod}(A)$ whose objects are maximal Cohen-Macaulay modules.

We are interested in the case that A is a non-isolated singularity.

Definition 3.1.8. *Let* $(A, \mathfrak{m})$ *be a local ring with a residue field* $\mathbb{C}$ *and* $\mathfrak{p}$ *be a prime ideal of* A *so that there is an associated local ring* $A_{\mathfrak{p}}$ *with a maximal ideal* $\mathfrak{p}A_{\mathfrak{p}}$. *The prime ideal* $\mathfrak{p}$ *is said to be* regular *if* $\dim_{\mathbb{C}} \mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2 = \dim_{Kr} A_{\mathfrak{p}}$. *Otherwise, it is said to be* singular.

Suppose that in a local ring $(A, \mathfrak{m})$, *the maximal ideal* $\mathfrak{m}$ *is singular. The ring* $(A, \mathfrak{m})$ *is called an* isolated singularity *if every non-maximal prime ideal is regular. Otherwise it is called a* non-isolated singularity.

Example 3.1.9. *The maximal ideal* $\mathfrak{m} = (x, y, z)$ *of the ring* $A = \mathbb{C}[[x, y, z]]/(xyz)$ *is singular because*

$$\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim_{\mathbb{C}}((x, y, z)/(xyz))/((x^2, y^2, z^2, xy, yz, xz)/(xyz)) = \dim_{\mathbb{C}}(x, y, z)/(x^2, y^2, z^2, xy, yz, xz) = 3,$$

while $\dim_{Kr} A_{\mathfrak{m}} = \dim_{Kr} A = 2$.

The ideal $\mathfrak{p} = (x, y)$ *is non-maximal prime ideal, but is singular. Indeed,*

$$\dim_{\mathbb{C}} \mathfrak{p}/\mathfrak{p}^2 = \dim_{\mathbb{C}}(x, y)/(x^2, xy, y^2) = 2,$$

while $\dim_{Kr} A_{\mathfrak{p}} = 1$. *Hence the ring* $\mathbb{C}[[x, y, z]]/(xyz)$ *is non-isolated singularity.*

3.1.2 Matrix Factorization

The Eisenbud's theorem [Eis80] says that for a hypersurface singularity, maximal Cohen-Macaulay modules are equivalent to matrix factorizations. In this subsection we recall basic definitions and properties of matrix factorizations. We refer readers to the book of Yoshino [Yos90] for more details.

Let S be a regular local ring and W be a nonzero-divisor in S.

Definition 3.1.10. *A* matrix factorization *of W is a $\mathbb{Z}_2$-graded free S-module $X = X^0 \oplus X^1$ with an odd degree map $d : X \to X$ such that $d^2 = W\,\mathrm{id}_X$. A* morphism *from (X, d_X) and (Y, d_Y) is an S-module homomorphism from X to Y. A morphism f is decomposed into an* odd part *f^0 and an* even part *f^1 so that the set of morphisms $\mathrm{Hom}^{DG}_{\mathrm{MF}}((X, d_X), (Y, d_Y))$ has a natrual $\mathbb{Z}_2$-graded S-module structure. A* differential *d on the hom module is given by*

$$d(f) = d_Y \circ f + (-1)^{|f|} f \circ d_X,$$

where f is homogeneous of degree $|f|$.

From the definition, we have three categories for matrix factorization as follows.

- The DG-category $\mathrm{MF}^{DG}(S, W)$, whose morphism set is given as $\mathrm{Hom}^{DG}_{\mathrm{MF}}((X, d_X), (Y, d_Y))$.
- The ordinary category $\mathrm{MF}(S, W)$, whose morphism set is given as $Z^0(\mathrm{Hom}^{DG}_{\mathrm{MF}}((X, d_X), (Y, d_Y)))$.
- The homotopy category $\underline{\mathrm{MF}}(S, W)$, whose morphism set is given as $H^0(\mathrm{Hom}^{DG}_{\mathrm{MF}}((X, d_X), (Y, d_Y)))$.

Remark 3.1.11. *Note that any matrix factorization (X, d) is isomorphic to $(S^n \oplus S^n = S^n_{even} \oplus S^n_{odd}, \varphi : S^n_{even} \to S^n_{odd}, \psi : S^n_{odd} \to S^n_{even})$ for some n. We will denote this matrix factorization by (φ, ψ) and n is called the* rank *of it . Also note that for two matrix factorizations $(\varphi, \psi), (\varphi', \psi')$, a morphism is given by a pair of matrices (α, β) satisfying $\beta \circ \varphi = \varphi' \circ \alpha$ and $\alpha \circ \psi = \psi' \circ \beta$.*

$$\begin{tikzcd}
S^n \arrow[r, "\varphi"] \arrow[d, "\alpha"] & S^n \arrow[r, "\psi"] \arrow[d, "\beta"] & S^n \arrow[d, "\alpha"] \\
S^{n'} \arrow[r, "\varphi'"] & S^{n'} \arrow[r, "\psi'"] & S^{n'}
\end{tikzcd}$$

A morphism of matrix factorizations

$$\begin{tikzcd}
 & S^n \arrow[r, "\varphi"] \arrow[dl, "\xi"'] \arrow[d, "\alpha-\alpha'"] & S^n \arrow[r, "\psi"] \arrow[dl, "\eta"] \arrow[d, "\beta-\beta'"] & S^n \arrow[dl, "\xi"] \\
S^{n'} \arrow[r, "\psi'"] & S^{n'} \arrow[r, "\varphi'"] & S^{n'} &
\end{tikzcd}$$

A homotopy between morphisms

Let us consider a quotient ring $A = S/(W)$. Given a matrix factorization (φ, ψ), we have an induced 2-periodic acyclic chain complex of A-modules (see section 7.2.2 in [Yos90]).

$$\cdots \to A^n \xrightarrow{\underline{\varphi}} A^n \xrightarrow{\underline{\psi}} A^n \xrightarrow{\underline{\varphi}} A^n \xrightarrow{\underline{\psi}} A^n \xrightarrow{\underline{\varphi}} A^n \to \cdots$$

The cokernel $\mathrm{coker}\,\underline{\varphi}$ is a Cohen-Macaulay A-module, and it defines a functor

$$\mathrm{coker} : \mathrm{MF}(W) \to \mathrm{CM}(A)$$

$$(\varphi, \psi) \mapsto \mathrm{coker}\,\underline{\varphi}$$

Conversely, Theorem 6.1 of [Eis80] states that for any A-module M, its minimal free resolution is eventually periodic with periodicity 2. Moreover, the theorem also tells that the minimal free resolution is 2-periodic itself if and only if the module M is maximal Cohen-Macaulay. In this case, the resulting resolution gives rise to a matrix factorization. Thus the functor coker is essentially surjective.

The stable category of Cohen-Macaulay modules is similarly defined as follows.

Definition 3.1.12. *For two Cohen-Macaulay A-modules M and N, consider the set $I(M,N)$ of morphisms $f: M \to N$ that passes through a projective A-module P (namely, there are morphisms $g: M \to P$, $h: P \to N$ satisfying $f = h \circ g$). Then $I(M,N)$ is an ideal of $\mathrm{Hom}_{\mathrm{CM}(A)}(M,N)$, which allows us to define*

$$\mathrm{Hom}_{\underline{\mathrm{CM}}(A)}(M,N) := \mathrm{Hom}_{\mathrm{CM}(A)}(M,N)/I(M,N).$$

The stable category $\underline{\mathrm{CM}}(A)$ of $\mathrm{CM}(A)$ is the category whose objects are the same as $\mathrm{CM}(A)$ and the hom set between two objects M, N is given by $\mathrm{Hom}_{\underline{\mathrm{CM}}(A)}(M,N)$.

Now the theorem of Eisenbud can be stated as follows.

Theorem 3.1.13. *(Eisenbud's matrix factorization theorem [Eis80]) The induced functor*

$$\mathrm{coker}: \underline{\mathrm{MF}}(W) \to \underline{\mathrm{CM}}(A)$$

is an equivalence of categories.

Instead of $\underline{\mathrm{MF}}(W)$, we will work with the following (A_∞-analogue of) dg-category of matrix factorizations, to which A_∞-functor from the Fukaya category is defined. One can find the definitions of the A_∞-category and functor in the next section.

Definition 3.1.14. *The A_∞-category $\mathcal{MF}(W)$ has the same set of objects as $\mathrm{MF}(W)$, and its $\mathbb{Z}/2$-graded hom space is defined as*

$$\mathrm{Hom}_{\mathcal{MF}(W)}\left((\varphi',\psi'),(\varphi,\psi)\right) := \mathrm{Hom}^{\mathbb{Z}/2}_{\mathrm{MF}(W)}\left((\varphi,\psi),(\varphi',\psi')\right)$$

with A_∞-operations $\mathfrak{m}_k$ $(k = 1,2,\ldots)$:

$$\mathfrak{m}_1\left((\alpha,\beta)\right) = Df = d \circ f - (-1)^{|f|} f \circ d, \quad \mathfrak{m}_2(f_1,f_2) := (-1)^{|f_1|} f_1 \circ f_2$$

and $\mathfrak{m}_k = 0$ for $k \neq 1,2$. Here, a morphism defined in Definition 3.1.10 is of even degree and odd degree morphism is defined in a similar way but with a condition $\alpha\varphi_1 = \psi_2\beta$ and $\beta\psi_1 = \varphi_2\alpha$.

3.2 Burban-Drozd Triple Category

In this section, we recall the work of Burban and Drozd from [BD17] on the classification of maximal Cohen-Macaulay modules over degenerate cusps. They introduced the notions of *bunch of decorated chains*, their category of representations $\mathrm{rep}(\mathfrak{X}_A)$ and another equivalent category $\mathrm{Tri}(A)$. They showed that locally free maximal Cohen-Macaulay modules for xyz are classified by a band data, which in turn produces an element of $\mathrm{rep}(\mathfrak{X}_A)$ or $\mathrm{Tri}(A)$.

3.2.1 The category $\mathrm{Tri}(A)$

Let A be a ring. Then the set of all non zero-divisors S becomes a multiplicative set. The localization of A with respect to S, which is denoted by $Q(A)$, is called the *total ring* of A.

Definition 3.2.1. *Let A be a ring and $Q(A)$ be the total ring of A. The* normalization *of A is the integral closure of A in $Q(A)$. Let R be the normalization of A. The* conductor ideal *of R is defined as*

$$ann(R/A) = \{r \in R : rR \subseteq A\}.$$

From now on, let $(A, \mathfrak{m})$ be a reduced complete Cohen-Macaulay ring of Krull dimension two which is non-isolated singularity. Also, we denote by $\overline{A}, \overline{R}$ the quotient ring A/I and R/I, where I is the conductor ideal of the normalization R of A.

Example 3.2.2. *The ring $A = \mathbb{C}[[x,y,z]]/\big(xyz\big)$ is a reduced complete Cohen-Macaulay ring of Krull dimension two. Let R be a product ring $\mathbb{C}[[x_1,y_1]] \times \mathbb{C}[[y_2,z_2]] \times \mathbb{C}[[z_3,x_3]]$ and consider an embedding $\iota : A \to R$ which is given by $\iota(x) = x_1 + x_3, \iota(y) = y_1 + y_2, \iota(z) = z_2 + z_3$. Then R is integral over $\iota(A)$. For example, x_1 is a root of a monic polynomial $t^2 - (x_1 + x_3)t = 0$, where $x_1 + x_3 = \iota(x) \in \iota(A)$. Since each component of R is integrally closed, R is a normalization of A.*

The conductor ideal I is, as an A-module, $\big(xy, yz, xz\big)$. As an R-module, it is given by $\big(x_1y_1, y_2z_2, x_3z_3\big)$. Therefore, $\overline{A} = A/I = \mathbb{C}[[x,y,z]]\big/(xyz)$ and $Q(\overline{A}) = \mathbb{C}((x)) \times \mathbb{C}((y)) \times \mathbb{C}((z))$. Similarly, we have

$$Q(\overline{R}) = \mathbb{C}((x_1)) \times \mathbb{C}((x_3)) \times \mathbb{C}((y_1)) \times \mathbb{C}((y_2) \times \mathbb{C}((z_2)) \times \mathbb{C}((z_3)).$$

We recall the category $\mathrm{Tri}(A)$ introduced in [BD17] which is equivalent to $\mathrm{CM}(A)$.

Definition 3.2.3. *An object of* $\mathrm{Tri}(A)$ *is a triple $(\tilde{M}, V, \theta)$ consisting of the following data.*

- *A maximal Cohen-Macaulay R-module $\tilde{M}$.*
- *A finitely generated $Q(\overline{A})$-module V.*
- *A surjective $Q(\overline{R})$-module homomorphism*

$$\theta : Q(\overline{R}) \otimes_{Q(\overline{A})} V \to Q(\overline{R}) \otimes_R \tilde{M}$$

such that the following induced $Q(\overline{A})$-module homomorphism is injective:

$$V \to Q(\overline{R}) \otimes_{Q(\overline{A})} V \xrightarrow{\theta} Q(\overline{R}) \otimes_R \tilde{M}$$

A morphism between two triples $(\tilde{M}, V, \theta)$ *and* $(\tilde{M}', V', \theta')$ *is a pair of morphisms* $(\varphi : \tilde{M} \to \tilde{M}', \psi : V \to V')$ *with a suitable commutative diagram.*

Example 3.2.4. *The category* $\mathrm{Tri}(A)$ *for* $A = \mathbb{C}[[x, y, z]]/(xyz)$ *is computed as follows. It is known that any maximal Cohen-Macaulay* R*-module* $\tilde{M}$ *is of the form* $\mathbb{C}[[x_1, y_1]]^a \times \mathbb{C}[[y_2, z_2]]^b \times \mathbb{C}[[z_3, x_3]]^c$ *for some nonnegative integers* a, b, c *and a finitely generated* $Q(\overline{A})$*-module* V *is of the form* $\mathbb{C}((x))^d \times \mathbb{C}((y))^e \times \mathbb{C}((z))^f$ *for nonnegative integers* d, e, f. *Thus the morphism* $\theta : Q(\overline{R}) \otimes_{Q(\overline{A})} V \to Q(\overline{R}) \otimes_R \tilde{M}$ *is written as*

$$\begin{aligned}\theta :& \mathbb{C}((x_1))^d \times \mathbb{C}((x_3))^d \times \mathbb{C}((y_1))^e \times \mathbb{C}((y_2))^e \times \mathbb{C}((z_2))^f \times \mathbb{C}((z_3))^f \\ \to& \mathbb{C}((x_1))^a \times \mathbb{C}((x_3))^c \times \mathbb{C}((y_1))^a \times \mathbb{C}((y_2))^b \times \mathbb{C}((z_2))^b \times \mathbb{C}((z_3))^c.\end{aligned}$$

This morphism is the direct sum of six morphisms

$$\begin{aligned}&\theta_1^x : \mathbb{C}((x_1))^d \to \mathbb{C}((x_1))^a, \quad \theta_3^x : \mathbb{C}((x_3))^d \to \mathbb{C}((x_3))^c \\ &\theta_1^y : \mathbb{C}((y_1))^e \to \mathbb{C}((y_1))^a, \quad \theta_2^y : \mathbb{C}((y_2))^e \to \mathbb{C}((y_2))^b \\ &\theta_2^z : \mathbb{C}((z_2))^f \to \mathbb{C}((z_2))^b, \quad \theta_3^z : \mathbb{C}((z_3))^f \to \mathbb{C}((z_3))^c.\end{aligned}$$

Hence the morphism θ *can be identified with a collection of six matrices* $\Theta_1^x, \Theta_3^x, \Theta_1^y, \Theta_2^y, \Theta_2^z$, *and* Θ_3^z *over a field* $\mathbb{C}((t))$ *satisfying the following conditions :*

- *each matrices* $\Theta_1^x, \Theta_3^x, \Theta_1^y, \Theta_2^y, \Theta_2^z$ *and* Θ_3^z *have full row-rank.*
- *induced matrices* $\begin{pmatrix}\Theta_1^x \\ \Theta_3^x\end{pmatrix}, \begin{pmatrix}\Theta_1^y \\ \Theta_3^y\end{pmatrix}$ *and* $\begin{pmatrix}\Theta_2^z \\ \Theta_3^z\end{pmatrix}$ *have full column-rank.*

These data may be illustrated as in the diagram below.

$$\begin{tikzcd}
& \circ x \arrow[r, "\theta_3^x"] \arrow[dl, "\theta_1^x"'] & \bullet 3 & \\
\bullet 1 & & & \circ z \arrow[ul, "\theta_3^z"'] \arrow[dl, "\theta_2^z"] \\
& \circ y \arrow[ul, "\theta_1^y"] \arrow[r, "\theta_2^y"'] & \bullet 2 &
\end{tikzcd}$$

3.2.2 Categorical equivalence between $\mathrm{CM}(A)$ and $\mathrm{Tri}(A)$

In this subsection, we recall a construction of a functor from the category $\mathrm{CM}(A)$ to $\mathrm{Tri}(A)$ introduced in [BD17], which gives a categorical equivalence.

The first step is the *Macaulayfication*. This is a functor $(-)^\dagger$ from A-mod to $\mathrm{CM}(A)$. The following definition is borrowed from [BH98].

Proposition 3.2.5. *[BH98] For a Noetherian local Cohen-Macaulay ring A the following hold.*

1. *There is a unique, up to isomorphism, A-module K_A called the* canonical module *such that*

$$\mathrm{Ext}^i_A(\mathbb{C}, K_A) = \begin{cases} 0 \text{ if } i \neq \dim_{Kr}(A) \\ \mathbb{C} \text{ if } i = \dim_{Kr}(A) \end{cases} .$$

2. *For any finitely generated module M, the module $M^\dagger := \mathrm{Hom}_A(\mathrm{Hom}_A(M, K_A), K_A)$ is a maximal Cohen-Macaulay module over A and $M^\dagger$ is called the Macaulayfication of M. This gives a functor from A-mod to $\mathrm{CM}(A)$*

The functor $(-)^\dagger$ is left adjoint to the forgetful functor from $\mathrm{CM}(A)$ to A-mod. More precisely, for any finitely generated A-module M and any maximal Cohen-Macaulay A-module N, we have a natural isomorphism

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, N) \cong \mathrm{Hom}_{\mathrm{CM}(A)}(M^\dagger, N).$$

Under the situation of 3.2.1, we have two natural functors

- $\mathrm{CM}(A) \to \mathrm{CM}(R)$, sending a maximal Cohen-Macaulay module M over A to the Macaulayfication $R \boxtimes_A M = (R \otimes_A M)^\dagger$.
- $\mathrm{CM}(A) \to Q(\overline{A}) - \mathrm{mod}$, sending M to the tensor product $Q(\overline{A}) \otimes_A M$.

Then there is a natural $Q(\overline{R})$-module homomorphism

$$\theta_M : Q(\overline{R}) \otimes_{Q(\overline{A})} (Q(\overline{A}) \otimes_A M) = Q(\overline{R}) \otimes_R (R \otimes_A M) \to Q(\overline{R}) \otimes_R (R \otimes_A M)^\dagger.$$

We get a triple $(R \boxtimes_A M, Q(\overline{A}) \otimes_A M, \theta_M)$. The next result shows that this is an object of $\mathrm{Tri}(A)$.

Lemma 3.2.6. *[BD17, Lemma 3.2] The morphism θ_M is surjective. Moreover, the following canonical morphism of $Q(\overline{A})$-modules is injective:*

$$\tilde{\theta}_M : Q(\overline{A}) \otimes_A M \to Q(\overline{R}) \otimes_A M \xrightarrow{\theta_M} Q(\overline{R}) \otimes_R (R \boxtimes_A M)$$

Since the construction is natural to the module M, it defines a functor $\mathbb{F} : \mathrm{CM}(A) \to \mathrm{Tri}(A)$. The main result in Section 3 of [BD17] is that this functor is an equivalence of categories.

Theorem 3.2.7. *[BD17, Theorem 3.5] The functor $\mathbb{F} : \mathrm{CM}(A) \to \mathrm{Tri}(A)$, which sends a maximal Cohen-Macaulay module M to a triple $(R \boxtimes_A M, Q(\overline{A}) \otimes_A M, \theta_M)$, is an equivalence of categories.*

Now recall two full subcategories $\mathrm{CM}^{\mathrm{lf}}(A)$ of $\mathrm{CM}(A)$ and $\mathrm{Tri}^{\mathrm{lf}}(A)$ of $\mathrm{Tri}(A)$.

Definition 3.2.8. *A maximal Cohen-Macaulay A-module M is said to be* locally free on the punctured spectrum *if the localization $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for any non-maximal prime ideal $\mathfrak{p}$. We denote by $\mathrm{CM}^{\mathrm{lf}}(A)$ the full subcategory of $\mathrm{CM}(A)$ consisting of such A-modules.*

We abbreviate it as locally free from now on. Similarly, an object $(\tilde{M}, V, \theta)$ in $\mathrm{Tri}(A)$ is locally free if the morphism θ is an isomorphism. It is shown in Theorem 3.9 [BD17] that the functor $\mathbb{F}$ induces an equivalence between full subcategories of locally free objects between $\mathrm{CM}^{\mathrm{lf}}(A)$ and $\mathrm{Tri}^{\mathrm{lf}}(A)$.

3.2.3 Burban & Drozd's classification

We have mentioned that for a ring $A = \mathbb{C}[[x,y,z]]/(xyz)$, an object of $\mathrm{Tri}(A)$ is determined by the 6 matrices $(\Theta_1^x, \Theta_3^x, \Theta_1^y, \Theta_2^y, \Theta_2^z, \Theta_3^z)$ in Example 3.2.4. By Theorem 3.2.7, maximal Cohen-Macaulay A-modules are equivalent to this collection of matrices. In particular, locally free maximal Cohen-Macaulay A-modules are determined by six nonsingular square matrices. By choosing appropriate basis for each $\mathbb{C}((t))$-vector spaces, one can transform these matrices into a canonical form, which is given as follows.

Definition 3.2.9. *A* band data *consists of*

- *positive integers τ and μ,*
- *a nonzero complex number $\lambda \in \mathbb{C}^\times$,*
- *a collection of positive integers $\omega = ((a_1, b_1, c_1, d_1, e_1, f_2), (a_2, b_2, c_2, d_2, e_2, f_3), \cdots, (a_\tau, b_\tau, c_\tau, d_\tau, e_\tau, f_1))$ such that $min(f_i, a_i) = min(b_i, c_i) = min(d_i, e_i) = 1$ for all i.*

Given band data $(\tau, \lambda, \mu, \omega)$, define the corresponding canonical form as follows.

- *Let I_r be the $r \times r$ identity matrix and $J_r(\lambda)$ be the Jordan block of size $r \times r$ with the eigenvalue λ.*
- *Consider the following diagonal matrices*

$$A_i := t^{a_i} I_\mu, B_i := t^{b_i} I_\mu, C_i := t^{c_i} I_\mu, D_i := t^{d_i} I_\mu, E_i := t^{e_i} I_\mu, F_i := t^{f_i} I_\mu, H = t^{f_1} J_\mu(\lambda).$$

- *Then define the following matrices*

$$\Theta_1^x = diag(A_1, A_2, \cdots, A_\tau), \quad \Theta_1^y = diag(B_1, B_2, \cdots, B_\tau), \quad \Theta_2^y = diag(C_1, C_2, \cdots, C_\tau),$$
$$\Theta_2^z = diag(D_1, D_2, \cdots, D_\tau), \quad \Theta_3^z = diag(E_1, E_2, \cdots, E_\tau),$$

and a block matrix $\Theta_3^x := \begin{pmatrix} 0 & F_2 & 0 & \cdots & 0 \\ 0 & 0 & F_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F_\tau \\ H & 0 & 0 & \cdots & 0 \end{pmatrix}$

The maximal Cohen-Macaulay A-module corresponding to a band data is described explicitly in Definition 3.4.4.

We are interested in indecomposable maximal Cohen-Macaulay modules. They correspond to a special class of band data, called non-periodic band data.

Definition 3.2.10. *Let $(\tau, \lambda, \mu, \omega)$ be a band data. It is said to be* periodic *if there is some $1 < r < \tau$ such that $(a_i, b_i, c_i, d_i, e_i, f_{i+1}) = (a_{i+r}, b_{i+r}, c_{i+r}, d_{i+r}, e_{i+r}, f_{i+r+1})$ for each $1 \le i \le \tau$, where indices are considered as an element of $\mathbb{Z}_\tau$.*

Theorem 3.2.11. *[BD17, Theorem 8.2] Let A be a ring* $\mathbb{C}[[x,y,z]]/(xyz)$*. There is a one to one correspondence between the set of isomorphism classes of indecomposable objects of* $\mathrm{Tri}^{\mathrm{lf}}(A)$ *and the set of non-periodic band data.*

Corollary 3.2.12. *Locally free maximal Cohen-Macaulay modules over* $\mathbb{C}[[x,y,z]]/(xyz)$ *are classified by band data.*

3.3 Generator diagram and Macaulayfication

Burban-Drozd classified maximal Cohen-Macaulay A-modules for $A = \mathbb{C}[[x,y,z]]/(xyz)$ by the band data in Theorem 3.2.11. In fact, a given band data produces an A-module $\tilde{M}$ which may *not* be maximal Cohen-Macaulay, but it is known that any Noetherian A-module $\tilde{M}$ can be extended to a maximal Cohen-Macaulay module $\tilde{M}^\dagger$ that is called Macaulayfication of $\tilde{M}$. Once we obtain the maximal Cohen-Macaulay module, we can apply the Eisenbud's theorem to obtain the corresponding matrix factorizations.

Therefore, we need to carry out Macaulayfication of A-modules corresponding to band data, which turns out to be quite subtle process. In this section, we introduce a combinatorial method to carry out Macaulayfication for all A-modules in the list of Burban-Drozd. Namely we will introduce what we call a generator diagram and explain how to perform Macaulayfication using such a diagram. In this section, we give a gentle introduction to this method by explaining the rank 1 ($\tau\mu = 1$) cases in the list. Higher rank cases are considerably more complicated, and will be discussed in Section 3.6.

3.3.1 Macaulayfication and Macaulayfying Elements

Recall from Section 3.2.2 that Macaulayfication of $\tilde{M}$ is defined as $\tilde{M}^\dagger := \mathrm{Hom}_A(\mathrm{Hom}_A(\tilde{M}, K_A), K_A)$ for the canonical module K_A. It is more convenient to find its Macaulayfication using *Macaulayfying elements*.

Definition 3.3.1. *For an A-submodule $\tilde{M}$ of a free A-module A^r, if there is an element $F \in A^r \setminus \tilde{M}$ such that $xF, yF, zF \in \tilde{M}$, we call it a* Macaulayfying element *of $\tilde{M}$ in A^r.*

Proposition 3.3.2. *For an A-submodule $\tilde{M}$ of a free A-module A^r, the following hold:*

1. *$\tilde{M}$ is maximal Cohen-Macaulay if and only if there is no Macaulayfying element of $\tilde{M}$ in A^r. We have $\tilde{M}^\dagger \cong \tilde{M}$ in this case.*

2. *$\tilde{M}^\dagger \cong \langle \tilde{M}, F\rangle_A^\dagger$ holds for any Macaulayfying element F of $\tilde{M}$ in A^r.*

Proof. (1) Recall from Corollary 2.23 of [BD08] that $\tilde{M}$ is maximal Cohen-Macaulay if and only if $H^i_{\{\mathfrak{m}\}}\left(\tilde{M}\right) = 0$ for $i = 0,1$, where $\mathfrak{m} = \left(x,y,z\right)$ is the maximal ideal of A. By the long exact sequence

$$0 \to H^0_{\{\mathfrak{m}\}}\left(\tilde{M}\right) \to H^0_{\{\mathfrak{m}\}}\left(A^r\right) \to H^0_{\{\mathfrak{m}\}}\left(A^r/\tilde{M}\right) \to H^1_{\{\mathfrak{m}\}}\left(\tilde{M}\right) \to H^1_{\{\mathfrak{m}\}}\left(A^r\right) \to \cdots$$

and the fact that A^r is maximal Cohen-Macaulay, it is also equivalent to say that

$$H^0_{\{\mathfrak{m}\}}\left(A^r/\tilde{M}\right) \cong \left\{[F] \in A^r/\tilde{M} \;\middle|\; \mathfrak{m}^t[F] = 0 \text{ for some } t \in \mathbb{Z}_{\geq 1}\right\}$$

is trivial, which proves the first part. See Theorem 2.18 of [BD08] for the second statement.

(2) See Lemma 1.5 of [BD17]. □

We will compute the Macaulayfication of an A-submodule $\tilde{M}$ of a free A-module A^r by finding all Macaulayfying elements of $\tilde{M}$ in A^r.

3.3.2 Generator Diagram and Macaulayfying Elements

Here we illustrate a method to find Macaulayfying elements of an A-submodule of A^1 (which is also an ideal of A). For this, we arrange the monomials in A in a lattice form as in Figure 3.1, called the *lattice diagram* for A.

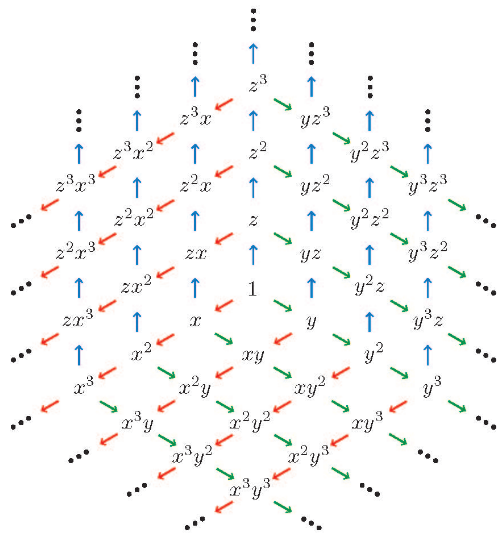


Figure 3.1: Lattice diagram for $A = \mathbb{C}[[x,y,z]]\big/(xyz)$

The colored arrows represent relations between them as elements of an A-module. Namely, the red, green and blue arrows indicate how each element changes when it is multiplied by x, y and z, respectively. If there is no corresponding arrow, this means that the element becomes zero.

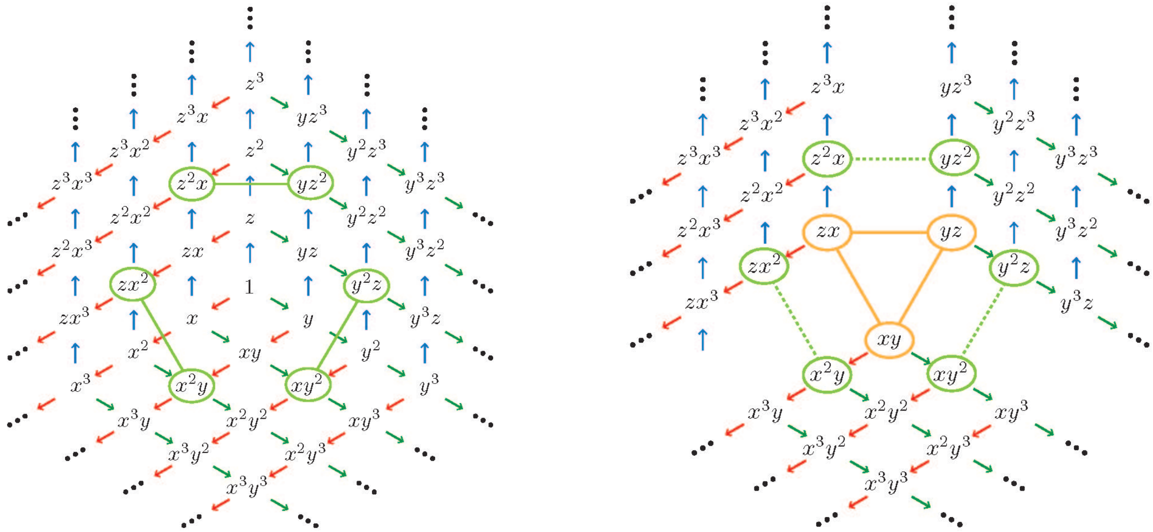
Now, let us explain how to find the Macaulayfication of the following A-submodule $\tilde{M}$ of A^1 generated by three elements as an example.

$$\tilde{M} = \langle zx^2 + x^2y, xy^2 + y^2z, yz^2 + z^2x \rangle_A .$$

We can express $\tilde{M}$ on the lattice by denoting its A-generators, which is described in Figure 3.2.(a). We call it a *generator diagram* of $\tilde{M}$.

In Figure 3.2, each A-generator is marked with two (light green) circles with an edge between them. Note that from these three generators, it is not hard to describe the module $\tilde{M}$ as a $\mathbb{C}$-vector space: Just multiply x, y or z to the generators repeatedly, which corresponds to moving the two circles (connected with an edge) along the red, green and blue arrows. A pair of circles may become one or disappear in this process, if the corresponding element after multiplication lies in the ideal (xyz). Thus, we may draw the generators only to describe $\tilde{M}$.

Let us show that $F := xy + yz + zx \in A$ is an Macaulayfying element of $\tilde{M}$. The element F is specified in the right side of the picture by the three yellow circles and edges. Clearly F is not in $\tilde{M}$. We can easily read from the diagram that $x \cdot F, y \cdot F, z \cdot F$ becomes three generators $zx^2 + x^2y, xy^2 + y^2z, yz^2 + z^2x$ of $\tilde{M}$ (by moving these circles and edges in three directions). This means that F is a Macaulayfying element of $\tilde{M}$ in A^1.



(a) $\tilde{M} = \langle zx^2 + x^2y, xy^2 + y^2z, yz^2 + z^2x\rangle_A$ (b) $\tilde{M}^\dagger = \langle xy + yz + zx\rangle_A$

Figure 3.2: Generator diagram in the degenerate case

We add F to $\tilde{M}$ to obtain $\langle \tilde{M}, F\rangle_A = \langle F\rangle_A$ as original generators are all generated by F. It is easy to check that there are no more Macaulayfying elements of $\langle F\rangle_A$. Therefore, Proposition 3.3.2 implies that $\tilde{M}^\dagger \cong \langle \tilde{M}, F\rangle_A^\dagger = \langle F\rangle_A^\dagger \cong \langle F\rangle_A$. Furthermore, it turns out that the map $A \to \langle F\rangle_A$, $a \mapsto aF$ is an A-module isomorphism, which forces $\tilde{M}^\dagger \cong A$ as A-modules.

3.3.3 Matrix Factorizations

Let us describe the Macaulayfication and the corresponding matrix factorization of the following A-modules:

Definition 3.3.3. *Rank* 1 *(modified) band data is given by $l, m, n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^\times$, denoted as*

$$((l, m, n), \lambda, 1)$$

We define the corresponding A-module as

$$\tilde{M}((l, m, n), \lambda, 1) = \langle x^2y^2, y^2z^2, z^2x^2, \lambda zx^{l^+ + 2} + x^{l^- + 2}y, xy^{m^+ + 2} + y^{m^- + 2}z, yz^{n^+ + 2} + z^{n^- + 2}x\rangle_A$$

where a^+, a^- (satisfying $a = a^+ - a^-$) for $a \in \mathbb{Z}$ are defined as

$$a^+ := \max\{0, a\} \quad \text{and} \quad a^- := \max\{0, -a\}$$

We denote its Macaulayfication by

$$M((l, m, n), \lambda, 1) = \tilde{M}((l, m, n), \lambda, 1)^\dagger$$

Remark 3.3.4. *A general definition of modified band data, and its corresponding module will be given in Section 3.4.1. It is called modified as we replace sextuple $(a_1, b_1, c_1, d_1, e_1, f_1)$ of the band data in Definition 3.2.9 by the triple (l, m, n).*

Now we compute the explicit Macaulayfication $M((l,m,n),\lambda,1)$ using the generator diagram. It turns out that we need to split into multiple cases according to the signs of l, m and n. We discuss each case separately and also find the corresponding matrix factorization as well.

Case 1: $l = m = n = 0$ and $\lambda = 1$. This is the "*degenerate*" case. The other cases will be called nondegenerate. This is the module that appeared in the previous subsection.

$$M((0,0,0),1,1) = \langle zx^2 + x^2y, xy^2 + y^2z, yz^2 + z^2x\rangle_A^\dagger = \langle xy + yz + zx\rangle_A \cong A$$

It gives a trivial matrix factorization $(xyz)\cdot 1 = 1\cdot(xyz) = xyz$ where we have $\operatorname{coker}\underline{(xyz)} = \operatorname{coker}\underline{(0)} \cong A$ as A-modules.

Case 2: $l,m,n \geq 0$ and nondegenerate.

We have $l^+ = l$, $m^+ = m$, $n^+ = n$ and $l^- = m^- = n^- = 0$ and hence

$$\tilde{M}((l,m,n),\lambda,1) = \langle \lambda zx^{l+2} + x^2y, xy^{m+2} + y^2z, yz^{n+2} + z^2x\rangle_A.$$

The corresponding generator diagram is shown in Figure 3.3a. Note that there are no Macaulayfying elements of $\tilde{M}((l,m,n),\lambda,1)$ in A^1 and therefore it is already maximal Cohen-Macaulay. That is,

$$M((l,m,n),\lambda,1) = \langle \lambda zx^{l+2} + x^2y, xy^{m+2} + y^2z, yz^{n+2} + z^2x\rangle_A.$$

We find a matrix factorization arising from this module. Denoting $G_1 := \lambda zx^{l+2} + x^2y$, $G_2 := xy^{m+2} + y^2z$ and $G_3 := yz^{n+2} + z^2x$, then with the help of the generator diagram, we can find following 3 relations (over A) among them:

$$\begin{cases} z\,G_1 \qquad\qquad - \lambda x^{l+1}\,G_3 = 0 \\ -y^{m+1}\,G_1 + x\,G_2 = 0 \\ -z^{n+1}\,G_2 + y\,G_3 = 0 \end{cases}$$

From this we get matrix factors (over $\mathbb{C}[[x,y,z]]$)

$$\varphi = \begin{pmatrix} z & -y^{m+1} & 0 \\ 0 & x & -z^{n+1} \\ -\lambda x^{l+1} & 0 & y \end{pmatrix} \quad\text{and}\quad \psi = \left(1 - \lambda x^l y^m z^n\right)^{-1} \begin{pmatrix} xy & y^{m+2} & y^{m+1}z^{n+1} \\ \lambda z^{n+1}x^{l+1} & yz & z^{n+2} \\ \lambda x^{l+2} & \lambda x^{l+1}y^{m+1} & zx \end{pmatrix}$$

of xyz which satisfy $\varphi\psi = \psi\varphi = xyzI_3$. Here we computed ψ using the matrix adjoint $\operatorname{adj}\varphi$ of φ which satisfies the relation $\varphi\cdot\operatorname{adj}\varphi = \operatorname{adj}\varphi\cdot\varphi = (\det\varphi)\,I_3$.

Then we have

$$\operatorname{coker}\underline{\varphi} \cong M((l,m,n),\lambda,1)$$

as A-modules. Indeed we will prove this rigorously in Theorem 3.6.1 in a more general situation.

Case 3: $l > 0$, $m < 0$ and $n \leq 0$.

We have

$$\tilde{M}((l,m,n),\lambda,1) = \langle z^2x^2, \lambda zx^{l+2} + x^2y, xy^2 + y^{-m+2}z, yz^2 + z^{-n+2}x\rangle_A.$$

and the generator diagram given in Figure 3.3b. It is enough to add one Macaulayfying element $\lambda zx^{l+1} + xy + y^{-m+1}z$ of $\tilde{M}((l,m,n),\lambda,1)$ in A^1, which is marked in the picture by yellow circles and edges among them:

$$M((l,m,n),\lambda,1) = \langle \lambda zx^{l+1} + xy + y^{-m+1}z, yz^2 + z^{-n+2}x, z^2x^2\rangle_A$$

where the order of the generators have been adjusted to obtain some desired form of matrix factorization. Note also that if $l = 1$ or $n = 0$, the third generator is redundant. But we do not exclude it to consistently get a 3×3 matrix in any cases below.

We can find relations among the above generators as before which gives the matrix factors

$$\varphi = \begin{pmatrix} z & 0 & 0 \\ -y^{-m} & x & 0 \\ -\lambda x^{l-1} & -z^{-n} & y \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} xy & 0 & 0 \\ y^{-m+1} & yz & 0 \\ y^{-m}z^{-n} + \lambda x^l & z^{-n+1} & zx \end{pmatrix}.$$

of xyz satisfying $\operatorname{coker}\varphi \cong M((l,m,n),\lambda,1)$ as A-modules. Also, when $l > 0$, $m < 0$ and $n > 0$, one can compute its Macaulayfication in a similar way.

Case 4: $l > 0$, $m = 0$ and $n < 0$.

The module and the generator diagram are given respectively by

$$\tilde{M}((l,0,n),\lambda,1) = \langle z^2x^2, \lambda zx^{l+2} + x^2y, xy^2 + y^2z, yz^2 + z^{-n+2}x\rangle_A.$$

and Figure 3.3c. Here we have a Macaulayfying element $\lambda zx^{l+1} + xy + yz + z^{-n+1}x$ of $\tilde{M}((l,0,n),\lambda,1)$ in A^1 and no more, which implies

$$M((l,0,n),\lambda,1) = \langle \lambda zx^{l+1} + xy + yz + z^{-n+1}x, yz^2 + z^{-n+2}x, z^2x^2\rangle_A.$$

This gives the matrix factors of xyz as

$$\varphi = \begin{pmatrix} z & 0 & 0 \\ -1 & x & 0 \\ -\lambda x^{l-1} & -z^{-n} & y \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} xy & 0 & 0 \\ y & yz & 0 \\ z^{-n} + \lambda x^l & z^{-n+1} & zx \end{pmatrix}.$$

satisfying $\operatorname{coker}\varphi \cong M((l,0,n),\lambda,1)$ as A-modules.

The other cases:

- The case $l, m, n \leq 0$ and nondegenerate can be treated in the same way as in Case 2.
- When the word (l,m,n) contains two nonzero elements with different signs, it is further subdivided. In the case $(0,-,+)$ and $(-,+,0)$, there appears $+ \to 0 \to -$ in cyclic order. These cases are equivalent to Case 4 (for $(+,0,-)$) if we change the order of x, y and z cyclically (and the auxiliary parameter λ should be handled appropriately), and we may rotate the corresponding generator diagram accordingly.
- In the remaining cases, the word has a subword of the form $+ \to -$ in cyclic order. In those cases, we can cycle the order of x, y and z and make some modifications to proceed as in Case 3.

3.3.4 Correction number and a uniform expression of matrix factorizations

Let us define the notion of '*correction number*' which enables us to write matrix factorizations in the above multiple cases into a single formula. This correction number turns out to be the data needed for mirror symmetry correspondence as well.

Definition 3.3.5. *Define the* correction number ε *of the band word* (l,m,n) *by*

$$\varepsilon := \varepsilon(l,m,n) := \begin{cases} 2 & \text{if} \quad l,m,n \geq 0, \\ 1 & \text{if} \quad \begin{cases} l>0,\ m>0,\ n<0 \ \text{or} \\ l>0,\ m<0,\ n>0 \ \text{or} \\ l<0,\ m>0,\ n>0, \end{cases} \\ 0 & \text{if} \quad \begin{cases} l>0,\ m\leq 0,\ n\leq 0,\ (m,n)\neq(0,0) \ \text{or} \\ l\leq 0,\ m>0,\ n\leq 0,\ (n,l)\neq(0,0) \ \text{or} \\ l\leq 0,\ m\leq 0,\ n>0,\ (l,m)\neq(0,0), \end{cases} \\ -1 & \text{if} \quad l,m,n\leq 0,\ (l,m,n)\neq(0,0,0) \end{cases}$$

Proposition 3.3.6. *We add the correction number to define a new triple* (l',m',n') *from* (l,m,n)*;*

$$\left(l',m',n'\right) = \left(l,m,n\right) + \left(\varepsilon(l,m,n),\varepsilon(l,m,n),\varepsilon(l,m,n)\right).$$

Then, for each non-degenerate modified band datum $((l,m,n),\lambda,1)$*, the Cohen-Macaulay module*

$$M\left((l,m,n),\lambda,1\right)$$

is equivalent to the following matrix factorization (φ,ψ) *over* $\mathbb{C}[[x,y,z]]$ *under Eisenbud's theorem.*

$$\varphi := \begin{pmatrix} z & -y^{m'-1} & -\lambda^{-1}x^{-l'} \\ -y^{-m'} & x & -z^{n'-1} \\ -\lambda x^{l'-1} & -z^{-n'} & y \end{pmatrix} \quad \text{and}$$

$$\psi := u^{-1}\begin{pmatrix} xy & y^{m'}+\lambda^{-1}z^{-n'}x^{-l'} & \lambda^{-1}x^{-l'+1}+y^{m'-1}z^{n'-1} \\ y^{-m'+1}+\lambda z^{n'-1}x^{l'-1} & yz & z^{n'}+\lambda^{-1}x^{-l'}y^{-m'} \\ \lambda x^{l'}+y^{-m'}z^{-n'} & z^{-n'+1}+\lambda x^{l'-1}y^{m'-1} & zx \end{pmatrix}$$

where

$$u := 1-\lambda x^{l'-2}y^{m'-2}z^{n'-2}-\lambda^{-1}x^{-l'-1}y^{-m'-1}z^{-n'-1}.$$

Here, we are using a non-standard notation that x^a, y^a or z^a is considered as zero if $a<0$.

Namely, we have

$$\varphi\cdot\psi = \psi\cdot\varphi = xyz\cdot I_3, \quad \operatorname{coker}\underline{\varphi} \cong M\left((l,m,n),\lambda,1\right) \quad \text{as } A\text{-modules}$$

Definition 3.3.7. *The matrix* φ *is called the* canonical form *of matrix factors arising from the band datum* $((l,m,n),\lambda,1)$ *and denote it by* $\varphi\left(\left(l',m',n'\right),\lambda,1\right)$.

These definitions and propositions will be generalized into the higher rank case in Section 3.6 and the rigorous proof will be given there.

3.4 Band Data and String Data

To discuss higher length cases, we study some properties of loop data in this section. We will also introduce a slightly modified version of the band data of Burban-Drozd [BD17] to compare it with the loop data.

3.4.1 Modified band datum

Let us denote the band datum of Burban-Drozd by $(w_{\mathrm{BD}}, \lambda, \mu)$ (see (**??**)). Let us define a modified version of band datum.

Definition 3.4.1. *A* modified band datum (w, λ, μ) *consists of the following:*

- *(band word)* $w = (l_1, m_1, n_1, l_2, m_2, n_2, \ldots, l_\tau, m_\tau, n_\tau) \in \mathbb{Z}^{3\tau}$ *for some* $\tau \in \mathbb{Z}_{\geq 1}$,
- *(eigenvalue)* $\lambda \in \mathbb{C}^\times$,
- *(multiplicity)* $\mu \in \mathbb{Z}_{\geq 1}$.

The length *of the band word* w *or the band datum* (w, λ, μ) *is defined as* 3τ, *and the* rank *of the band datum* (w, λ, μ) *is defined as* $\tau\mu$.

It is easy to see that two notions of band datum are equivalent.

Lemma 3.4.2. *Modified band word* w *can be obtained from* w_{BD} *by recoding the differences* $l_i := f_i - a_i$, $m_i := b_i - c_i$, $n_i := d_i - e_i$ *for* $i \in \mathbb{Z}_\tau$. *Conversely, we can recover* w_{BD} *from* w *by setting* $f_i := \max\{0, l_i\} + 1$, $a_i := \max\{0, -l_i\} + 1$, $b_i := \max\{0, m_i\} + 1$, $c_i := \max\{0, -m_i\} + 1$, $d_i := \max\{0, n_i\} + 1$ *and* $e_i := \max\{0, -n_i\} + 1$ *for* $i \in \mathbb{Z}_\tau$.

Since the word w (resp. w_{BD}) is in $\mathbb{Z}^{3\tau}$ (resp. $\mathbb{Z}^{6\tau}$) and it is easy to tell whether band datum is a modified version or not. Therefore, we will call the modified band datum (w, λ, μ), simply as a band datum.

The notions of *shift*, *subword*, *periodicity*, *period* and *concatenation* of a band word can be defined similarly following those of a normal loop word. Then two band words w and w_* are considered *equivalent* if they are the same as cyclic words, that is, $w_* = w^{(k)}$ for some $k \in \mathbb{Z}$. Note that the periodicity and the period of a band word are invariant under this equivalence relation. Two band data (w, λ, μ) and (w_*, λ_*, μ_*) are said to be *equivalent* if w and w_* are equivalent as band words, $\lambda = \lambda_*$ and $\mu = \mu_*$.

Definition 3.4.3. *Given a band datum* (w, λ, μ) *as above, define* $T(w, \lambda, \mu)$ *to be the following quiver representation:*

$$\begin{tikzcd}
& \mathbb{C}((t))^{\tau\mu} \arrow[r, "{\begin{pmatrix} 0 & t^{l_2^+ +1} I_\mu & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & t^{l_\tau^+ +1} I_\mu \\ t^{l_1^+ +1} J_\mu(\lambda) & 0 & \cdots & 0 \end{pmatrix}}"] \arrow[dl, "{\begin{pmatrix} t^{l_1^- +1} I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{l_\tau^- +1} I_\mu \end{pmatrix}}"'] & \mathbb{C}((t))^{\tau\mu} & \\
\mathbb{C}((t))^{\tau\mu} & & & \mathbb{C}((t))^{\tau\mu} \arrow[ul, "{\begin{pmatrix} t^{n_1^- +1} I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{n_\tau^- +1} I_\mu \end{pmatrix}}"'] \arrow[dl, "{\begin{pmatrix} t^{n_1^+ +1} I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{n_\tau^+ +1} I_\mu \end{pmatrix}}"] \\
& \mathbb{C}((t))^{\tau\mu} \arrow[ul, "{\begin{pmatrix} t^{m_1^+ +1} I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{m_\tau^+ +1} I_\mu \end{pmatrix}}"] \arrow[r, "{\begin{pmatrix} t^{m_1^- +1} I_\mu & & 0 \\ & \ddots & \\ 0 & & t^{m_\tau^- +1} I_\mu \end{pmatrix}}"'] & \mathbb{C}((t))^{\tau\mu} &
\end{tikzcd}$$

where I_μ is the $\mu \times \mu$ identity matrix and $J_\mu(\lambda)$ is the $\mu \times \mu$ Jordan block with the eigenvalue λ.

This generalizes the quiver in Example 3.2.4. This $T(w,\lambda,\mu)$ represents an object in the category $\mathrm{Tri}(A)$, which is equivalent to the category of maximal Cohen-Macaulay modules (Theorem 3.2.7). We describe the module corresponding to $T(w,\lambda,\mu)$ from [BD17].

Definition 3.4.4. *Given a band datum (w,λ,μ) as above, define $\tilde{M}(w,\lambda,\mu)$ to be the A-submodule of $A^{\tau\mu}$ generated by all columns of the following 6 matrices of size $\tau\mu \times \tau\mu$:*

$$x^2y^2 I_{\tau\mu}, \quad y^2z^2 I_{\tau\mu}, \quad z^2x^2 I_{\tau\mu}, \quad \pi_x(w,\lambda,\mu) := \begin{pmatrix} x^{l_1^- +2} y I_\mu & zx^{l_2^+ +2} I_\mu & \cdots & 0 \\ 0 & x^{l_2^- +2} y I_\mu & \ddots & \vdots \\ \vdots & \vdots & \ddots & zx^{l_\tau^+ +2} I_\mu \\ zx^{l_1^+ +2} J_\mu(\lambda) & 0 & \cdots & x^{l_\tau^- +2} y I_\mu \end{pmatrix}_{\tau\mu \times \tau\mu},$$

$$\pi_y(w,\lambda,\mu) := \begin{pmatrix} \left(xy^{m_1^+ +2} + y^{m_1^- +2} z\right) I_\mu & 0 & \cdots & 0 \\ 0 & \left(xy^{m_2^+ +2} + y^{m_2^- +2} z\right) I_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(xy^{m_\tau^+ +2} + y^{m_\tau^- +2} z\right) I_\mu \end{pmatrix}_{\tau\mu \times \tau\mu},$$

$$\pi_z(w,\lambda,\mu) := \begin{pmatrix} \left(yz^{n_1^+ +2} + z^{n_1^- +2} x\right) I_\mu & 0 & \cdots & 0 \\ 0 & \left(yz^{n_2^+ +2} + z^{n_2^- +2} x\right) I_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(yz^{n_\tau^+ +2} + z^{n_\tau^- +2} x\right) I_\mu \end{pmatrix}_{\tau\mu \times \tau\mu}.$$

Then we define $M(w,\lambda,\mu)$ to be the Macaulayfication $\tilde{M}(w,\lambda,\mu)^\dagger$ of $\tilde{M}(w,\lambda,\mu)$.

This generalizes the rank 1 case given in Definition 3.3.3. Now we can state one of the main results in [BD17] which establishes the relationship between band data and modules.

Theorem 3.4.5. *For a non-periodic band datum (w, λ, μ), the module $M(w, \lambda, \mu)$ is an indecomposable maximal Cohen-Macaulay module over A. Furthermore, it is locally free of rank $\tau\mu$ on the punctured spectrum where the length of w is 3τ.*

2. *Any indecomposable maximal Cohen-Macaulay module over A which is locally free on the punctured spectrum is isomorphic to $M(w, \lambda, \mu)$ for some non-periodic band datum (w, λ, μ).*

3. *For two band data (w, λ, μ) and (w_*, λ_*, μ_*), the corresponding modules $M(w, \lambda, \mu)$ and $M(w_*, \lambda_*, \mu_*)$ are isomorphic if and only if (w, λ, μ) and (w_*, λ_*, μ_*) are equivalent.*

3.5 Conversion Formula between Loop/Arc Data and Band/String Data

Definition 3.5.1 (Conversion from band data to loop data)**.** *Pick a band datum (w,λ,μ), with*

$$w = (w_1, w_2, w_3, w_4, w_5, w_6, \ldots, w_{3\tau-2}, w_{3\tau-1}, w_{3\tau}) \in \mathbb{Z}^{3\tau}.$$

We define the sign word $\delta = \delta(w) \in \{0,1\}^{3\tau}$, *the* correction word $\varepsilon = \varepsilon(w) \in \{-1,0,1,2\}^{3\tau}$ *of w and the loop word $w' \in \mathbb{Z}^{3\tau}$ converted from w below, where we regard the index j of w_j, δ_j, ε_j and w'_j to be in $\mathbb{Z}_{3\tau}$. First, each entry of the sign word $\delta = \delta(w)$ is defined as*

$$\delta_j := \begin{cases} 0 & \text{if } \begin{cases} w_j < 0, \text{ or} \\ w_j = 0 \text{ and at least one of the first non-zero entries adjacent to the} \\ \qquad \text{string of 0s containing } w_j \text{ (exists and) is negative,} \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

Next, each entry of the correction word $\varepsilon = \varepsilon(w)$ is defined as

$$\varepsilon_j := -1 + \delta_{j-1} + \delta_j + \delta_{j+1}.$$

Then each entry of the converted loop word w' is defined as

$$w'_j = w_j + \varepsilon_j$$

and the conversion *from the band datum to the loop datum is given by*

$$(w,\lambda,\mu) \mapsto (w' = w + \varepsilon(w), \lambda' = (-1)^{l_1+\cdots+l_\tau+\tau}\lambda, \mu = \mu)$$

where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$.

Lemma 3.5.2. *For a band word $w \in \mathbb{Z}^{3\tau}$ and its sign word $\delta = \delta(w) \in \{0,1\}^{3\tau}$, assume $w_k = \cdots = w_l = 0$ for some $k, l \in \mathbb{Z}_{3\tau}$. Then*

1. *$\delta_{k-1} = \delta_{l+1} = 1$ implies $\delta_k = \cdots = \delta_l = 1$, and*
2. *either $\delta_{k-1} = 0$ or $\delta_{l+1} = 0$ implies $\delta_k = \cdots = \delta_l = 0$.*

Proof. It is obvious from the definition of the sign word. □

Proposition 3.5.3. *The loop word w' converted from a band word w is always normal.*

Proof. Let w be any band word, $\varepsilon = \varepsilon(w)$ the correction word of w, and $w' = w + \varepsilon(w)$ the converted loop word. We will show that w' satisfies all of 4 conditions to be normal in order.

- *Condition 1* Assume that $w'_j = 1$ for some $j \in \mathbb{Z}_{3\tau}$. As ε_j takes its value in one of -1, 0, 1 and 2, the possible combination of w_j and ε_j are $(w_j,\varepsilon_j) = (2,-1)$, $(1,0)$, $(0,1)$ and $(-1,2)$. But the first one is impossible as $w_j = 2$ means $\delta_j = 1$ so that $\varepsilon_j \geq 0$. The last one is also ruled out as $w_j = -1$ yields $\delta_j = 0$

so that $\varepsilon_j \le 1$. In the third case, in order for $\varepsilon_j = 1$ to be hold, only one of δ_{j-1}, δ_j and δ_{j+1} is 0. But this cannot hold under $w_j = 0$ according to Lemma 4.1.2.

Thus only the second combination remains. In this case, in order to hold $w_j = 1$ and $\varepsilon_j = 0$, we must have $\delta_j = 1$ and $\delta_{j-1} = \delta_{j+1} = 0$. Then we have $w_{j-1} \le 0$. If $\delta_{j-2} = 0$, we have $\varepsilon_{j-1} = 0$ and hence $w'_{j-1} \le 0$. Otherwise, if $\delta_{j-2} = 1$, we have $\varepsilon_{j-1} = 1$ and $w_{j-1} \le -1$ by Lemma 4.1.2.(1) which gives $w'_{j-1} \le 0$. Therefore, $w'_{j-1} \le 0$ holds in any case and similarly we conclude that $w'_{j+1} \le 0$ also holds. This establishes the first normality condition of w'.

- *Condition 2* Assume that $w'_j = 0$ for some $j \in \mathbb{Z}_{3\tau}$. The possible combination of w_j and ε_j are

$$\left(w_j, \varepsilon_j\right) = (1,-1), (0,0), (-1,-1) \text{ and } (-2,-2).$$

As before, one can easily exclude the first and the last cases.

In the second case, because $\varepsilon_j = 0$, only one of δ_{j-1}, δ_j and δ_{j+1} is 1. As $w_j = 0$, it follows from Lemma 4.1.2.(2) that one of δ_{j-1} and δ_{j+1} is 1. Hence, we can assume without loss of generality that $\delta_{j-1} = 1$ and $\delta_j = \delta_{j+1} = 0$. Then we must have $w_{j-1} \ge 1$ by Lemma 4.1.2.(2) and $\varepsilon_{j-1} \ge 0$ and hence $w'_{j-1} \ge 1$. Also, we have $w_{j+1} \le 0$ and $\varepsilon_{j+1} \le 0$. If $w_{j+1} \le -1$, we get $w'_{j+1} \le -1$. Otherwise, if $w_{j+1} = 0$, Lemma 4.1.2.(1) gives $\delta_{j+2} = 0$ so that $\varepsilon_{j+1} = -1$ and $w'_{j+1} \le -1$ follow. Consequently, we have $w'_{j-1} \ge 1$ and $w'_{j+1} \le -1$. If $\delta_{j+1} = 1$, by symmetry, we get $w'_{j-1} \le -1$ and $w'_{j+1} \ge 1$, establishing the second normality condition of w'.

In the third case, $\varepsilon_j = 1$ and $\delta_j = 0$ yield $\delta_{j-1} = \delta_{j+1} = 1$. Then Lemma 4.1.2.(2) gives w_{j-1}, $w_{j+1} \ge 1$. Since $\varepsilon_{j-1}, \varepsilon_{j+1} \ge 0$, we have w'_{j-1}, $w'_{j+1} \ge 1$, establishing again the second normality condition of w'.

- *Condition 3* Assume there are integers $k, l \in \mathbb{Z}_{3\tau}$ such that $l \ne k+1$ and $w'_k = 0$, $w'_{k+1} = \cdots = w'_{l-1} = -1$, $w'_l = 0$. By discussion in the previous case, we have $w_k = w_l = 0$ and $\delta_{k-1} = \delta_{l+1} = 1$, $\delta_k = \delta_l = 0$. It can be easily checked that $w'_{k+1} = \cdots = w'_{l-1} = -1$ implies $\delta_{k+1} = \cdots = \delta_{l-1} = 0$, yielding $\varepsilon_{k+1} = \cdots = \varepsilon_{l-1} = -1$ and that $w_{k+1} = \cdots = w_{l-1} = 0$. Putting these together would contradict Lemma 4.1.2(1). Therefore, we conclude that there is no subword of the form $(0,-1,-1,\dots,-1,0)$ in w', establishing the third normality condition of w'.

- *Condition 4* Assume $w'_j = -1$ for all $j \in \mathbb{Z}_{3\tau}$. It can be easily checked that $\delta_j = 0$ for all j, yielding $\varepsilon_j = -1$ and $w_j = 0$. But then by definition we have $\delta_j = 1$ for any such j, which is a contradiction. Therefore, w' does not consist only of -1, establishing the last normality condition of w'.

□

Next we define the inverse of the above conversion formula.

Definition 3.5.4 (Conversion from normal loop data to band data)**.** *Pick a normal loop datum* (w', λ', μ), *with*

$$w' = (w'_1, w'_2, w'_3, w'_4, w'_5, w'_6, \dots, w'_{3\tau-2}, w'_{3\tau-1}, w'_{3\tau}) \in \mathbb{Z}^{3\tau}.$$

We define the sign word $\delta' = \delta'(w') \in \{0,1\}^{3\tau}$*, the* correction word $\varepsilon' = \varepsilon'(w') \in \{-1,0,1,2\}^{3\tau}$ *of w' and the band word $w \in \mathbb{Z}^{3\tau}$ converted from w' below, where we regard the index j of w'_j, s'_j, ε'_j and w_j to be in $\mathbb{Z}_{3\tau}$. First, each entry of the sign word $\delta' = \delta'(w')$ is defined as*

$$\delta'_j := \begin{cases} 0 & \text{if} \quad w'_j \leq 0 \\ 1 & \text{if} \quad w'_j > 0. \end{cases}$$

Next, each entry of the correction word $\varepsilon' = \varepsilon'(w')$ is defined as

$$\varepsilon'_j := -1 + \delta'_{j-1} + \delta'_j + \delta'_{j+1}.$$

Then each entry of the converted loop word w is defined as

$$w_j = w'_j - \varepsilon'_j$$

and the conversion *from the normal loop datum to the band datum is given by*

$$(w', \lambda', \mu) \mapsto (w = w' - \varepsilon'(w'), \lambda = (-1)^{l_1 + \cdots + l_\tau + \tau} \lambda', \mu = \mu)$$

where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$.

Example 3.5.5. *Consider a band datum (w, λ, μ) whose band word w is given as below.*

$$\begin{array}{rl}
w & = (\ 6\ ,\ 0\ ,\ 2\ ,-1\ ,\ 0\ ,-3\ ,\ 0\ ,\ 0\ ,\ 5\ ,\ 0\ ,-2\ ,\ 1\ ,-1\ ,\ 3\ ,\ 4\ ) \\
\delta(w) = \delta'\left(w'\right) & = (\ 1\ ,\ 1\ ,\ 1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ 1\ ,\ 1\ ) \\
\varepsilon(w) = \varepsilon'\left(w'\right) & = (\ 2\ ,\ 2\ ,\ 1\ ,\ 0\ ,-1\ ,-1\ ,-1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 1\ ,\ 1\ ,\ 2\ ) \\
w' & = (\ 8\ ,\ 2\ ,\ 3\ ,-1\ ,-1\ ,-4\ ,-1\ ,\ 0\ ,\ 5\ ,\ 0\ ,-2\ ,\ 1\ ,\ 0\ ,\ 4\ ,\ 6\ )
\end{array}$$

We computed the sign word $\delta(w)$, the correction word $\varepsilon(w)$ of w and the loop word $w' = w + \varepsilon(w)$ converted from w. Note that w' is presented in the normal form. Then we computed the sign word $\delta'\left(w'\right)$, the correction word $\varepsilon'\left(w'\right)$ of w'.

We underlined the spots of w in blue or red, respectively, according to whether the value of δ on them is 1 or 0. Likewise, the spots of w' are underlined according to the value of δ'. Observe that both w and w' have the same underline pattern, implying $\delta'\left(w'\right) = \delta(w)$ and hence $\varepsilon'\left(w'\right) = \varepsilon(w)$. Therefore, the band word $w' - \varepsilon'\left(w'\right)$ converted from w' is the same as the original band word w.

The parameter λ and the holonomy parameter λ' in this case are related by $\lambda' = (-1)^{6-1+0+0-1+5}\lambda = -\lambda$.

Proposition 3.5.6. *The conversion from the band datum to the loop datum and the conversion from the normal loop datum to the band datum are the inverses of each other.*

Proof. Let $\left(w, \lambda, \mu\right)$ be a band datum and

$$\left(w' = w + \varepsilon(w), \lambda' = (-1)^{l_1 + \cdots + l_\tau + \tau} \lambda, \mu\right)$$

the converted loop datum, where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$. Then let

$$\left(w'' = w' - \varepsilon'\left(w'\right), \lambda'' = (-1)^{l''_1+\cdots+l''_\tau+\tau}\lambda', \mu = \mu'\right)$$

be the band datum converted from (w', λ', μ), where $l''_i = w''_{3i-2}$ for $i \in \mathbb{Z}_\tau$. In order to show $(w'', \lambda'', \mu) = (w, \lambda, \mu)$, we notice that it is enough to show $w'' = w$, which is also equivalent to $\varepsilon(w) = \varepsilon'(w')$. By the construction of w and w', therefore, we only need to show that $\delta(w) = \delta'(w')$. Denoting $\delta = \delta(w)$ and $\delta' = \delta'(w')$, we can prove $\delta_j = \delta'_j$ for each $j \in \mathbb{Z}_{3\tau}$ as follows.

- *Case 1* $w_j \leq -1$

  In this case, we have $\delta_j = 0$ and hence $\varepsilon_j \leq 1$, implying $w'_j \leq 0$ so that $\delta'_j = 0$ follows.

- *Case 2* $w_j = 0$

  If $\delta_j = 0$, by Lemma 4.1.2.(1), at least one of δ_{j-1} and δ_{j+1} must be 0, implying $\varepsilon_j \leq 0$ so that $w'_j \leq 0$ and hence $\delta'_j = 0$. Otherwise, if $\delta_j = 1$, by Lemma 4.1.2.(2), both δ_{j-1} and δ_{j+1} must be 1, implying $\varepsilon_j = 2$ so that $w'_j = 2$ and hence $\delta'_j = 1$.

- *Case 3* $w_j \geq 1$

  We have $\delta_j = 1$ and hence $\varepsilon_j \geq 0$, implying $w'_j \geq 1$ so that $\delta'_j = 1$ follows.

Therefore, we proved that if we convert a given band datum to a loop datum and then convert it back to a band datum, it returns to itself.

Conversely, let (w', λ', μ) be a normal loop datum and

$$\left(w'' = w' - \varepsilon'(w), \lambda'' = (-1)^{l''_1+\cdots+l''_\tau+\tau}\lambda', \mu\right)$$

the converted band datum, where $l''_i = w''_{3i-2}$ for $i \in \mathbb{Z}_\tau$. Then let

$$\left(w''' = w'' - \varepsilon''\left(w''\right), \lambda''' = (-1)^{l''_1+\cdots+l''_\tau+\tau}\lambda'', \mu = \mu\right)$$

be the loop datum converted from (w'', λ'', μ). In order to show $(w''', \lambda''', \mu) = (w', \lambda', \mu)$, we only need to show $w''' = w'$, which again follows from $\delta'(w') = \delta''(w'')$. As above, denoting $\delta' = \delta'(w')$ and $\delta'' = \delta''(w'')$, we can prove $\delta'_j = \delta''_j$ for each $j \in \mathbb{Z}_{3\tau}$ by dividing the case.

- *Case 1* $w'_j \leq -2$

  We have $\delta'_j = 0$. As $\varepsilon'_j \geq -1$, we also have $w''_j \leq -1$ and hence $\delta''_j = 0$.

- *Case 2* $w'_j = -1$

  We have $\delta'_j = 0$. Since w' is normal, w' does not consist only of -1.

  If the first non-(-1) element to the left of w'_j is less than or equal to -2, namely, if there is an integer $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \leq -2$, $w'_{k+1} = \cdots = w'_j = -1$, we have $w''_k \leq -1$, $w''_{k+1}, \ldots, w''_j \leq 0$, implying $\delta''_j = 0$ by Lemma 4.1.2.(2).

If the first non-(-1) element to the left of w'_j is greater than or equal to 1, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \geq 1$, $w'_{k+1} = \cdots = w'_j = -1$, we have $\varepsilon'_{k+1} \geq 0$ and hence $w''_{k+1} \leq -1$, $w''_{k+2}, \ldots, w''_j \leq 0$, implying $\delta''_j = 0$ by Lemma 4.1.2.(2).

The above discussion also applies to the elements to the right of w'_j. It remains a case where both the first non-(-1) element to the left and the right of w'_j are 0, namely, there are $k, l \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $k < j < l$ and $w'_k = 0$, $w'_{k+1} = \cdots = w'_{l-1} = -1$, $w'_l = 0$. But this violates the third condition for w' to be normal. Thus we conclude that $\delta''_j = 0$ whenever $w'_j = -1$ and w' is normal.

- *Case 3* $w'_j = 0$

  We have $\delta'_j = 0$. Because w' is normal, one of $w'_{j-1} \leq -1$, $w'_{j+1} \geq 1$ or $w'_{j-1} \geq 1$, $w'_{j+1} \leq -1$ or w'_{j-1}, $w'_{j+1} \geq 1$ must hold. In the first case, we have $\varepsilon'_j = 0$, $w''_j = 0$ and $\delta''_{j-1} = 0$ bt arguments in the case 1 and 2. Hence, Lemma 4.1.2.(2) gives $\delta''_j = 0$. The second case can also be handled in the same way. In the third case, we have $\varepsilon'_j = 1$, $w''_j = -1$ and hence $\delta''_j = 0$.

- *Case 4* $w'_j = 1$

  We have $\delta'_j = 1$. As w' is normal, both w'_{j-1} and w'_{j+1} are less than or equal to 0. Therefore, we have $\varepsilon'_j = 0$, $w''_j = 1$ and hence $\delta''_j = 1$.

- *Case 5* $w'_j = 2$

  We have $\delta'_j = 1$. If w' consists only of 2, then w'' consists only of 0, whence $\delta''_j = 1$. Now assume that this is not the case.

  If the first non-2 element to the left of w'_j is less than or equal to 1, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \leq 1$, $w'_{k+1} = \cdots = w'_j = 2$, we automatically get $w'_k \leq 0$ by the first condition that w' is normal. Thus we have $\varepsilon'_{k+1} = 1$ and hence $w''_{k+1} = 1$, $w''_{k+2}, \ldots, w''_j \geq 0$. And if the first non-2 element to the left of w'_j is greater than or equal to 3, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \geq 3$, $w'_{k+1} = \cdots = w'_j = 2$, we have $w''_k \geq 1$, $w''_{k+1}, \ldots, w''_j \geq 0$. Therefore, in any cases, either $w''_j \geq 1$ or there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w''_k \geq 1$, $w''_{k+1}, \ldots, w''_j \geq 0$. This result also applies to the elements to the right of w'_j and therefore we conclude that either $w''_j \geq 1$, or there are $k, l \in \mathbb{Z}_{3\tau} \setminus \{j\}$ with $k < j < l$ such that $w''_k \geq 1$, $w''_{k+1}, \ldots, w''_{l-1} \geq 0$, $w''_l \geq 1$. In either case, we have $\delta''_j = 1$.

- *Case 6* $w'_j \geq 3$

  We have $\delta'_j = 1$. As $\varepsilon'_j \leq 2$, we also have $w''_j \geq 1$ and hence $\delta''_j = 1$.

We proved that if we convert a given normal loop datum to a band datum and then convert it back to a loop datum, it returns to itself. □

Remark 3.5.7. *We could have chosen different forms in the conversion formula from band words to loop words. To be more specific, we can take any sign word $\delta^* := \delta^*(w) \in \{0,1\}^{3\tau}$ in Definition 4.1.1 satisfying the following:*

- $\delta_j^* = 1 \quad$ *if* $\quad w_j > 0$,
- $\delta_j^* = 0 \quad$ *if* $\quad w_j < 0$,
- *if* $w_j = 0$ *and* $\delta_j^* < \delta_{j+1}^*$, *then the first non-zero entry to the left of* w_{j+1} *(exists and) is negative and the first non-zero entry to the right of* w_j *(exists and) is positive,*
- *if* $w_j = 0$ *and* $\delta_j^* > \delta_{j+1}^*$, *then the first non-zero entry to the left of* w_{j+1} *(exists and) is positive and the first non-zero entry to the right of* w_j *(exists and) is negative.*

Then we can prove that loop words obtained from the same band word should be equivalent to each other, no matter which sign word $\delta^*(w)$ *is used. In this case, all converted loop words satisfy the following* 'quasi-normal' *conditions:*

- *any subword of the form* $(a, 0, b)$ *in* w' *satisfies* $a \leq -1$, $b \geq 1$ *or* $a \geq 1$, $b \leq -1$ *or* $a, b \geq 1$,
- *any subword of the form* $(a, 1, b)$ *in* w' *satisfies* $a \geq 2$, $b \leq 0$ *or* $a \leq 0$, $b \geq 2$ *or* $a, b \leq 0$,
- w' *has no subword of the form* $(0, -1, -1, \ldots, -1, 0)$, *and*
- w' *has no subword of the form* $(1, 2, 2, \ldots, 2, 1)$.

The conversion formula from 'quasi-normal' loop words to band words remains the same as in Definition 4.1.4. If one convert a given band word to a loop word using any sign word and then convert it back to a band word, it returns to itself. Conversely, if one convert a given 'quasi-normal' loop word to a band word and then convert it back to a loop word using some sign word, it is equivalent to the original one.

3.6 Matrix Factorizations from Cohen-Macaulay Modules: Band-Type and String-Type

In this section, we work out the higher rank analogue of Section 3.3 to prove Theorem **??**.

Let us briefly recall our setting. Let $S := \mathbb{C}[[x,y,z]]$ be the formal power series ring of three variables and $A := \mathbb{C}[[x,y,z]]/(xyz)$ its quotient ring. Given a band datum $(w,\lambda,1)$ of length 3τ and multiplicity 1, let $\delta := \delta(w)$ and $\varepsilon := \varepsilon(w)$ be the sign word and the correction word of w, respectively and $(w',\lambda',1)$ the loop datum obtained by the conversion formula, which are described in Definition 4.1.1. Then the module $M(w,\lambda,1)$ corresponding to $(w,\lambda,1)$ and the Lagrangian $L(w',\lambda',1)$ corresponding to $(w',\lambda',1)$ give matrix factors of xyz over S. To compare them, we proposed the *canonical form* $\varphi(w',\lambda,1)$ of matrices arising from $(w,\lambda,1)$ as

$$\varphi(w',\lambda,1) := \begin{pmatrix} z & -y^{m'_1-1} & 0 & 0 & \cdots & 0 & -\lambda^{-1}x^{-l'_1} \\ -y^{-m'_1} & x & -z^{n'_1-1} & 0 & \cdots & 0 & 0 \\ 0 & -z^{-n'_1} & y & -x^{l'_2-1} & \cdots & 0 & 0 \\ 0 & 0 & -x^{-l'_2} & z & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -y^{m'_\tau-1} & 0 \\ 0 & 0 & 0 & \cdots & -y^{-m'_\tau} & x & -z^{n'_\tau-1} \\ -\lambda x^{l'_1-1} & 0 & 0 & \cdots & 0 & -z^{-n'_\tau} & y \end{pmatrix}_{3\tau\times 3\tau}.$$

As always, we denote $l'_i := w'_{3i-2}$, $m'_i := w'_{3i-1}$, $n'_i := w'_{3i}$ and regard x^a, y^a or z^a to be zero if $a<0$. The entries of the matrix $\varphi(w',\lambda,1)$ are considered to be in S and then it yields an obvious S-module homomorphism $\varphi(w',\lambda,1): S^{3\tau} \to S^{3\tau}$ between free S-modules.

Then we denote by $\underline{\varphi}(w',\lambda,1)$ the matrix $\varphi(w',\lambda,1)$ modulo xyz. That is, $\underline{\varphi}(w',\lambda,1)$ is the same form as $\varphi(w',\lambda,1)$ but entries are considered to be elements of A. This yields an A-module homomorphism $\underline{\varphi}(w',\lambda,1): A^{3\tau} \to A^{3\tau}$ between free A-modules.

Note that each A-module M also has a natural S-module structure, and we will denote by M_S the corresponding S-module. Namely, the underlying set and abelian group structure of M_S is the same as M whereas the scalar multiplication is defined by $f\cdot u := [f]\cdot u$, where $f\in S$, $[f]\in A = S/(xyz)$. and u in the left hand side is an element of M_S and u in the right hand side is the same element of M. Readers should note that $xyz\cdot u = 0$ for any $u\in M_S$ even if it is an S-module operation. Note also that a subset of M or M_S generates the whole set with the A-module structure if and only if it does with the S-module structure.

Our goal in this section is to show that the matrix factor of xyz over S arising from the module $M(w,\lambda,1)$ fits into the above canonical form $\varphi(w',\lambda,1)$.

Theorem 3.6.1. *For a nondegenerate band datum $(w,\lambda,1)$ of length 3τ and multiplicity 1, we can construct the following free resolution of $M_S(w,\lambda,1)$ as an S-module:*

$$0 \longrightarrow S^{3\tau} \xrightarrow{\varphi(w',\lambda,1)} S^{3\tau} \xrightarrow{\pi} M_S(w,\lambda,1) \longrightarrow 0. \tag{3.6.1}$$

The map π will be constructed during the proof. In particular,

$$M(w,\lambda,1) \cong \operatorname{coker}\underline{\varphi}(w',\lambda,1)$$

holds as A-modules.

This together with the conversion formula proves Theorem **??**.

3.6.1 Notations and Properties

For rings $S := \mathbb{C}[[x, y, z]]$ and $A = \mathbb{C}[[x, y, z]]/(xyz)$, let

$$A^\tau = \mathbb{C}[[x_1, y_1, z_1]]/(x_1 y_1 z_1) \times \cdots \times \mathbb{C}[[x_\tau, y_\tau, z_\tau]]/(x_\tau y_\tau z_\tau)$$

be the free A-module of rank τ. We introduce an alternative notation for elements of S, A and A^τ. This is to represent the variables such as x, y and z at once.

Notation 3.6.2. *We denote by*

$$\chi_{3i-2} := x, \quad \chi_{3i-1} := y \quad \text{and} \quad \chi_{3i} := z$$

for $i \in \mathbb{Z}_\tau$ the elements in the ring S or A, depending on the context. Then we consider χ_j^a to be zero if $a < 0$, for any $j \in \mathbb{Z}_{3\tau}$.

2. *We abbreviate some monomials in A^τ as*

$$^{l}X_{3i-2}^{m} := x_i^l y_i^m, \quad ^{m}X_{3i-1}^{n} := y_i^m z_i^n \quad \text{and} \quad ^{n}X_{3i}^{l} := z_i^n x_i^l$$

for $i \in \mathbb{Z}_\tau$ and l, m, $n \in \mathbb{Z}_{\geq 1}$.

3. *To deal with the eigenvalue λ, we introduce the symbols*

$$\Lambda_j^+ := \begin{cases} \lambda & \text{if } j = 1 \text{ and } \delta_1 = 1, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \Lambda_j^- := \begin{cases} \lambda^{-1} & \text{if } j = 1 \text{ and } \delta_1 = 0, \\ 1 & \text{otherwise} \end{cases}$$

for $j \in \mathbb{Z}_{3\tau}$, depending on the band datum $(w, \lambda, 1)$.

For later use, we summarize some useful properties of new notations in the following proposition.

Proposition 3.6.3. *As elements of A^τ,*

$$^{a}X_j^b = \chi_j^a \chi_{j+1}^b \mathbf{e}_i$$

for $i \in \mathbb{Z}_\tau$, $j \in \mathbb{Z}_{3\tau}$ with $3i - 2 \leq j \leq 3i$ and a, $b \in \mathbb{Z}_{\geq 1}$, where $\mathbf{e}_i$ is the column vector in A^τ whose i-th entry is 1 and the rest are 0. Here the multiplication of $\chi_j^a \chi_{j+1}^b \in (S \text{ or } A)$ to $\mathbf{e}_i \in ((A^\tau)_S \text{ or } A^\tau)$ can be either an S-module or an A-module operation.

2. *We have the arithmetic rules*

$$\chi_{j+3i}\left({}^{a}X_j^b\right) = {}^{a+1}X_j^b, \quad \chi_{j+3i+1}\left({}^{a}X_j^b\right) = {}^{a}X_j^{b+1}, \quad \text{and} \quad \chi_{j+3i+2}\left({}^{a}X_j^b\right) = 0 \tag{3.6.2}$$

for any $i \in \mathbb{Z}_\tau$, $j \in \mathbb{Z}_{3\tau}$ and a, $b \in \mathbb{Z}_{\geq 1}$, regardless whether these are S-module or A-module operations.

Using new notations, we can rewrite the relevant modules and matrices.

Proposition 3.6.4. *$\tilde{M}(w,\lambda,1)$ is the A-submodule of A^τ generated by 6τ elements*

$$H_j := {}^{2}X_j^{2} \quad \textit{and} \quad G_j := \Lambda_j^+\left({}^{1}X_{j-1}^{w_j^+ +2}\right) + \Lambda_j^-\left({}^{w_j^- +2}X_j^{1}\right) \qquad \left(j \in \mathbb{Z}_{3\tau}\right).$$

2. *The canonical matrix $\varphi(w',\lambda,1)$ can be written as*

$$\varphi(w',\lambda,1) = \begin{pmatrix} \chi_{3\tau} & -\Lambda_2^+\chi_2^{w'_2-1} & 0 & 0 & \cdots & 0 & -\Lambda_1^-\chi_1^{-w'_1} \\ -\Lambda_2^-\chi_2^{-w'_2} & \chi_1 & -\Lambda_3^+\chi_3^{w'_3-1} & 0 & \cdots & 0 & 0 \\ 0 & -\Lambda_3^-\chi_3^{-w'_3} & \chi_2 & -\Lambda_4^+\chi_4^{w'_4-1} & \cdots & 0 & 0 \\ 0 & 0 & -\Lambda_4^-\chi_4^{-w'_4} & \chi_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -\Lambda_{3\tau-1}^+\chi_{3\tau-1}^{w'_{3\tau-1}-1} & 0 \\ 0 & 0 & 0 & \cdots & -\Lambda_{3\tau-1}^-\chi_{3\tau-1}^{-w'_{3\tau-1}} & \chi_{3\tau-2} & -\Lambda_{3\tau}^+\chi_{3\tau}^{w'_{3\tau}-1} \\ -\Lambda_1^+\chi_1^{w'_1-1} & 0 & 0 & \cdots & 0 & -\Lambda_{3\tau}^-\chi_{3\tau}^{-w'_{3\tau}} & \chi_{3\tau-1} \end{pmatrix}_{3\tau\times 3\tau}$$

where entries are elements of S. The matrix $\underline{\varphi}(w',\lambda,1)$ can be written in the same form but entries are considered to be in A.

Proof. The second statement can be checked easily. For the first one, recall that $\tilde{M}(w,\lambda,1)$ is the A-submodule of A^τ generated by all columns of the 6 matrices

$$x^2y^2I_\tau, \quad y^2z^2I_\tau, \quad z^2x^2I_\tau, \quad \pi_x(w,\lambda,1), \quad \pi_y(w,\lambda,1) \quad \text{and} \quad \pi_z(w,\lambda,1)$$

in $A^{\tau\times\tau}$, where the last three are given by

$$\begin{pmatrix} x^{l_1^-+2}y & zx^{l_2^++2} & \cdots & 0 \\ 0 & x^{l_2^-+2}y & \ddots & \vdots \\ \vdots & \vdots & \ddots & zx^{l_\tau^++2} \\ \lambda zx^{l_1^++2} & 0 & \cdots & x^{l_\tau^-+2}y \end{pmatrix}_{\tau\times\tau}, \quad \begin{pmatrix} xy^{m_1^++2}+y^{m_1^-+2}z & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & xy^{m_\tau^++2}+y^{m_\tau^-+2}z \end{pmatrix}_{\tau\times\tau}, \quad \begin{pmatrix} yz^{n_1^++2}+z^{n_1^-+2}x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & yz^{n_\tau^++2}+z^{n_\tau^-+2}x \end{pmatrix}_{\tau\times\tau}$$

in order. Now the i-th column of each matrix as an element of A^τ is

$${}^{2}X_{3i-2}^{2}, \quad {}^{2}X_{3i-1}^{2}, \quad {}^{2}X_{3i}^{2},$$

$$\Lambda_{3i-2}^+\left(\Lambda_{3i-2}^-\right)^{-1}\left({}^{1}X_{3i-3}^{w_{3i-2}^++2}\right) + {}^{w_{3i-2}^-+2}X_{3i-2}^{1}, \quad {}^{1}X_{3i-2}^{w_{3i-1}^++2} + {}^{w_{3i-1}^-+2}X_{3i-1}^{1} \quad \text{and} \quad {}^{1}X_{3i-1}^{w_{3i}^++2} + {}^{w_{3i}^-+2}X_{3i}^{1},$$

respectively, in order. □

Recall in linear algebra that the adjoint matrix $\operatorname{adj}B$ of a square matrix B is the transpose of the cofactor matrix of B whose (a,b)-entry is $(-1)^{a+b}$ times the (a,b)-minor of B. A useful property of the adjoint matrix is that it satisfies

$$B\cdot\operatorname{adj}B = \operatorname{adj}B\cdot B = (\det B)\,I \tag{3.6.3}$$

where I is the identity matrix of the same size as B. This is valid whenever the matrix B has entries in a commutative ring.

Proposition 3.6.5. *Let* $(w,\lambda,1)$ *be a band datum of length* 3τ *and multiplicity* 1.

1. *The determinant of* $\varphi := \varphi\left(w',\lambda,1\right)$ *is*

$$\det\varphi\left(w',\lambda,1\right) = x^{\tau}y^{\tau}z^{\tau}u$$

where

$$u := u\left(w',\lambda,1\right) := 1 - \prod_{j=1}^{3\tau}\Lambda_j^{+}\chi_j^{w'_j-2} - \prod_{j=1}^{3\tau}\Lambda_j^{-}\chi_j^{-w'_j-1}$$

is a unit in S *if and only if* $(w,\lambda,1)$ *is nondegenerate. In particular,* $u=1$ *unless* $w_j \geq 0$ *for all* j *or* $w_j \leq 0$ *for all* j.

2. *The* (a,b)*-entry of the adjoint matrix of* $\varphi\left(w',\lambda,1\right)$ *is*

$$\left(\operatorname{adj}\varphi\left(w',\lambda,1\right)\right)_{ab} = \begin{cases} \displaystyle\prod_{j=a}^{a-2}\chi_j & \textit{if} \quad a=b, \\ \displaystyle\left(\prod_{j=b}^{a-2}\chi_j\right)\left(\prod_{j=a+1}^{b}\Lambda_j^{+}\chi_j^{w'_j-1}\right) + \left(\prod_{j=a}^{b-2}\chi_j\right)\left(\prod_{j=b+1}^{a}\Lambda_j^{-}\chi_j^{-w'_j}\right) & \textit{if} \quad a\neq b \end{cases}$$

where we regard the product $\displaystyle\prod_{j=a}^{b}h_j$ *as* 1 *if* $b=a-1$ *and* $h_a\cdots h_{3\tau}h_1\cdots h_b$ *if* $b\leq a-2$.

Proof. It is a straightforward calculation using the fact $\chi_j^{w'_j-1}\chi_j^{-w'_j}=0$ for all j. □

In particular, Proposition 3.6.5.(1) together with the equation 3.6.3 and the fact that S is an integral domain demonstrates that the map $\varphi : S^{3\tau} \to S^{3\tau}$ is injective. This establishes the exactness of the sequence 3.6.1 at the leftmost arrow.

Corollary 3.6.6. *For a nondegenerate band datum* $(w,\lambda,1)$ *of length* 3τ *and multiplicity* 1*, define a matrix* $\tilde{\psi} := \tilde{\psi}\left(w',\lambda,1\right) \in S^{3\tau\times 3\tau}$ *by*

$$\left(\tilde{\psi}\left(w',\lambda,1\right)\right)_{ab} := \begin{cases} \chi_a\chi_{a+1} & \textit{if} \quad a=b, \\ \displaystyle\prod_{j=a+1}^{b}\Lambda_j^{+}\chi_j^{w'_j-2+\mathbb{1}_{j\in\{a+1,b\}}} + \prod_{j=b+1}^{a}\Lambda_j^{-}\chi_j^{-w'_j-1+\mathbb{1}_{j\in\{b+1,a\}}} & \textit{if} \quad a\neq b \end{cases}$$

where we regard the product symbol as in Proposition 3.6.5.(2) and $\mathbb{1}_{j\in\{a,b\}}$ *is* 1 *if* $j\in\{a,b\}$ *and* 0 *otherwise. Then the matrix* $\psi := \psi\left(w',\lambda,1\right) \in S^{3\tau\times 3\tau}$ *defined by*

$$\psi\left(w',\lambda,1\right) := \left(u\left(w',\lambda,1\right)\right)^{-1}\tilde{\psi}\left(w',\lambda,1\right)$$

satisfies

$$\varphi\psi = \psi\varphi = xyzI_{3\tau}.$$

That is, both φ and ψ are matrix factors of xyz over S.

Proof. One can check that

$$\operatorname{adj}\varphi = x^{\tau-1}y^{\tau-1}z^{\tau-1}\tilde{\psi}.$$

Substituting this and $\det\varphi = x^{\tau}y^{\tau}z^{\tau}u$ into the equation 3.6.3 gives

$$x^{\tau-1}y^{\tau-1}z^{\tau-1}\varphi\tilde{\psi} = x^{\tau-1}y^{\tau-1}z^{\tau-1}\tilde{\psi}\varphi = x^{\tau}y^{\tau}z^{\tau}uI_{3\tau}.$$

We may cancel out the common terms $x^{\tau-1}y^{\tau-1}z^{\tau-1}$ in both sides since S is an integral domain, which yields the desired result. □

3.6.2 Generators, Relations and Macaulayfying Elements

Recall that $\tilde{M} := \tilde{M}(w,\lambda,\mu)$ is generated by elements $G_1,\ldots,G_{3\tau}$ together with $H_1,\ldots,H_{3\tau}$ in A^{τ}. Here we investigate the relations among them by plotting them on the lattice diagram for A^{τ}. Figure 3.4 illustrates a part of generator diagram for $\tilde{M}$ representing G_j, H_j and G_{j+1} in each case depending on the value of δ_j and δ_{j+1}.

In each case, we can observe the relations among G_j's and determine whether H_j is spanned by them or not, as follows:

- $\delta_j = \delta_{j+1} = 1$: $\quad \Lambda^+_{j+1}\chi_{j+1}^{w_{j+1}+1}G_j = \chi_j G_{j+1}$, $\quad H_j = \chi_{j+1}G_j$
- $\delta_j = 1, \delta_{j+1} = 0$: $\quad \chi_{j+1}G_j = \chi_j G_{j+1}$, $\quad H_j = \chi_{j+1}G_j = \chi_j G_{j+1}$
- $\delta_j = \delta_{j+1} = 0$: $\quad \chi_{j+1}G_j = \Lambda^-_j\chi_j^{-w_j+1}G_{j+1}$, $\quad H_j = \chi_j G_{j+1}$
- $\delta_j = 0, \delta_{j+1} = 1$: $\quad \Lambda^+_{j+1}\chi_{j+1}^{w_{j+1}+1}G_j = \Lambda^-_j\chi_j^{-w_j+1}G_{j+1}$, $\quad H_j$ may not be spanned by $G_1,\ldots,G_{3\tau}$

The relations among G_j's can be summarized as

$$\Lambda^+_{j+1}\chi_{j+1}^{w^+_{j+1}+1}G_j = \Lambda^-_j\chi_j^{w^-_j+1}G_{j+1} \tag{3.6.4}$$

for $j \in \mathbb{Z}_{3\tau}$ in any cases, noting that $\Lambda^-_j = 1$, $w_j \geq 0$ when $\delta_j = 1$ and $\Lambda^+_j = 1$, $w_j \leq 0$ when $\delta_j = 0$.

On the other hand, H_j is not spanned by $G_1,\ldots,G_{3\tau}$ only if $\delta_j = 0$ and $\delta_{j+1} = 1$ hold simultaneously. The converse is not true in general, as $w_j = 0$ may still hold in this case. As a consequence, $\tilde{M}$ is generated by $G_1,\ldots,G_{3\tau}$ together with those H_j's. In such a case, we find in Figure 3.4d the relations

$$-\chi_{j-2}G_j + \Lambda^-_j\chi_j^{-w_j}H_j = 0, \quad \chi_{j-1}H_j = 0 \quad \text{and} \quad \Lambda^+_{j+1}\chi_{j+1}^{w_{j+1}}H_j - \chi_j G_{j+1} = 0 \tag{3.6.5}$$

among H_j and G_j's. Note that these relations are '*enough*' in the sense that any other relations containing H_j are S-linear combinations of these. Namely, if sH_j can be represented as an S-linear combination of G_j's for some $s \in S$, then s should be an S-linear combination of $\chi_j^{-w_j}$, χ_{j-1} and $\chi_{j+1}^{w_{j+1}}$ and the whole equation should be an S-linear combination of the above 3 relations.

Next, we find Macaulayfying elements of $\tilde{M}$ in A^τ. For any $\imath \in \mathbb{Z}_{3\tau}$ with $\delta_\imath = 1$ and $\delta_{\imath+1} = 0$, there are $\jmath \in \mathbb{Z}_{3\tau} \setminus \{\imath\}$ and $\kappa \in \{0, \ldots, \jmath - \imath - 1\}$ such that

$$w_\imath \geq 1, \quad w_{\imath+1} = \cdots = w_{\imath+\kappa} = 0, \quad w_{\imath+\kappa+1} \leq -1, \quad w_{\imath+\kappa+2}, \ldots, w_\jmath \leq 0 \quad \text{and} \quad w_{\jmath+1} \geq 1. \tag{3.6.6}$$

One should be careful about the indices. If $\kappa = 0$, then the second condition in 3.6.6 becomes an empty condition. If $\kappa = \jmath - \imath - 1$, then the fourth one is empty.

In this case, we have the identities

$$\delta_\imath = 1, \quad \delta_{\imath+1} = \cdots = \delta_{\imath+\kappa+1} = \cdots = \delta_\jmath = 0, \quad \delta_{\jmath+1} = 1 \tag{3.6.7}$$

by definition of the sign word δ. Figure 3.5 shows the relevant part of the generator diagram for $\tilde{M}$.

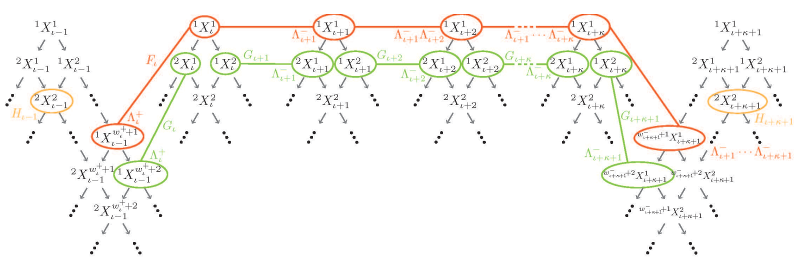


Figure 3.5: A Macaulayfying element of $\tilde{M}$ in $A^{3\tau}$

One may notice that the element $F_\imath \in A^{3\tau} \setminus \tilde{M}$ marked with orange color is a Macaulayfying element of $\tilde{M}$ in $A^{3\tau}$, which can be written as

$$\begin{aligned} F_\imath := {} & \Lambda_\imath^+ \left({}^1X_{\imath-1}^{w_\imath+1} \right) + {}^1X_\imath^1 + \Lambda_{\imath+1}^- \left({}^1X_{\imath+1}^1 \right) + \Lambda_{\imath+1}^- \Lambda_{\imath+2}^- \left({}^1X_{\imath+2}^1 \right) \\ & + \cdots + \left(\Lambda_{\imath+1}^- \cdots \Lambda_{\imath+\kappa}^- \right) \left({}^1X_{\imath+\kappa}^1 \right) + \left(\Lambda_{\imath+1}^- \cdots \Lambda_{\imath+\kappa+1}^- \right) \left({}^{-w_{\imath+\kappa+1}+1}X_{\imath+\kappa+1}^1 \right) \\ = {} & \Lambda_\imath^+ \left({}^1X_{\imath-1}^{w_\imath+1} \right) + \sum_{a=0}^{\kappa} \left(\prod_{b=1}^{a} \Lambda_{\imath+b}^- \right) \left({}^1X_{\imath+a}^1 \right) + \left(\prod_{b=1}^{\kappa+1} \Lambda_{\imath+b}^- \right) \left({}^{-w_{\imath+\kappa+1}+1}X_{\imath+\kappa+1}^1 \right). \end{aligned} \tag{3.6.8}$$

Indeed, we can express $\chi_{\imath+1}F_\imath$, $\chi_{\imath+2}F_\imath$ and $\chi_{\imath+3}F_\imath$ as S-linear combinations of

$$G''_{\imath-1}, \; G_\imath, \; G_{\imath+1}, \; G'_{\imath+2}, \; \ldots, \; G'_{\imath+\kappa+1}$$

where

$$G''_{\imath-1} := \begin{cases} H_{\imath-1} & \text{if } \delta_{\imath-1} = 0 \\ G_{\imath-1} & \text{if } \delta_{\imath-1} = 1 \end{cases}, \quad G'_\imath := G_\imath, \quad \ldots, \quad G'_\jmath := G_\jmath \quad \text{and} \quad G'_{\jmath+1} := H_\jmath, \tag{3.6.9}$$

confirming that $xF_\imath$, $yF_\imath$ and $zF_\imath$ are elements of $\tilde{M}$. Note that $G'_\imath = G_\imath$ and $G'_{\imath+1} = G_{\imath+1}$ always hold. Moreover, each of $G''_{\imath-1}$ and $G'_{\imath+a}$ is one of G_j's or one of H_j's that is not generated by G_j's, by the discussion in the above paragraph and identities $\delta_\imath = 1$, $\delta_\jmath = 0$ and $\delta_{\jmath+1} = 1$. For later use, we write down the explicit formulas as follows:

$$\begin{cases} \chi_\imath F_\imath = & G_\imath & + \zeta^{-1}_{\imath,-1,2} G'_{\imath+3} + \cdots + \zeta^{-1}_{\imath,-1,\kappa+1} G'_{\imath+\kappa+1} \\ \chi_{\imath+1} F_\imath = & G_{\imath+1} & + \zeta^{0}_{\imath,-1,2} G'_{\imath+3} + \cdots + \zeta^{0}_{\imath,-1,\kappa+1} G'_{\imath+\kappa+1} \\ \chi_{\imath+2} F_\imath = \Lambda^+_\imath \chi_\imath^{w'_\imath - 1} G''_{\imath-1} & + \Lambda^-_{\imath+1} \chi_{\imath+1}^{-w'_{\imath+1}} G'_{\imath+2} & + \zeta^{1}_{\imath,-1,2} G'_{\imath+3} + \cdots + \zeta^{1}_{\imath,-1,\kappa+1} G'_{\imath+\kappa+1} \end{cases} \tag{3.6.10}$$

where

$$\zeta^c_{\imath,a,b} := \begin{cases} \left(\prod_{d=a+1}^{b} \Lambda^-_{\imath+d} \right) \chi_{\imath+b}^{-w'_{\imath+b}-1} & \text{if } c \equiv b \pmod 3 \\ 0 & \text{otherwise} \end{cases} \tag{3.6.11}$$

for $b \in \{2, \dots, \kappa+1\}$, $a \in \{-1, \dots, b-1\}$ and $c \in \mathbb{Z}$. These can be observed in Figure 3.5 or computed straightforwardly from equation 3.6.8 applying the arithmetic rules 3.6.2, the (in)equalities 3.6.6, the identities 3.6.7 and the conversion formula $w'_j = w_j + \delta_{j-1} + \delta_j + \delta_{j+1} - 1$ in Definition 4.1.1.

We remark here on some basic properties of the symbols $\zeta^c_{\imath,a,b}$. First note that the monomial $\chi_{\imath+b}^{-w'_{\imath+b}-1}$ is 1 for $b \in \{2, \dots, \kappa\}$ and is nonzero for $b = \kappa + 1$. This follows from the observations $w'_{\imath+2} = \cdots = w'_{\imath+\kappa} = -1$ and $w'_{\imath+\kappa+1} \leq -1$. $\zeta^c_{\imath,a,b}$ is just an element of $\mathbb{C}$, namely $\prod_{d=a+1}^{b} \Lambda^-_{\imath+d}$, unless $b = \kappa + 1$. Based on this fact, we find that the symbols satisfy algebraic relations

$$\zeta^{b+d}_{\imath,a,b} \zeta^c_{\imath,b,c} = \zeta^{c+d}_{\imath,a,c} \tag{3.6.12}$$

for any $c \in \{2, \dots, \kappa-2\}$, $b \in \{2, \dots, c-1\}$, $a \in \{-1, \dots, b-1\}$ and $d \in \mathbb{Z}$, i.e. whenever all terms are defined.

Other than the form of $F_\imath$, there seem to be no further Macaulayfying elements. This will be revealed to be true in the proof of Theorem 3.6.1.

3.6.3 Proof of the Theorem

Let $M_0 := M_0(w, \lambda, 1)$ be the A-submodule of A^τ generated by the 3τ elements $G_1, \dots, G_{3\tau}$. Then $\tilde{M}$ is generated by elements of M_0 together with $H_1, \dots, H_{3\tau}$ in A^τ. And then M is generated by elements of $\tilde{M}$ together with the Macaulayfying elements of $\tilde{M}$ in A^τ.

The overall strategy of the proof is as follows. We will complete it in four steps. In step 1, we will first find a (part of) free resolution of $(M_0)_S$ as an S-module. In step 3, we modify it to get a (part of) free resolution of $\tilde{M}_S$. In step 4, we fix it again to finally establish the desired free resolution 3.6.1 of M_S. But for some technical reasons, we treat the special cases where $w_j \geq 0$ for all j or $w_j \leq 0$ for all j separately in step 2.

In step 3 and step 4, we need some technical lemmas, which allow us to modify the free resolution of a module when we add or replace its generators. Lemma 3.6.7 describes the '*matrix expansion*' process according to adding a generator and lemma 3.6.8 gives the '*matrix reduction*' process according to replacing a redundant generator. There is also a geometric version of matrix reduction in lemma 2.2.9 and remark 2.2.10.

Lemma 3.6.7. *Let $\pi_0 \in A^{a\times b}$ and $\varphi_0 \in S^{b\times c}$ be matrices such that the sequence*

$$S^c \xrightarrow{\varphi_0} S^b \xrightarrow{\pi_0} (A_S)^a \longrightarrow 0$$

of S-modules is exact and $B \in A^{a\times 1}$ and $C^T \in S^{1\times d}$, $D \in S^{b\times d}$ be matrices such that the enlarged matrices

$$\pi_1 := \left(\begin{array}{c|c} \quad \pi_0 \quad & B \end{array}\right) \in A^{a\times(b+1)} \quad \textit{and} \quad \varphi_1 := \left(\begin{array}{c|c} \varphi_0 & D \\ \hline 0 & C^T \end{array}\right) \in S^{(b+1)\times(c+d)}$$

satisfy $\pi_1\varphi_1 = 0 \in A^{a\times(c+d)}$. Assume further that $\operatorname{im} C^T \subseteq S$ contains the conductor

$$\operatorname{ann}_S\left(\operatorname{im}\pi_1/\operatorname{im}\pi_0\right) := \{s \in S \,|\, s(\operatorname{im}\pi_1) \subseteq \operatorname{im}\pi_0\} = \{s \in S \,|\, sB \subseteq \operatorname{im}\pi_0\}$$

of $\operatorname{im}\pi_0$ in $\operatorname{im}\pi_1$. Then we have the following modified exact sequence of S-modules:

$$S^{c+d} \xrightarrow{\varphi_1} S^{b+1} \xrightarrow{\pi_1} (A_S)^a \longrightarrow 0.$$

Proof. Note that $\operatorname{im}\pi_1 = (A_S)^a$ immediately follows from $\operatorname{im}\pi_0 = (A_S)^a$. The inclusion $\operatorname{im}\varphi_1 \subseteq \ker\pi_1$ is also immediate from $\pi_1\varphi_1 = 0$. For the opposite inclusion, assume that $v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in S^{b+1}$ is an element of $\ker\pi_1$ for some $v_1 \in S^b$ and $v_2 \in S$. Then $\pi_1 v = 0 \in A^a$ yields $\pi_0 v_1 + B v_2 = 0 \in A^a$. Thus, we have $v_2 B \in \operatorname{im}\pi_0$ and the assumption implies $v_2 \in \operatorname{im} C^T$. Put $v_2 = C^T u_2$ for some $u_2 \in S^d$ and substitute it to the previous equation, which gives $\pi_0 v_1 + BC^T u_2 = 0 \in A^a$. Note that the assumption $\pi_1\varphi_1 = 0$ implies the equation $\pi_0 D + BC^T = 0 \in A^{a\times d}$. Combining these, we get $\pi_0 v_1 - \pi_0 D u_2 = 0 \in A^a$, or equivalently, $v_1 - Du_2 \in \ker\pi_0 = \operatorname{im}\varphi_0$. Thus we have $v_1 = \varphi_0 u_1 + Du_2$ for some $u_1 \in S^c$. Now put $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in S^{c+d}$ then one can check $v = \varphi_1 u \in \operatorname{im}\varphi_1$, implying $\ker\pi_1 \subseteq \operatorname{im}\varphi_1$. □

Lemma 3.6.8. *Let $\pi_1 \in A^{a\times b}$, $B \in A^{a\times 1}$, $C \in S^{b\times c}$, $D \in S^{b\times 1}$ and $E^T \in S^{1\times c}$ be matrices and $u \in S$ be a unit such that for matrices*

$$\pi_0 := \left(\begin{array}{c|c} \quad \pi_1 \quad & B \end{array}\right) \in A^{a\times(b+1)} \quad \textit{and} \quad \varphi_0 := \left(\begin{array}{c|c} C & D \\ \hline E^T & u \end{array}\right) \in S^{(b+1)\times(c+1)}$$

the sequence

$$S^{c+1} \xrightarrow{\varphi_0} S^{b+1} \xrightarrow{\pi_0} (A_S)^a \longrightarrow 0$$

of S-modules is exact. Setting $\varphi_1 := C - Du^{-1}E^T \in S^{b\times c}$, we have the following modified exact sequence of S-modules:

$$S^c \xrightarrow{\varphi_1} S^b \xrightarrow{\pi_1} (A_S)^a \longrightarrow 0.$$

Proof. Consider the following diagram of S-modules:

$$\begin{tikzcd}
S^c \oplus S \arrow[r, "\varphi_0 = \begin{pmatrix} C & D \\ E^T & u \end{pmatrix}"] \arrow[d, "\wr\mid"', "\begin{pmatrix} I_c & 0 \\ u^{-1}E^T & 1 \end{pmatrix}"'] & S^b \oplus S \arrow[r, "\pi_0 = (\pi_1 \ B)"] \arrow[d, "\wr\mid"', "\begin{pmatrix} I_b & -Du^{-1} \\ 0 & u^{-1} \end{pmatrix}"] & (A_S)^a \arrow[r] \arrow[d, "\wr\mid"', "I_a"] & 0 \arrow[d, equal] \\
S^c \oplus S \arrow[r, "\varphi_1 = \begin{pmatrix} C - Du^{-1}E^T & 0 \\ 0 & 1 \end{pmatrix}"'] & S^b \oplus S \arrow[r, "(\pi_1 \ 0)"] & (A_S)^a \arrow[r] & 0
\end{tikzcd}$$

One can immediately check by matrix calculations that both squares in the diagram commute. Also, note that the vertical maps are all isomorphisms, hence the exactness of the top row yields the exactness of the bottom row, which also implies the exactness of the desired sequence. □

Proof of Theorem 3.6.1. We now prove the theorem, step by step as illustrated above.

Step 1: Find a (part of) free resolution of $(M_0)_S$.

Note that $M_0 = \langle G_1, \ldots, G_{3\tau} \rangle_A \in A^\tau$ yields an S-module $(M_0)_S = \langle G_1, \ldots, G_{3\tau} \rangle_S \in (A_S)^\tau$, where A_S is the same as A but considered as an S-module. That is, M_0 and $(M_0)_S$ are the same as underlying sets and the subset $\{G_1, \ldots, G_{3\tau}\}$ A-generates M_0 and at the same time S-generates $(M_0)_S$.

Consider the $\tau \times 3\tau$ matrix $\pi_0 := \pi_0(w, \lambda, 1) \in A^{\tau \times 3\tau}$ whose j-th column is $G_j \in A^\tau$ for $j \in \mathbb{Z}_{3\tau}$. That is,

$$\pi_0 := \left(G_1 \middle| G_2 \middle| G_3 \middle| \cdots \middle| G_{3\tau-1} \middle| G_{3\tau} \right)_{\tau \times 3\tau}.$$

Then regard the matrix as an S-module homomorphism $\pi_0 : S^{3\tau} \to (A_S)^\tau$, which is well-defined using the natural isomorphism $\mathrm{Hom}_S(S, A_S) \cong A_S$ where A_S is the same as A but considered as an S-module. As a result, the S-module $(M_0)_S$ is the image of $\pi_0 : S^{3\tau} \to (A_S)^\tau$. Restricting the codomain, we also denote the map by $\pi_0 : S^{3\tau} \to (M_0)_S$, which becomes automatically surjective.

We need to figure out the kernel of π_0 to further resolve $(M_0)_S$. It is equivalent to find relations among generators G_j of $(M_0)_S$, or the columns of π_0. Based on the observations

$$\Lambda_j^+ \chi_j^{w_j^+ + 1} G_{j-1} = \Lambda_{j-1}^- \chi_{j-1}^{w_{j-1}^- + 1} G_j \quad \text{and} \quad \chi_{j-2}\chi_{j-1} G_j = 0 \quad (j \in \mathbb{Z}_{3\tau}) \tag{3.6.13}$$

from equation 3.6.4 in Subsection 3.6.2, we define two matrices below. First, the matrix $\varphi_0 := \varphi_0(w, \lambda, 1) \in S^{3\tau \times 3\tau}$ is defined by

$$\varphi_0 := \begin{array}{c} \\ G^1 \\ G^2 \\ G^3 \\ \vdots \\ G^{3\tau-1} \\ G^{3\tau} \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} R_1 & R_2 & R_3 & \cdots & R_{3\tau-1} & R_{3\tau} \end{array} \\ \begin{pmatrix}
\Lambda_{3\tau}^- \chi_{3\tau}^{w_{3\tau}^- + 1} & -\Lambda_2^+ \chi_2^{w_2^+ + 1} & 0 & \cdots & 0 & 0 \\
0 & \Lambda_1^- \chi_1^{w_1^- + 1} & -\Lambda_3^+ \chi_3^{w_3^+ + 1} & \cdots & 0 & 0 \\
0 & 0 & \Lambda_2^- \chi_2^{w_2^- + 1} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & -\Lambda_{3\tau-1}^+ \chi_{3\tau-1}^{w_{3\tau-1}^+ + 1} & 0 \\
0 & 0 & 0 & \cdots & \Lambda_{3\tau-2}^- \chi_{3\tau-2}^{w_{3\tau-2}^- + 1} & -\Lambda_{3\tau}^+ \chi_{3\tau}^{w_{3\tau}^+ + 1} \\
-\Lambda_1^+ \chi_1^{w_1^+ + 1} & 0 & 0 & \cdots & 0 & \Lambda_{3\tau-1}^- \chi_{3\tau-1}^{w_{3\tau-1}^- + 1}
\end{pmatrix}_{3\tau \times 3\tau} \end{array} . \tag{3.6.14}$$

We denote the j-th row and the k-th column of φ_0 as G^j and R_k, respectively, for each $j, k \in \mathbb{Z}_{3\tau}$. This is considering that column R^k represents the k-th relation of G_j's given in the left part of 3.6.13, when the entry in row G^j is multiplied to G_j for each j. This proves $\pi_0\varphi_0 = 0$.

Next, we define the matrix $\varphi_{0\#} \in S^{3\tau\times 3\tau}$ by

$$\varphi_{0\#} := \begin{array}{c} \begin{array}{cccccc} R_{1\#} & R_{2\#} & R_{3\#} & \cdots & R_{(3\tau-1)\#} & R_{3\tau\#} \end{array} \\ \big(\chi_{3\tau-1}\chi_{3\tau}\mathbf{e}_{G^1} \,\big|\, \chi_{3\tau}\chi_1\mathbf{e}_{G^2} \,\big|\, \chi_1\chi_2\mathbf{e}_{G^3} \,\big|\, \cdots \,\big|\, \chi_{3\tau-3}\chi_{3\tau-2}\mathbf{e}_{G^{3\tau-1}} \,\big|\, \chi_{3\tau-2}\chi_{3\tau-1}\mathbf{e}_{G^{3\tau}} \big)_{3\tau\times 3\tau} \end{array}$$

where $\mathbf{e}_{G^j}$ is the column vector in $S^{3\tau}$ whose unique nonzero entry is 1 and lies in the same position corresponding to row G^j, namely the j-th row. We denote the j-th row and the k-th column of $\varphi_{0\#}$ by $G^{j\#}$ and $R_{k\#}$, respectively, for each $j, k \in \mathbb{Z}_{3\tau}$. Then column $R_{k\#}$ corresponds to the k-th relation of G_j's in the right part of 3.6.13, when the entry in row $G^{j\#}$ is multiplied to G_j for each j. This yields $\pi_0\varphi_{0\#} = 0$.

Now we claim that the following sequence of S-modules is exact:

$$S^{6\tau} \xrightarrow{(\varphi_0|\varphi_{0\#})} S^{3\tau} \xrightarrow{\pi_0} (M_0)_S \longrightarrow 0. \tag{3.6.15}$$

Since we already know $\pi_0\left(\varphi_0 \,\middle|\, \varphi_{0\#}\right) = 0$, we only need to show that $\ker\pi_0 \subseteq \operatorname{im}\left(\varphi_0 \,\middle|\, \varphi_{0\#}\right)$. Let $a = (a_1, \dots, a_{3\tau}) \in S^{3\tau}$ be an element of $\ker\pi_0$. Then, as the j-th column of π_0 is G_j as an element of A^τ, the equation $\pi_0 a = 0$ is equivalent to the relation

$$a_1G_1 + \cdots + a_{3\tau}G_{3\tau} = 0,$$

which can be also written as

$$\sum_{j=1}^{3\tau}\left(\Lambda_{j+1}^{+}\chi_{j+1}^{w_{j+1}^{+}+1}a_{j+1} + \Lambda_j^{-}\chi_j^{w_j^{-}+1}a_j\right)\left({}^1X_j^1\right) = 0. \tag{3.6.16}$$

This also implies that each summand vanishes.

Note that each $a_j \in S$ can be uniquely expressed as

$$a_j = a_{j,0} + a_{j,1}\chi_{j-1} + a_{j,2}\chi_j + a_{j,3}\chi_{j+1} + a_{j,4}\chi_{j-1}\chi_j + a_{j,5}\chi_j\chi_{j+1} + a_{j,6}\chi_{j+1}\chi_{j-1} + a_{j,7}xyz \tag{3.6.17}$$

for some $a_{j,0} \in \mathbb{C}$, $a_{j,1} \in \mathbb{C}[[\chi_{j-1}]]$, $a_{j,2} \in \mathbb{C}[[\chi_j]]$, $a_{j,3} \in \mathbb{C}[[\chi_{j+1}]]$, $a_{j,4} \in \mathbb{C}[[\chi_{j-1},\chi_j]]$, $a_{j,5} \in \mathbb{C}[[\chi_j,\chi_{j+1}]]$, $a_{j,6} \in \mathbb{C}[[\chi_{j+1},\chi_{j-1}]]$ and $a_{j,7} \in S$. Substituting this into the j-th summand in 3.6.16, simplifying using proposition 3.6.3, and comparing coefficients of each monomials yield the following:

$$a_{j,0} = a_{j,2} = a_{j+1,0} = a_{j+1,2} = 0 \quad \text{and}$$

$$\Lambda_{j+1}^{+}\chi_{j+1}^{w_{j+1}^{+}+1}\left(a_{j+1,1}\chi_j + a_{j+1,4}\chi_j\chi_{j+1}\right) + \Lambda_j^{-}\chi_j^{w_j^{-}+1}\left(a_{j,3}\chi_{j+1} + a_{j,5}\chi_j\chi_{j+1}\right) = 0.$$

From the latter one, we can write

$$a_{j,3}\chi_{j+1} + a_{j,5}\chi_j\chi_{j+1} = -\Lambda_{j+1}^{+}\chi_{j+1}^{w_{j+1}^{+}+1}b_{j+1} \quad \text{and} \quad a_{j+1,1}\chi_j + a_{j+1,4}\chi_j\chi_{j+1} = \Lambda_j^{-}\chi_j^{w_j^{-}+1}b_{j+1}$$

for some $b_{j+1} \in S$.

Substituting into equation 3.6.3 what we have earned so far, we get

$$a_j = \Lambda_{j-1}^{-}\chi_{j-1}^{w_{j-1}^{-}+1}b_j - \Lambda_{j+1}^{+}\chi_{j+1}^{w_{j+1}^{+}+1}b_{j+1} + \left(a_{j,6} + a_{j,7}\chi_j\right)\chi_{j+1}\chi_{j-1}$$

for any $j \in \{1,\ldots,3\tau\}$, or equivalently,

$$a = \sum_{j=1}^{3\tau}\left(b_j R_j + \left(a_{j,6} + a_{j,7}\chi_j\right)R_{j\#}\right).$$

Thus a is an S-linear combination of the columns R_k and $R_{k\#}$, that is, $a \in \operatorname{im}\left(\varphi_0 \,\middle|\, \varphi_{0\#}\right)$. Consequently, the sequence 3.6.15 is exact.

Step 2: Establish a free resolution of M_S in special cases.

Here we first consider the case where $w_j \geq 0$ for all j. Then we have $w_j^+ = w_j$, $w_j^- = 0$, $w_j' = w_j + 2$ and $\Lambda_j^- = 1$ for all j. Relations $H_j = \chi_{j+1}G_j$ show that $\tilde{M}$ is generated by only 3τ elements G_j $\left(j \in \mathbb{Z}_{3\tau}\right)$ in A^τ and hence $M_0 = \tilde{M}$. We denote π_0 by $\tilde{\pi}$. One can also notice $\varphi_0 = \varphi$.

Meanwhile, one can observe in Corollary 3.6.6 that ψ_{ab} is a multiple of $\chi_{a+1}\chi_b$ for any a, $b \in \mathbb{Z}_{3\tau}$. Therefore, the b-th column of ψ is a multiple of χ_b and can be written as $\chi_b v_b$ for some $v_b \in S^{3\tau}$. Using the equation $\varphi\psi = xyzI_{3\tau}$, we now have

$$\chi_b \varphi v_b = xyz\mathbf{e}_{G^b}.$$

Since S is an integral domain, we may cancel out common terms χ_b in both sides. The result is

$$\varphi v_b = \chi_{b-2}\chi_{b-1}\mathbf{e}_{G^b},$$

where the right side is the b-th column of $\varphi_{0\#}$. This proves $\operatorname{im}\varphi_{0+} \subseteq \operatorname{im}\varphi$ and the exact sequence 3.6.15 yields a new exact sequence

$$0 \longrightarrow S^{3\tau} \xrightarrow{\varphi} S^{3\tau} \xrightarrow{\tilde{\pi}} \tilde{M}_S \longrightarrow 0.$$

From this we deduce $\tilde{M} \cong \operatorname{coker}\underline{\varphi}$ as A-modules, which guarantees that $\tilde{M}$ is already a maximal Cohen-Macaulay A-module and hence $\tilde{M} = M$. Setting $\pi := \tilde{\pi}$, we get the desired free resolution 3.6.1 in the theorem.

We can handle the case where $w_j \leq 0$ for all j in a similar way but using some coordinate translation in the matrix. We leave it to the readers. For the other cases, we move on to the next step.

Step 3: Get a (part of) free resolution of $\tilde{M}_S$ in the other cases.

In the rest cases, some of w_j are positive and some are negative. Therefore, some of δ_j are 1 and some are 0. Let's say the value of δ_j changes 2ξ times for some $\xi \in \mathbb{Z}_{\geq 1}$ with $2\xi \leq 3\tau$. Then there are 2ξ different integers $\imath_1$, $\jmath_1,\ldots,\imath_\xi$, $\jmath_\xi$ in cyclic order in $\mathbb{Z}_{3\tau}$ such that

$$\delta_{\jmath_\xi+1} = \cdots = \delta_{\imath_1} = 1, \quad \delta_{\imath_1+1} = \cdots = \delta_{\jmath_1} = 0, \quad \delta_{\jmath_1+1} = \cdots = \delta_{\imath_2} = 1, \quad \ldots, \quad \delta_{\imath_\xi+1} = \cdots = \delta_{\jmath_\xi} = 0.$$

By the discussion in the first paragraph in Subsection 3.6.2, $\tilde{M}$ is generated by elements of $M_0 = \langle G_1, \dots, G_{3\tau}\rangle_A$ together with $H_{J_1}, \dots, H_{J_\xi}$. We denote it by $M_1 := \tilde{M}$. To resolve it, we enlarge the matrix $\pi_0 \in A^{\tau\times 3\tau}$ to a new matrix $\pi_1 := \pi_1(w, \lambda, 1) \in A^{\tau\times(3\tau+\xi)}$ by inserting the new column $H_{J_\nu} \in A^{\tau}$ between the columns G_{J_ν} and $G_{J_\nu+1}$ of π_0 for each $\nu \in \{1, \dots, \xi\}$. As a result, we get

$$\pi_1 := \left(G_1 \middle| \cdots \middle| G_{J_1} \middle| H_{J_1} \middle| G_{J_1+1} \middle| \cdots \middle| G_{J_2} \middle| H_{J_2} \middle| G_{J_2+1} \middle| \cdots \middle| G_{J_\xi} \middle| H_{J_\xi} \middle| G_{J_\xi+1} \middle| \cdots \middle| G_{3\tau} \right)_{\tau\times(3\tau+\xi)}.$$

Then it can be viewed as an S-module map $\pi_1 : S^{3\tau+\xi} \to (A_S)^{\tau}$ whose image is $(M_1)_S = \tilde{M}_S$. Restricting the codomain, we get the surjective map $\pi_1 : S^{3\tau+\xi} \to (M_1)_S$.

Next, denote by

$$\varphi_0\left[G^{j_1} : G^{j_2}; R_{k_1} : R_{k_2}\right]$$

the submatrix of φ_0 taking rows from G^{j_1} to G^{j_2} and columns from R_{k_1} to R_{k_2}. Now, for each $\nu \in \{1, \dots, \xi\}$, look at submatrices

$$(\varphi_0)^-_\nu := \varphi_0\left[G^{\iota_\nu+2} : G^{J_\nu+1}; R_{\iota_\nu+2} : R_{J_\nu+1}\right] \quad \text{and} \quad (\varphi_0)^+_{\nu+1} := \varphi_0\left[G^{J_\nu} : G^{\iota_{\nu+1}-1}; R_{J_\nu+1} : R_{\iota_{\nu+1}}\right]$$

of φ_0, which are highlighted in Figure 3.6 respectively as a red box and a blue box. Check that the entries are the same but only expressions have changed from the original definition 3.6.14 of φ_0, using the inequalities $w_{\iota_\nu+1}, \dots, w_{J_\nu} \le 0$ and $w_{J_\nu+1}, \dots, w_{\iota_{\nu+1}} \ge 0$.

$$\begin{array}{c|ccccccccccc}
 & R_{\iota_\nu} & R_{\iota_\nu+1} & R_{\iota_\nu+2} & & \cdots & R_{J_\nu} & R_{J_\nu+1} & R_{J_\nu+2} & \cdots & & R_{\iota_{\nu+1}} \\
\hline
G^{\iota_\nu-1} & -\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w_{\iota_\nu}+1} & (\varphi_0)^+_\nu & & & & & & & & & \\
G^{\iota_\nu} & \chi_{\iota_\nu-1} & -\chi_{\iota_\nu+1} & & & & & & & & & \\
G^{\iota_\nu+1} & & \chi_{\iota_\nu} & -\chi_{\iota_\nu+2} & & & & & & & & \\
G^{\iota_\nu+2} & & & \Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}+1} & -\chi_{\iota_\nu+3} & \cdots & 0 & 0 & & & & \\
 & & & 0 & \Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w_{\iota_\nu+2}+1} & \ddots & \vdots & \vdots & & & & \\
\vdots & & & \vdots & \vdots & \ddots & -\chi_{J_\nu} & 0 & & & (\varphi_0)^+_{\nu+1} & \\
G^{J_\nu} & & & 0 & 0 & \cdots & \Lambda^-_{J_\nu-1}\chi_{J_\nu-1}^{-w_{J_\nu-1}+1} & -\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}+1} & 0 & \cdots & 0 & 0 \\
G^{J_\nu+1} & & & 0 & 0 & \cdots & 0 & \Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}+1} & -\Lambda^+_{J_\nu+2}\chi_{J_\nu+2}^{w_{J_\nu+2}+1} & \cdots & 0 & 0 \\
 & & & & (\varphi_0)^-_\nu & & & 0 & \chi_{J_\nu+1} & \ddots & \vdots & \vdots \\
\vdots & & & & & & & \vdots & \vdots & \ddots & -\Lambda^+_{\iota_{\nu+1}-1}\chi_{\iota_{\nu+1}-1}^{w_{\iota_{\nu+1}-1}+1} & 0 \\
G^{\iota_{\nu+1}-1} & & & & & & & 0 & 0 & \cdots & \chi_{\iota_{\nu+1}-2} & -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}+1}
\end{array}$$

Figure 3.6: Submatrix $\varphi_0\left[G^{\iota_\nu-1} : G^{\iota_{\nu+1}-1}; R_{\iota_\nu} : R_{\iota_{\nu+1}}\right]$ of φ_0

Note that the diagonal entries of $\left(\varphi_0\right)^-_\nu$ and $\left(\varphi_0\right)^+_{\nu+1}$ can be read off respectively as

$$\begin{aligned} \operatorname{diag}\left(\varphi_0\right)^-_\nu &= \left(\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}-\delta_{\iota_\nu+1}+1}, \dots, \Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}-\delta_{J_\nu}+1}\right) \in S^{J_\nu-\iota_\nu} \quad \text{and} \\ \operatorname{diag}\left(\varphi_0\right)^+_{\nu+1} &= \left(-\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}+\delta_{J_\nu+1}}, \dots, -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}+\delta_{\iota_{\nu+1}}+1}\right) \in S^{\iota_{\nu+1}-J_\nu} \end{aligned} \tag{3.6.18}$$

by the identities $\delta_{\iota_\nu+1} = \cdots = \delta_{J_\nu} = 0$ and $\delta_{J_\nu+1} = \cdots = \delta_{\iota_{\nu+1}} = 1$, where $\operatorname{diag} B$ denotes the tuple consisting of the main diagonal entries of B for any matrix B.

Now we modify the matrix $\varphi_0 \in S^{3\tau\times 3\tau}$ to a new matrix $\varphi_1 \in S^{(3\tau+\xi)\times(3\tau+2\xi)}$ by performing the following procedure for each $\nu \in \{1,\dots,\xi\}$:

- Multiply the columns $R_{\iota_\nu+1}, \dots, R_{J_\nu}$ by -1,
- Insert a new row H^{J_ν} consisting of zeros between the rows G^{J_ν} and $G^{J_\nu+1}$, and then
- Replace column $R_{J_\nu+1}$ with the new three columns T_{J_ν}, $T_{J_\nu+1}$ and $T_{J_\nu+2}$

as described in Figure 3.7. We keep the same names for the original rows and columns, although their size and entries may have changed. But submatrices $\left(\varphi_0\right)^-_\nu$ and $\left(\varphi_0\right)^+_{\nu+1}$ of φ_0 are readjusted to submatrices

$$\left(\varphi_1\right)^-_\nu := \varphi_1\left[G^{\iota_\nu+2} : H^{J_\nu}; R_{\iota_\nu+2} : T_{J_\nu}\right] \quad \text{and} \quad \left(\varphi_1\right)^+_{\nu+1} := \varphi_1\left[H^{J_\nu} : G^{\iota_{\nu+1}-1}; T_{J_\nu+2} : R_{\iota_{\nu+1}}\right]$$

of φ_1, respectively. Check that the replaced column $R_{J_\nu+1}$ is an S-linear combination of the new columns, namely

$$-\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}} T_{J_\nu} + \Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}} T_{J_\nu+2},$$

meaning that the presence of the replaced column does not affect the image of φ_1. We also shifted indices in the upper diagonal entries of $\left(\varphi_1\right)^-_\nu$ by 3, which does not actually change the entries, to make them consecutive with those in the lower diagonal of $\left(\varphi_1\right)^+_{\nu+1}$.

The construction of φ_1 is based on the relations

$$-\chi_{J_\nu-2}G_{J_\nu} + \Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}} H_{J_\nu} = 0, \quad \chi_{J_\nu-1}H_{J_\nu} = 0 \quad \text{and} \quad \Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}} H_{J_\nu} - \chi_{J_\nu}G_{J_\nu+1} = 0$$

on G_{J_ν}, H_{J_ν} and $G_{J_\nu+1}$ for $\nu \in \{1,\dots,\xi\}$, already observed in equation 3.6.5. This enables the matrix equation $\pi_1\varphi_1 = 0$ to hold.

Comparing the exponents of the diagonal entries of $\left(\varphi_1\right)^-_\nu$ and $\left(\varphi_1\right)^+_{\nu+1}$ with those of $\left(\varphi_0\right)^-_\nu$ and $\left(\varphi_0\right)^+_{\nu+1}$, only the last one of $\left(\varphi_1\right)^-_\nu$ and the first one of $\left(\varphi_1\right)^+_{\nu+1}$ have been changed. Now we can rewrite 3.6.18 for those as

$$\begin{aligned} \operatorname{diag}\left(\varphi_1\right)^-_\nu &= \left(-\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}-\delta_{\iota_\nu+1}-\delta_{\iota_\nu+2}+1}, \dots, -\Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}-\delta_{J_\nu}-\delta_{J_\nu+1}+1}\right) \in S^{J_\nu-\iota_\nu} \quad \text{and} \\ \operatorname{diag}\left(\varphi_1\right)^+_{\nu+1} &= \left(-\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}+\delta_{J_\nu}+\delta_{J_\nu+1}-1}, \dots, -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}+\delta_{\iota_{\nu+1}-1}+\delta_{\iota_{\nu+1}}-1}\right) \in S^{\iota_{\nu+1}-J_\nu} \end{aligned}$$

using $\delta_{\iota_\nu+1} = \cdots = \delta_{J_\nu} = 0$ and $\delta_{J_\nu+1} = \cdots = \delta_{\iota_{\nu+1}} = 1$ again. Taking one step further, by the conversion formula

$$w'_j = w_j + \delta_{j-1} + \delta_j + \delta_{j+1} - 1 \quad \left(j \in \mathbb{Z}_{3\tau}\right).$$

in Definition 4.1.1, these can also be written as

$$
\begin{aligned}
\operatorname{diag}\left(\varphi_1\right)_\nu^- &= \left(-\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}-\delta_{\iota_\nu}-\delta_{\iota_\nu+1}-\delta_{\iota_\nu+2}+2}, -\Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w_{\iota_\nu+2}-\delta_{\iota_\nu+1}-\delta_{\iota_\nu+2}-\delta_{\iota_\nu+3}+1}, \dots, -\Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}-\delta_{J_\nu-1}-\delta_{J_\nu}-\delta_{J_\nu+1}+1}\right)\\
&= \left(-\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}+1}, -\Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w'_{\iota_\nu+2}}, \dots, -\Lambda^-_{J_\nu}\chi_{J_\nu}^{-w'_{J_\nu}}\right) \in S^{J_\nu-\iota_\nu} \quad \text{and}\\
\operatorname{diag}\left(\varphi_1\right)_{\nu+1}^+ &= \left(-\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}+\delta_{J_\nu}+\delta_{J_\nu+1}+\delta_{J_\nu+2}-2}, \dots, -\Lambda^+_{\iota_{\nu+1}-1}\chi_{\iota_{\nu+1}-1}^{w_{\iota_{\nu+1}-1}+\delta_{\iota_{\nu+1}-2}+\delta_{\iota_{\nu+1}-1}+\delta_{\iota_{\nu+1}}-2}, -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}+\delta_{\iota_{\nu+1}-1}+\delta_{\iota_{\nu+1}}+\delta_{\iota_{\nu+1}+1}-1}\right)\\
&= \left(-\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w'_{J_\nu+1}-1}, \dots, -\Lambda^+_{\iota_{\nu+1}-1}\chi_{\iota_{\nu+1}-1}^{w'_{\iota_{\nu+1}-1}-1}, -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w'_{\iota_{\nu+1}}}\right) \in S^{\iota_\nu-J_{\nu-1}}.
\end{aligned}
\tag{3.6.19}
$$

We also enlarge the matrix $\varphi_{0\#} \in S^{3\tau\times 3\tau}$ to a new matrix $\varphi_{1\#} \in S^{(3\tau+\xi)\times 3\tau}$ by inserting a new row $H^{J_\nu\#}$ consisting of zeros between the rows $G^{(J_\nu+1)\#}$ and $G^{(J_\nu+2)\#}$ of $\varphi_{0\#}$ for each $\nu \in \{1,\dots,\xi\}$. We use the same name $R_{k\#}$ for the k-th column. Then it is easy to see that $\pi_1\varphi_{1\#} = 0$. Moreover, we have the inclusion $\operatorname{im}\varphi_{1\#} \subseteq \operatorname{im}\varphi_1$. Indeed, the equations

$$
\begin{aligned}
R_{(\iota_\nu+1)\#} &= \chi_{\iota_\nu-1}\chi_{\iota_\nu}\mathbf{e}_{G^{\iota_\nu+1}} &&= \chi_{\iota_\nu}R_{\iota_\nu+2} + \Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}}R_{(\iota_\nu+2)\#}\\
&\ \vdots && \ \vdots\\
R_{(J_\nu-1)\#} &= \chi_{J_\nu-3}\chi_{J_\nu-2}\mathbf{e}_{G^{J_\nu-1}} &&= \chi_{J_\nu-2}R_{J_\nu} + \Lambda^-_{J_\nu-1}\chi_{J_\nu-1}^{-w_{J_\nu-1}}R_{J_\nu\#}\\
R_{J_\nu\#} &= \chi_{J_\nu-2}\chi_{J_\nu-1}\mathbf{e}_{G^{J_\nu}} &&= \chi_{J_\nu-1}T_{J_\nu} + \Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}}T_{J_\nu+1}\\
R_{(J_\nu+1)\#} &= \chi_{J_\nu-1}\chi_{J_\nu}\mathbf{e}_{G^{J_\nu+1}} &&= \Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}}T_{J_\nu+1} + \chi_{J_\nu-1}T_{J_\nu+2}\\
R_{(J_\nu+2)\#} &= \chi_{J_\nu}\chi_{J_\nu+1}\mathbf{e}_{G^{J_\nu+2}} &&= \Lambda^+_{J_\nu+2}\chi_{J_\nu+2}^{w_{J_\nu+2}}R_{(J_\nu+1)\#} + \chi_{J_\nu}R_{J_\nu+2}\\
&\ \vdots && \ \vdots\\
R_{\iota_{\nu+1}\#} &= \chi_{\iota_{\nu+1}-2}\chi_{\iota_{\nu+1}-1}\mathbf{e}_{G^{\iota_{\nu+1}}} &&= \Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}}R_{(\iota_{\nu+1}-1)\#} + \chi_{\iota_{\nu+1}-2}R_{\iota_{\nu+1}}
\end{aligned}
\tag{3.6.20}
$$

show that the columns $R_{(\iota_\nu+1)\#}$ through $R_{\iota_{\nu+1}\#}$ of $\varphi_{1\#}$ are spanned by the columns of φ_1. This holds for all $\nu \in \{1,\dots,\xi\}$. Thus, we have $\operatorname{im}\varphi_{1\#} \subseteq \operatorname{im}\varphi_1$.

Now we have the matrix equation $\pi_1\left(\varphi_1 \middle| \varphi_{1\#}\right) = 0$. We also checked in the below of equation 3.6.5 in Subsection 3.6.2 that any element $s \in S$ satisfying $sH_{J_\nu} \in (M_0)_S$ is S-generated by three elements

$$-\Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}}, \quad \chi_{J_\nu-1} \quad \text{and} \quad -\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}}$$

in S, which are the nonzero entries in the newly added row of φ_1. Thus, we can apply Lemma 3.6.7 for each $\nu \in \{1,\dots,\xi\}$ to get a part of free resolution of $(M_1)_S = \tilde{M}_S$ as follows:

$$S^{6\tau+2\xi} \xrightarrow{(\varphi_1|\varphi_{1\#})} S^{3\tau+\xi} \xrightarrow{\pi_1} (M_1)_S \longrightarrow 0.$$

Using $\operatorname{im}\varphi_{1\#} \subseteq \operatorname{im}\varphi_1$, we can drop $\varphi_{1\#}$ and have the following simplified one:

$$S^{3\tau+2\xi} \xrightarrow{\varphi_1} S^{3\tau+\xi} \xrightarrow{\pi_1} (M_1)_S \longrightarrow 0.$$

Step 4: Continue to establish a free resolution of M_S.

Recall that we have 2ξ different numbers $\imath_1, \jmath_1, \ldots, \imath_\xi, \jmath_\xi \in \mathbb{Z}_{3\tau}$ in cyclic order such that

$$\delta_{\jmath_\xi+1} = \cdots = \delta_{\imath_1} = 1, \quad \delta_{\imath_1+1} = \cdots = \delta_{\jmath_1} = 0, \quad \delta_{\jmath_1+1} = \cdots = \delta_{\imath_2} = 1, \quad \ldots, \quad \delta_{\imath_\xi+1} = \cdots = \delta_{\jmath_\xi} = 0.$$

For each $\nu \in \{1, \ldots, \xi\}$, we know that $w_{\imath_\nu} \geq 1$ and all $w_{\imath_\nu+1}, \ldots, w_{\jmath_\nu}$ must be non-positive while at least one must be negative, by the definition of the sign word δ_j. There is $\kappa_\nu \in \{0, \ldots, \jmath_\nu - \imath_\nu - 1\}$ for each ν such that

$$w_{\imath_\nu} \geq 1, \quad w_{\imath_\nu+1} = \cdots = w_{\imath_\nu+\kappa_\nu} = 0 \quad \text{and} \quad w_{\imath_\nu+\kappa_\nu+1} \leq -1.$$

Then we have a Macaulayfying element

$$F_{\imath_\nu} := \Lambda^+_{\imath_\nu}\left({}^1X^{w_{\imath_\nu}+1}_{\imath_\nu-1}\right) + \sum_{a=0}^{\kappa_\nu}\left(\prod_{b=1}^{a}\Lambda^-_{\imath_\nu+b}\right)\left({}^1X^1_{\imath_\nu+a}\right) + \left(\prod_{b=1}^{\kappa_\nu+1}\Lambda^-_{\imath_\nu+b}\right)\left({}^{-w_{\imath_\nu+\kappa_\nu+1}+1}X^1_{\imath_\nu+\kappa_\nu+1}\right)$$

of $M_1 = \tilde{M}$ in A^τ for each ν, by the observation in Subsection 3.6.2.

We know by Proposition 3.3.2.(2) that $M = (M_1)^\dagger \cong \langle M_1, F_{\imath_1}, \ldots, F_{\imath_\xi}\rangle^\dagger_A$. Let $M_2 := \langle M_1, F_{\imath_1}, \ldots, F_{\imath_\xi}\rangle_A \subseteq A^\tau$, for a while. Later we will find out that M_2 is actually maximal Cohen-Macaulay and hence $M_2 = M$. Therefore, the goal is to construct a free resolution of $(M_2)_S$.

We further enlarge the matrix $\pi_1 \in A^{\tau\times(3\tau+\xi)}$ constructed in Step 3 to a new matrix $\pi_2 \in A^{\tau\times(3\tau+2\xi)}$ by inserting column $F_{\imath_\nu} \in A^\tau$ between the columns $G''_{\imath_\nu-1}$ and $G_{\imath_\nu}$ of π_1 for each $\nu \in \{1, \ldots, \xi\}$. Here $G''_{\imath_\nu-1}$ is $H_{\imath_\nu-1}$ if $\delta_{\imath_\nu-1} = 0$, or equivalently $\jmath_{\nu-1} + 1 = \imath_\nu$, and is $G_{\imath_\nu-1}$ otherwise, as in equation 3.6.9. As a result, we get

$$\pi_2 := \left(G_1 \middle| \cdots \middle| G_{\imath_1-1} \middle| F_{\imath_1} \middle| G_{\imath_1} \middle| \cdots \middle| G_{\jmath_1} \middle| H_{\jmath_1} \middle| G_{\jmath_1+1} \middle| \cdots \middle| G_{\imath_\xi-1} \middle| F_{\imath_\xi} \middle| G_{\imath_\xi} \middle| \cdots \middle| G_{\jmath_\xi} \middle| H_{\jmath_\xi} \middle| G_{\jmath_\xi+1} \middle| \cdots \middle| G_{3\tau} \right)_{\tau\times(3\tau+2\xi)}.$$

Then it can be viewed as an S-module map $\pi_2 : S^{3\tau+2\xi} \to (A_S)^\tau$ whose image is $(M_2)_S$. Restricting the codomain, we get the surjective map $\pi_2 : S^{3\tau+2\xi} \to (M_2)_S$.

$$
\begin{array}{c|ccccccccccccc}
 & T_{J_{\nu-1}+2} & R_{J_{\nu-1}+2} & \cdots & R_{\iota_\nu-1} & R_{\iota_\nu} & R_{\iota_\nu+1} & R_{\iota_\nu+2} & R_{\iota_\nu+3} & \cdots & R_{J_\nu} & T_{J_\nu} & T_{J_\nu+1} & T_{J_\nu+2} \\
\hline
H^{J_{\nu-1}} & -\Lambda^+_{J_{\nu-1}+1}\chi^{w'_{J_{\nu-1}+1}-1}_{J_{\nu-1}+1} & 0 & \cdots & 0 & 0 & & & & & & & & \\
G^{J_{\nu-1}+1} & \chi_{J_{\nu-1}} & -\Lambda^+_{J_{\nu-1}+2}\chi^{w'_{J_{\nu-1}+2}-1}_{J_{\nu-1}+2} & \cdots & 0 & 0 & & & & & & & & \\
 & 0 & \chi_{J_{\nu-1}+1} & \ddots & \vdots & \vdots & (\varphi_1)^+_\nu & & & & & & & \\
\vdots & \vdots & \vdots & \ddots & -\Lambda^+_{\iota_\nu-1}\chi^{w'_{\iota_\nu-1}-1}_{\iota_\nu-1} & 0 & & & & & & & & \\
G^{\iota_\nu-1} & 0 & 0 & \cdots & \chi_{\iota_\nu-2} & -\Lambda^+_{\iota_\nu}\chi^{w'_{\iota_\nu}}_{\iota_\nu} & & & & & & & & \\
\hline
G^{\iota_\nu} & & & & & \chi_{\iota_\nu-1} & \chi_{\iota_\nu+1} & & & & & & & \\
G^{\iota_\nu+1} & & & & & & -\chi_{\iota_\nu} & \chi_{\iota_\nu+2} & & & & & & \\
G^{\iota_\nu+2} & & & & & & & -\Lambda^-_{\iota_\nu+1}\chi^{-w'_{\iota_\nu+1}+1}_{\iota_\nu+1} & \chi_{\iota_\nu} & \cdots & 0 & 0 & & \\
 & & & & & & & 0 & -\Lambda^-_{\iota_\nu+2}\chi^{-w'_{\iota_\nu+2}}_{\iota_\nu+2} & \ddots & \vdots & \vdots & & \\
\vdots & & & & & & (\varphi_1)^-_\nu & \vdots & \vdots & \ddots & \chi_{J_\nu-3} & 0 & & \\
G^{J_\nu} & & & & & & & 0 & 0 & \cdots & -\Lambda^-_{J_\nu-1}\chi^{-w'_{J_\nu-1}}_{J_\nu-1} & \chi_{J_\nu-2} & & (\varphi_1)^+_{\nu+1} \\
H^{J_\nu} & & & & & & & 0 & 0 & \cdots & 0 & -\Lambda^-_{J_\nu}\chi^{-w'_{J_\nu}}_{J_\nu} & \chi_{J_\nu-1} & -\Lambda^+_{J_\nu+1}\chi^{w'_{J_\nu+1}}_{J_\nu+1}
\end{array}
$$

Figure 3.8: Submatrix $\varphi_1\left[H^{J_{\nu-1}} : H^{J_\nu}; T_{J_{\nu-1}+2} : T_{J_\nu+2}\right]$ of φ_1

Next, for each $\nu \in \{1,\dots,\xi\}$, look at the submatrix $\varphi_1\left[H^{J_{\nu-1}} : H^{J_\nu}; T_{J_{\nu-1}+2} : T_{J_\nu}\right]$ of φ_1 containing $(\varphi_1)^+_\nu$ and $(\varphi_1)^-_\nu$ as shown in Figure 3.8, whose diagonal entries have been calculated in 3.6.19.

We modify the matrix $\varphi_1 \in S^{(3\tau+\xi)\times(3\tau+2\xi)}$ to a new matrix $\varphi_2 := \varphi_2(w,\lambda,1) \in S^{(3\tau+2\xi)\times(3\tau+5\xi)}$ in the following way. For each $\nu \in \{1,\dots,\xi\}$,

- Insert a new row F^{ι_ν} consisting of zeros above the row G^{ι_ν}, and then
- Insert the new three columns $Q_{\iota_\nu-1}$, Q_{ι_ν} and $Q_{\iota_\nu+1}$ to the left of the column R_{ι_ν} (or $T_{J_{\nu-1}+2}$ in the case $J_{\nu-1}+1 = \iota_\nu$)

as described in Figure 3.9, where the entries of the new columns that do not appear in the figure are all zeros. In the figure, the rows indicate

$$
\left(G^{\iota_\nu-1}\right)'' := \begin{cases} H^{J_{\nu-1}} & \text{if } J_{\nu-1}+1 = \iota_\nu \\ G^{\iota_\nu-1} & \text{otherwise} \end{cases}, \quad \left(G^{\iota_\nu}\right)' := G^{\iota_\nu}, \quad \dots, \quad \left(G^{J_\nu}\right)' := G^{J_\nu} \quad \text{and} \quad \left(G^{J_\nu+1}\right)' := H^{J_\nu}
$$

depending on the case, imitating the definition of $G''_{\iota_\nu-1}$ and $G'_{\iota_\nu+a}$ given in 3.6.9, and the columns indicate

$$R''_{\iota_\nu} := \begin{cases} T_{J_{\nu-1}+2} & \text{if } J_{\nu-1}+1=\iota_\nu \\ R_{\iota_\nu} & \text{otherwise} \end{cases}, \quad R'_{\iota_\nu+1} := R_{\iota_\nu+1}, \quad \ldots, \quad R'_{J_\nu} := R_{J_\nu} \quad \text{and} \quad R'_{J_\nu+1} := T_{J_\nu},$$

similarly. Note that we always have $(G^{\iota_\nu})' = G^{\iota_\nu}$, $\left(G^{\iota_\nu+1}\right)' = G^{\iota_\nu+1}$ and $R'_{\iota_\nu+1} = R_{\iota_\nu+1}$. The symbols $\zeta^c_{\iota_\nu,a,b}$ are as defined in 3.6.11.

The construction of φ_2 is based on the formulas 3.6.10, which makes the matrix equation $\pi_2\varphi_2 = 0$ hold. It is obvious that any element $s \in S$ satisfying $sF_{\iota_\nu} \in (M_1)_S = \tilde{M}_S$ is in the ideal of S generated by three elements x, y and z in S, namely the maximal ideal of S. Since these elements coincide with the nonzero entries χ_{ι_ν}, $\chi_{\iota_\nu+1}$ and $\chi_{\iota_\nu+2}$ in the newly added row of φ_2, we can apply Lemma 3.6.7 for each $\nu \in \{1,\ldots,\xi\}$ to get a part of free resolution of $(M_2)_S$:

$$S^{3\tau+5\xi} \xrightarrow{\varphi_2} S^{3\tau+2\xi} \xrightarrow{\pi_2} (M_2)_S \longrightarrow 0. \tag{3.6.21}$$

Then some appropriate coordinate changes on the codomain, i.e. elementary row operations, enable us to remove the terms $-\zeta^c_{\iota_\nu,-1,b}$ in φ_2. The detailed recipe for each $\nu \in \{1,\ldots,\xi\}$ is given as follows:

- Set $\left(G^{J_\nu+1}\right)^* := \left(G^{J_\nu+1}\right)'$, $(G^{J_\nu})^* := (G^{J_\nu})'$, $\ldots$, $\left(G^{\iota_\nu+\kappa_\nu+3}\right)^* := \left(G^{\iota_\nu+\kappa_\nu+3}\right)'$,
- Add $-\zeta^{b-1}_{\iota_\nu,b-1,b+2}$ times the row $\left(G^{\iota_\nu+b}\right)'$ to the row $\left(G^{\iota_\nu+b+3}\right)'$ and rename the changed row $\left(G^{\iota_\nu+b+3}\right)^*$ for each $b \in \{\kappa_\nu-1, \kappa_\nu-2, \ldots, 1, 0\}$ in order, and then
- Set $\left(G^{\iota_\nu+2}\right)^* := \left(G^{\iota_\nu+2}\right)'$, $\left(G^{\iota_\nu+1}\right)^* := G^{\iota_\nu+1}$ and $(G^{\iota_\nu})^* := G^{\iota_\nu}$.

We perform those for all ν and denote the resulting matrix by $\varphi_3 := \varphi_3(w,\lambda,1) \in S^{(3\tau+2\xi)\times(3\tau+5\xi)}$. We keep the original names for columns, although they may also have changed. For each ν, the procedure eliminates the all terms $-\zeta^c_{\iota_\nu,-1,b+2}$ $\left(\text{in the row } \left(G^{\iota_\nu+b+3}\right)'\right)$ in φ_2 in the order from bottom to top, using the identities $\zeta^c_{\iota_\nu,-1,b-1}\zeta^{b-1}_{\iota_\nu,b-1,b+2} = \zeta^c_{\iota_\nu,-1,b+2}$ observed in equation 3.6.12. At the same time, it yields some '*unwanted terms*'. They are described in Figure 3.10 in pink color, where we calculated as, for example,

$$\left(-\zeta^{b-1}_{\iota_\nu,b-1,b+2}\right)\left(-\Lambda^-_{\iota_\nu+b-1}\chi^{-w'_{\iota_\nu+b-1}}_{\iota_\nu+b-1}\right) = \zeta^{b-1}_{\iota_\nu,b-1,b+2}\zeta^{b-1}_{\iota_\nu,b-2,b-1}\chi_{\iota_\nu+b-1} = \zeta^{b-1}_{\iota_\nu,b-2,b+2}\chi_{\iota_\nu+b+2},$$

for $b \in \{3,\ldots,\kappa_\nu-1\}$ using the definition of $\zeta^{b-1}_{\iota_\nu,b-2,b-1}$ in 3.6.11 and the identity 3.6.12 again.

We should carry out the coordinate changes on the domain, i.e. elementary column operations on $\pi_2 \in A^{\tau\times(3\tau+2\xi)}$ corresponding to the row operations taken to φ_2, to maintain the exactness of the sequence that has been obtained. We proceed for each $\nu \in \{1,\ldots,\xi\}$ as follows:

- Set $G^*_{J_\nu+1} := G'_{J_\nu+1}$, $G^*_{J_\nu} := G'_{J_\nu}$, $\ldots$, $G^*_{\iota_\nu+\kappa_\nu} := G'_{\iota_\nu+\kappa_\nu}$, and then
- Add $\zeta^{b-1}_{\iota_\nu,b-1,b+2}$ times the column $G^*_{\iota_\nu+b+3}$ to the column $G'_{\iota_\nu+b}$ and rename the changed column $G^*_{\iota_\nu+b}$ for each $b \in \{\kappa_\nu-1, \kappa_\nu-2, \ldots, 1, 0\}$ in order.

Denoting the transformed matrix by $\pi_3 := \pi_3(w,\lambda,1) \in A^{\tau\times(3\tau+2\xi)}$, we can express it as

$$\pi_3 := \left(G_1 \middle| \cdots \middle| G_{\imath_1 - 1} \middle| F_{\imath_1} \middle| G^*_{\imath_1} \middle| \cdots \middle| G^*_{\jmath_1} \middle| H_{\jmath_1} \middle| G_{\jmath_1+1} \middle| \cdots \middle| G_{\imath_\xi - 1} \middle| F_{\imath_\xi} \middle| G^*_{\imath_\xi} \middle| \cdots \middle| G^*_{\jmath_\xi} \middle| H_{\jmath_\xi} \middle| G_{\jmath_\xi+1} \middle| \cdots \middle| G_{3\tau} \right)_{\tau\times(3\tau+2\xi)}$$

where the explicit form of columns are given by

$$G^*_{\imath_\nu+b} = \begin{cases} G_{\imath_\nu+b} + \displaystyle\sum_{c=b+3}^{\kappa_\nu+2} \zeta^{b-1}_{\imath_\nu,b-1,c-1} G'_{\imath_\nu+c} & \text{if } 0 \le b \le \kappa_\nu - 1, \\ G'_{\imath_\nu+b} = \begin{cases} G_{\imath_\nu+b} \\ H_{\jmath_\nu} \end{cases} & \begin{array}{l} \text{if } \kappa_\nu \le b \le \jmath_\nu - \imath_\nu, \\ \text{if } b = \jmath_\nu - \imath_\nu + 1. \end{array} \end{cases}$$

Then we may check $\operatorname{im}\varphi_3 = \ker\pi_3$ from the fact that $\operatorname{im}\varphi_2 = \ker\pi_2$ and the above constructions of φ_3 and π_3. Moreover, the construction of π_3 yields $\operatorname{im}\pi_3 = \operatorname{im}\pi_2$. Thus, setting $M_3 := M_2$, we get the following modified exact sequence of S-modules from 3.6.21:

$$S^{3\tau+5\xi} \xrightarrow{\varphi_3} S^{3\tau+2\xi} \xrightarrow{\pi_3} (M_3)_S \longrightarrow 0. \tag{3.6.22}$$

To get rid of the '*unwanted terms*' in the matrix φ_3, we execute coordinate changes on its domain, i.e. elementary column operations on it. First, to remove the bottom-right entry, consider the column

$$\zeta^{\kappa_\nu-2}_{\imath_\nu,\kappa_\nu-2,\kappa_\nu+1}\chi_{\imath_\nu+\kappa_\nu}\mathbf{e}_{(G^{\imath_\nu+\kappa_\nu+2})'} = \left(\prod_{d=\kappa_\nu-1}^{\kappa_\nu+1}\Lambda^-_{\imath+d}\right)\chi_{\imath_\nu+\kappa_\nu+1}^{-w'_{\imath_\nu+\kappa_\nu+1}-1}\chi_{\imath_\nu+\kappa_\nu}\mathbf{e}_{(G^{\imath_\nu+\kappa_\nu+2})'}.$$

When $\imath_\nu+\kappa_\nu+1 = \jmath_\nu$, it is just an S-multiple of the column $T_{\jmath_\nu+1} = \chi_{\jmath_\nu-1}\mathbf{e}_{H^{\jmath_\nu}}$ in φ_3. Otherwise, when $\imath_\nu+\kappa_\nu+2 \le \jmath_\nu$, it is an S-multiple of the column $\chi_{\imath_\nu+\kappa_\nu}\chi_{\imath_\nu+\kappa_\nu+1}\mathbf{e}_{G^{\imath_\nu+\kappa_\nu+2}}$, which is an S-linear combination of columns $R'_{\imath_\nu+\kappa_\nu+3}, \ldots, R'_{\jmath_\nu+1}$ and $T_{\jmath_\nu+1}$ in φ_3, by the same argument used in equations 3.6.20. In any cases, therefore, the mentioned column is an S-linear combination of columns in φ_3.

Using this fact, we provide the explicit manual to remove all '*unwanted terms*' in φ_3 as follows: For each $\nu \in \{1,\ldots,\xi\}$,

- Add the column $\zeta^{\kappa_\nu-2}_{\imath_\nu,\kappa_\nu-2,\kappa_\nu+1}\chi_{\imath_\nu+\kappa_\nu}\mathbf{e}_{(G^{\imath_\nu+\kappa_\nu+2})'}$ to the column $R'_{\imath_\nu+\kappa_\nu}$ using elementary column operations,
- Add $\zeta^{b-2}_{\imath_\nu,b-2,b+1}$ times the column $R'_{\imath_\nu+b+3}$ to the column $R'_{\imath_\nu+b}$ for each $b \in \{\kappa_\nu-1, \kappa_\nu-2, \ldots, 2, 1\}$, and then
- Add $\Lambda^-_{\imath_\nu+1}\chi_{\imath_\nu+1}^{-w'_{\imath_\nu+1}}$ times the column $R'_{\imath_\nu+3}$ to the column $R''_{\imath_\nu}$.

We denote the transformed matrix by $\varphi_4 := \varphi_4(w,\lambda,1) \in S^{(3\tau+2\xi)\times(3\tau+5\xi)}$ and keep the original names for its rows and columns.

One can check indeed that the process eliminates the entries of φ_3 which are colored in pink in Figure 3.10. The bottom-right entry disappears in the first stage. The entries other than this and the top-left entry are removed in the second stage in order from right to left, using identities such as

$$\zeta^{b-2}_{\imath_\nu,b-2,b+1}\left(-\Lambda^-_{\imath_\nu+b+2}\chi_{\imath_\nu+b+2}^{-w'_{\imath_\nu+b+2}}\right) = \zeta^{b-2}_{\imath_\nu,b-2,b+1}\zeta^{b+2}_{\imath_\nu,b+1,b+2}\chi_{\imath_\nu+b+2} = \zeta^{b-1}_{\imath_\nu,b-2,b+2}\chi_{\imath_\nu+b+2}$$

for $b\in\{1,\dots,\kappa_\nu-1\}$, which follow from the definition of $\zeta^{b+2}_{\iota_\nu,b+1,b+2}$ in 3.6.11 and the identity 3.6.12. Then in the final stage the top-left one goes away, but a new term $\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}}\chi_{\iota_\nu}$ is created instead in the place marked with $(*)$.

As coordinate changes on the domain of φ_3 do not affect the exactness of sequence 3.6.22, just setting $\pi_4:=\pi_3\in A^{\tau\times(3\tau+2\xi)}$ and $M_4:=M_3$, we get the following exact sequence:

$$S^{3\tau+5\xi}\xrightarrow{\varphi_4}S^{3\tau+2\xi}\xrightarrow{\pi_4}(M_4)_S\longrightarrow 0.$$

Here we may abandon three columns R''_{ι_ν}, $R_{\iota_\nu+1}$ and $R'_{\iota_\nu+2}$ in φ_4 because they can be expressed as S-linear combinations of other columns, not contributing to $\operatorname{im}\varphi_4$. To be specific, we have

$$\begin{aligned}R''_{\iota_\nu}&=\chi_{\iota_\nu}Q_{\iota_\nu+1}-\chi_{\iota_\nu-1}Q_{\iota_\nu-1},\\ R_{\iota_\nu+1}&=-\chi_{\iota_\nu+1}Q_{\iota_\nu-1}+\chi_{\iota_\nu}Q_{\iota_\nu},\\ R'_{\iota_\nu+2}&=-\chi_{\iota_\nu+2}Q_{\iota_\nu}+\chi_{\iota_\nu+1}Q_{\iota_\nu+1}+\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}-1}\chi_{\iota_\nu+1}\mathbf{e}_{(G^{\iota_\nu-1})''}.\end{aligned}$$

Note that the column $\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}-1}\chi_{\iota_\nu+1}\mathbf{e}_{(G^{\iota_\nu-1})''}$ is just an S-multiple of the column $T_{J_{\nu-1}+1}=\chi_{J_{\nu-1}-1}\mathbf{e}_{H^{J_{\nu-1}}}$ in φ_4 if $J_{\nu-1}+1=\iota_\nu$, and an S-multiple of the column $\chi_{\iota_\nu-3}\chi_{\iota_\nu-2}\mathbf{e}_{G^{\iota_\nu-1}}$ otherwise, which is an S-linear combination of columns $T_{J_{\nu-1}+1}$, $T_{J_{\nu-1}+2}$, $R_{J_{\nu-1}+2},\dots,R_{\iota_\nu-1}$ in φ_4 by the same argument used in equations 3.6.20.

Then we further delete rows $(G^{\iota_\nu})^*$, $\left(G^{\iota_\nu+1}\right)^*$ and columns $Q_{\iota_\nu-1}$, Q_{ι_ν} in φ_4. Denote the surviving matrix as $\varphi_5:=\varphi_5(w,\lambda,1)\in S^{3\tau\times3\tau}$ after removing two rows and five columns in $\varphi_4\in S^{(3\tau+2\xi)\times(3\tau+5\xi)}$ for each $\nu\in\{1,\dots,\xi\}$. Then, finally, we find that φ_5 is in fact exactly equal to φ, comparing the submatrix $\varphi_5\left[H^{J_{\nu-1}}:H^{J_\nu};T_{J_{\nu-1}+2}:T_{J_\nu+2}\right]$ of φ_5 described in Figure 3.11 with the corresponding part of φ. We name the rows and columns of φ the same as the corresponding rows and columns of φ_5, and define submatrices φ^+_ν and φ^-_ν of φ as boxed in the figure.

To get an exact sequence containing $\varphi=\varphi_5$, we also have to erase columns $G^*_{\iota_\nu}$ and $G^*_{\iota_\nu+1}$ in $\pi_4\in A^{\tau\times(3\tau+2\xi)}$, corresponding to the removed rows in φ_4. As a result, we get the reduced matrix $\pi_5:=\pi_5(w,\lambda,1)\in A^{\tau\times3\tau}$ given by

$$\pi_5:=\left(G_1\middle|\cdots\middle|G_{\iota_1-1}\middle|F_{\iota_1}\middle|G^*_{\iota_1+2}\middle|\cdots\middle|G^*_{J_1}\middle|H_{J_1}\middle|G_{J_1+1}\middle|\cdots\middle|G_{\iota_\xi-1}\middle|F_{\iota_\xi}\middle|G^*_{\iota_\xi+2}\middle|\cdots\middle|G^*_{J_\xi}\middle|H_{J_\xi}\middle|G_{J_\xi+1}\middle|\cdots\middle|G_{3\tau}\right)_{\tau\times3\tau}$$

Consequently, setting $M_5:=M_4$, we get the following (part of) free resolution of $(M_5)_S$ by Lemma 3.6.8:

$$S^{3\tau}\xrightarrow{\varphi=\varphi_5}S^{3\tau}\xrightarrow{\pi_5}(M_5)_S\longrightarrow 0.$$

To finish the proof, we recall in Proposition 3.6.5.(1) that $\det\varphi$ is a unit in S when $(w,\lambda,1)$ is nondegenerate. This implies that the map $\varphi:S^{3\tau}\to S^{3\tau}$ is injective. Indeed, if $\varphi u=0$ for some $u\in S^{3\tau}$, we multiply $\operatorname{adj}\varphi$ to both sides, then we have $\operatorname{adj}\varphi\cdot\varphi u=\left(\det\varphi\right)u=0$ by equation 3.6.3, yielding $u=0$ when $\det\varphi$ is a unit. Therefore, by Theorem 3.1.13, we know that M_5 is isomorphic to $\operatorname{coker}\underline{\varphi}$ as an A-module and it is actually a maximal Cohen-Macaulay A-module. But we also have $M_5=M_4=M_3=M_2$ and $M=(M_2)^\dagger$ by construction. Taking these together, we conclude that $M = M_5$ and thus establish the free resolution of M_S with $\pi := \pi(w,\lambda,1) := \pi_5 \in A^{\tau\times 3\tau}$:

$$0 \longrightarrow S^{3\tau} \xrightarrow{\varphi} S^{3\tau} \xrightarrow{\pi} M_S \longrightarrow 0.$$

$\square$

3.6.4 Higher multiplicity cases

Theorem 3.6.9. *For a band datum (w,λ,μ), let w' be the corresponding loop word. Define the corresponding matrix as*

$$\begin{aligned}
\varphi(w',\lambda,\mu) \quad &:= \quad \varphi(w',\lambda,1)\otimes I_\mu - x^{l'_1-1}P_{3\tau}^T\otimes\left(J_\mu(\lambda)-\lambda I_\mu\right) - x^{-l'_1}P_{3\tau}\otimes\left(J_\mu(\lambda)^{-1}-\lambda^{-1}I_\mu\right)\\
&= \quad \begin{pmatrix}
zI_\mu & -y^{m'_1-1}I_\mu & 0 & 0 & \cdots & 0 & -x^{-l'_1}J_\mu(\lambda)^{-1}\\
-y^{-m'_1}I_\mu & xI_\mu & -z^{n'_1-1}I_\mu & 0 & \cdots & 0 & 0\\
0 & -z^{-n'_1}I_\mu & yI_\mu & -x^{l'_2-1}I_\mu & \cdots & 0 & 0\\
0 & 0 & -x^{-l'_2}I_\mu & zI_\mu & \ddots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \ddots & \ddots & -y^{m'_\tau-1}I_\mu & 0\\
0 & 0 & 0 & \cdots & -y^{-m'_\tau}I_\mu & xI_\mu & -z^{n'_\tau-1}I_\mu\\
-x^{l'_1-1}J_\mu(\lambda) & 0 & 0 & \cdots & 0 & -z^{-n'_\tau}I_\mu & yI_\mu
\end{pmatrix}_{3\tau\mu\times 3\tau\mu}
\end{aligned}$$

where $P_{3\tau} := \begin{pmatrix} 0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 0 & 0\\ 0 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & 0\end{pmatrix}_{3\tau\times 3\tau}$, $N_\mu := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ 0 & 0 & 0 & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & 1\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}_{\mu\times\mu}$ *and $J_\mu(\lambda) := \lambda I_\mu + N_\mu$ is the Jordan block of eigenvalue λ and size $\mu\times\mu$. Then $\varphi(w',\lambda,\mu)$ is a matrix factor of xyz and satisfies*

$$M(w,\lambda,\mu) \cong \operatorname{coker}\underline{\varphi}(w',\lambda,\mu).$$

Definition 3.6.10. *Let R be a $\Bbbk$-algebra. For a one-parameter family*

$$\varphi(\lambda) = \sum_{i=-N}^{\infty}\varphi_i\lambda^i \in R^{m\times n}((\lambda))$$

of R-valued matrices and a $\Bbbk$-valued matrix $\Lambda\in\Bbbk^{\mu\times\mu}$, we define a new matrix $\varphi(\Lambda)$ by

$$\varphi(\Lambda) := \sum_{i=-N}^{\infty}\varphi_i\otimes\Lambda^i \in R^{m\mu\times n\mu}.$$

Remark 3.6.11. *(1) $\pi_\chi(w,\lambda,\mu) = \pi_\chi(w,J_\mu(\lambda))$ where $\chi\in\{x,y,z\}$.*

(2) $\varphi(w',\lambda,\mu) = \varphi(w',J_\mu(\lambda))$.

Lemma 3.6.12. *Let $\varphi(\lambda) \in R^{m\times n}((\lambda))$, $\psi(\lambda) \in R^{l\times n}((\lambda))$ be $R((\lambda))$-valued matrices satisfying*

(i) $\psi(\lambda)\varphi(\lambda) = 0$ as $R((\lambda))$-valued matrices,

(ii) $\varphi(\lambda) = \ker\psi(\lambda)$ for all eigenvalues λ of Λ.

Then we have $\varphi(\Lambda) = \ker\psi(\Lambda)$.

Proof of Theorem 3.6.9. Let

$$\tilde{\pi}\left(w,\lambda,\mu\right) := \left(\, x^2y^2I_{\tau\mu}\middle| y^2z^2I_{\tau\mu}\middle| z^2x^2I_{\tau\mu}\middle|\pi_x\left(w,J_\mu(\lambda)\right)\middle|\pi_y\left(w,J_\mu(\lambda)\right)\middle|\pi_z\left(w,J_\mu(\lambda)\right)\right)_{\tau\mu\times 6\tau\mu}$$

be an A-valued matrix. Then we have

$$\tilde{M}(w,\lambda) = \tilde{\pi}(w,\lambda,1) \subset A^\tau.$$

Recall that we found all of its Macaulayfying elements $F_{\iota_1},\ldots,F_{\iota_\xi}$ in A^τ. Here, a Macaulayfying element of $\tilde{M}(w,\lambda)$ in A^τ is an element $F \in A^\tau \setminus \tilde{M}(w,\lambda)$ satisfying

$$\chi F \in \tilde{M}(w,\lambda) \quad \text{for any } \chi \in \{x,y,z\}.$$

It turned out that such elements appear in the form of Laurent series $F(\lambda) \in A^\tau((\lambda))$ in λ and satisfy

$$\chi F(\lambda) = \tilde{\pi}(w,\lambda)\, a_\chi(\lambda) \quad \text{for any } \chi \in \{x,y,z\}.$$

for some $a_\chi(\lambda) \in A^{6\tau\mu}((\lambda))$. See equation (9.10) in [CJKR].

Using those elements, we got the Macaulayfication of $\tilde{M}(w,\lambda)$ as

$$\begin{aligned} M(w,\lambda) := \tilde{M}(w,\lambda)^\dagger &= \langle \tilde{M}(w,\lambda), F_{\iota_1}(\lambda),\ldots,F_{\iota_\xi}(\lambda)\rangle \\ &= \pi(w,\lambda) \subset A^\tau \end{aligned}$$

for some matrix $\pi(w,\lambda) \in A^{\tau\times 3\tau}$ and also constructed its free resolution

$$0 \longrightarrow S^{3\tau} \xrightarrow{\varphi(w',\lambda)} S^{3\tau} \xrightarrow{\pi(w,\lambda)} M_S(w,\lambda) \longrightarrow 0$$

as an S-module.

Now we consider

$$\tilde{M}\left(w,\lambda,\mu\right) = \tilde{\pi}\left(w,\lambda,\mu\right) \subset A^{\tau\mu}.$$

and its Macaulayfication. For each Macaulayfying element $F(\lambda) = \sum_{\imath=-N}^{\infty} F_i\lambda^i \in A^\tau$ of $\tilde{M}(w,\lambda,1)$ in A^τ, we associate a matrix $F\left(J_\mu(\lambda)\right) := \sum_{i=-N}^{\infty} F_i \otimes J_\mu(\lambda)^i \in A^{\tau\mu\times\mu}$. Then it satisfies

$$\chi F\left(J_\mu(\lambda)\right) = \tilde{\pi}\left(w,\lambda,\mu\right) a_\chi\left(J_\mu(\lambda)\right) \quad \text{for any } \chi \in \{x,y,z\},$$

so that each column of $F\left(J_\mu(\lambda)\right)$ is a Macaulayfying element of $\tilde{M}\left(w,\lambda,\mu\right)$ in $A^{\tau\mu}$. From this we get

$$M\left(w,\lambda,\mu\right) := \tilde{M}\left(w,\lambda,\mu\right)^\dagger = \langle \tilde{M}\left(w,\lambda,\mu\right), F_{\iota_1}\left(J_\mu(\lambda)\right),\ldots,F_{\iota_\xi}\left(J_\mu(\lambda)\right)\rangle^\dagger$$

and

$$\langle \tilde{M}\left(w,\lambda,\mu\right), F_{\iota_1}\left(J_\mu(\lambda)\right),\ldots,F_{\iota_\xi}\left(J_\mu(\lambda)\right)\rangle = \pi\left(w,J_\mu(\lambda)\right) \subset A^{\tau\mu}.$$

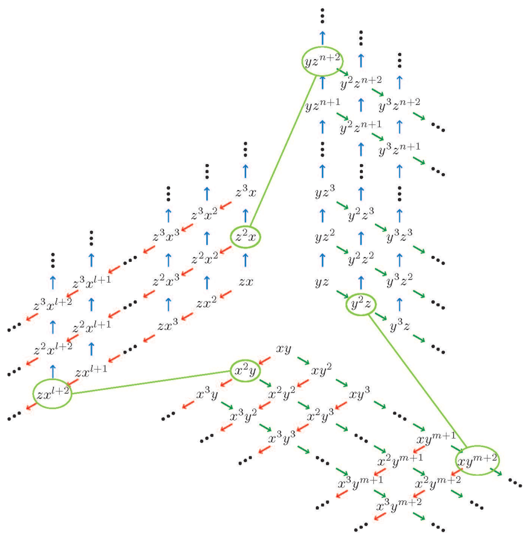
On the other hand, applying the lemma on the above resolution, we have a free resolution

$$0 \longrightarrow S^{3\tau\mu} \xrightarrow{\varphi(w',J_\mu(\lambda))} S^{3\tau\mu} \xrightarrow{\pi(w,J_\mu(\lambda))} \pi\left(w,J_\mu(\lambda)\right)_S \longrightarrow 0 \tag{3.6.23}$$

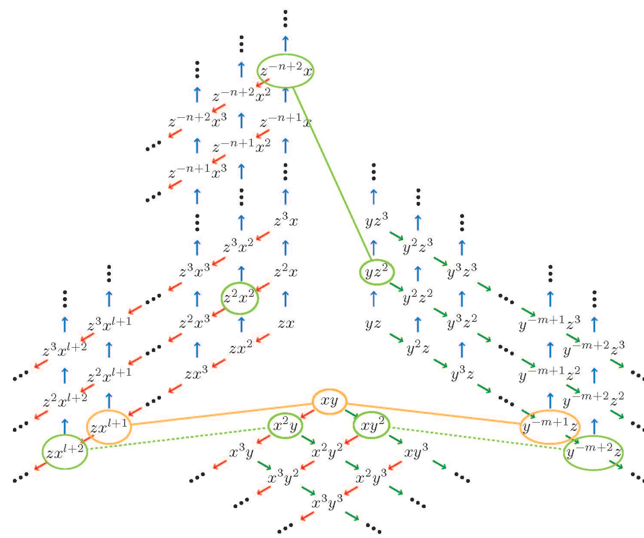
of $\pi\left(w,J_\mu(\lambda)\right)$ as an S-module. Finally, we know that $\varphi\left(w',J_\mu(\lambda)\right)$ is a matrix factor of xyz, namely,

$$\varphi\left(w',J_\mu(\lambda)\right)\psi\left(w',J_\mu(\lambda)\right) = xyzI_{3\tau\mu},$$

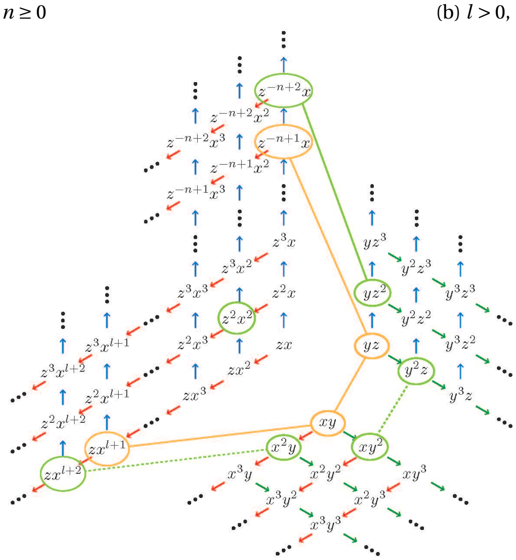
which implies that $\pi\left(w,J_\mu(\lambda)\right)$ is already Cohen-Macaulay and hence equals $M\left(w,\lambda,\mu\right)$. □



(a) $l \geq 0$, $m \geq 0$, $n \geq 0$

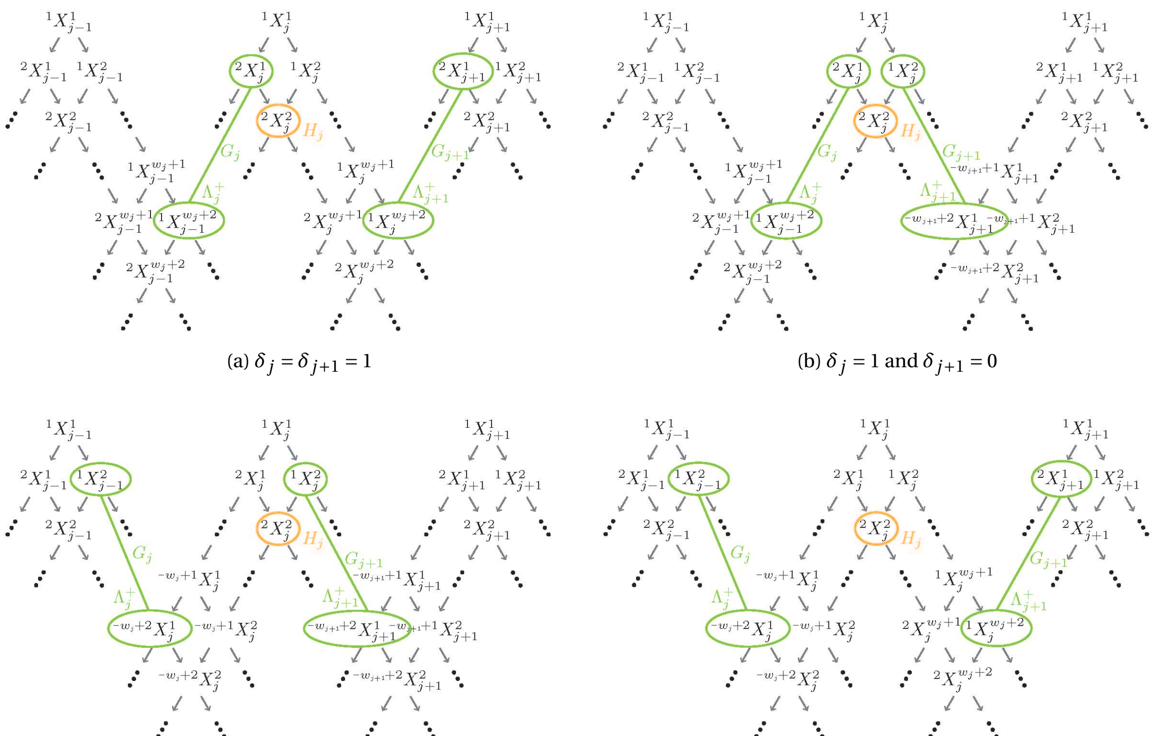


(b) $l > 0$, $m < 0$, $n \leq 0$



(c) $l > 0$, $m = 0$, $n < 0$

Figure 3.3: Generator diagram in some cases



(a) $\delta_j = \delta_{j+1} = 1$

(b) $\delta_j = 1$ and $\delta_{j+1} = 0$

(c) $\delta_j = \delta_{j+1} = 0$

(d) $\delta_j = 0$ and $\delta_{j+1} = 1$

Figure 3.4: A part of generator diagram for $\tilde{M}$ containing G_j, H_j and G_{j+1} in each case

$$
\begin{array}{cccccccccccccc}
 & R_{\iota_\nu} & R_{\iota_\nu+1} & R_{\iota_\nu+2} & R_{\iota_\nu+3} & \cdots & R_{J_\nu} & T_{J_\nu} & T_{J_\nu+1} & T_{J_\nu+2} & R_{J_\nu+2} & \cdots & R_{\iota_{\nu+1}-1} & R_{\iota_{\nu+1}} \\
G^{\iota_\nu-1} & -\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w_{\iota_\nu}+1} & (\varphi_1)^+_\nu & & & & & & & & & & & \\
G^{\iota_\nu} & \chi_{\iota_\nu-1} & \chi_{\iota_\nu+1} & & & & & & & & & & & \\
G^{\iota_\nu+1} & & -\chi_{\iota_\nu} & \chi_{\iota_\nu+2} & & & & & & & & & & \\
G^{\iota_\nu+2} & & & -\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w_{\iota_\nu+1}+1} & \chi_{\iota_\nu} & \cdots & 0 & 0 & & & & & & \\
\vdots & & & 0 & -\Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w_{\iota_\nu+2}+1} & \ddots & \vdots & \vdots & & & & & & \\
 & & & \vdots & \vdots & \ddots & \chi_{J_\nu-3} & 0 & & & & & & \\
G^{J_\nu} & & & 0 & 0 & \cdots & -\Lambda^-_{J_\nu-1}\chi_{J_\nu-1}^{-w_{J_\nu-1}+1} & \chi_{J_\nu-2} & & & & & (\varphi_1)^+_{\nu+1} & \\
H^{J_\nu} & & & 0 & 0 & \cdots & 0 & -\Lambda^-_{J_\nu}\chi_{J_\nu}^{-w_{J_\nu}} & \chi_{J_\nu-1} & -\Lambda^+_{J_\nu+1}\chi_{J_\nu+1}^{w_{J_\nu+1}} & 0 & \cdots & 0 & 0 \\
G^{J_\nu+1} & & & & (\varphi_1)^-_\nu & & & & & \chi_{J_\nu} & -\Lambda^+_{J_\nu+2}\chi_{J_\nu+2}^{w_{J_\nu+2}+1} & \cdots & 0 & 0 \\
\vdots & & & & & & & & & 0 & \chi_{J_\nu+1} & \ddots & \vdots & \vdots \\
 & & & & & & & & & \vdots & \vdots & \ddots & -\Lambda^+_{\iota_{\nu+1}-1}\chi_{\iota_{\nu+1}-1}^{w_{\iota_{\nu+1}-1}+1} & 0 \\
G^{\iota_{\nu+1}-1} & & & & & & & & & 0 & 0 & \cdots & \chi_{\iota_{\nu+1}-2} & -\Lambda^+_{\iota_{\nu+1}}\chi_{\iota_{\nu+1}}^{w_{\iota_{\nu+1}}+1}
\end{array}
$$

Figure 3.7: Submatrix $\varphi_1\left[G^{\iota_\nu-1}:G^{\iota_{\nu+1}-1};R_{\iota_\nu}:R_{\iota_{\nu+1}}\right]$ of φ_1

$$
\begin{array}{ccccccccccccc}
 & Q_{\iota_\nu-1} & Q_{\iota_\nu} & Q_{\iota_\nu+1} & R''_{\iota_\nu} & R_{\iota_\nu+1} & R'_{\iota_\nu+2} & & & \cdots & & R'_{\iota_\nu+\kappa_\nu+1} & R'_{\iota_\nu+\kappa_\nu+2} \\
(G^{\iota_\nu-1})'' & 0 & 0 & -\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}-1} & -\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}} & (\varphi_1)^+_\nu & & & & & & & \\
F^{\iota_\nu} & \chi_{\iota_\nu} & \chi_{\iota_\nu+1} & \chi_{\iota_\nu+2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
G^{\iota_\nu} & -1 & 0 & 0 & \chi_{\iota_\nu-1} & \chi_{\iota_\nu+1} & & & & & & & \\
G^{\iota_\nu+1} & 0 & -1 & 0 & & -\chi_{\iota_\nu} & \chi_{\iota_\nu+2} & & & & & & \\
(G^{\iota_\nu+2})' & 0 & 0 & -\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}} & & & -\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}+1} & \chi_{\iota_\nu} & 0 & 0 & \cdots & 0 & 0 \\
(G^{\iota_\nu+3})' & -\zeta^{-1}_{\iota_\nu,-1,2} & -\zeta^{0}_{\iota_\nu,-1,2} & -\zeta^{1}_{\iota_\nu,-1,2} & & & 0 & -\Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w'_{\iota_\nu+2}} & \chi_{\iota_\nu+1} & 0 & \cdots & 0 & 0 \\
(G^{\iota_\nu+4})' & -\zeta^{-1}_{\iota_\nu,-1,3} & -\zeta^{0}_{\iota_\nu,-1,3} & -\zeta^{1}_{\iota_\nu,-1,3} & & & 0 & 0 & -\Lambda^-_{\iota_\nu+3}\chi_{\iota_\nu+3}^{-w'_{\iota_\nu+3}} & \chi_{\iota_\nu+2} & \cdots & 0 & 0 \\
 & -\zeta^{-1}_{\iota_\nu,-1,4} & -\zeta^{0}_{\iota_\nu,-1,4} & -\zeta^{1}_{\iota_\nu,-1,4} & & & 0 & 0 & 0 & -\Lambda^-_{\iota_\nu+4}\chi_{\iota_\nu+4}^{-w'_{\iota_\nu+4}} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & & \vdots & \ddots & \ddots & \ddots & \ddots & \chi_{\iota_\nu+\kappa_\nu-2} & 0 \\
(G^{\iota_\nu+\kappa_\nu+1})' & -\zeta^{-1}_{\iota_\nu,-1,\kappa_\nu} & -\zeta^{0}_{\iota_\nu,-1,\kappa_\nu} & -\zeta^{1}_{\iota_\nu,-1,\kappa_\nu} & & & 0 & \cdots & 0 & 0 & 0 & -\Lambda^-_{\iota_\nu+\kappa_\nu}\chi_{\iota_\nu+\kappa_\nu}^{-w'_{\iota_\nu+\kappa_\nu}} & \chi_{\iota_\nu+\kappa_\nu-1} \\
 & & & & (\varphi_1)^-_\nu & & & & & & & & \\
(G^{\iota_\nu+\kappa_\nu+2})' & -\zeta^{-1}_{\iota_\nu,-1,\kappa_\nu+1} & -\zeta^{0}_{\iota_\nu,-1,\kappa_\nu+1} & -\zeta^{1}_{\iota_\nu,-1,\kappa_\nu+1} & & & 0 & \cdots & 0 & 0 & 0 & 0 & -\Lambda^-_{\iota_\nu+\kappa_\nu+1}\chi_{\iota_\nu+\kappa_\nu+1}^{-w'_{\iota_\nu+\kappa_\nu+1}}
\end{array}
$$

Figure 3.9: Submatrix $\varphi_2\left[\left(G^{\iota_\nu-1}\right)'':\left(G^{\iota_\nu+\kappa_\nu+2}\right)';Q_{\iota_\nu-1}:R'_{\iota_\nu+\kappa_\nu+2}\right]$ of φ_2

$$
\begin{array}{c|cccccccccccc}
 & Q_{\iota_\nu-1} & Q_{\iota_\nu} & Q_{\iota_\nu+1} & R''_{\iota_\nu} & R_{\iota_\nu+1} & R'_{\iota_\nu+2} & & \cdots & & & R'_{\iota_\nu+\kappa_\nu+1} & R'_{\iota_\nu+\kappa_\nu+2} \\
\hline
(G^{\iota_\nu-1})'' & 0 & 0 & -\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}-1} & \boxed{-\Lambda^+_{\iota_\nu}\chi_{\iota_\nu}^{w'_{\iota_\nu}}} & (\varphi_1)^+_\nu & & & & & & & \\
F^{\iota_\nu} & \chi_{\iota_\nu} & \chi_{\iota_\nu+1} & \chi_{\iota_\nu+2} & & & & & & & & & \\
(G^{\iota_\nu})^* & -1 & 0 & 0 & \chi_{\iota_\nu-1} & \chi_{\iota_\nu+1} & & & & & & & \\
(G^{\iota_\nu+1})^* & 0 & -1 & 0 & & -\chi_{\iota_\nu} & \chi_{\iota_\nu+2} & & & & & & \\
(G^{\iota_\nu+2})^* & 0 & 0 & -\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}} & 0^{(*)} & & -\Lambda^-_{\iota_\nu+1}\chi_{\iota_\nu+1}^{-w'_{\iota_\nu+1}+1} & \chi_{\iota_\nu} & 0 & 0 & \cdots & 0 & 0 \\
(G^{\iota_\nu+3})^* & 0 & 0 & 0 & \begin{array}{c}-\zeta^{-1}_{\iota_\nu,-1,2}\\ \chi_{\iota_\nu+2}\end{array} & \begin{array}{c}-\zeta^{-1}_{\iota_\nu,-1,2}\\ \chi_{\iota_\nu+1}\end{array} & 0 & -\Lambda^-_{\iota_\nu+2}\chi_{\iota_\nu+2}^{-w'_{\iota_\nu+2}} & \chi_{\iota_\nu+1} & 0 & \cdots & 0 & 0 \\
(G^{\iota_\nu+4})^* & 0 & 0 & 0 & & \begin{array}{c}\zeta^{0}_{\iota_\nu,-1,3}\\ \chi_{\iota_\nu+3}\end{array} & \begin{array}{c}-\zeta^{0}_{\iota_\nu,0,3}\\ \chi_{\iota_\nu+2}\end{array} & 0 & -\Lambda^-_{\iota_\nu+3}\chi_{\iota_\nu+3}^{-w'_{\iota_\nu+3}} & \chi_{\iota_\nu+2} & \cdots & 0 & 0 \\
 & 0 & 0 & 0 & & & \begin{array}{c}\zeta^{1}_{\iota_\nu,0,4}\\ \chi_{\iota_\nu+4}\end{array} & \begin{array}{c}-\zeta^{1}_{\iota_\nu,1,4}\\ \chi_{\iota_\nu+3}\end{array} & 0 & -\Lambda^-_{\iota_\nu+4}\chi_{\iota_\nu+4}^{-w'_{\iota_\nu+4}} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & & \vdots & \ddots & \ddots & \ddots & \ddots & \chi_{\iota_\nu+\kappa_\nu-2} & 0 \\
(G^{\iota_\nu+\kappa_\nu+1})^* & 0 & 0 & 0 & & & 0 & \cdots & \begin{array}{c}\zeta^{\kappa_\nu-3}_{\iota_\nu,\kappa_\nu-4,\kappa_\nu}\\ \chi_{\iota_\nu+\kappa_\nu}\end{array} & \begin{array}{c}-\zeta^{\kappa_\nu-3}_{\iota_\nu,\kappa_\nu-3,\kappa_\nu}\\ \chi_{\iota_\nu+\kappa_\nu-1}\end{array} & 0 & -\Lambda^-_{\iota_\nu+\kappa_\nu}\chi_{\iota_\nu+\kappa_\nu}^{-w'_{\iota_\nu+\kappa_\nu}} & \chi_{\iota_\nu+\kappa_\nu-1} \\
 & & & & & (\varphi_1)^-_\nu & & & & & & & \\
(G^{\iota_\nu+\kappa_\nu+2})^* & 0 & 0 & 0 & & & 0 & \cdots & 0 & \begin{array}{c}\zeta^{\kappa_\nu-2}_{\iota_\nu,\kappa_\nu-3,\kappa_\nu+1}\\ \chi_{\iota_\nu+\kappa_\nu+1}\end{array} & \begin{array}{c}-\zeta^{\kappa_\nu-2}_{\iota_\nu,\kappa_\nu-2,\kappa_\nu+1}\\ \chi_{\iota_\nu+\kappa_\nu}\end{array} & 0 & -\Lambda^-_{\iota_\nu+\kappa_\nu+1}\chi_{\iota_\nu+\kappa_\nu+1}^{-w'_{\iota_\nu+\kappa_\nu+1}}
\end{array}
$$

Figure 3.10: Submatrix $\varphi_3\left[\left(G^{\iota_\nu-1}\right)'' : \left(G^{\iota_\nu+\kappa_\nu+2}\right)^* ; Q_{\iota_\nu-1} : R'_{\iota_\nu+\kappa_\nu+2}\right]$ of φ_3

$$
\begin{array}{c|ccccccccccc}
 & T_{J_{\nu-1}+2} & R_{J_{\nu-1}+2} & \cdots & R_{\iota_\nu-1} & Q_{\iota_\nu+1} & R_{\iota_\nu+3} & \cdots & R_{J_\nu} & T_{J_\nu} & T_{J_\nu+1} & T_{J_\nu+2} \\
\hline
H^{J_{\nu-1}} & -\Lambda^+_{J_{\nu-1}+1}\chi^{w'_{J_{\nu-1}+1}-1}_{J_{\nu-1}+1} & 0 & \cdots & 0 & 0 & & & & & & \\
G^{J_{\nu-1}+1} & \chi_{J_{\nu-1}} & -\Lambda^+_{J_{\nu-1}+2}\chi^{w'_{J_{\nu-1}+2}-1}_{J_{\nu-1}+2} & \cdots & 0 & 0 & & & & & & \\
\vdots & 0 & \chi_{J_{\nu-1}+1} & \ddots & \vdots & \vdots & \varphi^+_\nu & & & & & \\
 & \vdots & \vdots & \ddots & -\Lambda^+_{\iota_\nu-1}\chi^{w'_{\iota_\nu-1}-1}_{\iota_\nu-1} & 0 & & & & & & \\
G^{\iota_\nu-1} & 0 & 0 & \cdots & \chi_{\iota_\nu-2} & -\Lambda^+_{\iota_\nu}\chi^{w'_{\iota_\nu}-1}_{\iota_\nu} & & & & & & \\
F^{\iota_\nu} & & & & & \chi_{\iota_\nu-1} & & & & & & \\
(G^{\iota_\nu+2})^* & & & & & -\Lambda^-_{\iota_\nu+1}\chi^{-w'_{\iota_\nu+1}}_{\iota_\nu+1} & \chi_{\iota_\nu} & \cdots & 0 & 0 & & \\
\vdots & & & & & 0 & -\Lambda^-_{\iota_\nu+2}\chi^{-w'_{\iota_\nu+2}}_{\iota_\nu+2} & \ddots & \vdots & \vdots & & \\
 & & & & \varphi^-_\nu & \vdots & \vdots & \ddots & \chi_{J_\nu-3} & 0 & & \\
(G^{J_\nu})^* & & & & & 0 & 0 & \cdots & -\Lambda^-_{J_\nu-1}\chi^{-w'_{J_\nu-1}}_{J_\nu-1} & \chi_{J_\nu-2} & & \varphi^+_{\nu+1} \\
H^{J_\nu} & & & & & 0 & 0 & \cdots & 0 & -\Lambda^-_{J_\nu}\chi^{-w'_{J_\nu}}_{J_\nu} & \chi_{J_\nu-1} & -\Lambda^+_{J_\nu+1}\chi^{w'_{J_\nu+1}-1}_{J_\nu+1}
\end{array}
$$

Figure 3.11: Submatrix $\varphi\left[H^{J_{\nu-1}} : H^{J_\nu}; T_{J_{\nu-1}+2} : T_{J_\nu+2}\right]$ of φ

Chapter 4

Mirror Symmetry Correspondence between Modules and Lagrangians

4.1 Loop/Arc-Type Lagrangians ↔ Band/String-Type Cohen-Macaulay Modules

We have defined the conversion formula from rank 1 band data (l, m, n) to normal loop data (l', m', n') in Proposition 3.3.6, and showed that it defines a bijection in Theorem **??**. In this section, we discuss the conversion formula for higher rank cases that appeared in Theorem **??**.

Definition 4.1.1 (Conversion from band data to loop data)**.** *Pick a band datum (w, λ, μ), with*

$$w = (w_1, w_2, w_3, w_4, w_5, w_6, \dots, w_{3\tau-2}, w_{3\tau-1}, w_{3\tau}) \in \mathbb{Z}^{3\tau}.$$

We define the sign word $\delta = \delta(w) \in \{0,1\}^{3\tau}$, *the* correction word $\varepsilon = \varepsilon(w) \in \{-1, 0, 1, 2\}^{3\tau}$ *of w and the loop word $w' \in \mathbb{Z}^{3\tau}$ converted from w below, where we regard the index j of w_j, δ_j, ε_j and w'_j to be in $\mathbb{Z}_{3\tau}$. First, each entry of the sign word $\delta = \delta(w)$ is defined as*

$$\delta_j := \begin{cases} 0 & \text{if} \quad \begin{cases} w_j < 0, \text{ or} \\ w_j = 0 \text{ and at least one of the first non-zero entries adjacent to the} \\ \qquad \text{string of 0s containing } w_j \text{ (exists and) is negative,} \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

Next, each entry of the correction word $\varepsilon = \varepsilon(w)$ is defined as

$$\varepsilon_j := -1 + \delta_{j-1} + \delta_j + \delta_{j+1}.$$

Then each entry of the converted loop word w' is defined as

$$w'_j = w_j + \varepsilon_j$$

and the conversion *from the band datum to the loop datum is given by*

$$(w, \lambda, \mu) \mapsto (w' = w + \varepsilon(w), \lambda' = (-1)^{l_1 + \dots + l_\tau + \tau} \lambda, \mu = \mu)$$

where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$.

Lemma 4.1.2. *For a band word $w \in \mathbb{Z}^{3\tau}$ and its sign word $\delta = \delta(w) \in \{0,1\}^{3\tau}$, assume $w_k = \cdots = w_l = 0$ for some $k, l \in \mathbb{Z}_{3\tau}$. Then*

1. *$\delta_{k-1} = \delta_{l+1} = 1$ implies $\delta_k = \cdots = \delta_l = 1$, and*
2. *either $\delta_{k-1} = 0$ or $\delta_{l+1} = 0$ implies $\delta_k = \cdots = \delta_l = 0$.*

Proof. It is obvious from the definition of the sign word. □

Proposition 4.1.3. *The loop word w' converted from a band word w is always normal.*

Proof. Let w be any band word, $\varepsilon = \varepsilon(w)$ the correction word of w, and $w' = w + \varepsilon(w)$ the converted loop word. We will show that w' satisfies all of 4 conditions to be normal in order.

- *Condition 1* Assume that $w'_j = 1$ for some $j \in \mathbb{Z}_{3\tau}$. As ε_j takes its value in one of -1, 0, 1 and 2, the possible combination of w_j and ε_j are $(w_j, \varepsilon_j) = (2,-1)$, $(1,0)$, $(0,1)$ and $(-1,2)$. But the first one is impossible as $w_j = 2$ means $\delta_j = 1$ so that $\varepsilon_j \geq 0$. The last one is also ruled out as $w_j = -1$ yields $\delta_j = 0$ so that $\varepsilon_j \leq 1$. In the third case, in order for $\varepsilon_j = 1$ to be hold, only one of δ_{j-1}, δ_j and δ_{j+1} is 0. But this cannot hold under $w_j = 0$ according to Lemma 4.1.2.

  Thus only the second combination remains. In this case, in order to hold $w_j = 1$ and $\varepsilon_j = 0$, we must have $\delta_j = 1$ and $\delta_{j-1} = \delta_{j+1} = 0$. Then we have $w_{j-1} \leq 0$. If $\delta_{j-2} = 0$, we have $\varepsilon_{j-1} = 0$ and hence $w'_{j-1} \leq 0$. Otherwise, if $\delta_{j-2} = 1$, we have $\varepsilon_{j-1} = 1$ and $w_{j-1} \leq -1$ by Lemma 4.1.2.(1) which gives $w'_{j-1} \leq 0$. Therefore, $w'_{j-1} \leq 0$ holds in any case and similarly we conclude that $w'_{j+1} \leq 0$ also holds. This establishes the first normality condition of w'.

- *Condition 2* Assume that $w'_j = 0$ for some $j \in \mathbb{Z}_{3\tau}$. The possible combination of w_j and ε_j are

  $$(w_j, \varepsilon_j) = (1,-1), (0,0), (-1,-1) \text{ and } (-2,-2).$$

  As before, one can easily exclude the first and the last cases.

  In the second case, because $\varepsilon_j = 0$, only one of δ_{j-1}, δ_j and δ_{j+1} is 1. As $w_j = 0$, it follows from Lemma 4.1.2.(2) that one of δ_{j-1} and δ_{j+1} is 1. Hence, we can assume without loss of generality that $\delta_{j-1} = 1$ and $\delta_j = \delta_{j+1} = 0$. Then we must have $w_{j-1} \geq 1$ by Lemma 4.1.2.(2) and $\varepsilon_{j-1} \geq 0$ and hence $w'_{j-1} \geq 1$. Also, we have $w_{j+1} \leq 0$ and $\varepsilon_{j+1} \leq 0$. If $w_{j+1} \leq -1$, we get $w'_{j+1} \leq -1$. Otherwise, if $w_{j+1} = 0$, Lemma 4.1.2.(1) gives $\delta_{j+2} = 0$ so that $\varepsilon_{j+1} = -1$ and $w'_{j+1} \leq -1$ follow. Consequently, we have $w'_{j-1} \geq 1$ and $w'_{j+1} \leq -1$. If $\delta_{j+1} = 1$, by symmetry, we get $w'_{j-1} \leq -1$ and $w'_{j+1} \geq 1$, establishing the second normality condition of w'.

  In the third case, $\varepsilon_j = 1$ and $\delta_j = 0$ yield $\delta_{j-1} = \delta_{j+1} = 1$. Then Lemma 4.1.2.(2) gives w_{j-1}, $w_{j+1} \geq 1$. Since $\varepsilon_{j-1}, \varepsilon_{j+1} \geq 0$, we have w'_{j-1}, $w'_{j+1} \geq 1$, establishing again the second normality condition of w'.

- *Condition 3* Assume there are integers $k, l \in \mathbb{Z}_{3\tau}$ such that $l \neq k+1$ and $w'_k = 0$, $w'_{k+1} = \cdots = w'_{l-1} = -1$, $w'_l = 0$. By discussion in the previous case, we have $w_k = w_l = 0$ and $\delta_{k-1} = \delta_{l+1} = 1$, $\delta_k = \delta_l = 0$. It can be easily checked that $w'_{k+1} = \cdots = w'_{l-1} = -1$ implies $\delta_{k+1} = \cdots = \delta_{l-1} = 0$, yielding $\varepsilon_{k+1} = \cdots =$

$\varepsilon_{l-1} = -1$ and that $w_{k+1} = \cdots = w_{l-1} = 0$. Putting these together would contradict Lemma 4.1.2(1). Therefore, we conclude that there is no subword of the form $(0,-1,-1,\ldots,-1,0)$ in w', establishing the third normality condition of w'.

- *Condition 4* Assume $w'_j = -1$ for all $j \in \mathbb{Z}_{3\tau}$. It can be easily checked that $\delta_j = 0$ for all j, yielding $\varepsilon_j = -1$ and $w_j = 0$. But then by definition we have $\delta_j = 1$ for any such j, which is a contradiction. Therefore, w' does not consist only of -1, establishing the last normality condition of w'.

$\square$

Next we define the inverse of the above conversion formula.

Definition 4.1.4 (Conversion from normal loop data to band data)**.** *Pick a normal loop datum (w',λ',μ), with*

$$w' = (w'_1, w'_2, w'_3, w'_4, w'_5, w'_6, \ldots, w'_{3\tau-2}, w'_{3\tau-1}, w'_{3\tau}) \in \mathbb{Z}^{3\tau}.$$

We define the sign word $\delta' = \delta'(w') \in \{0,1\}^{3\tau}$*, the* correction word $\varepsilon' = \varepsilon'(w') \in \{-1,0,1,2\}^{3\tau}$ *of w' and the band word $w \in \mathbb{Z}^{3\tau}$ converted from w' below, where we regard the index j of w'_j, s'_j, ε'_j and w_j to be in $\mathbb{Z}_{3\tau}$. First, each entry of the sign word $\delta' = \delta'(w')$ is defined as*

$$\delta'_j := \begin{cases} 0 & \text{if} \quad w'_j \le 0 \\ 1 & \text{if} \quad w'_j > 0. \end{cases}$$

Next, each entry of the correction word $\varepsilon' = \varepsilon'(w')$ is defined as

$$\varepsilon'_j := -1 + \delta'_{j-1} + \delta'_j + \delta'_{j+1}.$$

Then each entry of the converted loop word w is defined as

$$w_j = w'_j - \varepsilon'_j$$

and the conversion *from the normal loop datum to the band datum is given by*

$$(w',\lambda',\mu) \mapsto (w = w' - \varepsilon'(w'), \lambda = (-1)^{l_1+\cdots+l_\tau+\tau}\lambda', \mu = \mu)$$

where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$.

Example 4.1.5. *Consider a band datum (w,λ,μ) whose band word w is given as below.*

$$\begin{array}{rl}
w & = (\underline{6, 0, 2}, \underline{-1, 0, -3, 0, 0}, \underline{5}, \underline{0, -2}, \underline{1}, \underline{-1}, \underline{3, 4}) \\
\delta(w) = \delta'(w') & = (1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1) \\
\varepsilon(w) = \varepsilon'(w') & = (2, 2, 1, 0, -1, -1, -1, 0, 0, 0, 0, 0, 1, 1, 2) \\
w' & = (\underline{8, 2, 3}, \underline{-1, -1, -4, -1, 0}, \underline{5}, \underline{0, -2}, \underline{1}, \underline{0}, \underline{4, 6})
\end{array}$$

We computed the sign word $\delta(w)$, the correction word $\varepsilon(w)$ of w and the loop word $w' = w + \varepsilon(w)$ converted from w. Note that w' is presented in the normal form. Then we computed the sign word $\delta'\left(w'\right)$, the correction word $\varepsilon'\left(w'\right)$ of w'.

We underlined the spots of w in blue or red, respectively, according to whether the value of δ on them is 1 or 0. Likewise, the spots of w' are underlined according to the value of δ'. Observe that both w and w' have the same underline pattern, implying $\delta'\left(w'\right) = \delta(w)$ and hence $\varepsilon'\left(w'\right) = \varepsilon(w)$. Therefore, the band word $w' - \varepsilon'\left(w'\right)$ converted from w' is the same as the original band word w.

The parameter λ and the holonomy parameter λ' in this case are related by $\lambda' = (-1)^{6-1+0+0-1+5}\lambda = -\lambda$.

Proposition 4.1.6. *The conversion from the band datum to the loop datum and the conversion from the normal loop datum to the band datum are the inverses of each other.*

Proof. Let $\left(w, \lambda, \mu\right)$ be a band datum and

$$\left(w' = w + \varepsilon(w), \lambda' = (-1)^{l_1+\cdots+l_\tau+\tau}\lambda, \mu\right)$$

the converted loop datum, where $l_i = w_{3i-2}$ for $i \in \mathbb{Z}_\tau$. Then let

$$\left(w'' = w' - \varepsilon'\left(w'\right), \lambda'' = (-1)^{l''_1+\cdots+l''_\tau+\tau}\lambda', \mu = \mu'\right)$$

be the band datum converted from (w', λ', μ), where $l''_i = w''_{3i-2}$ for $i \in \mathbb{Z}_\tau$. In order to show $(w'', \lambda'', \mu) = (w, \lambda, \mu)$, we notice that it is enough to show $w'' = w$, which is also equivalent to $\varepsilon(w) = \varepsilon'(w')$. By the construction of w and w', therefore, we only need to show that $\delta(w) = \delta'(w')$. Denoting $\delta = \delta(w)$ and $\delta' = \delta'(w')$, we can prove $\delta_j = \delta'_j$ for each $j \in \mathbb{Z}_{3\tau}$ as follows.

- *Case 1* $w_j \le -1$

  In this case, we have $\delta_j = 0$ and hence $\varepsilon_j \le 1$, implying $w'_j \le 0$ so that $\delta'_j = 0$ follows.

- *Case 2* $w_j = 0$

  If $\delta_j = 0$, by Lemma 4.1.2.(1), at least one of δ_{j-1} and δ_{j+1} must be 0, implying $\varepsilon_j \le 0$ so that $w'_j \le 0$ and hence $\delta'_j = 0$. Otherwise, if $\delta_j = 1$, by Lemma 4.1.2.(2), both δ_{j-1} and δ_{j+1} must be 1, implying $\varepsilon_j = 2$ so that $w'_j = 2$ and hence $\delta'_j = 1$.

- *Case 3* $w_j \ge 1$

  We have $\delta_j = 1$ and hence $\varepsilon_j \ge 0$, implying $w'_j \ge 1$ so that $\delta'_j = 1$ follows.

Therefore, we proved that if we convert a given band datum to a loop datum and then convert it back to a band datum, it returns to itself.

Conversely, let (w', λ', μ) be a normal loop datum and

$$\left(w'' = w' - \varepsilon'\left(w\right), \lambda'' = (-1)^{l''_1+\cdots+l''_\tau+\tau}\lambda', \mu\right)$$

the converted band datum, where $l''_i = w''_{3i-2}$ for $i \in \mathbb{Z}_\tau$. Then let

$$\left(w''' = w'' - \varepsilon''\left(w''\right), \lambda''' = (-1)^{l''_1 + \cdots + l''_\tau + \tau} \lambda'', \mu = \mu\right)$$

be the loop datum converted from (w'', λ'', μ). In order to show $(w''', \lambda''', \mu) = (w', \lambda', \mu)$, we only need to show $w''' = w'$, which again follows from $\delta'(w') = \delta''(w'')$. As above, denoting $\delta' = \delta'(w')$ and $\delta'' = \delta''(w'')$, we can prove $\delta'_j = \delta''_j$ for each $j \in \mathbb{Z}_{3\tau}$ by dividing the case.

- *Case 1* $w'_j \leq -2$

  We have $\delta'_j = 0$. As $\varepsilon'_j \geq -1$, we also have $w''_j \leq -1$ and hence $\delta''_j = 0$.

- *Case 2* $w'_j = -1$

  We have $\delta'_j = 0$. Since w' is normal, w' does not consist only of -1.

  If the first non-(-1) element to the left of w'_j is less than or equal to -2, namely, if there is an integer $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \leq -2$, $w'_{k+1} = \cdots = w'_j = -1$, we have $w''_k \leq -1$, $w''_{k+1}, \ldots, w''_j \leq 0$, implying $\delta''_j = 0$ by Lemma 4.1.2.(2).

  If the first non-(-1) element to the left of w'_j is greater than or equal to 1, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \geq 1$, $w'_{k+1} = \cdots = w'_j = -1$, we have $\varepsilon'_{k+1} \geq 0$ and hence $w''_{k+1} \leq -1$, $w''_{k+2}, \ldots, w''_j \leq 0$, implying $\delta''_j = 0$ by Lemma 4.1.2.(2).

  The above discussion also applies to the elements to the right of w'_j. It remains a case where both the first non-(-1) element to the left and the right of w'_j are 0, namely, there are k, $l \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $k < j < l$ and $w'_k = 0$, $w'_{k+1} = \cdots = w'_{l-1} = -1$, $w'_l = 0$. But this violates the third condition for w' to be normal. Thus we conclude that $\delta''_j = 0$ whenever $w'_j = -1$ and w' is normal.

- *Case 3* $w'_j = 0$

  We have $\delta'_j = 0$. Because w' is normal, one of $w'_{j-1} \leq -1$, $w'_{j+1} \geq 1$ or $w'_{j-1} \geq 1$, $w'_{j+1} \leq -1$ or w'_{j-1}, $w'_{j+1} \geq 1$ must hold. In the first case, we have $\varepsilon'_j = 0$, $w''_j = 0$ and $\delta''_{j-1} = 0$ bt arguments in the case 1 and 2. Hence, Lemma 4.1.2.(2) gives $\delta''_j = 0$. The second case can also be handled in the same way. In the third case, we have $\varepsilon'_j = 1$, $w''_j = -1$ and hence $\delta''_j = 0$.

- *Case 4* $w'_j = 1$

  We have $\delta'_j = 1$. As w' is normal, both w'_{j-1} and w'_{j+1} are less than or equal to 0. Therefore, we have $\varepsilon'_j = 0$, $w''_j = 1$ and hence $\delta''_j = 1$.

- *Case 5* $w'_j = 2$

  We have $\delta'_j = 1$. If w' consists only of 2, then w'' consists only of 0, whence $\delta''_j = 1$. Now assume that this is not the case.

  If the first non-2 element to the left of w'_j is less than or equal to 1, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \le 1$, $w'_{k+1} = \cdots = w'_j = 2$, we automatically get $w'_k \le 0$ by the first condition that w' is normal. Thus we have $\varepsilon'_{k+1} = 1$ and hence $w''_{k+1} = 1$, $w''_{k+2}, \dots, w''_j \ge 0$. And if the first non-2 element to the left of w'_j is greater than or equal to 3, namely, if there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w'_k \ge 3$, $w'_{k+1} = \cdots = w'_j = 2$, we have $w''_k \ge 1$, $w''_{k+1}, \dots, w''_j \ge 0$. Therefore, in any cases, either $w''_j \ge 1$ or there is $k \in \mathbb{Z}_{3\tau} \setminus \{j\}$ such that $w''_k \ge 1$, $w''_{k+1}, \dots, w''_j \ge 0$. This result also applies to the elements to the right of w'_j and therefore we conclude that either $w''_j \ge 1$, or there are $k, l \in \mathbb{Z}_{3\tau} \setminus \{j\}$ with $k < j < l$ such that $w''_k \ge 1$, $w''_{k+1}, \dots, w''_{l-1} \ge 0$, $w''_l \ge 1$. In either case, we have $\delta''_j = 1$.

- *Case 6* $w'_j \ge 3$

  We have $\delta'_j = 1$. As $\varepsilon'_j \le 2$, we also have $w''_j \ge 1$ and hence $\delta''_j = 1$.

We proved that if we convert a given normal loop datum to a band datum and then convert it back to a loop datum, it returns to itself. □

Remark 4.1.7. *We could have chosen different forms in the conversion formula from band words to loop words. To be more specific, we can take any sign word $\delta^* := \delta^*(w) \in \{0,1\}^{3\tau}$ in Definition 4.1.1 satisfying the following:*

- $\delta^*_j = 1 \quad$ *if* $\; w_j > 0$,
- $\delta^*_j = 0 \quad$ *if* $\; w_j < 0$,
- *if $w_j = 0$ and $\delta^*_j < \delta^*_{j+1}$, then the first non-zero entry to the left of w_{j+1} (exists and) is negative and the first non-zero entry to the right of w_j (exists and) is positive,*
- *if $w_j = 0$ and $\delta^*_j > \delta^*_{j+1}$, then the first non-zero entry to the left of w_{j+1} (exists and) is positive and the first non-zero entry to the right of w_j (exists and) is negative.*

Then we can prove that loop words obtained from the same band word should be equivalent to each other, no matter which sign word $\delta^*(w)$ is used. In this case, all converted loop words satisfy the following 'quasi-normal' *conditions:*

- *any subword of the form $(a,0,b)$ in w' satisfies $a \le -1$, $b \ge 1$ or $a \ge 1$, $b \le -1$ or $a, b \ge 1$,*
- *any subword of the form $(a,1,b)$ in w' satisfies $a \ge 2$, $b \le 0$ or $a \le 0$, $b \ge 2$ or $a, b \le 0$,*
- *w' has no subword of the form $(0,-1,-1,\dots,-1,0)$, and*
- *w' has no subword of the form $(1,2,2,\dots,2,1)$.*

The conversion formula from 'quasi-normal' loop words to band words remains the same as in Definition 4.1.4. If one convert a given band word to a loop word using any sign word and then convert it back to a band word, it returns to itself. Conversely, if one convert a given 'quasi-normal' loop word to a band word and then convert it back to a loop word using some sign word, it is equivalent to the original one.

Finally, we can state our main theorem.

Theorem 4.1.8. *The following diagram commutes.*

$$\begin{tikzcd}
 & \varphi\left(w',\lambda,1\right)\in \underline{\mathrm{MF}}(xyz) \arrow[dl, shift right, "\substack{\text{coker}\\ \textit{(Eisenbud, 1980)}}"', "\cong"] & \\
M(w,\lambda,1)\in \underline{\mathrm{CM}}(A) \arrow[ur, shift right, "\textit{resolve}"'] & & WF(\mathscr{P})\ni L\left(w',\lambda',1\right) \arrow[ul, "\cong"', "\textit{localized mirror functor}"] \\
(w,\lambda,1)\ \textit{band data} \arrow[u, "\substack{\textit{classification Thm}\\ \textit{(Burban-Drozd)}}"] \arrow[rr, leftrightarrow, "\textit{conversion formula}"] & & \textit{loop data}\ \left(w',\lambda',1\right) \arrow[u, "\textit{classify objects}"']
\end{tikzcd}$$

Namely, given a non-degenerate band datum $(w,\lambda,1)$ and the corresponding loop datum $\left(w',\lambda',1\right)$,

(1) $M(w,\lambda,1)$ corresponds to $\varphi\left(w',\lambda,1\right)$ under Eisenbud's theorem, i.e.,

$$M(w,\lambda,1)\cong \operatorname{coker}\underline{\varphi}\left(w',\lambda,1\right) \quad \textit{in } \underline{\mathrm{CM}}(A),$$

(2) $L\left(w',\lambda',1\right)$ corresponds to $\varphi\left(w',\lambda,1\right)$ under the localized mirror functor, i.e.,

$$\mathscr{F}^{\mathbb{L}}\left(L\left(w',\lambda',1\right)\right)\cong\varphi\left(w',\lambda,1\right) \quad \textit{in } \underline{\mathrm{MF}}(xyz).$$

Proof. We will prove (1) and (2) in Section 3.6 and 2.4, respectively. See Theorem 3.6.1 and 2.4.1. □

Remark 4.1.9. *The above theorem is between the band data and the loop data. On the other hand, there are indecomposable maximal Cohen-Macaulay modules that are not locally free on the punctured spectrum, and they correspond to the string data (Burban-Drozd). On the mirror side, these correspond to non-compact Lagrangians which start and end at punctures (i.e. Lagrangian immersions of $\mathbb{R}$). Most of the proof in this paper would carry over to these cases without much difficulty, and hence we do not discuss them.*

Remark 4.1.10. *In the sequel, we will generalize the above theorem in the following sense.*

(1) The above theorem is for the multiplicity 1 ($\mu=1$). We will prove the corresponding statements for the higher multiplicity, namely for (w,λ,μ) with $\mu\geq 2$ as well. The mirror Lagrangian will be given by twisted complexes (of length μ) of a single Lagrangian $L(w',\lambda,1)$.

(2) The above theorem holds even for periodic words. (Recall that the definition of the band and loop data excludes periodic ones). They correspond to some direct sums of indecomposable objects corresponding to non-periodic words in each category.

(3) The theorem holds for degenerate band/loop data. For degenerate cases, the correspondences become somewhat subtle, and we will explain how to handle them in the sequel.

Theorem 4.1.11. *The following diagram commutes.*

$$
\begin{tikzcd}
 & \varphi\left(w',\lambda,\mu\right) \in \underline{\mathrm{MF}}\left(xyz\right) \arrow[dl, shift right, "\substack{\mathrm{coker} \\ \textit{(Eisenbud, 1980)}}"', "\cong"] & \\
M\left(w,\lambda,\mu\right) \in \underline{\mathrm{CM}}\left(A\right) \arrow[ur, shift right, "\textit{resolve}"'] & & WF\left(\mathscr{P}\right) \ni L\left(w',\lambda',\mu\right) \arrow[ul, "\cong", "\textit{localized mirror functor}"'] \\
\left(w,\lambda,\mu\right)\ \textit{band data} \arrow[u, "\substack{\textit{classification Thm} \\ \textit{(Burban-Drozd)}}"] \arrow[rr, leftrightarrow, "\textit{conversion formula}"] & & \textit{loop data}\ \left(w',\lambda',\mu\right) \arrow[u, "\textit{classify objects}"']
\end{tikzcd}
$$

Namely, given a non-degenerate band datum $\left(w,\lambda,\mu\right)$ *and the corresponding loop datum* $\left(w',\lambda',\mu\right)$*,*
(1) $M\left(w,\lambda,\mu\right)$ *corresponds to* $\varphi\left(w',\lambda,\mu\right)$ *under Eisenbud's theorem, i.e.,*

$$M\left(w,\lambda,\mu\right) \cong \mathrm{coker}\,\underline{\varphi}\left(w',\lambda,\mu\right) \quad \textit{in } \underline{\mathrm{CM}}\left(A\right),$$

(2) $L\left(w',\lambda',\mu\right)$ *corresponds to* $\varphi\left(w',\lambda,\mu\right)$ *under the localized mirror functor, i.e.,*

$$\mathscr{F}^{\mathbb{L}}\left(L\left(w',\lambda',\mu\right)\right) \cong \varphi\left(w',\lambda,\mu\right) \quad \textit{in } \underline{\mathrm{MF}}\left(xyz\right).$$

Corollary 4.1.12. *There is a one-to-one correspondence*

$$\left\{\textit{indecomposable Cohen-Macaulay modules over } A\right\} / \sim_{\textit{isomorphism}}$$

$$\xleftrightarrow{1:1} \left\{\textit{indecomposable flat connections on vector bundles over geodesics in } \mathscr{P}\right\} / \sim_{\textit{gauge equivalence}}.$$

4.2 Applications

In this section, we give two applications (Proposition 4.2.1 and 4.2.2) of the correspondence between Lagrangians and Cohen-Macaulay modules. It relates algebraic operations of matrices or modules with symmetry of mirror geometry.

Pair-of-pants surface $\mathscr{P}$ has many obvious symmetries. Here we focus on its $\mathbb{Z}_2$-symmetry by flipping back and forth. We can invert a loop/arc L under this symmetry. Then its mirror matrix factorization $\mathscr{F}^{\mathbb{L}}(L) = \left(\Phi^{\mathbb{L}}(L), \Psi^{\mathbb{L}}(L)\right)$ is transposed into $\mathscr{F}^{\mathbb{L}}(L)^T = \left(\Phi^{\mathbb{L}}(L)^T, \Psi^{\mathbb{L}}(L)^T\right)$. Taking cokernel of $\Phi^{\mathbb{L}}(L)^T$ is then corresponds to the dual of coker $\Phi^{\mathbb{L}}(L)$. Therefore we have the following proposition.

Proposition 4.2.1. *Under homological mirror symmetry, the following operations are compatible with each other:*

- *Inversion of Lagrangians in* $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$,
- *Transpose of matrix factorizations in* $\underline{\mathrm{MF}}(xyz)$
- *Dual of Cohen-Macaulay modules in* $\underline{\mathrm{CM}}(A)$

By reversing orientation of a loop/arc L, its mirror matrix factors change their position so that

$$\mathscr{F}^{\mathbb{L}}(L[1]) = \left(\Psi^{\mathbb{L}}(L), \Phi^{\mathbb{L}}(L)\right).$$

Because cokernels of two complementary matrix factors are Auslander-Reiten translation of each other, we have the following proposition.

Proposition 4.2.2. *Under homological mirror symmetry, the following operations are compatible with each other:*

- *Orientation-reversing of Lagrangians in* $D^{\pi}(W\,\mathrm{Fuk}(\mathscr{P}))$,
- *Position-changing of matrix factorizations in* $\underline{\mathrm{MF}}(xyz)$
- *Auslander-Reiten translation of Cohen-Macaulay modules in* $\underline{\mathrm{CM}}(A)$

Chapter 5

Degenerate Vector Bundles over Degenerate Cusp Singularities

5.1 Burban-Drozd Triples and Decorated Quiver Representations

Let $A := \Bbbk[[x,y,z]]/(xyz)$ be a degenerate cusp singularity and $R := \Bbbk[[x_2,y_1]] \times \Bbbk[[y_2,z_1]] \times \Bbbk[[z_2,x_1]]$ be its normalization with an inclusion map given by $x \mapsto x_1 + x_2$, $y \mapsto y_1 + y_2$ and $z \mapsto z_1 + z_2$. The construction in this thesis is also applicable to $A_n := \Bbbk[[x_1,\ldots,x_n]]/\left(x_i x_j \,\middle|\, |i-j| \geq 2,\ i,j \in \mathbb{Z}/n\right)$ and its normalization $R_n := \Bbbk[[u_1,v_1]] \times \cdots \times \Bbbk[[u_n,v_n]]$, but we focus on A_3 case here for simplicity.

Definition 5.1.1. *A* Burban-Drozd triple $\left(\tilde{M}, V, \theta\right)$ *on A consists of*

- *Cohen-Macaulay R-module $\tilde{M}$,*
- *Noetherian $Q\left(\bar{A}\right)$-module V, and*
- *$Q\left(\bar{R}\right)$-module epimorphism $\theta : Q\left(\bar{R}\right) \otimes_{Q(\bar{A})} V \to Q\left(\bar{R}\right) \otimes_R \tilde{M}$ which (canonically) induces a $Q\left(\bar{A}\right)$-module monomorphism $V \to Q\left(\bar{R}\right) \otimes_{Q(\bar{A})} V \to Q\left(\bar{R}\right) \otimes_R \tilde{M}$.*

A morphism $(\varphi, \psi) : \left(\tilde{M}, V, \theta\right) \to \left(\tilde{M}', V', \theta'\right)$ *between triples consists of*

- *R-module map $\varphi : \tilde{M} \to \tilde{M}'$, and*
- *$Q\left(\bar{A}\right)$-module map $\psi : V \to V'$*

such that the following diagram is commutative:

$$\begin{tikzcd} Q\left(\bar{R}\right) \otimes_{Q(\bar{A})} V \arrow[r, "\theta"] \arrow[d, "\mathbb{1}\otimes\psi"] & Q\left(\bar{R}\right) \otimes_R \tilde{M} \arrow[d, "\mathbb{1}\otimes\varphi"] \\ Q\left(\bar{R}\right) \otimes_{Q(\bar{A})} V' \arrow[r, "\theta'"] & Q\left(\bar{R}\right) \otimes_R \tilde{M}' \end{tikzcd}.$$

We denote by $\mathrm{Tri}(A)$ *the* category of Burban-Drozd triples *on A.*

Definition 5.1.2. *A* representation $\Theta = \left\{\theta^{\mathrm{x}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{zx}}, \theta^{\mathrm{x}}_{\mathrm{zx}}\right\}$ *on the decorated quiver* Q_A *is a collection of* $\Bbbk((t))$*-vector spaces and* $\Bbbk((t))$*-linear maps*

$$\begin{tikzcd}
 & & \Bbbk((t))^{l_{\mathrm{z}}} \arrow[dll, "\theta^{\mathrm{z}}_{\mathrm{zx}}"'] \arrow[drr, "\theta^{\mathrm{z}}_{\mathrm{yz}}"] & & \\
\Bbbk((t))^{d_{\mathrm{zx}}} & & & & \Bbbk((t))^{d_{\mathrm{yz}}} \\
 & & \Theta & & \\
\Bbbk((t))^{l_{\mathrm{x}}} \arrow[uu, "\theta^{\mathrm{x}}_{\mathrm{zx}}"] \arrow[drr, "\theta^{\mathrm{x}}_{\mathrm{xy}}"'] & & & & \Bbbk((t))^{l_{\mathrm{y}}} \arrow[uu, "\theta^{\mathrm{y}}_{\mathrm{yz}}"'] \arrow[dll, "\theta^{\mathrm{y}}_{\mathrm{xy}}"] \\
 & & \Bbbk((t))^{d_{\mathrm{xy}}} & &
\end{tikzcd} \tag{5.1.1}$$

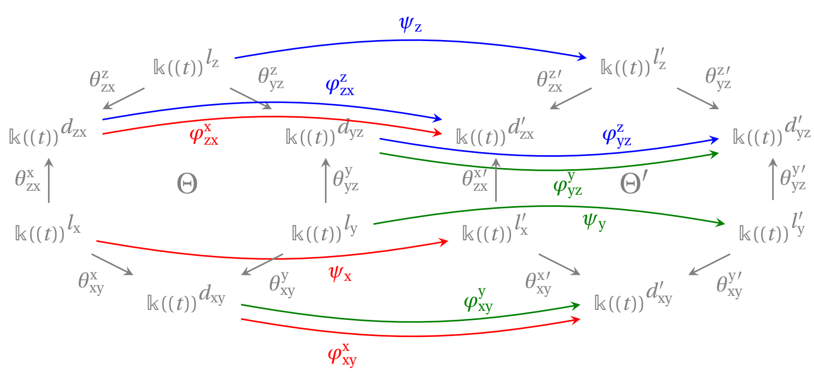
such that

$$- \ \theta^{\bullet}_{\blacktriangle\bullet} : \Bbbk((t))^{l_\bullet} \to \Bbbk((t))^{d_{\blacktriangle\bullet}},\ \theta^{\bullet}_{\bullet\blacksquare} : \Bbbk((t))^{l_\bullet} \to \Bbbk((t))^{d_{\bullet\blacksquare}} \ \text{ are surjective,} \tag{5.1.2}$$

$$- \ \begin{pmatrix} \theta^{\bullet}_{\blacktriangle\bullet} \\ \theta^{\bullet}_{\bullet\blacksquare} \end{pmatrix} : \Bbbk((t))^{l_\bullet} \to \Bbbk((t))^{d_{\blacktriangle\bullet}+d_{\bullet\blacksquare}} \ \text{ is injective} \tag{5.1.3}$$

for $(\blacktriangle, \bullet, \blacksquare) = (\mathrm{x}, \mathrm{y}, \mathrm{z}), (\mathrm{y}, \mathrm{z}, \mathrm{x}),$ *or* $(\mathrm{z}, \mathrm{x}, \mathrm{y})$.

A morphism $\Phi = \{\varphi_{\mathrm{xy}}, \varphi_{\mathrm{yz}}, \varphi_{\mathrm{zx}}, \psi_{\mathrm{x}}, \psi_{\mathrm{y}}, \psi_{\mathrm{z}}\} : \Theta \to \Theta'$ *between representations is a collection of* $\Bbbk[[u,v]]$*-linear maps* $\varphi_{\bullet\blacksquare} : \Bbbk[[u,v]]^{d_{\bullet\blacksquare}} \to \Bbbk[[u,v]]^{d'_{\bullet\blacksquare}}$ *and* $\Bbbk((t))$*-linear maps* $\psi_\bullet : \Bbbk((t))^{l_\bullet} \to \Bbbk((t))^{l'_\bullet}$



satisfying the 6 *commuting rules*

$$- \ \varphi^{\bullet}_{\bullet\blacksquare} \circ \theta^{\bullet}_{\bullet\blacksquare} = \theta^{\bullet\prime}_{\bullet\blacksquare} \circ \psi_\bullet \quad \text{where } \varphi^{\bullet}_{\bullet\blacksquare} := \varphi_{\bullet\blacksquare}(t,0) : \Bbbk((t))^{d_{\bullet\blacksquare}} \to \Bbbk((t))^{d'_{\bullet\blacksquare}}, \tag{5.1.4}$$

$$- \ \varphi^{\bullet}_{\blacktriangle\bullet} \circ \theta^{\bullet}_{\blacktriangle\bullet} = \theta^{\bullet\prime}_{\blacktriangle\bullet} \circ \psi_\bullet \quad \text{where } \varphi^{\bullet}_{\blacktriangle\bullet} := \varphi_{\blacktriangle\bullet}(0,t) : \Bbbk((t))^{d_{\blacktriangle\bullet}} \to \Bbbk((t))^{d'_{\blacktriangle\bullet}} \tag{5.1.5}$$

for $(\blacktriangle, \bullet, \blacksquare) = (\mathrm{x}, \mathrm{y}, \mathrm{z}), (\mathrm{y}, \mathrm{z}, \mathrm{x}),$ *or* $(\mathrm{z}, \mathrm{x}, \mathrm{y})$.

We denote by $\operatorname{Rep}(Q_A)$ *the* category of representation *on* Q_A.

Remark 5.1.3. *Our definition of* $\operatorname{Rep}(Q_A)$ *is almost the same as* $\operatorname{Rep}(\mathfrak{X}_A)$ *in [BD17], but the morphism rules at targets are slightly different. Namely, in* $\operatorname{Rep}(\mathfrak{X}_A)$*, the maps* $\varphi_{\bullet\blacksquare} : \left(\Bbbk[[u,v]]/(uv)\right)^{d_{\bullet\blacksquare}} \to \left(\Bbbk[[u,v]]/(uv)\right)^{d'_{\bullet\blacksquare}}$

are $\Bbbk[[u,v]]\big/(uv)$*-linear maps which satisfy a similar commuting relation as above. So we have a natural functor* $\mathrm{Rep}(Q_A) \to \mathrm{Rep}(\mathfrak{X}_A)$ *which is full and essentially surjective but not faithful.*

This alternative definition is necessary in our next discussion where we will define functors (in particular to send morphisms) from $\mathrm{Rep}(Q_A)$ *to other categories and derive their equivalences. In the meantime, the functor* $\mathrm{Rep}(Q_A) \to \mathrm{Rep}(\mathfrak{X}_A)$ *preserves indecomposability and isomorphism classes of objects, so the classification result on* $\mathrm{Rep}(\mathfrak{X}_A)$ *in [BD17] directly applies to* $\mathrm{Rep}(Q_A)$ *as well.*

Remark 5.1.4. *Given a representation* $\Theta = \left\{\theta^{\mathrm{x}}_{\mathrm{zx}}, \theta^{\mathrm{x}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{zx}}\right\}$ *on* Q_A *as in Diagram 5.1.1, we associate a Burban-Drozd triple* $\left(\tilde{M}, V, \theta\right)$ *on* A *as follows:*

- $V := \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}}$
- $\tilde{M} := \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$,
- $\theta := \theta^{\mathrm{x}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{zx}} \times \theta^{\mathrm{x}}_{\mathrm{zx}}$
  $: \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((x))^{l_{\mathrm{x}}}$
  $\to \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}$.

Given a morphism $\Phi = \left\{\varphi_{\mathrm{xy}}, \varphi_{\mathrm{yz}}, \varphi_{\mathrm{zx}}, \psi_{\mathrm{x}}, \psi_{\mathrm{y}}, \psi_{\mathrm{z}}\right\} : \Theta \to \Theta'$ *between representations, we associate a morphism* $\left(\varphi, \psi\right) : \left(\tilde{M}, V, \theta\right) \to \left(\tilde{M}', V', \theta'\right)$ *between triples as follows:*

- $\varphi := \varphi_{\mathrm{xy}} \times \varphi_{\mathrm{yz}} \times \varphi_{\mathrm{zx}} : \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}} \to \Bbbk[[x,y]]^{d'_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d'_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d'_{\mathrm{zx}}}$, *and*
- $\psi := \psi_{\mathrm{x}} \times \psi_{\mathrm{y}} \times \psi_{\mathrm{z}} : \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \to \Bbbk((x))^{l'_{\mathrm{x}}} \times \Bbbk((y))^{l'_{\mathrm{y}}} \times \Bbbk((z))^{l'_{\mathrm{z}}}$.

Then we can check that those assignments indeed define a functor $\mathrm{Rep}(Q_A) \to \mathrm{Tri}(A)$. *We can also show that this induces an equivalence of categories. Indeed, any Cohen-Macaulay* R*-module* $\tilde{M}$ *and Noetherian* $Q\left(\bar{A}\right)$*-module* V *can be expressed in the above form for some* $d_{\mathrm{xy}}, d_{\mathrm{yz}}, d_{\mathrm{zx}}, l_{\mathrm{x}}, l_{\mathrm{y}}, l_{\mathrm{z}} \in \mathbb{Z}_{\geq 0}$*, and then any* $Q\left(\bar{R}\right)$*-module map* θ *decomposes into above form. It is also similar for morphisms* $\left(\varphi, \psi\right)$*. This gives a (non-canonical) quasi-inverse of the above functor.*

5.2 Geometry of $X = \mathrm{Spec}(A)$

Geometrically, the affine scheme $X = \mathrm{Spec}(A)$ is a gluing of three '(infinitesimal) planes' $\mathrm{Spec}\left(\Bbbk[[x,y]]\right)$, $\mathrm{Spec}\left(\Bbbk[[y,z]]\right)$, $\mathrm{Spec}\left(\Bbbk[[z,x]]\right)$ along three '(infinitesimal) axes' $\mathrm{Spec}\left(\Bbbk((x))\right)$, $\mathrm{Spec}\left(\Bbbk((y))\right)$, $\mathrm{Spec}\left(\Bbbk((z))\right)$ in the following sense. We consider six 'inclusion maps' from axes to planes, for example, $\widehat{\iota^{\mathrm{x}}_{\mathrm{xy}}} : \mathrm{Spec}\left(\Bbbk((x))\right) \to \mathrm{Spec}\left(\Bbbk[[x,y]]\right)$ induced from the dual ring homomorphisms $\widetilde{\iota^{\mathrm{x}}_{\mathrm{xy}}} : \Bbbk[[x,y]] \to \Bbbk((x))$, $f(x,y) \mapsto f(x,0)$, etc. Then they form a hexagon diagram $\widehat{I}$ as in top left in Figure 5.1.

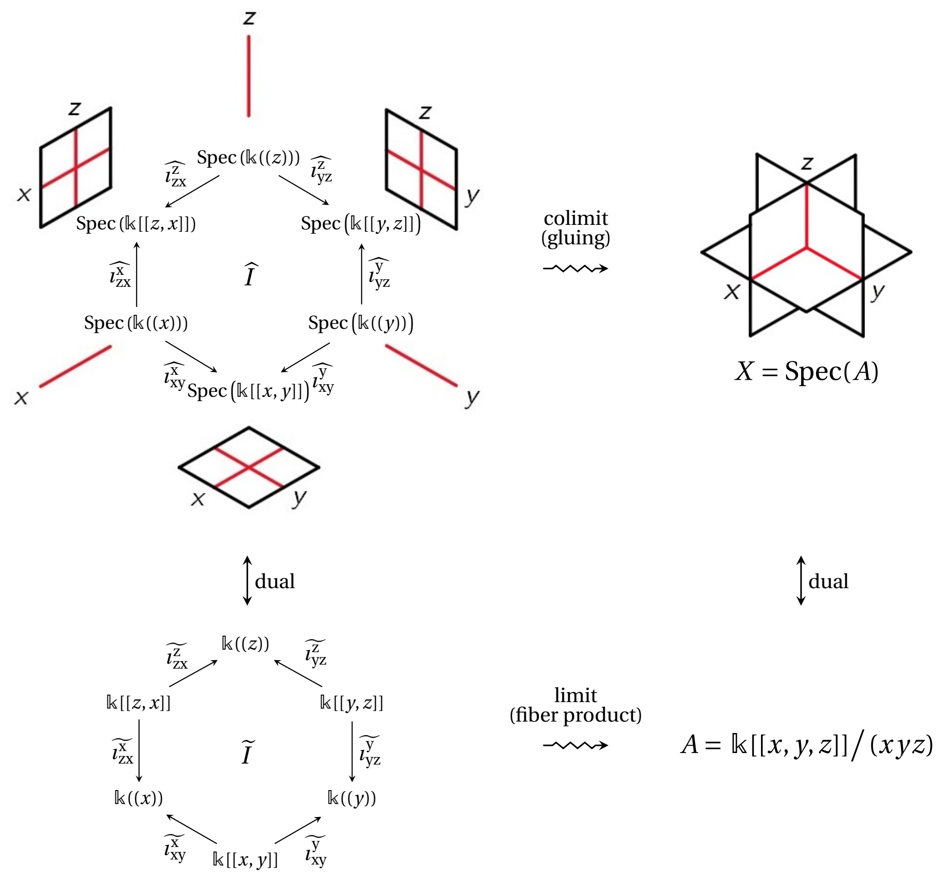


Figure 5.1: Geometric Construction of $X = \mathrm{Spec}(A)$

Proposition 5.2.1. *(1) The affine scheme $X = \mathrm{Spec}(A)$ is the (categorical) colimit of the hexagon diagram $\widehat{I}$ of affine schemes in top left in Figure 5.1.*

(2) The commutative ring $A = \Bbbk[[x,y,z]]/(xyz)$ is the (categorical) limit of the hexagon diagram $\widetilde{I}$ of commutative rings in bottom left in Figure 5.1, which can be written as

$$A \cong \left\{ (f,g,h) \in \Bbbk[[x,y]] \times \Bbbk[[y,z]] \times \Bbbk[[z,x]] \,\middle|\, h(0,x) = f(x,0),\ f(0,y) = g(y,0),\ g(0,z) = h(z,0) \right\}.$$

5.3 Degenerate Vector Bundles

In this section, we introduce the notion of *degenerate vector bundles.*

Definition 5.3.1. *Given a representation* Θ *on* Q_A *as in 5.1.1, we construct a hexagon diagram* $\widehat{\Theta}$ *of affine* X*-schemes as shown in Figure 5.2.*

- *At* 6 *vertices, we attach trivial vector bundles*

$$\begin{array}{ccc} \operatorname{Spec}\left(\Bbbk[[x,y]]\right)\times\mathbb{A}^{d_{\mathrm{xy}}}, & \operatorname{Spec}\left(\Bbbk[[y,z]]\right)\times\mathbb{A}^{d_{\mathrm{yz}}}, & \operatorname{Spec}\left(\Bbbk[[z,x]]\right)\times\mathbb{A}^{d_{\mathrm{zx}}}, \\ \operatorname{Spec}\left(\Bbbk((x))\right)\times\mathbb{A}^{l_{\mathrm{x}}}, & \operatorname{Spec}\left(\Bbbk((y))\right)\times\mathbb{A}^{l_{\mathrm{y}}}, & \operatorname{Spec}\left(\Bbbk((z))\right)\times\mathbb{A}^{l_{\mathrm{z}}}, \end{array}$$

  whose ranks are determined by the dimension of $\Bbbk((t))$*-vector spaces in* Θ*. (Figure 5.2 illustrates the case where* $\left(d_{\mathrm{xy}},d_{\mathrm{yz}},d_{\mathrm{zx}},l_{\mathrm{x}},l_{\mathrm{y}},l_{\mathrm{z}}\right)=(0,1,1,1,1,2)$*.) They are also affine* X*-schemes via the natural inclusions* $\operatorname{Spec}\left(\Bbbk[[\bullet,\blacksquare]]\right)\to X$ *and* $\operatorname{Spec}\left(\Bbbk((\bullet))\right)\to X$*.*

- *At* 6 *arrows, we attach vector bundle maps (with respect to the inclusion maps between base spaces in* $\widehat{I}$*) determined by* $\Bbbk((t))$*-linear maps in* Θ*. For example,*

$$\widehat{\theta^{\mathrm{x}}_{\mathrm{xy}}}:\operatorname{Spec}\left(\Bbbk((x))\right)\times\mathbb{A}^{l_{\mathrm{x}}}\to\operatorname{Spec}\left(\Bbbk[[x,y]]\right)\times\mathbb{A}^{d_{\mathrm{xy}}}$$

  is induced from the dual ring homomorphism

$$\begin{aligned} \widetilde{\theta^{\mathrm{x}}_{\mathrm{xy}}}:\Bbbk[[x,y]]\left[s^1_{\mathrm{xy}},\dots,s^{d_{\mathrm{xy}}}_{\mathrm{xy}}\right] &\to \Bbbk((x))\left[r^1_{\mathrm{x}},\dots,r^{l_{\mathrm{x}}}_{\mathrm{x}}\right] \\ F\left(x,y,s^1_{\mathrm{xy}},\dots,s^{d_{\mathrm{xy}}}_{\mathrm{xy}}\right) &\mapsto F\left(x,0,\theta^{\mathrm{x}}_{\mathrm{xy}}(x)\left(r^1_{\mathrm{x}},\dots,r^{l_{\mathrm{x}}}_{\mathrm{x}}\right)\right) \end{aligned}$$

  and so on. They are also X*-morphisms between* X*-schemes.*

  Then we define the degenerate vector bundle $\mathscr{E}(\Theta)$ *over* X *associated to* Θ *as the colimit*

$$\mathscr{E}(\Theta):=\operatorname{colim}\widehat{\Theta}$$

  of the diagram $\widehat{\Theta}$ *in the category of affine* X*-schemes. In general, we call an* X*-scheme which is isomorphic to* $\mathscr{E}(\Theta)$ *for some* $\Theta\in\operatorname{Rep}(Q_A)$ *as a* degenerate vector bundle *over* X*.*

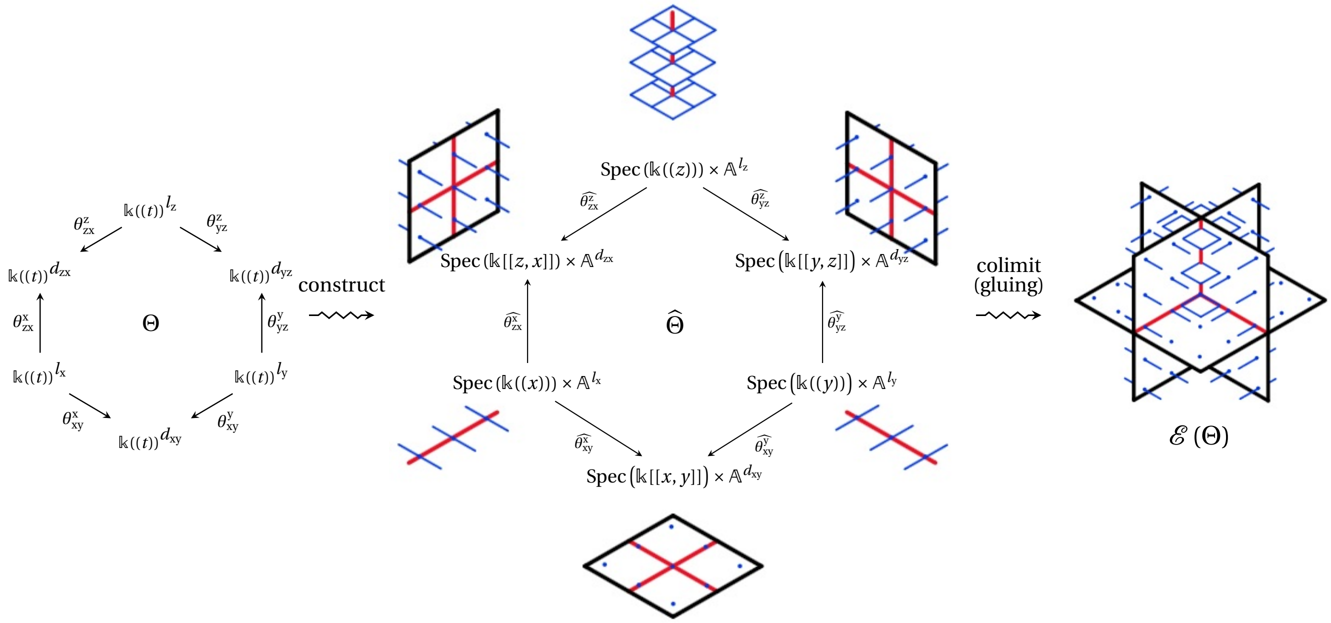


Figure 5.2: Degenerate Vector Bundle $\mathscr{E}(\Theta)$ associated to the Representation Θ

Note that the natural projection maps of the trivial vector bundles form the hexagonal prism diagram as shown in Figure 5.3. The fact that the attached maps above are vector bundle maps respecting the diagram $\widetilde{I}$ ensures that all vertical rectangles commute. Via natural inclusions $\operatorname{Spec}(\Bbbk[[\bullet,\blacksquare]]) \to X$ and $\operatorname{Spec}(\Bbbk((\bullet))) \to X$, we can assume that they have a common target X. Then the universal property of the colimit yields the *projection* map

$$\widehat{\pi} : \mathscr{E}(\Theta) \to X.$$

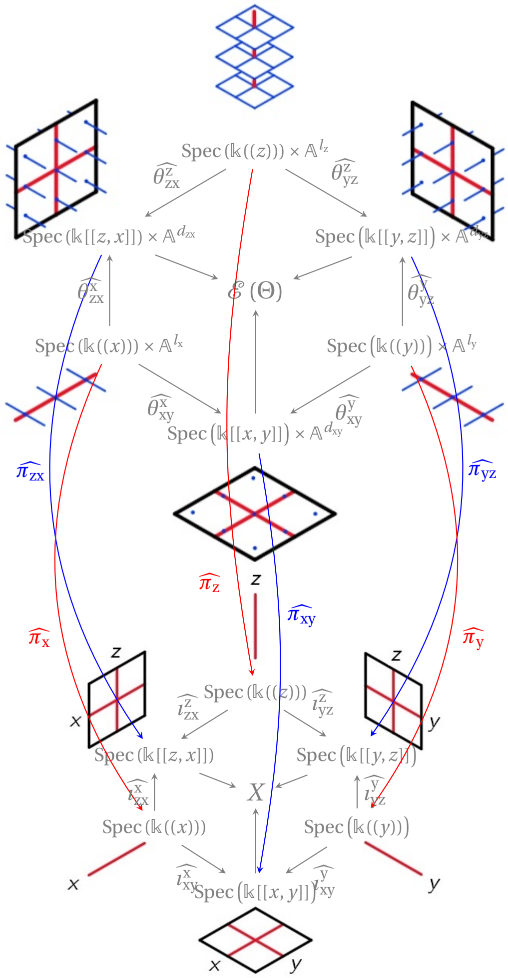


Figure 5.3: Projections $\widehat{\pi_{xy}}, \widehat{\pi_{yz}}, \widehat{\pi_{zx}}$ and $\widehat{\pi_x}, \widehat{\pi_y}, \widehat{\pi_z}$ defining the projection $\widehat{\pi} : \mathscr{E}(\Theta) \to X$

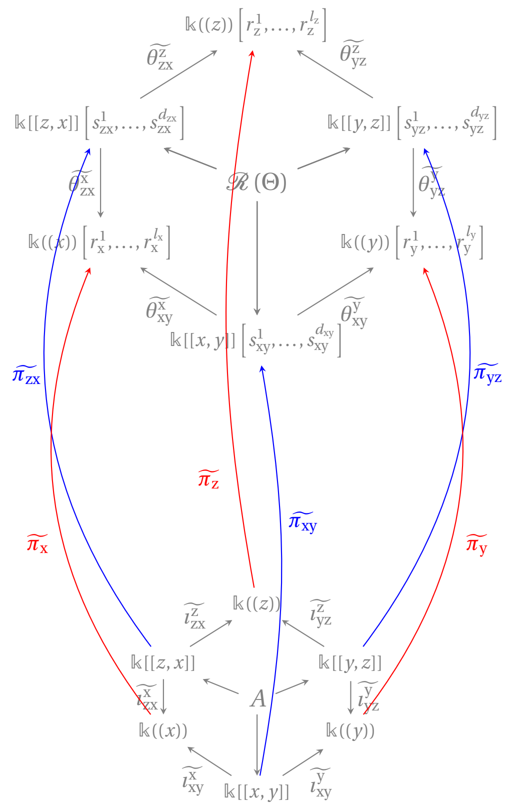


Figure 5.4: Dual projections $\widetilde{\pi_{xy}}, \widetilde{\pi_{yz}}, \widetilde{\pi_{zx}}$ and $\widetilde{\pi_x}, \widetilde{\pi_y}, \widetilde{\pi_z}$ defining the dual projection $\widetilde{\pi} : A \to \mathscr{R}(\Theta)$

Remark 5.3.2. *Conceptually, we can view the degenerate vector bundle $\mathscr{E}(\Theta)$ as a gluing of three vector bundles over 'planes' along three vector bundles over 'lines'. A 'point' $\left(x, \left(r_x^1, \ldots, r_x^{l_x}\right)\right)$ on the 'fiber' over x in the 'x-axis', for example, is identified with the 'point' $\left((x,0), \theta_{xy}^x(x)\left(r_x^1, \ldots, r_x^{l_x}\right)\right)$ on the 'fiber' over $(x,0)$ in the 'xy-plane' and at the same time with the 'point' $\left((0,x), \theta_{zx}^z(x)\left(r_x^1, \ldots, r_x^{l_x}\right)\right)$ on the 'fiber' over $(0,x)$ in the 'zx-plane'. As it has different ranks on each component, it can not be a vector bundle in the traditional sense in most cases.*

We can provide a concrete realization of $\mathscr{E}(\Theta)$ by using the dual language of commutative rings. Specifically, consider the hexagon diagram $\widetilde{\Theta}$ of commutative rings as shown in Figure 5.5, which is dual to the diagram $\widehat{\Theta}$ of affine schemes. Therefore, denoting the limit of $\widetilde{\Theta}$ as

$$\mathscr{R}(\Theta) := \lim \widetilde{\Theta},$$

we have $\mathscr{E}(\Theta) = \operatorname{Spec}\mathscr{R}(\Theta)$. The ring $\mathscr{R}(\Theta)$ has a concrete description as in the next proposition.

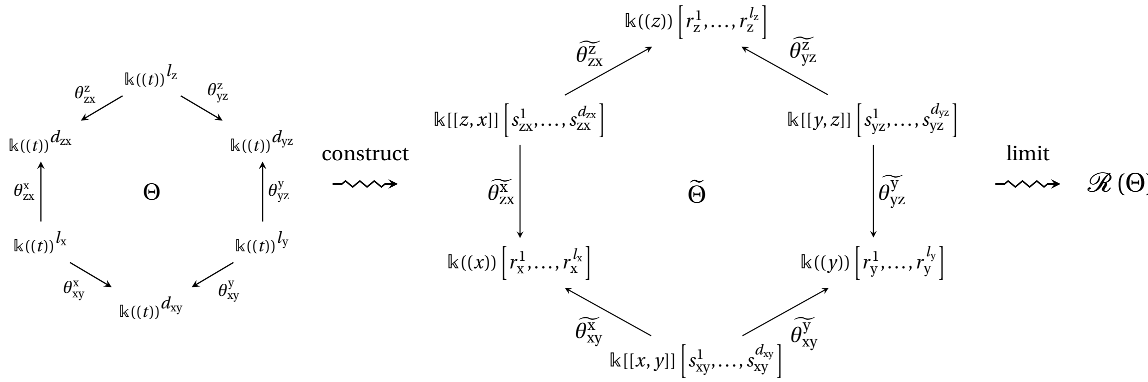


Figure 5.5: Ring $\mathscr{R}(\Theta)$ associated to the Representation Θ

Proposition 5.3.3. *The (categorical) limit of the hexagon diagram $\widetilde{\Theta}$ of commutative rings in Figure 5.5 can be written as*

$$\mathscr{R}(\Theta) = \Big\{(F, G, H) \in \Bbbk[[x,y]]\left[s_{\mathrm{xy}}^1, \dots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right] \times \Bbbk[[y,z]]\left[s_{\mathrm{yz}}^1, \dots, s_{\mathrm{yz}}^{d_{\mathrm{yz}}}\right] \times \Bbbk[[z,x]]\left[s_{\mathrm{zx}}^1, \dots, s_{\mathrm{zx}}^{d_{\mathrm{zx}}}\right]$$

$$\Big|\ \widetilde{\theta_{\mathrm{zx}}^{\mathrm{x}}}(H) = \widetilde{\theta_{\mathrm{xy}}^{\mathrm{x}}}(F),\ \widetilde{\theta_{\mathrm{xy}}^{\mathrm{y}}}(F) = \widetilde{\theta_{\mathrm{yz}}^{\mathrm{y}}}(G),\ \widetilde{\theta_{\mathrm{yz}}^{\mathrm{z}}}(G) = \widetilde{\theta_{\mathrm{zx}}^{\mathrm{z}}}(H)\Big\}.$$

Proof. (Need to write) □

Note that the natural projection maps of trivial vector bundles are induced from the dual ring homomorphisms. For example,

$$\widehat{\pi_{\mathrm{xy}}} : \mathrm{Spec}\left(\Bbbk[[x,y]]\right) \times \mathbb{A}^{d_{\mathrm{xy}}} \to \mathrm{Spec}\left(\Bbbk[[x,y]]\right) \quad \text{and} \quad \widehat{\pi_{\mathrm{x}}} : \mathrm{Spec}\left(\Bbbk((x))\right) \times \mathbb{A}^{l_{\mathrm{x}}} \to \mathrm{Spec}\left(\Bbbk((x))\right)$$

are respectively induced from the natural inclusions of rings

$$\begin{array}{rlcrl} \widetilde{\pi_{\mathrm{xy}}} : \Bbbk[[x,y]] & \to \Bbbk[[x,y]]\left[s_{\mathrm{xy}}^1, \dots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right] & \text{and} & \widetilde{\pi_{\mathrm{x}}} : \Bbbk((x)) & \to \Bbbk((x))\left[r_{\mathrm{x}}^1, \dots, r_{\mathrm{x}}^{l_{\mathrm{x}}}\right] \\ f(x,y) & \mapsto f(x,y) & & p(x) & \mapsto p(x) \end{array}$$

and so on. They form the hexagonal prism diagram in Figure 5.4, and the universal property of the limit yields the *dual projection* map between rings. Under the expression of $\mathscr{R}(\Theta)$ given in Proposition 5.3.3, it is written as

$$\widetilde{\pi} : A \to \mathscr{R}(\Theta),\ p(x,y,z) \mapsto \left(p(x,y,0), p(0,y,z), p(x,0,z)\right).$$

Next we define morphisms between degenerate vector bundles. Recall that a morphism from Θ to Θ' in $\mathrm{Rep}(Q_A)$ is a collection $\Phi = \{\varphi_{\mathrm{xy}}, \varphi_{\mathrm{yz}}, \varphi_{\mathrm{zx}}, \psi_{\mathrm{x}}, \psi_{\mathrm{y}}, \psi_{\mathrm{z}}\}$ of $\Bbbk[[u,v]]$-linear maps $\varphi_{\bullet\blacksquare} : \Bbbk[[u,v]]^{d_{\bullet\blacksquare}} \to \Bbbk[[u,v]]^{d'_{\bullet\blacksquare}}$ and $\Bbbk((t))$-linear maps $\psi_{\bullet} : \Bbbk((t))^{l_\bullet} \to \Bbbk((t))^{l'_\bullet}$ satisfying the commuting rules 5.1.4.

Definition 5.3.4. *Given a morphism $\Phi = \{\varphi_{\mathrm{xy}}, \varphi_{\mathrm{yz}}, \varphi_{\mathrm{zx}}, \psi_{\mathrm{x}}, \psi_{\mathrm{y}}, \psi_{\mathrm{z}}\} : \Theta \to \Theta'$ in $\mathrm{Rep}(Q_A)$, we construct a hexagonal prism diagram $\widehat{\Phi}$ of X-schemes from $\widehat{\Theta}$ to $\widehat{\Theta'}$ as shown in Figure 5.6.*

- *Between trivial vector bundles having the common base space* $\mathrm{Spec}\,(\Bbbk[[\bullet,\blacksquare]])$, *we attach vector bundle maps determined by* $\Bbbk[[u,v]]$*-linear maps in* Φ. *For example,*

$$\widehat{\varphi_{\mathrm{xy}}}:\mathrm{Spec}\left(\Bbbk[[x,y]]\right)\times\mathbb{A}^{d_{\mathrm{xy}}}\to\mathrm{Spec}\left(\Bbbk[[x,y]]\right)\times\mathbb{A}^{d'_{\mathrm{xy}}}$$

is induced from the dual ring homomorphism

$$\begin{aligned}\widetilde{\varphi_{\mathrm{xy}}}:\Bbbk[[x,y]]\left[s^1_{\mathrm{xy}},\dots,s^{d'_{\mathrm{xy}}}_{\mathrm{xy}}\right]&\to\Bbbk[[x,y]]\left[s^1_{\mathrm{xy}},\dots,s^{d_{\mathrm{xy}}}_{\mathrm{xy}}\right]\\ F\left(x,y,s^1_{\mathrm{xy}},\dots,s^{d'_{\mathrm{xy}}}_{\mathrm{xy}}\right)&\mapsto F\left(x,y,\varphi_{\mathrm{xy}}(x,y)\left(s^1_{\mathrm{xy}},\dots,s^{d_{\mathrm{xy}}}_{\mathrm{xy}}\right)\right)\end{aligned}$$

and so on.

- *Between trivial vector bundles having the common base space* $\mathrm{Spec}\,(\Bbbk((\bullet)))$, *we attach vector bundle maps determined by* $\Bbbk((t))$*-linear maps in* Φ. *For example,*

$$\widehat{\psi_{\mathrm{x}}}:\mathrm{Spec}\,(\Bbbk((x)))\times\mathbb{A}^{l_{\mathrm{x}}}\to\mathrm{Spec}\,(\Bbbk((x)))\times\mathbb{A}^{l'_{\mathrm{x}}}$$

is induced from the dual ring homomorphism

$$\begin{aligned}\widetilde{\psi_{\mathrm{x}}}:\Bbbk((x))\left[r^1_{\mathrm{x}},\dots,r^{l'_{\mathrm{x}}}_{\mathrm{x}}\right]&\to\Bbbk((x))\left[r^1_{\mathrm{x}},\dots,r^{l_{\mathrm{x}}}_{\mathrm{x}}\right]\\ P\left(x,r^1_{\mathrm{x}},\dots,r^{l'_{\mathrm{x}}}_{\mathrm{x}}\right)&\mapsto P\left(x,\psi_{\mathrm{x}}(x)\left(r^1_{\mathrm{x}},\dots,r^{l_{\mathrm{x}}}_{\mathrm{x}}\right)\right)\end{aligned}$$

and so on.

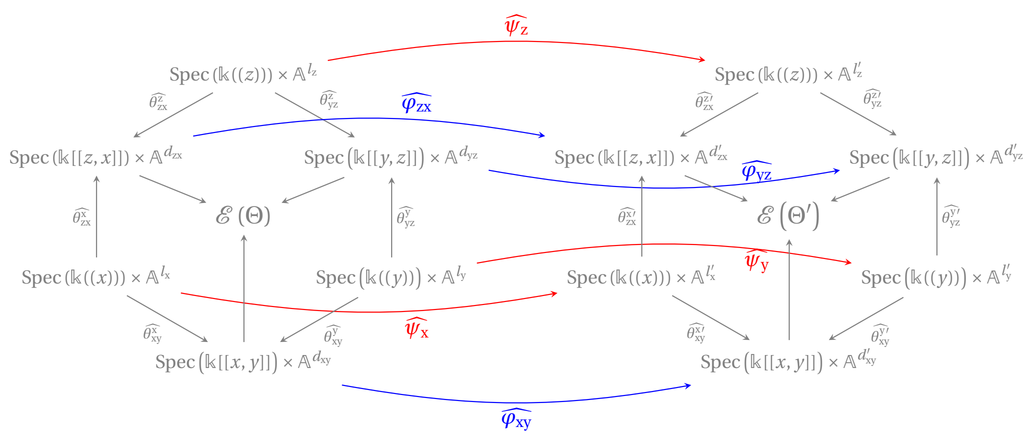


Figure 5.6: Natural transformation $\widehat{\Phi}:\widehat{\Theta}\to\widehat{\Theta'}$ associated to a morphism $\Phi:\Theta\to\Theta'$ in $\mathrm{Rep}\,(Q_A)$

Then the commuting rules of Φ *described in Equation 5.1.4 yield the commuting of* $\widehat{\Phi}$

$$-\widehat{\varphi_{\bullet\blacksquare}}\circ\widehat{\theta^{\bullet}_{\bullet\blacksquare}}=\widehat{\theta^{\bullet\prime}_{\bullet\blacksquare}}\circ\widehat{\psi_{\bullet}}\quad\text{and}\quad\widehat{\varphi_{\blacktriangle\bullet}}\circ\widehat{\theta^{\bullet}_{\blacktriangle\bullet}}=\widehat{\theta^{\bullet\prime}_{\blacktriangle\bullet}}\circ\widehat{\psi_{\bullet}}\quad\text{for}\quad(\blacktriangle,\bullet,\blacksquare)=(\mathrm{x},\mathrm{y},\mathrm{z}),(\mathrm{y},\mathrm{z},\mathrm{x}),\ \text{or}\ (\mathrm{z},\mathrm{x},\mathrm{y}),$$

corresponding to 6 *horizontal rectangles in Figure 5.6. (See also Remark 5.3.6.) The functoriality of taking colimit gives an* X*-morphism*

$$\mathscr{E}(\Phi):=\mathrm{colim}\,\widehat{\Phi}:\mathscr{E}(\Theta)\to\mathscr{E}\left(\Theta'\right)$$

between X*-schemes.*

Definition 5.3.5. *The* category of degenerate vector bundles *over X, denoted as* $\mathrm{DVB}(X)$*, is a subcategory of the category of X-schemes consisting of X-schemes which are isomorphic to $\mathscr{E}(\Theta)$ for some $\Theta \in \mathrm{Rep}(Q_A)$. We define the morphism space from $\mathscr{E}(\Theta)$ to $\mathscr{E}(\Theta')$ in* $\mathrm{DVB}(X)$ *as*

$$\mathrm{Hom}_{\mathrm{DVB}(X)}\left(\mathscr{E}(\Theta),\mathscr{E}(\Theta')\right) := \left\{\mathscr{E}(\Phi) \,\middle|\, \Phi \in \mathrm{Hom}_{\mathrm{Rep}(Q_A)}\left(\Theta,\Theta'\right)\right\}.$$

The assignments

$$\mathrm{Rep}(Q_A) \to \mathrm{DVB}(X), \quad \Theta \mapsto \mathscr{E}(\Theta)$$

in Definition 5.3.1 and

$$\mathrm{Hom}_{\mathrm{Rep}(Q_A)}\left(\Theta,\Theta'\right) \to \mathrm{Hom}_{\mathrm{DVB}(X)}\left(\mathscr{E}(\Theta),\mathscr{E}(\Theta')\right), \quad \Phi \mapsto \mathscr{E}(\Phi)$$

in Definition 5.3.4 define a covariant functor

$$\mathscr{E} : \mathrm{Rep}(Q_A) \xrightarrow{\simeq} \mathrm{DVB}(X),$$

which we call the gluing functor. *It is obvious from the definition of* $\mathrm{DVB}(X)$ *that $\mathscr{E}$ induces an equivalence of categories.*

Remark 5.3.6. *The dual construction of definition 5.3.4 can be described in the category of commutative rings. Namely, given a morphism $\Phi : \Theta \to \Theta'$ in* $\mathrm{Rep}(Q_A)$*, we construct a hexagonal prism diagram $\widetilde{\Phi} : \widetilde{\Theta'} \to \widetilde{\Theta}$ of commutative rings as shown in Figure 5.7, which is dual to the above construction $\widehat{\Phi} : \widehat{\Theta} \to \widehat{\Theta'}$.*

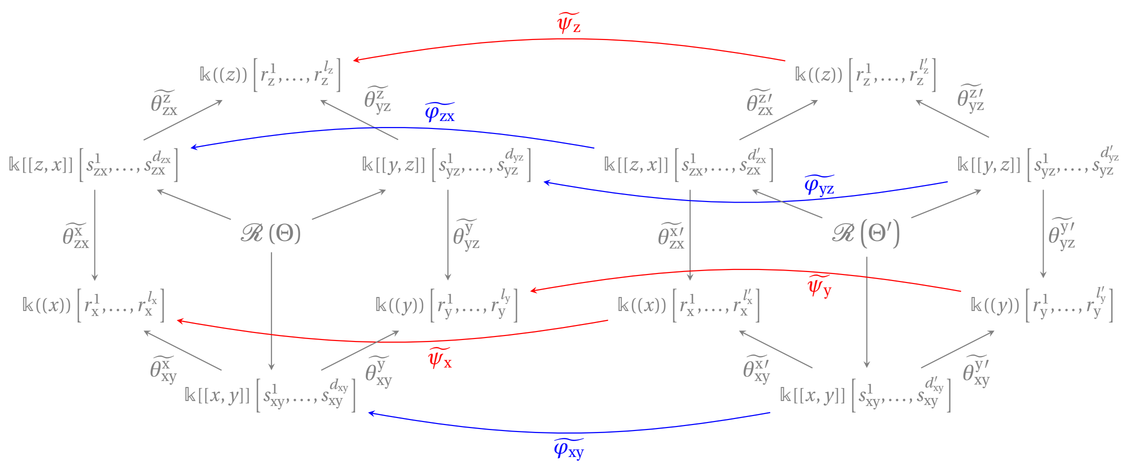


Figure 5.7: Natural transformation $\widetilde{\Phi} : \widetilde{\Theta'} \to \widetilde{\Theta}$ associated to a morphism $\Phi : \Theta \to \Theta'$ in $\mathrm{Rep}(Q_A)$

We can directly check here the commuting of $\widetilde{\Phi}$:

$$-\widetilde{\theta^{\bullet}_{\bullet\blacksquare}} \circ \widetilde{\varphi_{\bullet\blacksquare}} = \widetilde{\psi_{\bullet}} \circ \widetilde{\theta^{\bullet\prime}_{\bullet\blacksquare}} \quad \textit{and} \quad \widetilde{\theta^{\bullet}_{\blacktriangle\bullet}} \circ \widetilde{\varphi_{\blacktriangle\bullet}} = \widetilde{\psi_{\bullet}} \circ \widetilde{\theta^{\bullet\prime}_{\blacktriangle\bullet}} \quad \textit{for} \quad (\blacktriangle,\bullet,\blacksquare) = (\mathrm{x},\mathrm{y},\mathrm{z}),(\mathrm{y},\mathrm{z},\mathrm{x}),\ \textit{or}\ (\mathrm{z},\mathrm{x},\mathrm{y}),$$

corresponding to 6 *horizontal rectangles in Figure 5.7. The functoriality of taking limit gives the morphism between commutative rings, which can be written as*

$$\begin{aligned}\mathscr{R}(\Phi) := \lim \widetilde{\Phi} : \mathscr{R}(\Theta') &\to \mathscr{R}(\Theta) \\ \left(F',G',H'\right) &\mapsto \left(\widetilde{\varphi_{xy}}(F'),\widetilde{\varphi_{yz}}(G'),\widetilde{\varphi_{zx}}(H')\right).\end{aligned}$$

in the context of Proposition 5.3.3. Thus we have a contravariant functor $\mathscr{R} : \mathrm{Rep}\,(Q_A) \to \mathrm{CRing}$ *to the category of commutative rings.*

5.4 Global Sections

A section of the degenerate vector bundle $\mathscr{E}(\Theta)$ over X is an X-morphism $\widehat{s}: X \to \mathscr{E}(\Theta)$ between X-schemes, that is, a morphism of schemes satisfying $\widehat{\pi} \circ \widehat{s} = \mathrm{id}_X$. In this section we will discuss how to construct sections, and we will demonstrate that it covers all sections.

Definition 5.4.1. *To construct a section $\widehat{s}: X \to \mathscr{E}(\Theta)$ of a degenerate vector bundle $\mathscr{E}(\Theta)$, we consider a hexagonal prism diagram as shown in Figure 5.8.*

- *The sections $\widehat{s_{\mathrm{xy}}}$, $\widehat{s_{\mathrm{yz}}}$, and $\widehat{s_{\mathrm{zx}}}$ of* $\mathrm{Spec}\left(\Bbbk[[x,y]]\right) \times \mathbb{A}^{d_{\mathrm{xy}}}$, $\mathrm{Spec}\left(\Bbbk[[y,z]]\right) \times \mathbb{A}^{d_{\mathrm{yz}}}$, *and* $\mathrm{Spec}\left(\Bbbk[[z,x]]\right) \times \mathbb{A}^{d_{\mathrm{zx}}}$ *are determined by $s_{\mathrm{xy}} \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}}$, $s_{\mathrm{xy}} \in \Bbbk[[y,z]]^{d_{\mathrm{yz}}}$, and $s_{\mathrm{xy}} \in \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$, respectively. For example, a section*

$$\widehat{s_{\mathrm{xy}}}: \mathrm{Spec}\left(\Bbbk[[x,y]]\right) \to \mathrm{Spec}\left(\Bbbk[[x,y]]\right) \times \mathbb{A}^{d_{\mathrm{xy}}}$$

is induced from the dual ring homomorphism

$$\begin{aligned} \widetilde{s_{\mathrm{xy}}}: \Bbbk[[x,y]]\left[s_{\mathrm{xy}}^1, \ldots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right] &\to \Bbbk[[x,y]] \\ F\left(x, y, s_{\mathrm{xy}}^1, \ldots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right) &\mapsto F\left(x, y, s_{\mathrm{xy}}^1(x,y), \ldots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}(x,y)\right) \end{aligned}$$

where $s_{\mathrm{xy}} = \left(s_{\mathrm{xy}}^1, \ldots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right) \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}}$.

- *Similarly, the sections $\widehat{r_{\mathrm{x}}}$, $\widehat{r_{\mathrm{y}}}$, and $\widehat{r_{\mathrm{z}}}$ of* $\mathrm{Spec}\left(\Bbbk((x))\right) \times \mathbb{A}^{l_{\mathrm{x}}}$, $\mathrm{Spec}\left(\Bbbk((y))\right) \times \mathbb{A}^{l_{\mathrm{y}}}$, *and* $\mathrm{Spec}\left(\Bbbk((z))\right) \times \mathbb{A}^{l_{\mathrm{z}}}$ *are determined by $r_{\mathrm{x}} \in \Bbbk((x))^{l_{\mathrm{x}}}$, $r_{\mathrm{y}} \in \Bbbk((y))^{l_{\mathrm{y}}}$, and $r_{\mathrm{z}} \in \Bbbk((z))^{l_{\mathrm{z}}}$, respectively.*

The commuting of 6 *vertical rectangles in Figure 5.8 yields the* 6 gluing conditions

$$\begin{aligned} s_{\mathrm{zx}}(0,x) &= \theta_{\mathrm{x}}^{\mathrm{zx}}(x)\, r_{\mathrm{x}}(x), \quad & s_{\mathrm{xy}}(x,0) &= \theta_{\mathrm{x}}^{\mathrm{xy}}(x)\, r_{\mathrm{x}}(x), \\ s_{\mathrm{xy}}(0,y) &= \theta_{\mathrm{y}}^{\mathrm{xy}}(y)\, r_{\mathrm{y}}(y), \quad & s_{\mathrm{yz}}(y,0) &= \theta_{\mathrm{y}}^{\mathrm{yz}}(y)\, r_{\mathrm{y}}(y), \\ s_{\mathrm{yz}}(0,z) &= \theta_{\mathrm{z}}^{\mathrm{yz}}(z)\, r_{\mathrm{z}}(z), \quad & s_{\mathrm{zx}}(z,0) &= \theta_{\mathrm{z}}^{\mathrm{zx}}(z)\, r_{\mathrm{z}}(z). \end{aligned} \tag{5.4.1}$$

In that case, we can assume that such sections have a common target $\mathscr{E}(\Theta)$. Then the universal property of the colimit yields a section

$$\widehat{s}: X \to \mathscr{E}(\Theta)$$

of $\mathscr{E}(\Theta)$. This means that for any tuple

$$\left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}, r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right) \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}} \times \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}}$$

satisfying the gluing conditions, there exists a corresponding section $\widehat{s}$ associated with it.

Remark 5.4.2. *Conceptually, we can understand a section $\widetilde{s}$ of $\mathscr{E}(\Theta)$ associate with $\left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}, r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right)$ as a gluing of sections over individual trivial vector bundles. In view of Remark 5.3.2, the gluing conditions of these sections ensure the consistency of the sections along the three axes, as they are merged together under the gluing rule $\widehat{\Theta}$ of $\mathscr{E}(\Theta)$.*

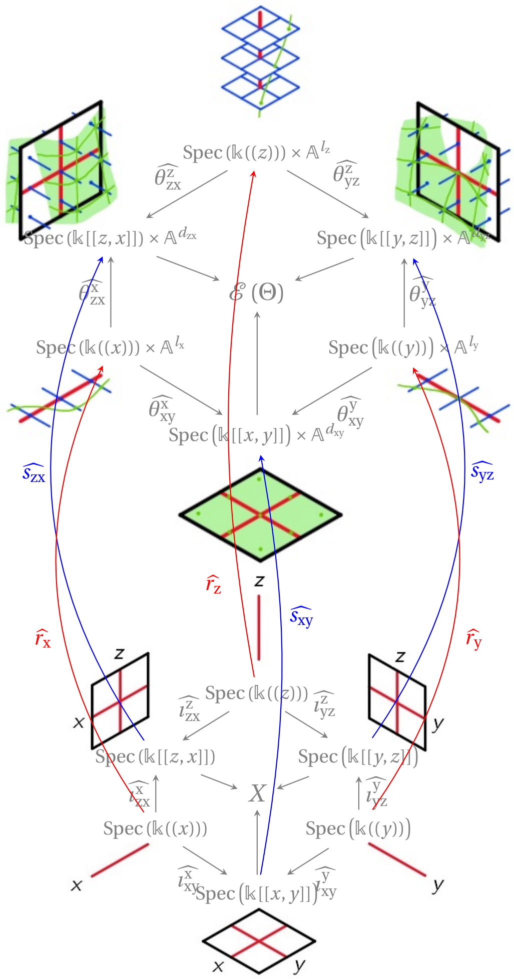


Figure 5.8: Sections $\widehat{s_{xy}}, \widehat{s_{yz}}, \widehat{s_{zx}}$ and $\widehat{r_x}, \widehat{r_y}, \widehat{r_z}$ defining a section $\widehat{s} : X \to \mathscr{E}(\Theta)$

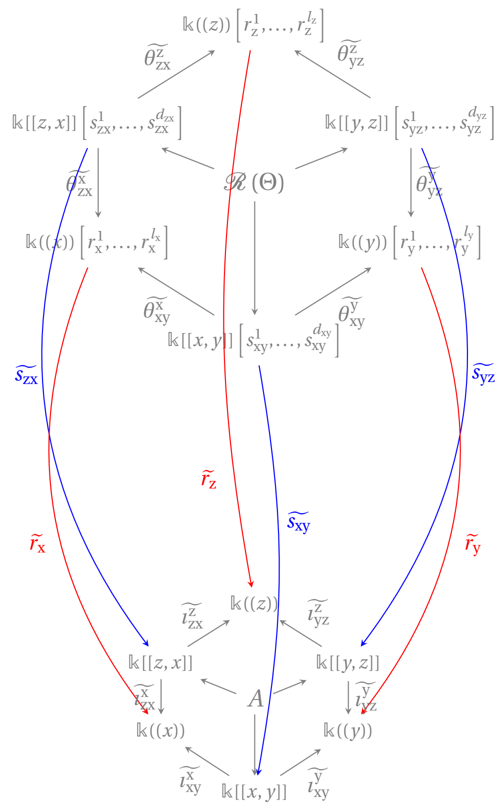


Figure 5.9: Dual sections $\widetilde{s_{xy}}, \widetilde{s_{yz}}, \widetilde{s_{zx}}$ and $\widetilde{r_x}, \widetilde{r_y}, \widetilde{r_z}$ defining a dual section $\widetilde{s} : \mathscr{R}(\Theta) \to A$

Remark 5.4.3. *In the dual hexagonal prism diagram as shown in Figure 5.9, the universal property of the limit yields the* dual section

$$\widetilde{s} : \mathscr{R}(\Theta) \to A$$

$$\left(F\left(x,y,s^1_{xy},\ldots,s^{d_{xy}}_{xy}\right),G\left(y,z,s^1_{yz},\ldots,s^{d_{yz}}_{yz}\right),H\left(z,x,s^1_{zx},\ldots,s^{d_{zx}}_{zx}\right)\right) \longmapsto (F(x,y,s_{xy}(x,y)),(G(y,z,s_{yz}(y,z)),(H(z,x,s_{zx}(z,x))),$$

which is written in the expression of $\mathscr{R}(\Theta)$ in Proposition 5.3.3 and A in Proposition 5.2.1(2).

Proposition 5.4.4. *Any section of the degenerate vector bundle $\mathscr{E}(\Theta)$ over X is induced from a tuple*

$$\left(s_{xy}, s_{yz}, s_{zx}, r_x, r_y, r_z\right) \in \Bbbk[[x,y]]^{d_{xy}} \times \Bbbk[[y,z]]^{d_{yz}} \times \Bbbk[[z,x]]^{d_{zx}} \times \Bbbk((x))^{l_x} \times \Bbbk((y))^{l_y} \times \Bbbk((z))^{l_z}$$

satisfying the gluing conditions as in Definition 5.4.1.

Suppose that we have a tuple

$$\left(s_{xy}, s_{yz}, s_{zx}, r_x, r_y, r_z\right) \in \Bbbk[[x,y]]^{d_{xy}} \times \Bbbk[[y,z]]^{d_{yz}} \times \Bbbk[[z,x]]^{d_{zx}} \times \Bbbk((x))^{l_x} \times \Bbbk((y))^{l_y} \times \Bbbk((z))^{l_z}$$

satisfying the gluing conditions. The assumption

$$-\begin{pmatrix}\theta^{\bullet}_{\blacktriangle\bullet}\\ \theta^{\bullet}_{\bullet\blacksquare}\end{pmatrix}:\Bbbk((t))^{l_\bullet}\to\Bbbk((t))^{d_{\blacktriangle\bullet}+d_{\bullet\blacksquare}}\text{ is injective for }(\blacktriangle,\bullet,\blacksquare)=(\mathrm{x},\mathrm{y},\mathrm{z}),(\mathrm{y},\mathrm{z},\mathrm{x}),\text{ or }(\mathrm{z},\mathrm{x},\mathrm{y})$$

imposed on $\Theta\in\mathrm{Rep}(Q_A)$ in Equation 5.1.2 implies that $\left(r_{\mathrm{x}},r_{\mathrm{y}},r_{\mathrm{z}}\right)$ is uniquely determined by $\left(s_{\mathrm{xy}},s_{\mathrm{yz}},s_{\mathrm{zx}}\right)$. Therefore, a section can be naturally identified with an element

$$s:=\left(s_{\mathrm{xy}},s_{\mathrm{yz}},s_{\mathrm{zx}}\right)\in\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}}.$$

In this respect, we can identify the set of all sections of $\widehat{\pi}:\mathscr{E}(\Theta)\to X$ with

$$\Gamma(\mathscr{E}(\Theta)):=\left\{s:=\left(s_{\mathrm{xy}},s_{\mathrm{yz}},s_{\mathrm{zx}}\right)\in\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}}\ \middle|\ \begin{array}{l} s_{\mathrm{zx}}(0,x)=\theta^{\mathrm{zx}}_{\mathrm{x}}(x)\,r_{\mathrm{x}}(x),\quad s_{\mathrm{xy}}(x,0)=\theta^{\mathrm{xy}}_{\mathrm{x}}(x)\,r_{\mathrm{x}}(x),\\ s_{\mathrm{xy}}(0,y)=\theta^{\mathrm{xy}}_{\mathrm{y}}(y)\,r_{\mathrm{y}}(y),\quad s_{\mathrm{yz}}(y,0)=\theta^{\mathrm{yz}}_{\mathrm{y}}(y)\,r_{\mathrm{y}}(y),\\ s_{\mathrm{yz}}(0,z)=\theta^{\mathrm{yz}}_{\mathrm{z}}(z)\,r_{\mathrm{z}}(z),\quad s_{\mathrm{zx}}(z,0)=\theta^{\mathrm{zx}}_{\mathrm{z}}(z)\,r_{\mathrm{z}}(z)\\ \text{for some }\left(r_{\mathrm{x}},r_{\mathrm{y}},r_{\mathrm{z}}\right)\in\Bbbk((x))^{l_{\mathrm{x}}}\times\Bbbk((y))^{l_{\mathrm{y}}}\times\Bbbk((z))^{l_{\mathrm{z}}}\end{array}\right\},$$

which is a subset of $\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}}$.

Note that the set of all sections has a natural A-module structure. Namely, each of

$$xs=\left(xs_{\mathrm{xy}},0,xs_{\mathrm{zx}}\right),\quad ys=\left(ys_{\mathrm{xy}},ys_{\mathrm{yz}},0\right),\quad zs=\left(0,zs_{\mathrm{yz}},zs_{\mathrm{zx}}\right)$$

satisfies the gluing conditions with $\left(r_{\mathrm{x}},r_{\mathrm{y}},r_{\mathrm{z}}\right)$ multiplied by x, y or z, respectively. Thus, $\Gamma(\mathscr{E}(\Theta))$ is an A-submodule of $\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}}$.

Remark 5.4.5. *By the above discussion,* $\Gamma(\mathscr{E}(\Theta))$ *is determined by the following pull-back diagram.*

$$\begin{tikzcd}
\Gamma(\mathscr{E}(\Theta)) \arrow[r] \arrow[d] & \Bbbk((x))^{l_{\mathrm{x}}}\times\Bbbk((y))^{l_{\mathrm{y}}}\times\Bbbk((z))^{l_{\mathrm{z}}} \arrow[d, "\Theta"] \\
\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}} \arrow[r, "\pi"] & \Bbbk((x))^{d_{\mathrm{xy}}}\times\Bbbk((y))^{d_{\mathrm{xy}}}\times\Bbbk((y))^{d_{\mathrm{yz}}}\times\Bbbk((z))^{d_{\mathrm{yz}}}\times\Bbbk((z))^{d_{\mathrm{zx}}}\times\Bbbk((x))^{d_{\mathrm{zx}}}
\end{tikzcd}$$

Thus, algebraically the gluing conditions naturally arises from the diagram. It has already been developed and exploited several times in the literature, e.g., [BD17] defined a functor $\mathrm{Tri}'(A)\to\mathrm{M}(A)$ *and [LW12] defined a functor* $\mathrm{BD}'(A)\to\mathrm{CM}(A)$ *using this construction.*

Lemma 5.4.6. $\Gamma(\mathscr{E}(\Theta))$ *is a Cohen-Macaulay* A*-module.*

Proof. We show that $(a_1,a_2):=\left(x+y+z,xy+yz+zx\right)$ is a regular sequence of $\Gamma:=\Gamma(\mathscr{E}(\Theta))$, which directly yields $\operatorname{depth}\Gamma=2$ and hence $\Gamma\in\mathrm{CM}(A)$. Since $x+y+z$ is not a zero divisor of $\Bbbk[[x,y]]^{d_{\mathrm{xy}}}\times\Bbbk[[y,z]]^{d_{\mathrm{yz}}}\times\Bbbk[[z,x]]^{d_{\mathrm{zx}}}$, it is also not a zero divisor of Γ. To show $xy+yz+zx$ is not a zero divisor of $\Gamma/\left(x+y+z\right)\Gamma$ is equivalent to show

$$s\in\Gamma,\quad\left(xy+yz+zx\right)s\in\left(x+y+z\right)\Gamma\quad\Rightarrow\quad s\in\left(x+y+z\right)\Gamma.$$

By assumption, we have

$$\left(xys_{\mathrm{xy}}, yzs_{\mathrm{yz}}, zxs_{\mathrm{zx}}\right) = \left(\left(x+y\right)u_{\mathrm{xy}}, \left(y+z\right)u_{\mathrm{yz}}, (z+x)\,u_{\mathrm{zx}}\right)$$

for $s = \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right) \in \Gamma$ and for some $u = \left(u_{\mathrm{xy}}, u_{\mathrm{yz}}, u_{\mathrm{zx}}\right) \in \Gamma$. Because each $\mathbb{k}[[x,y]]$, $\mathbb{k}[[y,z]]$ and $\mathbb{k}[[z,x]]$ is a UFD, we have

$$s_{\mathrm{xy}} = \left(x+y\right)s'_{\mathrm{xy}}, \quad s_{\mathrm{yz}} = \left(y+z\right)s'_{\mathrm{yz}}, \quad s_{\mathrm{zx}} = (z+x)\,s'_{\mathrm{zx}}$$

for some $s' := \left(s'_{\mathrm{xy}}, s'_{\mathrm{yz}}, s'_{\mathrm{zx}}\right) \in \mathbb{k}[[x,y]]^{d_{\mathrm{xy}}} \times \mathbb{k}[[y,z]]^{d_{\mathrm{yz}}} \times \mathbb{k}[[z,x]]^{d_{\mathrm{zx}}}$ and hence $s = \left(x+y+z\right)s'$.

Gluing conditions for $s \in \Gamma$ imply

$$\begin{aligned} xs'_{\mathrm{zx}}(0,x) &= \theta^{\mathrm{zx}}_{\mathrm{x}}(x)\,r_{\mathrm{x}}(x), \quad & xs'_{\mathrm{xy}}(x,0) &= \theta^{\mathrm{xy}}_{\mathrm{x}}(x)\,r_{\mathrm{x}}(x), \\ ys'_{\mathrm{xy}}\left(0,y\right) &= \theta^{\mathrm{xy}}_{\mathrm{y}}\left(y\right)r_{\mathrm{y}}\left(y\right), \quad & ys'_{\mathrm{yz}}\left(y,0\right) &= \theta^{\mathrm{yz}}_{\mathrm{y}}\left(y\right)r_{\mathrm{y}}\left(y\right), \\ zs'_{\mathrm{yz}}(0,z) &= \theta^{\mathrm{yz}}_{\mathrm{z}}(z)\,r_{\mathrm{z}}(z), \quad & zs'_{\mathrm{zx}}(z,0) &= \theta^{\mathrm{zx}}_{\mathrm{z}}(z)\,r_{\mathrm{z}}(z) \end{aligned}$$

for some $\left(r_{\mathrm{x}}, r_{\mathrm{y}}, r_{\mathrm{z}}\right) \in \mathbb{k}((x))^{l_{\mathrm{x}}} \times \mathbb{k}((y))^{l_{\mathrm{y}}} \times \mathbb{k}((z))^{l_{\mathrm{z}}}$. Dividing each equation by x, y or z, we know that s' also satisfies the gluing conditions with $r' = \left(r_{\mathrm{x}}/x, r_{\mathrm{y}}/y, r_{\mathrm{z}}/z\right) \in \mathbb{k}((x))^{l_{\mathrm{x}}} \times \mathbb{k}((y))^{l_{\mathrm{y}}} \times \mathbb{k}((z))^{l_{\mathrm{z}}}$. Therefore, we have $s' \in \Gamma$ and hence $s \in \left(x+y+z\right)\Gamma$. □

5.5 Equivalence of Categories

Theorem 5.5.1.

$$\begin{tikzcd} & \mathrm{DVB}(X) \arrow[dl, "\Gamma"', "\substack{\textit{global}\\ \textit{section}}"] & \\ \mathrm{CM}(A) \arrow[rr, "\mathbb{F}_{\mathrm{BD}}"] & & \mathrm{Rep}(Q_A) \arrow[ul, "\mathscr{E}\ \textit{gluing functor}"'] \end{tikzcd}$$

(1) $(\Gamma \circ \mathscr{E}) \circ \mathbb{F}_{\mathrm{BD}} \simeq \mathrm{id}_{\mathrm{CM}(A)}$

(2) $\mathbb{F}_{\mathrm{BD}} \circ (\Gamma \circ \mathscr{E}) \simeq \mathrm{id}_{\mathrm{Rep}(Q_A)}$

Therefore, categories $\mathrm{CM}(A)$ *and* $\mathrm{Rep}(Q_A)$ *are equivalent. As the gluing functor* $\mathscr{E}$ *was established as an equivalence in Definition 5.3.5, the category* $\mathrm{DVB}(X)$ *is also equivalent to them.*

Proof. (1) For any $M \in \mathrm{CM}(A)$,

$$\mathbb{F}_{\mathrm{BD}}(M) = \left(\tilde{M} := R \boxtimes_A M, V_M := Q\left(\bar{A}\right) \otimes_A M, \theta_M : Q\left(\bar{R}\right) \otimes_A M \to Q\left(\bar{R}\right) \otimes_R \tilde{M}\right).$$

We choose trivializations

- $Q\left(\bar{A}\right) \otimes_A M \cong \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}}$,
- $\tilde{M} \cong \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$,

which yield another trivializations

- $Q\left(\bar{R}\right) \otimes_A M \cong \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((x))^{l_{\mathrm{x}}}$,
- $Q\left(\bar{R}\right) \otimes_R \tilde{M} \cong \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}$,

and expression of θ_M as

- $\theta_M \cong \theta^{\mathrm{x}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{zx}} \times \theta^{\mathrm{x}}_{\mathrm{zx}}$
  $: \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((x))^{l_{\mathrm{x}}}$
  $\to \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}$,

which gives a representation $\Theta_M = \left(\theta^{\mathrm{x}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{zx}}, \theta^{\mathrm{x}}_{\mathrm{zx}}\right) \in \mathrm{Rep}(Q_A)$ of M on Q_A.

To prove that $\Gamma(\mathscr{E}(\Theta_M))$ is naturally isomorphic to M, we identify it as an A-submodule of

$$\tilde{M} \cong \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}.$$

Note that there is a natural injection $\iota_M : M \to \tilde{M}$. Therefore, it is enough to show the equality

$$\Gamma(\mathscr{E}(\Theta_M)) = \mathrm{im}\, \iota_M. \tag{5.5.1}$$

In the following commuting diagram, both rows are exact.

$$\begin{tikzcd} 0 \arrow[r] & \ker\pi \arrow[r] \arrow[d, "\iota_M|_{\ker\pi}"] & M \arrow[r, "\pi"] \arrow[d, "\iota_M"] & Q(\bar{A})\otimes_A M \arrow[d, "\tilde{\theta_M}=\iota\otimes\iota_M"] \\ 0 \arrow[r] & I\tilde{M} \arrow[r] & \tilde{M} \arrow[r, "\tilde{\pi}"] & Q(\bar{R})\otimes_R \tilde{M} \end{tikzcd} \tag{5.5.2}$$

For any $s=\left(s_{\mathrm{xy}},s_{\mathrm{yz}},s_{\mathrm{zx}}\right)\in\tilde{M}$,

$$\tilde{\pi}(s)=\left(s_{\mathrm{xy}}(x,0),s_{\mathrm{xy}}(0,y),s_{\mathrm{yz}}(y,0),s_{\mathrm{yz}}(0,z),s_{\mathrm{zx}}(z,0),s_{\mathrm{zx}}(0,x)\right)\in Q\left(\bar{R}\right)\otimes_R\tilde{M}$$

and for any $r=\left(r_{\mathrm{x}},r_{\mathrm{y}},r_{\mathrm{z}}\right)\in Q\left(\bar{A}\right)\otimes_A M$,

$$(\iota\otimes\mathbb{1}_M)(r)=\left(r_{\mathrm{x}},r_{\mathrm{y}},r_{\mathrm{y}},r_{\mathrm{z}},r_{\mathrm{z}},r_{\mathrm{x}}\right)\in Q\left(\bar{R}\right)\otimes_A M$$

and hence

$$\tilde{\theta_M}(r)=\left(\theta^{\mathrm{x}}_{\mathrm{xy}}r_{\mathrm{x}},\theta^{\mathrm{y}}_{\mathrm{xy}}r_{\mathrm{y}},\theta^{\mathrm{y}}_{\mathrm{yz}}r_{\mathrm{y}},\theta^{\mathrm{z}}_{\mathrm{yz}}r_{\mathrm{z}},\theta^{\mathrm{z}}_{\mathrm{zx}}r_{\mathrm{z}},\theta^{\mathrm{x}}_{\mathrm{zx}}r_{\mathrm{x}}\right)\in Q\left(\bar{R}\right)\otimes_R\tilde{M}.$$

Therefore, $s\in\tilde{M}$ satisfies the gluing conditions specified in Equation 5.4.1 if and only if

$$\tilde{\pi}(s)=\tilde{\theta_M}(r)$$

for some $r\in Q\left(\bar{A}\right)\otimes_A M$.

The $(\supset)$ part of Equation 5.5.1 immediately follows, as $\iota_M(a)$ satisfies the gluing condition $\tilde{\pi}(s)=\tilde{\theta_M}(r)$ for any $a\in M$.

To establish $(\subset)$ part, we need a further claim that $I\tilde{M}\subset\operatorname{im}\iota_M$. We have a natural commuting diagram

$$\begin{tikzcd} \operatorname{Hom}_A(\operatorname{Hom}_A(M,A),A)\cong M \arrow[r, hook, "\iota_M"] & \tilde{M}\cong\operatorname{Hom}_R(\operatorname{Hom}_A(M,R),R) \arrow[d] \\ \operatorname{Hom}_A(\operatorname{Hom}_A(M,R),A) \arrow[u] \arrow[r] & \operatorname{Hom}_A(\operatorname{Hom}_A(M,R),R) \end{tikzcd}$$

for $M\in\mathrm{CM}(A)$. For $a\in I=\{a\in A|aR\subset A\}\cong\operatorname{Hom}_A(R,A)$ and $f\in\tilde{M}\cong\operatorname{Hom}_R\left(\hat{M},R\right)$ where $\hat{M}:=\operatorname{Hom}_A(M,R)$, we have $af\in\operatorname{Hom}_A\left(\hat{M},R\right)$, $\operatorname{im}\left(af\right)\subset A$ and hence $af\in\operatorname{Hom}_A\left(\hat{M},A\right)$. Then commuting of the diagram yields $af\in\operatorname{im}\iota_M$, proving the claim.

Now let $s\in\tilde{M}$ satisfy the gluing condition $\tilde{\pi}(s)=\tilde{\theta_M}(r)$ for some $r\in Q\left(\bar{A}\right)\otimes_A M$. Note that $x^n r\in\operatorname{im}\pi$ for sufficiently large $n\in\mathbb{Z}$. Let's say $\pi(a_{\mathrm{x}})=x^n r$ for some $m_{\mathrm{x}}\in M$. Then

$$\tilde{\pi}\left(x^n s\right)=\tilde{\theta_M}\left(x^n r\right)=\tilde{\theta_M}(\pi(m_{\mathrm{x}}))=\tilde{\pi}(\iota_M(m_{\mathrm{x}}))$$

yields

$$x^n s-\iota_M(m_{\mathrm{x}})\in I\tilde{M}\subset\operatorname{im}\iota_M$$

and hence $x^n s\in\operatorname{im}\iota_M$. Similarly, we have $x^n s$, $y^n s$, $z^n s\in\operatorname{im}\iota_M$ for sufficiently large $n\in\mathbb{Z}$. Because M is Cohen-Macaulay, by Proposition 3.3.2, finally we have $s\in\operatorname{im}\iota_M$.

Naturallity follows from commuting of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \downarrow \iota_M & & \downarrow \iota_M \\ \tilde{M} & \xrightarrow{\tilde{\alpha}} & \tilde{N} \end{array}$$

for $M, N \in \mathrm{CM}(A)$.

(2) Let $\Theta = \left(\theta^{\mathrm{x}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{xy}}, \theta^{\mathrm{y}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{yz}}, \theta^{\mathrm{z}}_{\mathrm{zx}}, \theta^{\mathrm{x}}_{\mathrm{zx}}\right) \in \mathrm{Rep}(Q_A)$ be a representation on Q_A.

Step 1: Characterize elements of $\Gamma(\mathscr{E}(\Theta))$.

We divide the sections in $\Gamma(\mathscr{E}(\Theta))$ into the following three types.

- First type: The sections written in the forms

$$xys = \left(xys_{\mathrm{xy}}, 0, 0\right), \quad yzs = \left(0, yzs_{\mathrm{yz}}, 0\right), \quad \text{or} \quad zxs = (0, 0, zxs_{\mathrm{zx}})$$

for some $s := \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right) \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$, whose support lie on the complement of $\mathrm{Spec}\left(A/(xy)\right)$, $\mathrm{Spec}\left(A/(yz)\right)$, or $\mathrm{Spec}(A/(zx))$ in $X = \mathrm{Spec}(A)$, respectively.

- Second type: The sections written in the forms

$$\begin{aligned}
s_{\mathrm{x}}(xr_{\mathrm{x}}) &:= \left(x\theta^{\mathrm{x}}_{\mathrm{xy}}(x)\, r_{\mathrm{x}}, 0, x\theta^{\mathrm{x}}_{\mathrm{zx}}(x)\, r_{\mathrm{x}}\right) \quad \text{where} \quad \theta^{\mathrm{x}}_{\mathrm{zx}}(x)\, r_{\mathrm{x}} \in \Bbbk[[x]]^{d_{\mathrm{zx}}},\ \theta^{\mathrm{x}}_{\mathrm{xy}}(x)\, r_{\mathrm{x}} \in \Bbbk[[x]]^{d_{\mathrm{xy}}}, \\
s_{\mathrm{y}}(yr_{\mathrm{y}}) &:= \left(y\theta^{\mathrm{y}}_{\mathrm{xy}}(y)\, r_{\mathrm{y}}, y\theta^{\mathrm{y}}_{\mathrm{yz}}(y)\, r_{\mathrm{y}}, 0\right) \quad \text{where} \quad \theta^{\mathrm{y}}_{\mathrm{xy}}(y)\, r_{\mathrm{y}} \in \Bbbk[[y]]^{d_{\mathrm{xy}}},\ \theta^{\mathrm{y}}_{\mathrm{yz}}(y)\, r_{\mathrm{y}} \in \Bbbk[[y]]^{d_{\mathrm{yz}}}, \text{ or} \\
s_{\mathrm{z}}(zr_{\mathrm{z}}) &:= \left(0, z\theta^{\mathrm{z}}_{\mathrm{yz}}(z)\, r_{\mathrm{z}}, z\theta^{\mathrm{z}}_{\mathrm{zx}}(z)\, r_{\mathrm{z}}\right) \quad \text{where} \quad \theta^{\mathrm{z}}_{\mathrm{yz}}(z)\, r_{\mathrm{z}} \in \Bbbk[[z]]^{d_{\mathrm{yz}}},\ \theta^{\mathrm{z}}_{\mathrm{zx}}(z)\, r_{\mathrm{z}} \in \Bbbk[[z]]^{d_{\mathrm{zx}}}
\end{aligned}$$

for some $r_{\mathrm{x}} \in \Bbbk((x))^{l_{\mathrm{x}}}, r_{\mathrm{y}} \in \Bbbk((y))^{l_{\mathrm{y}}}, r_{\mathrm{z}} \in \Bbbk((z))^{l_{\mathrm{z}}}$, whose support lie on the complement of $\mathrm{Spec}(A/(z))$, $\mathrm{Spec}(A/(x))$, or $\mathrm{Spec}\left(A/(y)\right)$ in $X = \mathrm{Spec}(A)$, respectively.

- Third type: For any other section $s \in \Gamma(\mathscr{E}(\Theta))$, each of xs, ys and zs can be expressed as an A-linear combination of sections of the first or second type. For example, xs is expressed as

$$\begin{aligned}
xs &= \left(xs_{\mathrm{xy}}(x,y), 0, xs_{\mathrm{zx}}(z,x)\right) \\
&= \left(xs_{\mathrm{xy}}(x,0), 0, xs_{\mathrm{zx}}(0,x)\right) + \left(xys'_{\mathrm{xy}}(x,y), 0, 0\right) + \left(0, 0, zxs'_{\mathrm{zx}}(z,x)\right) \\
&= s_{\mathrm{x}}(xr_{\mathrm{x}}) + xys' + zxs'
\end{aligned}$$

for some $r_{\mathrm{x}} \in \Bbbk((x))^{l_{\mathrm{x}}}$ and $s'_{\mathrm{xy}} \in \Bbbk[[x,y]]^{d_{\mathrm{xy}}}$, $s'_{\mathrm{zx}} \in \Bbbk[[z,x]]^{d_{\mathrm{zx}}}$, where we decompose s_{xy} and s_{zx} as

$$\begin{aligned}
s_{\mathrm{xy}}(x,y) &= s_{\mathrm{xy}}(x,0) + \left(s_{\mathrm{xy}}(x,y) - s_{\mathrm{xy}}(x,0)\right) \\
&= s_{\mathrm{xy}}(x,0) + ys'_{\mathrm{xy}}(x,y), \\
s_{\mathrm{zx}}(z,x) &= s_{\mathrm{zx}}(0,x) + (s_{\mathrm{zx}}(z,x) - s_{\mathrm{zx}}(0,x)) \\
&= s_{\mathrm{zx}}(0,x) + zs'_{\mathrm{zx}}(z,x).
\end{aligned}$$

Step 2: Find a basis for each component of $V_{\Gamma(\mathscr{E}(\Theta))} = Q\left(\bar{A}\right) \otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((x))^{l_x} \times \Bbbk((y))^{l_y} \times \Bbbk((z))^{l_z}$.

For any section $s \in \Gamma(\mathscr{E}(\Theta))$, we have

$$1 \otimes s = x^{-1} \otimes xs \in \Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta))$$

and the discussion in Step 1 implies that the $\Bbbk((x))$-vector space $\Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta))$ is generated by elements of the form

$$1 \otimes xys, \quad 1 \otimes yzs, \quad 1 \otimes zxs, \quad 1 \otimes s_x(xr_x), \quad 1 \otimes s_y\left(yr_y\right), \quad 1 \otimes s_z(zr_z).$$

We show the first three vanish. For any $s_{xy} \in \Bbbk[[x,y]]^{d_{xy}}$, we have $s_{xy}(x,0) \in \Bbbk[[x]]^{d_{xy}}$. As θ^x_{xy} is surjective, there is a $r_x \in \Bbbk((x))^{l_x}$ satisfying $s_{xy}(x,0) = \theta^x_{xy}(x)\, r_x$. Take $n \in \mathbb{Z}_{\geq 0}$ so that $x^n\theta^x_{xy}(x)\, r_x \in \Bbbk[[x]]^{d_{xy}}$, $x^n\theta^x_{zx}(x)\, r_x \in \Bbbk[[x]]^{d_{zx}}$ and hence $s_x\left(x^{n+1}r_x\right) = \left(x^{n+1}\theta^x_{xy}(x)\, r_x, 0, x^{n+1}\theta^x_{zx}(x)\, r_x\right)$ is included in $\Gamma(\mathscr{E}(\Theta))$. Then the calculation

$$\begin{aligned}
x^n\left(xys\right) &= x^n\left(xys_{xy}(x,0), 0, 0\right) + x^n\left(xy^2 s'_{xy}(x,y), 0, 0\right) \\
&= x^n\left(xy\theta^x_{xy}(x)\, r_x, 0, 0\right) + x^{n+1}y^2 s' \\
&= y\left(x^{n+1}\theta^x_{xy}(x)\, r_x, 0, x^{n+1}\theta^x_{zx}(x)\, r_x\right) + x^{n+1}y^2 s' \\
&= y\left(s_x\left(x^{n+1}r_x\right) + x^{n+1}ys'\right) \in y\Gamma(\mathscr{E}(\Theta))
\end{aligned} \tag{5.5.3}$$

shows that

$$1 \otimes xys = x^{-n} \otimes x^n\left(xys\right) = y \cdot x^{-n} \otimes \left(s_x\left(x^{n+1}r_x\right) + x^{n+1}ys'\right)$$

vanishes in $\Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta))$ since y acts as 0 on $\Bbbk((x))$. Similarly $1 \otimes yzs$ and $1 \otimes zxs$ also vanish.

The last two also vanish, for example,

$$1 \otimes s_y\left(yr_y\right) = x^{-1} \otimes \left(xy\theta^y_{xy}(y)\, r_y, 0, 0\right) = x^{-1}\left(1 \otimes \left(xys_{xy}, 0, 0\right)\right) = x^{-1}\left(1 \otimes xys\right) = 0 \in \Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta)).$$

Consequently, $\Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta))$ is generated by elements of the form $1 \otimes s_x(xr_x)$, or equivalently, by l_x elements

$$x^{-n} \otimes s_x\left(x^n\mathbf{e}_1\right), \ldots, x^{-n} \otimes s_x\left(x^n\mathbf{e}_{l_x}\right)$$

where $\mathbf{e}_1 := (1,0,\ldots,0), \ldots, \mathbf{e}_{l_x} := (0,\ldots,0,1) \in \Bbbk((x))^{l_x}$ all satisfy $s_x(x^n\mathbf{e}_i) \in \Gamma(\mathscr{E}(\Theta))$ for some $n \in \mathbb{Z}$. They indeed form a basis, whose linearly independence follows from injectivity of $\begin{pmatrix}\theta^x_{zx} \\ \theta^x_{xy}\end{pmatrix} : \Bbbk((t))^{l_x} \to \Bbbk((t))^{d_{zx}+d_{xy}}$.

Therefore, we get an explicit isomorphism $\Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((x))^{l_x}$ and this is also similar for $\Bbbk((y)) \otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((y))^{l_y}$ and $\Bbbk((z)) \otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((y))^{l_z}$.

Step 3: Describe the map $\iota_{\Gamma(\mathscr{E}(\Theta))} : \Gamma(\mathscr{E}(\Theta)) \to \widetilde{\Gamma(\mathscr{E}(\Theta))} \cong \Bbbk[[x,y]]^{d_{xy}} \times \Bbbk[[y,z]]^{d_{yz}} \times \Bbbk[[z,x]]^{d_{zx}}$.

We use the decomposition

$$\begin{aligned}
\widetilde{\Gamma(\mathscr{E}(\Theta))} &\cong \mathrm{Hom}_R\left(\mathrm{Hom}_A\left(\Gamma(\mathscr{E}(\Theta)), R\right), R\right) \\
&\cong \quad \mathrm{Hom}_{\Bbbk[[x,y]]}\left(\mathrm{Hom}_A\left(\Gamma(\mathscr{E}(\Theta)), \Bbbk[[x,y]]\right), \Bbbk[[x,y]]\right) \\
&\quad \times \mathrm{Hom}_{\Bbbk[[y,z]]}\left(\mathrm{Hom}_A\left(\Gamma(\mathscr{E}(\Theta)), \Bbbk[[y,z]]\right), \Bbbk[[y,z]]\right) \\
&\quad \times \mathrm{Hom}_{\Bbbk[[z,x]]}\left(\mathrm{Hom}_A\left(\Gamma(\mathscr{E}(\Theta)), \Bbbk[[z,x]]\right), \Bbbk[[z,x]]\right).
\end{aligned}$$

Note that for any $f \in \mathrm{Hom}_A\left(\Gamma\left(\mathscr{E}\left(\Theta\right)\right), \Bbbk[[x,y]]\right)$ and $xys = \left(xys_{\mathrm{xy}}, 0, 0\right) \in \Gamma\left(\mathscr{E}\left(\Theta\right)\right)$, we have from 5.5.3 that

$$x^n f\left(xys\right) = yf\left(s_{\mathrm{x}}\left(x^{n+1}r_{\mathrm{x}}\right) + x^{n+1}ys'\right)$$

and hence $y|f\left(xys\right)$. Similarly, we also have $x|f\left(xys\right)$ and therefore $xy|f\left(xys\right)$.

It enables us to define a $\Bbbk[[x,y]]$-module isomorphism

$$\begin{aligned}\mathrm{Hom}_A\left(\Gamma\left(\mathscr{E}\left(\Theta\right)\right), \Bbbk[[x,y]]\right) &\to \Bbbk[[x,y]]^{d_{\mathrm{xy}}}\\ f &\mapsto \left(f\left(\left(xy\mathbf{e}_1, 0, 0\right)\right)/xy, \dots, f\left(\left(xy\mathbf{e}_{d_{\mathrm{xy}}}, 0, 0\right)\right)/xy\right)\end{aligned}$$

with an inverse

$$\begin{aligned}\Bbbk[[x,y]]^{d_{\mathrm{xy}}} &\to \mathrm{Hom}_A\left(\Gamma\left(\mathscr{E}\left(\Theta\right)\right), \Bbbk[[x,y]]\right)\\ \left(s_1^*, \dots, s_{d_{\mathrm{xy}}}^*\right) &\mapsto \left(f : \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right) \mapsto s_1^* s_{\mathrm{xy}}^1 + \cdots + s_{d_{\mathrm{xy}}}^* s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right).\end{aligned}$$

Therefore, we have an isomorphism

$$\mathrm{Hom}_{\Bbbk[[x,y]]}\left(\mathrm{Hom}_A\left(\Gamma\left(\mathscr{E}\left(\Theta\right)\right), \Bbbk[[x,y]]\right), \Bbbk[[x,y]]\right) \cong \mathrm{Hom}_{\Bbbk[[x,y]]}\left(\Bbbk[[x,y]]^{d_{\mathrm{xy}}}, \Bbbk[[x,y]]\right) \cong \Bbbk[[x,y]]^{d_{\mathrm{xy}}}$$

which is composed with a natural map

$$\Gamma\left(\mathscr{E}\left(\Theta\right)\right) \to \mathrm{Hom}_{\Bbbk[[x,y]]}\left(\mathrm{Hom}_A\left(\Gamma\left(\mathscr{E}\left(\Theta\right)\right), \Bbbk[[x,y]]\right), \Bbbk[[x,y]]\right),$$

to yield

$$\begin{aligned}\Gamma\left(\mathscr{E}\left(\Theta\right)\right) &\to \Bbbk[[x,y]]^{d_{\mathrm{xy}}}\\ \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right) &\mapsto \left(s_{\mathrm{xy}}^1, \dots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right).\end{aligned}$$

Finally, we have an injection

$$\begin{aligned}\iota_{\Gamma(\mathscr{E}(\Theta))} : \Gamma\left(\mathscr{E}\left(\Theta\right)\right) \to \widetilde{\Gamma\left(\mathscr{E}\left(\Theta\right)\right)} &\cong \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}\\ \left(s_{\mathrm{xy}}, s_{\mathrm{yz}}, s_{\mathrm{zx}}\right) \mapsto \quad &\left(\left(s_{\mathrm{xy}}^1, \dots, s_{\mathrm{xy}}^{d_{\mathrm{xy}}}\right), \left(s_{\mathrm{yz}}^1, \dots, s_{\mathrm{yz}}^{d_{\mathrm{yz}}}\right), \left(s_{\mathrm{zx}}^1, \dots, s_{\mathrm{zx}}^{d_{\mathrm{zx}}}\right)\right).\end{aligned}$$

Step 4: Describe the map

$$\begin{aligned}\theta_{\Gamma(\mathscr{E}(\Theta))} : Q\left(\bar{R}\right) \otimes_A \Gamma\left(\mathscr{E}\left(\Theta\right)\right) &\cong \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((x))^{l_{\mathrm{x}}}\\ \to Q\left(\bar{R}\right) \otimes_R \widetilde{\Gamma\left(\mathscr{E}\left(\Theta\right)\right)} &\cong \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}.\end{aligned}$$

Under the natural ring homomorphisms

$$\begin{tikzcd}
A = \Bbbk[[x,y,z]]/(xyz) \arrow[r, two heads] \arrow[d, hook] & \bar{A} = \Bbbk[[x,y,z]]/(xy,yz,zx) \arrow[r, hook] \arrow[d, hook] & Q\left(\bar{A}\right) \cong \Bbbk((x)) \times \Bbbk((y)) \times \Bbbk((z)) \arrow[d, hook] \\
R = \Bbbk[[x,y]] \times R[[y,z]] \times R[[z,x]] \arrow[r, two heads] & \bar{R} = \Bbbk[[x,y]]/(xy) \times \Bbbk[[y,z]]/(yz) \times \Bbbk[[z,x]]/(zx) \arrow[r, hook] & Q\left(\bar{R}\right) \cong \Bbbk((x)) \times \Bbbk((y)) \times \Bbbk((y)) \times \Bbbk((z)) \times \Bbbk((z)) \times \Bbbk((x))
\end{tikzcd}$$

we have

$$\begin{tikzcd}
\Gamma(\mathscr{E}(\Theta)) \arrow[rr] \arrow[dd, hook, "\iota_{\Gamma(\mathscr{E}(\Theta))}"] & & Q(\bar{A})\otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \arrow[d] \\
 & & Q(\bar{R})\otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \arrow[d, "\theta_{\Gamma(\mathscr{E}(\Theta))}"] \\
\widehat{\Gamma(\mathscr{E}(\Theta))} \cong \Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}} \arrow[rr] & & Q(\bar{R})\otimes_R \widehat{\Gamma(\mathscr{E}(\Theta))} \cong \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}
\end{tikzcd}$$

where the map $\theta_{\Gamma(\mathscr{E}(\Theta))}$ decomposes into 6 maps

$$\begin{aligned}
&\left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{x}}_{\mathrm{xy}} : \Bbbk((x))^{l_{\mathrm{x}}} \to \Bbbk((x))^{d_{\mathrm{xy}}}, \quad \left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{y}}_{\mathrm{yz}} : \Bbbk((y))^{l_{\mathrm{y}}} \to \Bbbk((y))^{d_{\mathrm{yz}}}, \quad \left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{z}}_{\mathrm{zx}} : \Bbbk((z))^{l_{\mathrm{z}}} \to \Bbbk((z))^{d_{\mathrm{zx}}}, \\
&\left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{y}}_{\mathrm{xy}} : \Bbbk((y))^{l_{\mathrm{y}}} \to \Bbbk((y))^{d_{\mathrm{xy}}}, \quad \left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{z}}_{\mathrm{yz}} : \Bbbk((z))^{l_{\mathrm{z}}} \to \Bbbk((z))^{d_{\mathrm{yz}}}, \quad \left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{x}}_{\mathrm{zx}} : \Bbbk((x))^{l_{\mathrm{x}}} \to \Bbbk((x))^{d_{\mathrm{zx}}}.
\end{aligned}$$

We claim that $\left(\theta_{\Gamma(\mathscr{E}(\Theta))}\right)^{\mathrm{x}}_{\mathrm{xy}}$ coincides with the originally given $\theta^{\mathrm{x}}_{\mathrm{xy}}$ and similar for others. By discussion in Step 2, $\Bbbk((x)) \otimes_A \Gamma(\mathscr{E}(\Theta)) \cong \Bbbk((x))^{l_{\mathrm{x}}}$ has a basis consisting of

$$x^{-n} \otimes s_{\mathrm{x}}\left(x^n \mathbf{e}_1\right), \ldots, x^{-n} \otimes s_{\mathrm{x}}\left(x^n \mathbf{e}_{l_{\mathrm{x}}}\right),$$

where $s_{\mathrm{x}}(x^n \mathbf{e}_i) = \left(x^n \theta^{\mathrm{x}}_{\mathrm{xy}}(x)\,\mathbf{e}_i, 0, x^n \theta^{\mathrm{x}}_{\mathrm{zx}}(x)\,\mathbf{e}_i\right) \in \Gamma(\mathscr{E}(\Theta))$. The result in Step 3 reveals

$$\begin{aligned}
\theta_{\Gamma(\mathscr{E}(\Theta))}\left(x^{-n} \otimes s_{\mathrm{x}}\left(x^n \mathbf{e}_i\right)\right) &= x^{-n} \otimes \iota_{\Gamma(\mathscr{E}(\Theta))}\left(s_{\mathrm{x}}\left(x^n \mathbf{e}_i\right)\right) \\
&= x^{-n} \otimes x^n \theta^{\mathrm{x}}_{\mathrm{xy}}(x)\,\mathbf{e}_i \qquad \in \Bbbk((x)) \otimes \left(\Bbbk[[x,y]]^{d_{\mathrm{xy}}} \times \Bbbk[[y,z]]^{d_{\mathrm{yz}}} \times \Bbbk[[z,x]]^{d_{\mathrm{zx}}}\right) \\
&= \theta^{\mathrm{x}}_{\mathrm{xy}}(x)\,\mathbf{e}_i \qquad \in \Bbbk((x))^{d_{\mathrm{xy}}},
\end{aligned}$$

which proves the claim. Therefore, we have a decomposition of $\theta_{\Gamma(\mathscr{E}(\Theta))}$ as

$$\begin{aligned}
\theta_{\Gamma(\mathscr{E}(\Theta))} \cong\, & \theta^{\mathrm{x}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{xy}} \times \theta^{\mathrm{y}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{yz}} \times \theta^{\mathrm{z}}_{\mathrm{zx}} \times \theta^{\mathrm{x}}_{\mathrm{zx}} \\
& : \Bbbk((x))^{l_{\mathrm{x}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((y))^{l_{\mathrm{y}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((z))^{l_{\mathrm{z}}} \times \Bbbk((x))^{l_{\mathrm{x}}} \\
& \to \Bbbk((x))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{xy}}} \times \Bbbk((y))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{yz}}} \times \Bbbk((z))^{d_{\mathrm{zx}}} \times \Bbbk((x))^{d_{\mathrm{zx}}}.
\end{aligned}$$

□

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