이학 박사 학위논문

Primitively n-universal quadratic forms of minimal rank

(최소 랭크의 원시 n보편 이차형식)

2023년 8월

서울대학교 대학원

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이 논문을 이학 박사 학위논문으로 제출함

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Primitively n-universal quadratic forms of minimal rank

A dissertation

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Abstract

For a prime p and a positive integer n, an integral quadratic form over the ring $\mathbb{Z}_p$ is called primitively n-universal if it primitively represents all integral quadratic forms of rank n over $\mathbb{Z}_p$. In [7], Earnest and Gunawardana provided some criteria for determining whether or not a given integral quadratic form over $\mathbb{Z}_p$ is primitively 1-universal. In this thesis, we prove that the minimal rank of primitively n-universal integral quadratic form over $\mathbb{Z}_p$ is $2n$, if p is an odd prime or if $p = 2$ and $n \geq 5$. Moreover, we obtain a complete classification of primitively 2-universal integral quadratic forms over $\mathbb{Z}_p$ of minimal rank.

For a positive integer n, a positive definite integral quadratic form is called primitively n-universal if it primitively represents all positive definite integral quadratic forms of rank n. It was proved in [11] that there are exactly 107 primitively 1-universal quaternary integral quadratic forms up to isometry. In this thesis, we prove that the minimal rank of primitively 2-universal integral quadratic forms is six, and we prove that there are exactly 201 primitively 2-universal senary integral quadratic forms up to isometry.

Key words: Primitive n-universality

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Chapter 1

Introduction

A quadratic form of rank n is a quadratic homogeneous polynomial

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} f_{ij} x_i x_j \qquad (f_{ij} = f_{ji} \in \mathbb{Q}),$$

where the corresponding symmetric matrix $M_f = (f_{ij})$ is nondegenerate. We say f is integral if M_f is an integral matrix, and we say f is positive definite if M_f is positive definite. Throughout this thesis, we always assume that any quadratic form is integral and positive definite.

For two (positive definite integral) quadratic forms f and g of rank n and m, respectively, we say f is represented by g if there is an integral matrix $T \in M_{m,n}(\mathbb{Z})$ such that $M_f = T^t M_g T$. We say f is isometric to g if the above matrix T is invertible. We further say f is primitively represented by g if the above matrix T can be extended to an invertible matrix in $GL_m(\mathbb{Z})$ by adding suitable $(m-n)$ columns. In particular, a positive integer a is primitively represented by g if and only if there are integers $x_1, \ldots, x_m$ such that

$$g(x_1, \ldots, x_m) = a \quad \text{and} \quad \gcd(x_1, \ldots, x_m) = 1.$$

For a positive integer n, a quadratic form is called (primitively) n-universal if it (primitively, respectively) represents all quadratic forms of rank n. Lagrange's four-square theorem states that the quaternary quadratic form corresponding to the identity matrix I_4 is 1-universal. The complete classification of 1-universal quadratic forms up to isometry has been done by Ramanujan, Dickson, Conway–Schneeberger, and Bhargava (see [20], [6], and [1]). In 1998, Kim, Kim, and Oh in [13] proved that there are exactly eleven 2-universal quinary quadratic forms up to isometry. For some more information on n-universal quadratic forms, see [12] or [16].

For a ring R, a quadratic R-form of rank n is a quadratic homogeneous polynomial

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} f_{ij} x_i x_j \qquad (f_{ij} = f_{ji} \in R),$$

where the corresponding symmetric matrix $M_f = (f_{ij}) \in M_n(R)$ is nondegenerate. An integral quadratic form is a quadratic $\mathbb{Z}$-form, and a quadratic S-form can be viewed as a quadratic R-form whenever S is a subring of R. For two quadratic R-forms f and g of rank n and m, respectively, we say f is represented by g (over R) if there is a matrix $T \in M_{m,n}(R)$ over R such that $M_f = T^t M_g T$. We say f is isometric to g (over R) if the above matrix T is invertible over R. We further say g is primitively represented by f (over R) if the above matrix T can be extended to an invertible matrix in $GL_m(R)$ by adding suitable $(m-n)$ columns in R^m.

Clearly, if f is (primitively) represented by g over $\mathbb{Z}$, then f is also (primitively, respectively) represented by g over $\mathbb{Z}_p$ for any prime p. However, the converse is not true in general. In fact, there is an effective criterion whether or not f is represented by g over $\mathbb{Z}_p$ for any prime p (for this, see [18]). However, as far as the author knows, there is no known effective criterion whether or not f is primitively represented by g over $\mathbb{Z}_p$.

Finding primitively 1-universal quadratic forms was first considered by Budarina in [2]. She classified all primitively 1-universal quaternary quadratic forms satisfying some special local properties. Later, she also classified in [3] all primitively 2-universal quadratic forms that is of class number one and has odd squarefree discriminant. Recently, Earnest and Gunawardana classified in [7] all quadratic $\mathbb{Z}_p$-forms that primitively represent all unary quadratic $\mathbb{Z}_p$-forms for any prime p including $p = 2$. Furthermore, they gave a complete list of all quaternary 1-universal quadratic forms that are almost primitively 1-universal. Here, a quadratic form is called almost primitively 1-universal if it represents almost all positive integers primitively. Recently, Ju, Kim, Kim, Kim and Oh in [11] finally proved that there are exactly 107 primitively 1-universal quaternary quadratic forms up to isometry.

In this thesis, we study the minimal rank of primitively n-universal quadratic forms and the classification of primitively n-universal quadratic forms of minimal rank, over $\mathbb{Z}$ and $\mathbb{Z}_p$ for a prime p. Most results were done by joint work with Prof. Byeong-Kweon Oh.

In Chapter 2, we summarize basic facts and preliminary results about representations of quadratic spaces and lattices. We also introduce some results on n-universal and primitively n-universal quadratic forms over $\mathbb{Z}$ and $\mathbb{Z}_p$ for a prime p.

In Chapter 3, we discuss primitive n-universality over $\mathbb{Z}_p$ for a prime p. We first state a necessary condition for a $\mathbb{Z}_p$-lattice to be primitively n-universal. Next, we prove that the minimal rank of primitively n-universal quadartic forms over $\mathbb{Z}_p$ is $2n$ if p is odd or $n \geq 5$. Furthermore, it is $2n+1$ if $p = 2$ and $n = 2, 3$. Finally, we provide a complete classification of primitively n-universal quadratic forms of minimal rank, when p is odd and $n = 2, 3$, and when $p = 2$ and $n = 2$.

In Chapter 4, we discuss primitive n-universality over $\mathbb{Z}$. We prove that the minimal rank of primitively 2-universal quadratic forms over $\mathbb{Z}$ is six. Furthermore, we prove that there are exactly 201 primitively 2-universal senary quadratic forms up to isometry (see Table 4.1).

Chapter 2

Preliminaries

In this chapter, we introduce definitions, notations and known results which will be used throughout the thesis.

2.1 Representations of quadratic spaces

Let F be a field of characteristic not 2. By a *quadratic space* V over F we mean a finite dimensional vector space V over F equipped with a symmetric bilinear form B on V, i. e. a mapping

$$B : V \times V \to F$$

with the following properties:

$$B(\alpha x + y, z) = \alpha B(x, z) + B(y, z), \qquad B(x, y) = B(y, x)$$

for all $x, y, z \in V$ and all $\alpha \in F$. We define the quadratic form Q (associated with B) on V by $Q(x) = B(x, x)$ for all $x \in V$. We use Q and B to denote the quadratic form and the associated bilinear form on any quadratic space. We say that a quadratic space is *unary*, *binary*, *ternary*, *quaternary*, ..., *n-ary* according as its dimension is 1, 2, 3, 4, ..., n. The quadratic space V is said to *represent* a field element α if $\alpha \in Q(V)$. We say that V is *universal* if $Q(V) = F$.

Suppose that V and W are quadratic spaces. A linear map $\sigma \in L(V, W)$ is called a *representation* from V into W (with respect to the bilinear forms B on V and W) if

$$B(\sigma x, \sigma y) = B(x, y) \qquad \text{for any } x, y \in V.$$

We let $V \to W$ denote a representation. We say that V is *represented* by W if there is a representation $V \to W$. An injective representation is called an *isometry* of V into W. And V and W are said to be *isometric* if there exists an isometry σ of V onto W. We let $V \cong W$ denote an isometry of V onto W. The set of all isometries V into itself is written $O(V)$. It is a subgroup of $GL(V)$, called the *orthogonal group* of V with respect to the quadratic form Q.

Let V be an n-ary quadratic space. With each basis $x_1, \ldots, x_n$ for V, we associate an $n \times n$ symmetric matrix N whose (i, j) entry is $B(x_i, x_j)$. We call N the (*Gram*) *matrix* of V in the basis $x_1, \ldots, x_n$ and write

$$V \cong N \quad \text{in} \quad x_1, \ldots, x_n.$$

If there is a basis $x_1, \ldots, x_n$ for which this holds, then we say that V has the (Gram) matrix N and we write

$$V \cong N.$$

The *discriminant* of V, written dV, is defined to be the canonical image of $\det N$ in the quotient monoid $F/(F^\times)^2$. It is easily seen that the above definition of discriminant is actually independent of the choice of a basis.

Consider the quadratic space V. The *orthogonal sum* is the direct sum of subspaces $V_1, \ldots, V_r$, which are pairwise orthogonal, i. e. which satisfies

$$B(V_i, V_j) = 0 \quad \text{for} \quad 1 \leq i < j \leq r.$$

It is denoted $V_1 \perp \cdots \perp V_r$. If the orthogonal sum of subspaces $V_1, \ldots, V_r$ is equal to V, then we say that V has the (*orthogonal*) *splitting*

$$V = V_1 \perp \cdots \perp V_r$$

into subspaces $V_1, \ldots, V_r$. We call V_i the (*orthogonal*) *components* of the splitting. We say that a subspace U (*orthogonally*) *splits* V, or that it is a *component* of V, if there exists a subspace W such that

$$V = U \perp W.$$

Suppose that we are given quadratic spaces $V_1, \ldots, V_r$ over F. Then there is a unique symmetric bilinear form on the direct sum $V_1 \oplus \cdots \oplus V_r$ which induces the given bilinear forms on the V_i and under which the summands V_1,

$\ldots$, V_r are mutually orthogonal. For if B_1, $\ldots$, B_r are the respective given bilinear forms, define

$$B\Big(\sum x_i, \sum y_i\Big) = \sum B_i(x_i, y_i)$$

for typical vectors $\sum x_i$, $\sum y_i$ in $\bigoplus V_i$; it is easily seen that B has the required properties. In this case, $\bigoplus V_i$ equipped with such a B also is denoted by $V_1 \perp \cdots \perp V_r$ and is called an *orthogonal sum* of quadratic spaces V_1, $\ldots V_r$.

Given a symmetric $n \times n$ matrix N, we let $\langle N \rangle$ (or sometimes N itself) stand for an n-ary quadratic space which has the matrix N. Hence, for instance, the notation

$$N_1 \perp N_2$$

with N_1 and N_2 symmetric matrices over F denotes a quadratic space over F which has the matrix

$$\left(\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array}\right).$$

Similarly,

$$\langle \alpha_1 \rangle \perp \cdots \perp \langle \alpha_n \rangle$$

with all α_i in F denotes a quadratic space over F which has the matrix

$$\operatorname{diag}(\alpha_1, \ldots, \alpha_n).$$

We simply let $\langle \alpha_1, \ldots, \alpha_n \rangle$ stand for such a space. A basis $\mathcal{B}$ for the quadratic space V is called an *orthogonal basis* if the matrix of V in $\mathcal{B}$ is diagonal. Every nonzero quadratic space has an orthogonal basis.

For a subset S of the quadratic space V, we put

$$S^{\perp} = \{x \in V \mid B(x, S) = 0\}.$$

It is easily seen that $S^{\perp}$ is a subspace of V. For a subspace U of V, we call $U^{\perp}$ the *orthogonal complement* of U in V. We say that V is a *nondegenerate* quadratic space if $V^{\perp} = 0$, or equivalently if $dV \neq 0$. If U is a nondegenerate subspace of the quadratic space V, then it is well known that

$$V = U \perp U^{\perp}.$$

Theorem 2.1.1 (Witt). (a) *If U and W are isometric nondegenerate subspaces of a quadratic space V, then $U^{\perp}$ and $W^{\perp}$ are isometric.*

(b) *If V and V' are isometric nondegenerate quadratic spaces and U is any subspace of V, then for any isometry $\sigma : U \to V'$, there is an extension of σ to an isometry of V onto V'.*

Proof. (a) See [19, Theorem 42:16]. (b) See [19, Theorem 42:17]. □

Let x be an nonzero vector in the quadratic space V. We call x *isotropic* if $Q(x) = 0$, and we call it *anisotropic* otherwise. Let V be a nonzero quadratic space. We call V isotropic if it contains an isotropic vector, and we call it anisotropic otherwise. A quadratic space V is called a *hyperbolic plane* if it has the matrix

$$V \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in one of its bases, called a *hyperbolic basis* for V. A binary quadratic space V is a hyperbolic plane if and only if V is isotropic and nondegenerate, if and only if $-1 \in dV$. We let $\mathbb{H}$ stand for the hyperbolic plane.

We call a nonzero quadratic space is *totally isotropic* if each of its nonzero vector is isotropic. Let V be a nondegenerate quadratic space. Then any maximal totally isotropic subspace of V has the same dimension. This dimension is called the (*Witt*) *index* of V, and is written $\operatorname{ind} V$. If the index of V is r, then V is split by an orthogonal sum of r copies of hyperbolic planes. This implies

$$0 \leq 2 \operatorname{ind} V \leq \dim V.$$

We call V a *hyperbolic space* if $0 < 2r = \dim V$. Thus V is hyperbolic if and only if it is isometric to a nonempty orthogonal sum of hyperbolic planes.

Let a field F and nonzero scalars α, $\beta \in F$ are given. Take a four dimensional space U and a basis 1, i, j, k for U so that

$$U = F1 + Fi + Fj + Fk.$$

Define a multiplication on these basis vectors by the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	$\alpha 1$	k	αj
j	j	$-k$	$\beta 1$	$-\beta i$
k	k	$-\alpha j$	βi	$-\alpha\beta 1$

and extend this by linearity to a multiplication on U. Then U is an associative F-algebra with multiplicative identity 1. For each pair of nonzero scalars $\alpha,\ \beta \in F$, the algebra obtained by the preceding construction is called the *quaternion algebra*

$$\left(\frac{\alpha, \beta}{F}\right).$$

Consider a nondegenerate n-ary quadratic space V over the field F. Suppose that

$$V \cong \langle \alpha_1, \ldots, \alpha_n \rangle.$$

We define the *Hasse algebra*

$$S_F V := \bigotimes_{1 \le i \le j \le n} \left(\frac{\alpha_i, \alpha_j}{F}\right).$$

It may be shown that $S_F V$ is uniquely determined up to an algebra isomorphism. Hence it is an invariant of the quadratic space V.

A *global field* is either a finite extension of the field of rational numbers $\mathbb{Q}$, or a finite extension of the field $\mathbb{F}_q(t)$ of rational functions in one variable over a finite constant field $\mathbb{F}_q$. A *local field* is a composite object consisting of a place $\mathfrak{p}$ on F such that $\mathfrak{p}$ is complete and discrete, and the residue class field at $\mathfrak{p}$ is finite. It is known that the completion of a global field at any one of its nontrivial nonarchimedean places is a local field.

Suppose that F has a unique nontrivial place $\mathfrak{p}$, and suppose that F is either a local field at $\mathfrak{p}$, or $\mathfrak{p}$ is archimedean and complete. In any of these situation, it is well-known that the Brauer group of F is cyclic of order at most 2, and hence it may be identified with a subgroup of $\{\pm 1\}$. For a nondegenerate quadratic space V over F, we define the *Hasse symbol* $S_{\mathfrak{p}}V \in \{\pm 1\}$ to be the canonical image of the Hasse algebra in the Brauer group of F. Also, given nonzero scalars α, $\beta \in F$, we define the *Hilbert symbol* $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right) \in \{\pm 1\}$ to be the canonical image of a quaternion algebra $\left(\frac{\alpha,\beta}{F}\right)$ in the Brauer group of F. Then evidently

$$\left(\frac{\alpha,\beta}{\mathfrak{p}}\right) = \begin{cases} +1 & \text{if } \langle \alpha, \beta \rangle \text{ represents } 1, \\ -1 & \text{otherwise.} \end{cases}$$

The Hasse symbol may be computed from Hilbert symbols by the identity

$$S_{\mathfrak{p}}V = \prod_{1 \leq i \leq j \leq n} \left(\frac{\alpha_i, \alpha_j}{\mathfrak{p}}\right).$$

The following three theorems completely resolve the representability problem of positive definite quadratic spaces over $\mathbb{Q}$. For instance, we know that If the dimension of a positive definite quadratic space over $\mathbb{Q}$ is ≥ 4, then it represents any positive rational number in $\mathbb{Q}$. If V is a nondegenerate quadratic space over the global field F and $\mathfrak{p}$ is a nontrivial place on F, then we put $V_{\mathfrak{p}} = F_{\mathfrak{p}} \otimes V$. The Hasse symbol $S_{\mathfrak{p}}V_{\mathfrak{p}}$ will be written $S_{\mathfrak{p}}V$.

Theorem 2.1.2 (19, Theorem 63:20)**.** *Let F be a local field at the prime place $\mathfrak{p}$. Then nondegenerate quadratic spaces U and V over F are isometric if and only if*

$$\dim U = \dim V, \quad dU = dV, \quad S_{\mathfrak{p}}U = S_{\mathfrak{p}}V.$$

Theorem 2.1.3 (19, Theorem 63:21)**.** *Let U and V be nondegenerate quadratic spaces over a local field with $\nu = \dim V - \dim U \geq 0$. Then U is represented* *by V if and only if $\nu \geq 3$ or*

$$\begin{cases} V \cong U & \text{if } \nu = 0, \\ V \cong U \perp \langle dU \cdot dV \rangle & \text{if } \nu = 1, \\ V \cong U \perp \mathbb{H} & \text{if } \nu = 2 \text{ and } dV = -dU, \end{cases}$$

where $\mathbb{H}$ is the hyperbolic plane.

Theorem 2.1.4 (19, Theorem 66:3)**.** *Let U and V be nondegenerate quadratic spaces over the global field F. Then U is represented by V if and only if $U_{\mathfrak{p}}$ is represented by $V_{\mathfrak{p}}$ for all places $\mathfrak{p}$ on F.*

The following facts about isotropy of quadratic spaces over local fields are well known. If F is a local field and R is the ring of integers in F, we fix a nonsquare unit Δ in R such that $\Delta = 1 + 4\rho$ for some unit ρ in R. If $(F, R) = (\mathbb{Q}_p, \mathbb{Z}_p)$ for a prime p, we let $\Delta_p = \Delta$.

Theorem 2.1.5. *Let V be a nondegenerate n-ary quadratic space over F a local field at $\mathfrak{p}$.*

(a) *If $n = 3$, then V is isotropic if and only if $S_{\mathfrak{p}}V = (-1, -1)$.*

(b) *Suppose $n = 4$. If dV is nonsquare, then V is isotropic. If dV is a square, then V is isotropic if and only if $S_{\mathfrak{p}}V = (-1, -1)$. If V is anisotropic, then*

$$V \cong \langle 1, -\Delta, \pi, -\pi\Delta \rangle,$$

where π is a uniformizer of F.

(c) *If $n \geq 5$, then V is isotropic.*

Proof. (a) See [19, 58:6]. (b) See [19, 63:17]. (c) See [19, 63:19]. □

Let V be a nondegenerate quadratic space over the global field F and let Ω be the set of all nontrivial places on F. The Hilbert Reciprocity Law for F gives a reciprocity law for Hasse symbols, namely

$$\prod_{\mathfrak{p} \in \Omega} S_{\mathfrak{p}} V = 1,$$

for any nondegenerate quadratic space V over F.

2.2 Representations of quadratic lattices

Let F be a field and let R be a Dedekind domain defined by a Dedekind set of places S on F (for the definition, see [19, §22F]). Let V be a finite dimensional vector space over F. A *lattice* in V (with respect to R, or with respect to the defining set of places S) is a finitely generated R-submodule of V. For a lattice M in V, we define FM to be the F-span of M in V. We call M a lattice on V if $FM = V$. If F is a local field at $\mathfrak{p}$ then $S = \{\mathfrak{p}\}$ so that R is the ring of integers in F. If $F = \mathbb{Q}$ then we assume $S = \Omega \setminus \{\infty\}$ so that $R = \mathbb{Z}$. We are interested mainly in the cases when $F = \mathbb{Q}$ or $F = \mathbb{Q}_p$, so we suppose that R is a PID from now on. Thus, every lattice is free over R, and its rank over R is well-defined.

Let F be a field of characteristic not 2. A lattice L in a quadratic space V is called a *quadratic lattice*, for it inherits the symmetric bilinear form B and associated quadratic form Q from the ambient space V. We call a quadratic lattice L *unary*, *binary*, *ternary*, *quaternary*, $\ldots$, n-*ary* according as its rank is 1, 2, 3, 4, $\ldots$, n.

Suppose that L and M are lattices in quadratic spaces V and W, respectively. A *representation* from L into M is a representation $\sigma : FL \to FM$ such that $\sigma L \subseteq M$, and we denote it by $L \to M$. We say that L is represented by M if there is a representation $L \to M$. An *isometry* of L into M is an isometry $\sigma : FL \to FM$ such that $\sigma L \subseteq M$. We say that L and M are *isometric*, and write

$$L \cong M,$$

if there is an isometry $\sigma : FL \cong FM$ such that $\sigma L = M$. A *primitive representation* from L into M is a representation $\sigma : L \to M$ such that σL is a primitive sublattice in M. Suppose that L and M are lattices on the same quadratic space V. We say that L and M are in the same class if

$$M = \sigma L \quad \text{for some} \quad \sigma \in O(V).$$

This is clearly an equivalence relation on the set of all lattices on V, and we accordingly obtain a partition of this set into equivalence classes. We use

$$\operatorname{cls} L$$

to denote the class of L.

Let L be an n-ary lattice on the quadratic space V. Since L is free over R, there is an R-basis for L, and any such basis is also an F-basis for V. With each basis $x_1, \ldots, x_n$ for L, we associate the matrix $N = (B(x_i, x_j))$, i. e. the matrix of V in $x_1, \ldots, x_n$. We call N the (*Gram*) *matrix* of L in the basis $x_1, \ldots, x_n$ and write

$$L \cong N \quad \text{in} \quad x_1, \ldots, x_n.$$

If there is a basis $x_1, \ldots, x_n$ for which this holds, then we say that L has the (Gram) matrix N and we write

$$L \cong N.$$

The *discriminant* of L, written dL, is defined to be the canonical image of $\det N$ in the quotient monoid $F/(R^\times)^2$. It is easily seen that the above definition of discriminant is actually independent of the choice of a basis.

Consider the quadratic space V. The *orthogonal sum* is the direct sum of lattices $L_1, \ldots, L_r$ in V, which are pairwise orthogonal, i. e. which satisfies

$$B(L_i, L_j) = 0 \quad \text{for} \quad 1 \leq i < j \leq r.$$

It is denoted $L_1 \perp \cdots \perp L_r$. If a lattice L in V is the orthogonal sum of sublattices $L_1, \ldots, L_r$, then we say that L has the (*orthogonal*) *splitting*

$$L = L_1 \perp \cdots \perp L_r$$

into sublattices $L_1, \ldots, L_r$. We call L_i the (*orthogonal*) *components* of the splitting. We say that a sublattice K (*orthogonally*) *splits* L, or that it is a *component* of L, if there exists a sublattice M such that

$$L = K \perp M.$$

Suppose that we are given quadratic spaces V_i $(1 \leq i \leq r)$ over F and lattices L_i in V_i Then we know that there exists a quadratic space V over F such that

$$V \cong V_1 \perp \cdots \perp V_r.$$

Hence there always exists a quadratic space V which includes a lattice L such that

$$L \cong L_1 \perp \cdots \perp L_r.$$

Given a symmetric $n \times n$ matrix N, we have agreed to let $\langle N \rangle$ or N stand for an n-ary quadratic space having the matrix N. We also use the symbol $\langle N \rangle$ or N to denote a free n-ary quadratic lattice with the matrix N (in a suitable quadratic space). Hence, as for spaces we have

$$N_1 \perp N_2 \cong \left(\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array}\right)$$

for symmetric matrices N_1 and N_2 over F and

$$\langle \alpha_1, \ldots, \alpha_n \rangle \cong \langle \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \rangle \cong \langle \alpha_1 \rangle \perp \cdots \perp \langle \alpha_n \rangle$$

for field elements α_i in F. A basis $\mathcal{B}$ for the quadratic lattice L is called an *orthogonal basis* if the matrix of L in $\mathcal{B}$ is diagonal.

For a subset S of the quadratic lattice L, we put

$$S^{\perp} = \{x \in L \mid B(x, S) = 0\}.$$

Clearly $S^{\perp}$ (in L) is equal to the intersection of $S^{\perp}$ (in V) with L. Hence $S^{\perp}$ is a primitive sublattice of L. For a sublattice K of L, we call $K^{\perp}$ the *orthogonal* *complement* of K in L. We call L *nondegenerate* if $L^{\perp} = 0$, or equivalently if $dL \neq 0$.

Consider a lattice L in the quadratic space V. By the *scale*

$$\mathfrak{s}L$$

of L we mean the R-submodule $B(L, L)$ of F. We define the *norm*

$$\mathfrak{n}L$$

of L to be the R-submodule generated by the subset $Q(L)$ of F. We know that $\mathfrak{s}L$ and $\mathfrak{n}L$ are either a fractional ideal or 0, and

$$2\mathfrak{s}L \subseteq \mathfrak{n}L \subseteq \mathfrak{s}L.$$

It is clear that $\mathfrak{s}(L_{\mathfrak{p}}) = (\mathfrak{s}L)_{\mathfrak{p}}$ and $\mathfrak{n}(L_{\mathfrak{p}}) = (\mathfrak{n}L)_{\mathfrak{p}}$ for any $\mathfrak{p} \in S$.

Consider a n-ary lattice L in the quadratic space V. Suppose that the scale of L is the fractional ideal $(a) = aR$. Then we know that $\mathfrak{s}L = aR$ and $dL \subseteq a^n R$. If L actually satisfies

$$\mathfrak{s}L = aR \quad \text{and} \quad dL \subseteq a^n R^{\times},$$

then we call L *aR-modular* or simply *a-modular*. We call L *unimodular* if it is R-modular. We say that L is modular if it is a-modular for some a. Since R is a PID, a nonzero lattice L in a quadratic space V is a-modular if and only if $B(x, L) = (a)$ for every primitive vector x in L.

We have agreed to let $\mathbb{H}$ stand for a hyperbolic plane. We also use the symbol $\mathbb{H}$ to denote a free binary quadratic lattice with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in one of its bases, called a *hyperbolic basis* for $\mathbb{H}$. Note that $\mathfrak{s}\mathbb{H} = R$, $\mathfrak{n}\mathbb{H} = 2R$, and that $\mathbb{H}$ is a unimodular lattice on a hyperbolic plane. Moreover, we let $\mathbb{A}$ denote a lattice with the matrix $\mathbb{A}$. For $\alpha \in F$, by an expression $\alpha\mathbb{H}$ ($\alpha\mathbb{A}$) we mean a scaling of $\mathbb{H}$ ($\mathbb{A}$, resp.) by α. That is,

$$\alpha\mathbb{H} \cong \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha\mathbb{A} \cong \alpha \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that $\alpha\mathbb{H}$ and $\alpha\mathbb{A}$ are α-modular.

Proposition 2.2.1 (19, Proposition 82:15 and Corollary 82:15a)**.** *Let L be a lattice in a quadratic space V and J is an a-modular sublattice of L. Then J splits L if and only if $B(J, L) \subseteq (a)$. In particular, J splits L if $\mathfrak{s}L = (a)$.*

Consider a nonzero nondegenerate lattice L in the quadratic space V over the local field F. The above proposition implies that L splits into unary and binary modular lattices. If we group the modular components of the above splitting suitably, then L has a splitting

$$L = J_1 \perp \cdots \perp J_t$$

in which each component is modular and

$$\mathfrak{s}J_1 \supsetneq \cdots \supsetneq \mathfrak{s}J_t.$$

Any such splitting is called a *Jordan splitting* of L. We have therefore proved that every nonzero nondegenerate lattice L in a quadratic space V over a local field F has at least one Jordan splitting.

Theorem 2.2.2 (19, Theorem 91:9)**.** *Let L be a nonzero nondegenerate lattice in the quadratic space V over the local field, and let*

$$L = J_1 \perp \cdots \perp J_t, \qquad L = K_1 \perp \cdots \perp K_T$$

be two Jordan splittings of L. Then $t = T$. And for $1 \leq i \leq t$ we have $\mathfrak{s}J_i = \mathfrak{s}K_i$, $\operatorname{rank} J_i = \operatorname{rank} K_i$, and $\mathfrak{n}J_i = \mathfrak{s}J_i$ if and only if $\mathfrak{n}K_i = \mathfrak{s}K_i$.

Therefore given a nonzero nondegenerate quadratic lattice L, the number of Jordan components t, the i-th Jordan scale $\mathfrak{s}J_i$ and the i-th Jordan rank $\operatorname{rank} J_i$ are invariants of L. Consider nonzero nondegenerate lattice L and M in quadratic spaces V and W over the same field F. Let

$$L = J_1 \perp \cdots \perp J_t, \qquad M = K_1 \perp \cdots \perp K_T$$

be any Jordan splitting of respectively L and M. We say that the lattices L and M are of the same *Jordan type* if $t = T$ and, whenever $1 \leq i \leq t$, we have

$$\mathfrak{s}J_i = \mathfrak{s}K_i, \qquad \operatorname{rank} J_i = \operatorname{rank} K_i$$

and

$$\mathfrak{n}J_i = \mathfrak{s}J_i \quad \text{if and only if} \quad \mathfrak{n}K_i = \mathfrak{n}J_i.$$

The last theorem guarantees that the above conditions are independent of the choice of Jordan splittings, and hence the notion of Jordan type is well-defined. Sometimes it is convenient to index the Jordan components by its scale, namely

$$L = \mathop{\perp}_{i \in \mathbb{Z}} L_i$$

where each L_i is $\mathfrak{p}^i$-modular or 0, and all but finitely many summands are 0.

Theorem 2.2.3 (19, Theorem 92:1)**.** *Let L be a unimodular lattice with respect to R on the quadratic space over the nondyadic local field. Then*

$$L \cong \langle 1, \ldots, 1, \epsilon \rangle$$

for any unit ϵ in R satisfying $dL = \epsilon (R^{\times})^2$.

Theorem 2.2.4 (19, Theorem 92:2)**.** *Let L and M be lattices of the same Jordan type on the nondegenerate quadratic space over the nondyadic local field. Consider Jordan splittings*

$$L = J_1 \perp \cdots \perp J_t, \qquad M = K_1 \perp \cdots \perp K_t.$$

Then $L \cong M$ if and only if

$$dJ_i = dK_i \quad \textit{for} \quad 1 \leq i \leq t.$$

Theorem 2.2.5 (18, Theorem 1)**.** *Let ℓ and L be nonzero nondegenerate quadratic lattices over the nondyadic local field F. Consider Jordan splittings $\ell = \bigperp \ell_\lambda$, $L = \bigperp L_\lambda$ and define*

$$\mathfrak{l}_i := \bigperp \{\ell_\mu \mid \mathfrak{s}\ell_\mu \supseteq \mathfrak{p}^i\}, \qquad \mathfrak{L}_i := \bigperp \{L_\mu \mid \mathfrak{s}L_\mu \supseteq \mathfrak{p}^i\}$$

for $i \in \mathbb{Z}$. Then $\ell \to L$ if and only if

$$F\mathfrak{l}_i \to F\mathfrak{L}_i \qquad \textit{for all } i.$$

The above four theorem together with Witt theorem(Theorem 2.1.1) implies the following **cancellation law**: for nondegenerate lattices M, M_1, M_2

over the nondyadic local field, $M \perp M_1 \to M \perp M_2$ if and only if $M_1 \to M_2$. However, over a dyadic local field, the situation is much more complicated.

Now consider dyadic case. Let L be a lattice on a nondegenerate quadratic space over the dyadic local field F. We define the norm group of L to be an additive subgroup

$$\mathfrak{g}L = Q(L) + 2\mathfrak{s}L$$

of F. For a fractional ideal $\mathfrak{a}$ in F, we define $L^{\mathfrak{a}}$ as the sublattice

$$L^{\mathfrak{a}} = \{x \in L \mid B(x, L) \subseteq \mathfrak{a}\}$$

of L.

Theorem 2.2.6. *Suppose that a L on a nondegenerate quadratic space over a dyadic local field F has splittings*

$$L = M \perp M_1 = N \perp N_1$$

with M is isometric to N.

(a) *If $M \cong H$, then M_1 is isometric to N_1.*

(b) *If M is $\mathfrak{a}$-modular with $\mathfrak{g}M \subseteq \mathfrak{g}(M_1^{\mathfrak{a}})$ and $\mathfrak{g}M \subseteq \mathfrak{g}(N_1^{\mathfrak{a}})$, then M_1 is isometric to N_1.*

(c) *Suppose $F = \mathbb{Q}_2$. If $M \cong \langle \epsilon \rangle$ for a unit ϵ in $\mathbb{Z}_2$, $\mathfrak{s}(M_1) = \mathfrak{s}(N_1) \subseteq (2)$ and $\mathfrak{n}(M_1) = \mathfrak{n}(N_1)$, then M_1 is isometric to N_1.*

Proof. (a)(b) See [19, Theorem 93:14 and Corollary 93:14a]. (c) See [14, Theorem 5.3.6]. $\square$

Theorem 2.2.7 (19, Theorem 93:16)**.** *Let L and M be unimodular lattices on the same quadratic space over a dyadic local field. Then $L \cong M$ if and only if $\mathfrak{g}L = \mathfrak{g}M$. Hence $L \cong M$ if and only if $Q(L) = Q(M)$.*

There are no known effective criteria to determine representability between lattices over a general dyadic local field. A dyadic local field is called a 2-adic local field if 2 is unramified. Let L be a lattice in a nonzero nondegenerate quadratic space V over a 2-adic local field F with the Jordan splitting $L = L_1 \perp \cdots \perp L_t$. For $1 \leq i \leq t$, $\mathfrak{n}L_i = \mathfrak{s}L_i$ or $2\mathfrak{s}L_i$, and hence $\mathfrak{n}L_i$ is also an invariant of L. Put $\mathfrak{s}_i = \mathfrak{s}L_i$ and $\mathfrak{n}_i = \mathfrak{n}L_i$ $(1 \leq i \leq t)$. We call the quantities

$$t,\ \operatorname{rank} L_i,\ \mathfrak{s}_i,\ \mathfrak{n}_i\ (1 \leq i \leq t)$$

the *Jordan invariants* of L. Clearly two lattices have the same Jordan invariants if and only if they are of the same Jordan type. We put

$$u_i = \operatorname{ord}_2 \mathfrak{n}_i.$$

Theorem 2.2.8 (19, Theorem 93:29)**.** *Let L and M be lattices on a nonzero nondegenerate quadratic spaces over the 2-adic local field F and suppose that L and M have the same Jordan invariants. Consider Jordan splittings*

$$L = J_1 \perp \cdots \perp J_t, \qquad M = K_1 \perp \cdots \perp K_t$$

and put

$$L_{(i)} = J_1 \perp \cdots \perp J_i, \qquad M_{(i)} = K_1 \perp \cdots \perp K_i \qquad (1 \leq i \leq t).$$

Then $L \cong M$ if and only if the following coditions hold for $1 \leq i \leq t-1$:

(1) *$dL_{(i)}/dM_{(i)}$ is congruent to a unit square modulo $\mathfrak{n}_i\mathfrak{n}_{i+1}/\mathfrak{s}_i^2$,*

(2) *$FL_{(i)} \to FK_{(i)} \perp \langle 2^{u_i}\rangle$ when $\mathfrak{n}_{i+1} \subseteq 4\mathfrak{n}_i$.*

In order to describe representations of lattices over a 2-adic local field, we need more definitions. Let F be a 2-adic local field and let R be the ring of integers in F. A modular R-lattice M is called *proper* if $\mathfrak{n}M = \mathfrak{s}M$, and *improper* otherwise. Let ℓ and L be nonzero nondegenerate quadratic lattices over the 2-adic local field F. Consider Jordan splittings $\ell = \perp \ell_\lambda$ and $L = \perp L_\lambda$. We define

$$\begin{aligned} \mathfrak{l}_i &:= \bigperp\{\ell_\mu \mid \mathfrak{s}_\mu \supseteq 2^i R\}, & \mathfrak{L}_i &:= \bigperp\{L_\mu \mid \mathfrak{s}_\mu \supseteq 2^i R\}, \\ \mathfrak{l}_{[i]} &:= \mathfrak{l}_i \perp \bigperp\{\ell_\mu \mid \mathfrak{n}_\mu = 2\mathfrak{s}_\mu = 2^{i+2} R\}, & \mathfrak{L}_{(i)} &:= \bigperp\{L_\mu \mid \mathfrak{n}_\mu \supseteq 2^i R\} \end{aligned}$$

for $i \in \mathbb{Z}$.

We define Δ_i for L as follows: If L has a proper 2^{i+1}-modular component, $\Delta_i := 2^{i+1}R$; failing this, $\Delta_i := 2^{i+2}R$ if L has a proper 2^{i+2}-modular component; otherwise, $\Delta_i = 0$. We define δ_i for ℓ_i in the same manner. We put $D_i = d(\mathfrak{L}_i)R$ and $d_i = d(\mathfrak{l}_i)R$; if $\mathfrak{L}_i = 0$ then put $D_i = 0$ and the same when $\mathfrak{l}_i = 0$. Note that these definitions are independent of the Jordan decomposition of ℓ or L. For any R-submodule $\mathfrak{a}$ in F and a quadratic space U over F, we write $\mathfrak{a} \to U$ if $\mathfrak{a} = Q(x)R$ for some $x \in U$. Hence, $0 \to U$ means a vacuous condition.

Definition 2.2.9. We say that ℓ have a lower type than L if the followings hold for all i:

(1) $\dim \mathfrak{l}_i \le \dim \mathfrak{L}_i$,

(2) $d_i D_i \to \langle 1 \rangle$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(3) $\delta_i \subseteq \Delta_i + 2^{i+2}R$ and $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}R$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(4) $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}R$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_i D_i \to \langle 2^{i-1} \rangle$,

(5) $\delta_i \subseteq \Delta_i + 2^{i+2}R$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_i D_i \to \langle 2^{i} \rangle$.

For two nondegenerate quadratic lattices m and M over R such that $m \to M$, we denote by M/m the quadratic space over F such that $Fm \perp (M/m) \cong FM$. For any $\alpha \in F$ and a quadratic space U over F, we write $\overline{\alpha} \to U$ if either $\alpha \to U$ or $\Delta\alpha \to U$.

Theorem 2.2.10 (18, Theorem 3)**.** *Let ℓ have a lower type than L. Then $\ell \to L$ if and only if the following conditions hold for all i:*

$$\Delta_i \to \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]}, \tag{6}$$

$$\delta_i \to \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]}, \tag{7}$$

$$\mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]} \cong H \textit{ implies } \Delta_i\delta_i \subseteq \delta_i^2, \tag{8}$$

$$\overline{2^i} \to (2^i \perp \mathfrak{L}_{(i+1)})/\mathfrak{l}_i, \tag{9}$$

$$\overline{2^i} \to (2^i \perp \mathfrak{L}_{i+1})/\mathfrak{l}_{[i]}. \tag{10}$$

Remark 2.2.11. In the statement of condition (V) of [18, Theorem 3], there is a typo; "$\mathfrak{L}_{(i+1)}$" should be replaced by "$\mathfrak{L}_{i+1}$". The same for [18, Proposition 25].

Let L be a lattice on a quadratic space V over the global field F. We define the *genus* $\operatorname{gen} L$ of L on V to be the set of all lattices M on V with the following property: for each $\mathfrak{p} \in S$ there exists an isometry $\Sigma_{\mathfrak{p}} \in O(V_{\mathfrak{p}})$ such that $M_{\mathfrak{p}} = \Sigma_{\mathfrak{p}} L_{\mathfrak{p}}$. The set of all lattices on V is thereby partitioned into genera. We immediately have

$$\operatorname{gen} M = \operatorname{gen} L \quad \text{if and only if} \quad \operatorname{cls} M_{\mathfrak{p}} = \operatorname{cls} L_{\mathfrak{p}} \quad \forall \mathfrak{p} \in S.$$

Proposition 2.2.12. *Let L be a lattice on the quadratic space V over a global field F, let K be a nondegenerate lattice in V. If there is a representation $K_{\mathfrak{p}} \to L_{\mathfrak{p}}$ at each $\mathfrak{p} \in S$, then there is a representation $K \to L'$ of K into some lattice L' in* $\operatorname{gen} L$*. If there is a primitive prepresentation $K_{\mathfrak{p}} \to L_{\mathfrak{p}}$ at each $\mathfrak{p} \in S$, then there is a primitive representation $K \to L'$ of K into some lattice L' in* $\operatorname{gen} L$*.*

Proof. See [19, Example 102:5]. □

Let $\mathfrak{p}$ and $\mathfrak{q}$ be fractional ideals in F such that $2\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. Suppose that L and M are lattices in quadratic spaces V and W, respectively. We write

$$L \to M \bmod (\mathfrak{q}, \mathfrak{p})$$

if there is an R-linear map σ from L into M such that

$$Q(\sigma x) \equiv Q(x) \bmod \mathfrak{q} \quad \text{and} \quad B(\sigma x, \sigma y) \equiv Q(x, y) \bmod \mathfrak{p}$$

for any x, $y \in L$. If $\mathfrak{q} = \mathfrak{p}$, we write $L \to M \bmod \mathfrak{p}$. If σ is bijective, we write $L \cong M \bmod (\mathfrak{q}, \mathfrak{p})$ and $L \cong M \bmod \mathfrak{p}$, respectively.

2.3 n-universality and primitive n-universality

Let F be a global field or a local field and let R be a Dedekind domain defined by a Dedekind set of spots S on F. A lattice in a quadratic space V over F is called an (*integral*) *R-lattice* if its scale is included in R. A $\mathbb{Z}$-lattice L is called *positive definite* if $Q(x) > 0$ for all nonzero $x \in L$, or equivalently if its Gram matrix (in any basis) is positive definite. Hereafter, we always assume that any $\mathbb{Z}$-lattice is positive definite. An R-lattice is called *n-universal* if it represents all n-ary R-lattices. A 1-universal lattice is simply called *universal*. Clearly, if a $\mathbb{Z}$-lattice L is n-universal then L_p is n-universal for all prime p. We denote by $u(n)$ the minimal rank of n-universal positive definite $\mathbb{Z}$-lattices, and by $u_p(n)$ the minimal rank of n-universal $\mathbb{Z}_p$-lattices. Evidently we have

$$u(n) \geq u_p(n) \qquad \text{for all prime } p.$$

In the following four theorems, F is a local field and R is the ring of integers in F.

Theorem 2.3.1 (10, Proposition 3.3). *Let F be a nondyadic local field. Let $L = J_1 \perp \cdots \perp J_t$ be a Jordan splitting of an R-lattice such that $R = \mathfrak{s}J_1 \supsetneq \cdots \supsetneq \mathfrak{s}J_t$. Then L is 2-universal if and only if one of the following conditions hold:*

(A) $\operatorname{rank} J_1 \geq 5$.

(B) $J_1 \cong \langle 1, 1, 1, 1\rangle$.

(C) $J_1 \cong \langle 1, 1, 1, \Delta\rangle$ *and J_2 is $\mathfrak{p}$-modular with* $\operatorname{rank} J_2 \geq 1$.

(D) $\operatorname{rank} J_1 = 3$, *and* J_2 *is* $\mathfrak{p}$*-modular with* $\operatorname{rank} J_2 \geq 2$.

Theorem 2.3.2 (10, Proposition 3.4)**.** *Let F be a nondyadic local field. Let $L = J_1 \perp \cdots \perp J_t$ be a Jordan splitting of an R-lattice such that $R = \mathfrak{s}J_1 \supsetneq \cdots \supsetneq \mathfrak{s}J_t$ and let $n \geq 3$. Then L is n-universal if and only if one of the following conditions hold:*

(A) $\operatorname{rank} J_1 \geq k+3$.

(B) $\operatorname{rank} J_1 = k+2$ *and* J_2 *is* $\mathfrak{p}$*-modular with* $\operatorname{rank} J_2 \geq 1$.

(C) $\operatorname{rank} J_1 = k+1$ *and* J_2 *is* $\mathfrak{p}$*-modular with* $\operatorname{rank} J_2 \geq 2$.

Theorem 2.3.3 (9, Theorem 1.3)**.** *Let F be a 2-adic local field. Let $L = J_1 \perp \cdots \perp J_t$ be a Jordan splitting of an R-lattice such that $R \supseteq \mathfrak{s}J_1 \supsetneq \cdots \supsetneq \mathfrak{s}J_t$, and let n be an even integer ≥ 2. Then L is n-universal if and only if $\mathfrak{s}J_1 = \mathfrak{n}J_1 = R$ and one of the following conditions hold:*

(A) $\dim J_1 \geq k+3$.

(B) $\dim J_1 = k+2$ *and* $\mathfrak{s}J_2 = \mathfrak{n}J_2 = (2)$.

(C) $\dim J_1 = k+2$, $(-1)^{\frac{(\dim FJ_1 - 1)\dim FJ_1}{2}} d(FJ_1) \notin \{1, \Delta\}(F^\times)^2$ *and* $\mathfrak{n}J_2 = (4)$.

(D) $\dim J_1 = k+1$, $\dim J_2 \geq 2$ *and* $\mathfrak{s}J_2 = \mathfrak{n}J_2 = (2)$.

(E) $\dim J_1 = k+1$, $\dim J_2 = 1$, $\mathfrak{s}J_2 = \mathfrak{n}J_2 = (2)$ *and* $\mathfrak{n}J_3 \supseteq (8)$.

Theorem 2.3.4 (9, Theorem 6.16)**.** *Let F be a 2-adic local field. Let $L = J_1 \perp \cdots \perp J_t$ be a Jordan splitting of an R-lattice such that $R \supseteq \mathfrak{s}J_1 \supsetneq \cdots \supsetneq \mathfrak{s}J_t$, and let n be an odd integer ≥ 3. Then L is n-universal if and only if $\mathfrak{s}J_1 = \mathfrak{n}J_1 = R$ and one of the following conditions hold:*

(A) $\dim J_1 \geq k+3$.

(B) $\dim J_1 = k+2$ *and* $\mathfrak{n}J_2 \supseteq (4)$.

(C) $\dim J_1 = k+1$, $\mathfrak{s}J_2 = \mathfrak{n}J_2 = (2)$, *and one of the following cases happens:*

- (C1) $\dim J_2 \geq 2$;
- (C2) $\dim J_2 = 1$ *and* $\mathfrak{n}J_3 \supseteq (8)$.

(D) $\dim J_1 = k$, $\mathfrak{s}J_2 = \mathfrak{n}J_2 = (2)$, *and one of the following cases happens:*

- (D1) $\dim J_2 \geq 3$;
- (D2) $\dim J_2 = 2$ *and* $\mathfrak{s}J_3 = \mathfrak{n}J_3 = (4)$;
- (D3) $\dim J_2 = 1$, $\dim J_3 \geq 2$, *and* $\mathfrak{s}J_3 = \mathfrak{n}J_3 = (4)$;
- (D4) $\dim J_2 = \dim J_3 = 1$, $\mathfrak{s}J_3 = \mathfrak{n}J_3 = (4)$, *and* $\mathfrak{s}J_4 = \mathfrak{n}J_4 = (8)$.

Theorem 2.3.5 ("The Fifteen Theorem")**.** *A positive definite $\mathbb{Z}$-lattice is universal if and only if it represents the nine critical numbers*

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 14,\ \textit{and}\ 15.$$

If t is any one of the above critical numbers, then there is a $\mathbb{Z}$-lattice that represents every positive integer except t. There are exactly 204 universal quaternary $\mathbb{Z}$-lattices up to isometry.

Proof. See [1]. $\square$

Theorem 2.3.6 (13, Theorems 1 and 2)**.** *A positive definite $\mathbb{Z}$-lattice is 2-universal if and only if it represents the following six positive definite binary $\mathbb{Z}$-lattices:*

$$I_2, \quad \langle 2,3\rangle, \quad \langle 3,3\rangle, \quad \mathbb{A}, \quad \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right).$$

Moreover, this is a minimal set, that is, for any ℓ among the six $\mathbb{Z}$-lattices above, there is a positive $\mathbb{Z}$-lattice that represents the other five except ℓ. There are exactly eleven 2-universal positive definite quinary $\mathbb{Z}$-lattices up to isometry.

An R-lattice is called *primitively n-universal* if it primitively represents all n-ary R-lattices. A primitively 1-universal lattice is simply called *primitively universal*. An R-lattice is called *almost* (primitively) n-universal if it (primitively, resp.) represents almost all (that is, all but finitely many) n-ary R-lattices. We also define almost (primitively) universal lattices similarly. Clearly a (primitively) n-universal lattice is almost (primitively, resp.) n-universal. If a $\mathbb{Z}$-lattice L is almost (primitively) n-universal then L_p is (primitively, resp.) n-universal for all prime p. The converse is not true in general. However, the following Cassels' theorem serves a partial converse. It is known that the conclusion of the theorem is no longer true for $n = 3$ or if the word "primitively" is omitted.

Theorem 2.3.7 (4, Ch. 11, Theorem 1.6)**.** *Let L be a $\mathbb{Z}$-lattice of rank $n \geq 4$. Then there is an integer N with the following property:*

If $a \geq N$ is an integer which is primitively represented by L_p for all primes p, then a is primitively represented by L.

Recently Earnest and Gunawardana established a connection between the primitive universality and isotropy of $\mathbb{Z}_p$-lattices.

Theorem 2.3.8 (8, Corollary3.10)**.** *A primitively universal $\mathbb{Z}_p$-lattice is isotropic.*

Chapter 3

Primitively n-universal $\mathbb{Z}_p$-lattices of minimal rank

3.1 Generalities

In this section, we prove a necessary space condition of primitive n-universality. If M is a primitively n-universal quadratic $\mathbb{Z}_p$-lattice, then the space $\mathbb{Q}_pM$ must be represent an $2n$-dimensional hyperbolic space. In particular, we have $u_p^*(n) \geq 2n$.

Let K be a field complete with respect to an absolute value $|\cdot|$ satisfying the strong triangle inequality and let $\mathfrak{o} := \{x \in K : |x| \leq 1\}$.

Lemma 3.1.1 (5, Theorem 3.3)**.** *Let $n \geq 1$ and define a norm of $\mathbf{c} = (c_1, \ldots, c_n) \in K^n$ by $\|\mathbf{c}\| := \max_i |c_i|$. Denote the derivative matrix and Ja-*

cobian of $\mathbf{f}(\mathbf{X}) = \mathbf{f}(X_1, \dots, X_n) = (f_1(\mathbf{X}), \dots, f_n(\mathbf{X})) \in K[X_1, \dots, X_n]^n$ *by* $(D\mathbf{f})(\mathbf{X}) = \left(\frac{\partial f_i}{\partial X_j}\right)_{1 \le i,j \le n}$ *and* $J_{\mathbf{f}}(\mathbf{X}) = \det((D\mathbf{f})(\mathbf{X}))$.

Let $\mathbf{f} \in \mathfrak{o}[\mathbf{X}]^n$ *and* $\mathbf{a} \in \mathfrak{o}^n$ *satisfy* $\|\mathbf{f}(\mathbf{a})\| < |J_{\mathbf{f}}(\mathbf{a})|^2$. *Then there is a unique* $\boldsymbol{\alpha} \in \mathfrak{o}^n$ *such that* $\mathbf{f}(\boldsymbol{\alpha}) = \mathbf{0}$ *and* $\|\boldsymbol{\alpha} - \mathbf{a}\| < |J_{\mathbf{f}}(\mathbf{a})|$.

Corollary 3.1.2 (5, Theorem 3.8). *For* $m \ge n$, *let* $\mathbf{f} = (f_1, \dots, f_n) \in \mathfrak{o}[X_1, \dots, X_m]^n$ *and* $\mathbf{a} = (a_1, \dots, a_m) \in \mathfrak{o}^m$ *satisfy* $\|\mathbf{f}(\mathbf{a})\| < |J_{\mathbf{f},n}(\mathbf{a})|^2$ *where* $J_{\mathbf{f},n}(\mathbf{a}) = \det\left(\frac{\partial f_i}{\partial X_j}\right)_{1 \le i,j \le n}$. *Then there is an* $\boldsymbol{\alpha} \in \mathfrak{o}^n$ *such that*

$$\mathbf{f}(\alpha_1, \dots, \alpha_n, a_{n+1}, \dots, a_m) = \mathbf{0}$$

and $|\alpha_i - a_i| < |J_{\mathbf{f},n}(\mathbf{a})|$ *for* $i = 1, \dots, n$.

Lemma 3.1.3. *For* $m \ge n \ge 1$, *let* $F = (f_{ij})_{m \times m}$ *and* $G = (g_{ij})_{n \times n}$ *be symmetric matrices over* $\mathfrak{o}$, *and let* $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) = (a_{ij})_{m \times n}$ *be a matrix over* $\mathfrak{o}$ *such that* $A^t F A = G$. *Suppose that* F *has nonzero determinant and* A *is primitive. Then for any* $(h_1, \dots, h_n) \in \mathfrak{o}^n$ *satisfying* $\max_{1 \le i \le n} |g_{in} - h_i| < 4|\det F|^2$, *there is an* $\boldsymbol{\alpha} \in \mathfrak{o}^m$ *such that* $B = (\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \boldsymbol{\alpha})$ *is again primitive and* $B^t F B = (g'_{ij})$ *satisfies*

$$g'_{ij} = \begin{cases} h_j & \text{if } i = n, \\ h_i & \text{if } j = n, \\ g_{ij} & \text{otherwise.} \end{cases}$$

Proof. Regard $g_{1n} - h_1 = g_{n1} - h_1$, $\dots$, $g_{n-1,n} - h_{n-1} = g_{n,n-1} - h_{n-1}$ and $g_{nn} - h_n$ as n polynomials in m variables $a_{1n}, \dots, a_{mn}$. To apply the previous corollary, it suffices to show that the derivative matrix has an $n \times n$ subdeterminant whose absolute value is $\geq 2|\det F|$.

First note that the derivative matrix is $\mathrm{diag}(1, \ldots, 1, 2)A^t F$. Hence it suffices to show that $A^t F$ has an $n \times n$ subdeterminant whose absolute value is $\geq |\det F|$. By the primitivity of A, we may complete A^t to an element of $GL_m \mathfrak{o}$, namely

$$U = \left(\begin{array}{c} A^t \\ \hline C \end{array} \right).$$

By the theory of modules over PID, there is a $V \in GL_m \mathfrak{o}$ such that $UFV = T = (t_{ij})_{m \times m}$ is lower triangular. Clearly $A^t FV = (t_{ij})_{n \times m}$ has a submatrix (the leftmost one) whose determinant is $d = \prod_{i=1}^n t_{ii}$, then evidently $|d| = |\prod_{i=1}^n t_{ii}| \geq |\prod_{i=1}^m t_{ii}| = |\det UFV| = |\det F|$. Now observe that d is a linear combination of $n \times n$ subdeterminants of $A^t F$, hence there must exist at least one with absolute value $\geq |\det F|$, as desired. $\square$

Corollary 3.1.4. *Let M, N, N' be quadratic $\mathbb{Z}_p$-lattices such that M is nondegenerate. Suppose that N, N' has Gram matrices G, G', respectively such that $G - G' \subsetneq 4(dM)^2\mathbb{Z}_p$. Then M primitively represents N if and only if M primitively represents N'.*

Corollary 3.1.5. *Let M be a nondegenerate quadratic $\mathbb{Z}_p$-lattice. Then the followings are equivalent:*

(1) $\mathrm{ind}\, \mathbb{Q}_p M \geq n$.

(2) *M primitively represents some n-ary quadratic lattice N with $\mathfrak{s}N \subsetneq 4(dM)^2\mathbb{Z}_p$.*

(3) *M primitively represents every n-ary quadratic lattice N with $\mathfrak{s}N \subsetneq 4(dM)^2\mathbb{Z}_p$.*

Corollary 3.1.6. *If M is a primitively n-universal quadratic $\mathbb{Z}_p$-lattice then* $\operatorname{ind}\mathbb{Q}_pM \geq n$.

Corollary 3.1.7. *We have $u_p(n) \geq 2n$. Let M be a primitively n-universal $\mathbb{Z}_p$-lattice of rank $m \geq 2n$.*

(a) *If $m = 2n$, then $\mathbb{Q}_pM$ is hyperbolic.*

(b) *If $m = 2n+1$, then $\mathbb{Q}_pM \cong \mathbb{H}^n \perp \langle(-1)^n dM\rangle$.*

(c) *If $m = 2n+2$ and $dM = (-1)^{n+1}$, then $\mathbb{Q}_pM$ is hyperbolic.*

In particular, any primitively n-universal $(2n)$-ary $\mathbb{Z}_p$-lattice is an n-universal $\mathbb{Z}_p$-lattice on the hyperbolic space $\mathbb{H}^n$, and any primitively n-universal $(2n+1)$-ary $\mathbb{Z}_p$-lattice M is an n-universal $\mathbb{Z}_p$-lattice on the space $\mathbb{H}^n \perp \langle(-1)^n dM\rangle$.

3.2 Primitively n-universal $\mathbb{Z}_p$-lattices of minimal rank for an odd prime p

In this section, we prove that $u_p(n) = 2n$ for any odd prime p and any positive integer n. Also, we show that for any odd prime p, up to isometry, there are exactly one primitively 2-universal quaternary $\mathbb{Z}_p$-lattice and exactly two primitively 3-universal senary $\mathbb{Z}_p$-lattices.

Let $R = \mathbb{Z}$ or $\mathbb{Z}_p$ for a prime p. Recall that $\mathbb{H} \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the even unimodular R-lattice on the hyperbolic plane. If e, f denotes a hyperbolic basis for $\mathbb{H}$, then $Q(\alpha e + \beta f) = 2\alpha\beta$ for any α, $\beta \in R$. Now the proof of the following lemma is quite straightforward.

Lemma 3.2.1. *Let R be either $\mathbb{Z}$ or $\mathbb{Z}_p$ for a prime p.*

(a) *The R-lattice $\mathbb{H}$ primitively represents all even integers. In particular, $\mathbb{H}$ is primitively 1-universal over $\mathbb{Z}_p$ for any odd prime p.*

(b) *If an R-lattice J primitively represents an k-ary R-lattice ℓ, then $\mathbb{H} \perp J$ primitively represents all $(k+1)$-ary R-lattices of the form $\langle \alpha \rangle \perp \ell$ for any even integer α. In particular, $\mathbb{H} \perp \cdots \perp \mathbb{H}$ (n copies) is primitively n-universal over $\mathbb{Z}_p$ for any odd prime p.*

The above lemma shows that for any odd prime p, there is a primitively n-universal $\mathbb{Z}_p$-lattice of $2n$, namely $\mathbb{H}^n$. By combining it with Corollary 3.1.7, we conclude that the minimal rank of primitively n-universal quadratic $\mathbb{Z}_p$-lattices is exactly $2n$.

Lemma 3.2.2. *Let p be an odd prime.*

(a) *A quaternary $\mathbb{Z}_p$-lattice is primitively 2-universal if and only if it is isometric to $\mathbb{H}^2$.*

(b) *A senary $\mathbb{Z}_p$-lattice is primitively 3-universal if and only if it is isometric to either $\mathbb{H}^3$ or $\mathbb{H}^2 \perp p\mathbb{H}$.*

Proof. (a) According to Theorem 2.3.1 and Corollary 3.1.7, any 2-universal quaternary lattice on the hyperbolic space $\mathbb{H}^2$ is isometric to $I_4 \cong \mathbb{H}^2$.
(b) According to Theorem 2.3.2 and Corollary 3.1.7, any 3-universal senary lattice on the hyperbolic space $\mathbb{H}^3$ is isometric to either $\mathbb{H}^3$ or $\mathbb{H}^2 \perp p\mathbb{H}$. It is easily seen that $\mathbb{H}^2 \perp p\mathbb{H}$ also is primitively 3-universal. $\square$

According to Theorem 2.3.2 and Corollary 3.1.7, any 4-universal octonary $\mathbb{Z}_p$-lattice on the space $\mathbb{H}^4$ is isometric to one of the followings, where a is a nonnegative integer and ϵ is a unit in $\mathbb{Z}_p$. Thus they are the candidates of primitively 4-universal octonary $\mathbb{Z}_p$-lattices.

(A) $\mathbb{H}^3 \perp \langle -\epsilon, p^{2a}\epsilon \rangle$.

(B) $\mathbb{H}^3 \perp \langle -p\epsilon, p^{2a+1}\epsilon \rangle$.

(C) $\mathbb{H}^2 \perp \langle -\epsilon \rangle \perp p\mathbb{H} \perp \langle p^{2a+2}\epsilon \rangle$.

3.3 Primitively n-universal $\mathbb{Z}_2$-lattices of minimal rank

The following is a supplement of Lemma 3.2.1 for the case when $R = \mathbb{Z}_2$.

Lemma 3.3.1. (a) *If a $\mathbb{Z}_2$-lattice J is isotropic, then $\mathbb{H} \perp J$ primitively represents all binary $\mathbb{Z}_2$-lattices of the form $2^a\mathbb{H}$ for any nonnegative integer a.*

(b) *If a $\mathbb{Z}_2$-lattice J primitively represents $2^{a+1}\epsilon$ for a nonnegative integer a and a unit $\epsilon \in \mathbb{Z}_2$, then $\mathbb{H} \perp J$ primitively represents binary $\mathbb{Z}_2$-lattices of the form $2^a\mathbb{H}$ and $2^a\mathbb{A}$. In particular, $\mathbb{H} \perp \mathbb{H}$ primitively represents all binary $\mathbb{Z}_2$-lattices of the form $2^a\mathbb{H}$ and $2^a\mathbb{A}$ for any nonnegative integer a. Hence, $\mathbb{H} \perp \cdots \perp \mathbb{H}$ (n copies) primitively represents all n-ary $\mathbb{Z}_2$-lattices ℓ with $\mathfrak{n}\ell \subseteq 2\mathbb{Z}_2$.*

(c) *The binary $\mathbb{Z}_2$-lattice $\langle 1, -1\rangle$ represents all units in $\mathbb{Z}_2$ and all integers in $4\mathbb{Z}_2$. It primitively represents all units in $\mathbb{Z}_2$ and all integers in $8\mathbb{Z}_2$.*

(d) *For a unit $\epsilon \in \mathbb{Z}_2$, $\mathbb{H} \perp \langle\epsilon\rangle$ is isometric to $\langle 1, -1, \epsilon\rangle$. Hence $\mathbb{H} \perp \langle\epsilon\rangle$ primitively represents all binary lattices of the form $\langle\alpha, \epsilon\rangle$ for any integer $\alpha \in \mathbb{Z}_2$. In particular, $\mathbb{H} \perp \langle\epsilon\rangle$ is primitively 1-universal over $\mathbb{Z}_2$.*

(e) *For a unit $\epsilon \in \mathbb{Z}_2$, $\mathbb{H} \perp \mathbb{H} \perp \langle\epsilon\rangle$ primitively represents all ternary lattices of the form $\ell' \perp \langle\epsilon\rangle$ for any binary $\mathbb{Z}_2$-lattice ℓ'. In particular, $\mathbb{H} \perp \mathbb{H} \perp \langle\epsilon\rangle$ is primitively 2-universal over $\mathbb{Z}_2$.*

(f) *If a $\mathbb{Z}_2$-lattice J primitively represents some unit in $\mathbb{Z}_2$, then $\mathbb{H} \perp \cdots \perp \mathbb{H} \perp J$ (n copies of $\mathbb{H}$) primitively represents all $(n+m)$-ary $\mathbb{Z}_2$-lattices of the form $\ell' \perp \ell$ for any m-ary $\mathbb{Z}_2$-lattice ℓ primitively represented by J and for any n-ary $\mathbb{Z}_2$-lattice ℓ'. In particular, $\mathbb{H} \perp \cdots \perp \mathbb{H} \perp J$ is primitively n-universal over $\mathbb{Z}_2$.*

Proof. (a) Since J is isotropic, there is a primitive vector $x \in J$ with $Q(x) = 0$. Observe that $\mathbb{Z}_2[e, 2^a f + x] \cong 2^a\mathbb{H}$. (b) Pick a primitive vector $x \in J$ with

$Q(x) = 2^{a+1}\epsilon$. Then $\mathbb{Z}_2[\epsilon^{-1}e, -e + 2^a\epsilon f + x] \cong 2^a\mathbb{H}$ and

$$\mathbb{Z}_2[e + 2^a\epsilon f, e + x] \cong 2^a\epsilon\mathbb{A} \cong 2^a\mathbb{A}.$$

For the latter isometry, see [19, 93:11]. (c) Let e, f be a basis for the given lattice so that $Q(\alpha e+\beta f) = \alpha^2 - \beta^2$ for any α, $\beta \in \mathbb{Z}_2$. If exactly one of α, β is odd, then so is $\alpha^2 - \beta^2$. If both are odd, then $\alpha^2 - \beta^2 \in 8\mathbb{Z}_2$. If both are even, then $\alpha^2 - \beta^2 \in 4\mathbb{Z}_2$. Now observe that the given lattice primitively represents 1, $4 - 1 = 3$, $1 - 4 = -3$, -1, and $Q(\sqrt{1 + 8\alpha}e + f) = 8\alpha$ for any $\alpha \in \mathbb{Z}_2$. (d) See [19, 93:16]. (e) Combine (b) and (d). (f) By (b), we may assume that ℓ' is unimodular. If J primitively represents $\epsilon \in \mathbb{Z}_2^\times$ then $J \cong \langle\epsilon\rangle \perp \cdots$. Now apply (d) and (e) inductively. □

By the lemma, $\mathbb{H}^n \perp \langle\epsilon\rangle$ is prmitively n-universal for any unit ϵ in $\mathbb{Z}_2$. By combining it with Corollary 3.1.7, we conclude that $2n \le u_2^*(n) \le 2n + 1$.

3.3.1 Classification of primitively 2-universal $\mathbb{Z}_2$-lattices

In this subsection, we prove that $u_2^*(2) = 5$, and any primitively 2-universal quinary $\mathbb{Z}_2$-lattice is isometric to $\mathbb{H}^2 \perp \langle\epsilon\rangle$ for some unit ϵ in $\mathbb{Z}_2$.

We know that $u_2(2) = 5$ according to Theorem 2.3.3 or [17, Lemma 2.3]. Thus, we have $u_2^*(2) \ge u_2(2) = 5$. Therefore, the minimal rank of primitively 2-universal $\mathbb{Z}_2$-lattices is five. By Theorem 2.3.3 (or [17, Lemma 2.3]) and Corollary 3.1.7, any 2-universal quinary $\mathbb{Z}_2$-lattice L on the space $\mathbb{H}^2 \perp \langle dL\rangle$ is isometric to one of the following lattices. Thus they are the candidates of primitively 2-universal quinary $\mathbb{Z}_2$-lattices. Hereafter in this section, a, a_i

denote nonnegative integers, α, β, α_i denote integers in $\mathbb{Z}_2$, and ϵ, δ, ϵ_i denote units in $\mathbb{Z}_2$.

(A) $\mathbb{H}^2 \perp \langle \epsilon \rangle$ (B) $\mathbb{H}^2 \perp \langle 2\epsilon \rangle$ (C) $\mathbb{H} \perp \langle \epsilon, \epsilon, -4\epsilon \rangle$

(D) $\mathbb{H} \perp \langle \epsilon, 2, -2 \rangle$ (E) $\mathbb{H} \perp \langle -1, 2\epsilon, 4 \rangle$ (F) $\mathbb{H} \perp \langle \epsilon, -2\epsilon, 8\epsilon \rangle$

(G) $\mathbb{H} \perp \langle \epsilon, 2\epsilon, -8\epsilon \rangle$

Lemma 3.3.2. *A $\mathbb{Z}_2$-lattice L is primitively 2-universal if and only if $L \cong \mathbb{H}^2 \perp \langle \epsilon \rangle$ for some unit ϵ in $\mathbb{Z}_2$.*

Proof. It suffices to show the "only if" part. Moreover, it suffices to prove the existence of a binary $\mathbb{Z}_2$-lattice not primitively represented by each of the lattices (B)–(G) only for $\epsilon = 1$, since a $\mathbb{Z}_2$-lattice L is primitively 2-universal if and only if so is a scaling of L by ϵ for any unit ϵ in $\mathbb{Z}_2$.

(B) Let L be a quinary $\mathbb{Z}_2$-lattice such that $L \cong \langle 1, 3, 1, 3, 2 \rangle$ in a basis $e_1, \ldots, e_5$. We claim that $2\mathbb{A}$ is not primitively represented by L. Let $z = \sum_1^5 z_i e_i$ be a typical primitive vector in L such that

$$4 = Q(z) = z_1^2 + 3z_2^2 + z_3^2 + 3z_4^2 + 2z_5^2.$$

Since it is impossible that $z_1, \ldots, z_4$ are all even, we may assume that z_i is odd for some $1 \le i \le 4$. Since $M := \mathbb{Z}_2[e_i, z - z_i e_i]$ is unimodular, it splits L. Note that

$$M \cong \langle 1, 3 \rangle \quad \text{and} \quad M^\perp \cong \langle 1, 3, 2 \rangle.$$

This observation allows us to assume that $z_3 = z_4 = z_5 = 0$.

Now suppose to the contrary that we have two primitive vectors $z = z_1e_1 + z_2e_2$, $w = \sum_1^5 w_ie_i$ in L such that $Q(z) = 4 = Q(w)$ and $B(z, w) = 2$. Since $z_1z_2 \equiv 1 \pmod 2$ and $2 = B(z, w) = z_1w_1 + 3z_2w_2$, one may easily show that $Q(w_1e_1 + w_2e_2) \equiv 4 \pmod 8$ and $w_3e_3 + w_4e_4 + w_5e_5$ is primitive. However, any integer in $8\mathbb{Z}_2$ is not primitively represented by $\langle 1, 3, 2\rangle$, which is a contradiction. Hence, $2\mathbb{A}$ is not primitively represented by L.

(C) Let $L \cong \mathbb{H} \perp \langle 1, 1, -4\rangle \cong \langle 3\rangle \perp \langle 5, 5, 5, -4\rangle$. By Theorems 2.2.2 and 2.2.6,

$$\langle 3\rangle \perp M \cong \langle 3\rangle \perp \langle 5, 5, 5, -4\rangle \quad \text{implies} \quad M \cong \langle 5, 5, 5, -4\rangle.$$

If L were primitively 2-universal, then $\langle 5, 5, 5, -4\rangle$ must be primitively universal, which is false according to [7, Theorem 5.2]. In particular, 8 is not primitively represented by $\langle 5, 5, 5, -4\rangle$. Hence, L is not primitively 2-universal.

(D) Let $L \cong \mathbb{H} \perp \langle 1, 2, -2\rangle \cong \langle 5\rangle \perp \langle 1, 3, 2, -2\rangle$. By Theorems 2.2.2 and 2.2.6,

$$\langle 5\rangle \perp M \cong \langle 5\rangle \perp \langle 1, 3, 2, -2\rangle \quad \text{implies} \quad M \cong \langle 1, 3, 2, -2\rangle.$$

If L were primitively 2-universal, then $\langle 1, 3, 2, -2\rangle$ must be primitively universal, which is false according to [7, Theorem 5.2]. In particular, 8 is not primitively represented by $\langle 1, 3, 2, -2\rangle$. Hence, L is not primitively 2-universal.

(F)(G) Let $L \cong \mathbb{H} \perp \langle 1, \mp 2, \pm 8\rangle \cong \langle 5\rangle \perp \langle 1, 3, \mp 2, \pm 8\rangle$. By Theorems 2.2.2 and 2.2.6,

$$\langle 5\rangle \perp M \cong \langle 5\rangle \perp \langle 1, 3, \mp 2, \pm 8\rangle \quad \text{implies} \quad M \cong \langle 1, 3, \mp 2, \pm 8\rangle.$$

If L were primitively 2-universal, then $\langle 1, 3, \mp 2, \pm 8\rangle$ must be primitively universal, which is false according to [7, Theorem 5.2]. In particular, 32 is not primitively represented by $\langle 1,3,\mp 2,\pm 8\rangle$. Hence, L is not primitively 2-universal.

(E) We prove that $L \cong \mathbb{H} \perp \langle -1,2,4\rangle \cong \langle 1,3,3,2,4\rangle$ cannot primitively represent $\langle 10,16\rangle$. By Proposition 2.2.12, it is logically equivalent to prove that $\langle 10,16\rangle$ is not primitively represented by the genus of $\langle 1,2,3,3,4\rangle$ over $\mathbb{Z}$, since evidently $\langle 10,16\rangle$ is represented by $\langle 1,2,3,3,4\rangle$ at any odd prime, and such representation must be primitive since the discriminant of $\langle 10,16\rangle$ is square-free at such prime.

There are six classes in the genus:

$$\langle 1,1,2,3,12\rangle, \quad \langle 1,2,3,3,4\rangle, \quad \langle 1,4,6\rangle \perp \mathbb{A},$$
$$\langle 2,3\rangle \perp \begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&4 \end{pmatrix}, \quad \langle 1,6\rangle \perp \begin{pmatrix} 2&1&1\\ 1&3&0\\ 1&0&3 \end{pmatrix}, \quad \langle 1,2\rangle \perp \begin{pmatrix} 4&-2&1\\ -2&4&1\\ 1&1&4 \end{pmatrix}.$$

For each of above six lattices, one may easily check that $\langle 10,16\rangle$ is not primitively represented by it using a direct computation, for there are only finitely many possibilities. $\square$

3.3.2 The minimal rank of primitively 3-universal $\mathbb{Z}_2$-lattices

In this subsection, we prove that $u_2^*(3) = 7$.

We have to determine whether $u_2^*(3) = 6$ or 7. According to Theorem 2.3.4 and Corollary 3.1.7, any 3-universal senary $\mathbb{Z}_2$-lattice on the hyperbolic space $\mathbb{H}^3$ is isometric to one of the following eight lattices.

(A) $\mathbb{H}^2 \perp \langle 1,-1\rangle$ (B) $\mathbb{H}^2 \perp \langle -1,4\rangle$ (C) $\mathbb{H}^2 \perp \langle 1,-4\rangle$

(D) $\mathbb{H} \perp \langle 1,-1,2,-2\rangle$ (E) $\mathbb{H} \perp \langle 1,-1,-2,8\rangle$ (F) $\mathbb{H} \perp \langle 1,-1,2,-8\rangle$

(G) $\mathbb{H} \perp \langle -1, 2, -2, 4\rangle$ (H) $\mathbb{H} \perp \langle -1, -2, 4, 8\rangle$

Lemma 3.3.3. *No senary lattice is primitively* 3*-universal.*

Proof. It suffices to show that none of the eight 3-universal senary $\mathbb{Z}_2$-lattice is primitively 3-universal. First, Let L be one of (A), (B), (C), or (G). Since L is 2-universal, $\mathbb{A}$ is represented by L. For any sublattice $M \cong \mathbb{A}$ of L, M splits L and

$$M^{\perp} \cong \langle 1, 1, 1, 5\rangle,\ \langle 5, 5, 5, 4\rangle,\ \langle 3, 3, 3, -4\rangle, \text{ or } \langle 5, 2, 2, 4\rangle$$

by Theorems 2.2.2 and 2.2.6. Next, suppose L is one of (D), (E), (F), or (H). Then $\langle 1, 3\rangle$ is primitively represented by L. For any sublattice $M \cong \langle 1, 3\rangle$ of L, M splits L and

$$M^{\perp} \cong \langle 1, 3, 2, -2\rangle,\ \langle 1, 3, -2, 8\rangle,\ \langle 1, 3, 2, -8\rangle, \text{ or } \langle 3, -2, 4, 8\rangle$$

again by Theorems 2.2.2 and 2.2.6. In any case, if L were primitively 3-universal, then $M^{\perp}$ must be primitively universal. However, according to [7, Theorem 5.2], $M^{\perp}$ is not primitively universal. Thus, L is not primitively 3-universal. $\square$

According to Theorem 2.3.4 and Corollary 3.1.7, any 3-universal septenary $\mathbb{Z}_2$-lattice L on the space $\mathbb{H}^3 \perp \langle -dL\rangle$ is isometric to one of the following lattices, where lattices are numbered in accordance with Theorem 2.3.4. In this list, M stands for a binary unimodular $\mathbb{Z}_2$-lattice, N stands for a $\mathbb{Z}_2$-lattice of the form $\langle \epsilon, 2\delta\rangle$, and we assume that

$$-\alpha \in Q(\mathbb{Q}_2 M) \quad \text{and} \quad -\beta \in Q(\mathbb{Q}_2 N).$$

(A) $\mathbb{H} \perp \langle 1, -1\rangle \perp M \perp \langle \alpha\rangle$ ($\alpha \in \mathbb{Z}_2$).

(B) $\mathbb{H} \perp \langle 1, -1\rangle \perp N \perp \langle \beta\rangle$ ($\beta \in 2\mathbb{Z}_2$),

$\mathbb{H} \perp \langle 1, -1, \epsilon\rangle \perp 2\mathbb{H}$, or

$\mathbb{H} \perp M \perp \langle -1, 4\rangle \perp \langle \alpha\rangle$, $\mathbb{H} \perp M \perp \langle 1, -4\rangle \perp \langle \alpha\rangle$ ($\alpha \in 4\mathbb{Z}_2$).

(C1) $\langle 1, -1\rangle \perp M \perp \langle 2, -2\rangle \perp \langle \alpha\rangle$ ($\alpha \in 2\mathbb{Z}_2$).

(C2) $\mathbb{H} \perp \langle -1\rangle \perp N \perp \langle 4\rangle \perp \langle \beta\rangle$ ($\beta \in 4\mathbb{Z}_2$),

$\mathbb{H} \perp \langle 1, -1, 2\epsilon\rangle \perp 4\mathbb{H}$, or

$\langle 1, -1\rangle \perp M \perp \langle -2, 8\rangle \perp \langle \alpha\rangle$, $\langle 1, -1\rangle \perp M \perp \langle 2, -8\rangle \perp \langle \alpha\rangle$ ($\alpha \in 8\mathbb{Z}_2$).

(D1) $\mathbb{H} \perp N \perp 2\mathbb{H} \perp \langle \beta\rangle$ ($\beta \in 2\mathbb{Z}_2$).

(D2) $M \perp \langle -1, 2, -2, 4\rangle \perp \langle \alpha\rangle$ ($\alpha \in 4\mathbb{Z}_2$).

(D3) $\mathbb{H} \perp N \perp \langle 4, -4\rangle \perp \langle \beta\rangle$ ($\beta \in 4\mathbb{Z}_2$).

(D4) $M \perp \langle -1, -2, 4, 8\rangle \perp \langle \alpha\rangle$ ($\alpha \in 8\mathbb{Z}_2$).

3.3.3 Primitive 4-universality over $\mathbb{Z}_2$

In this subsection, we prove that three octonary $\mathbb{Z}_2$-lattices on the hyperbolic space $\mathbb{H}^4$ are almost primitively 4-universal, but not primitively 4-universal. They serve as key ingredients in the next subsection when we prove that $u_2^*(n) = 2n$ for any $n \geq 5$. Currently we do not know whether $u_2^*(4) = 8$ or 9.

According to Theorem 2.3.3 and Corollary 3.1.7, any 4-universal octonary $\mathbb{Z}_2$-lattice on the hyperbolic space $\mathbb{H}^4$ is isometric to one of the following lattices.

(A) $\mathbb{H}^3 \perp \langle -\epsilon, 2^{2a}\epsilon \rangle$

(B) $\mathbb{H}^2 \perp \langle 1, -1, -2\epsilon, 2^{2a+1}\epsilon \rangle$

(C) $\mathbb{H}^2 \perp \langle -\epsilon, -\epsilon, 4\epsilon, 2^{2a+2}\epsilon \rangle$

(D) $\mathbb{H}^2 \perp \langle -\epsilon, -2\epsilon, 4\epsilon, 2^{2a+3}\epsilon \rangle$

(E) $\mathbb{H}^2 \perp \langle -\epsilon, 2, -2, 2^{2a+2}\epsilon \rangle$

(F) $\mathbb{H}^2 \perp \langle -\epsilon, -2\epsilon, 8\epsilon, 2^{2a+4}\epsilon \rangle$

(G) $\mathbb{H}^2 \perp \langle 5\epsilon, 2\epsilon, -8\epsilon, 2^{2a+4} \cdot 3\epsilon \rangle$

Lemma 3.3.4. (a) $\mathbb{H}^3 \perp \langle 1, -1 \rangle$ *primitively represents all quaternary* $\mathbb{Z}_2$-*lattices except* $\mathbb{A} \perp 2\mathbb{A}$.

(b) $\mathbb{H}^2 \perp \langle 1, -1, 2, -2 \rangle$ *primitively represents all quaternary* $\mathbb{Z}_2$*-lattices except* $\langle 1, 3 \rangle \perp 4\mathbb{A}$ *and* $2\mathbb{A} \perp 4\mathbb{A}$.

(c) $\mathbb{H}^2 \perp \langle 1, -1 \rangle \perp 2\mathbb{H}$ *primitively represents all quaternary* $\mathbb{Z}_2$*-lattices except* $\mathbb{H} \perp \mathbb{A}$. *Note that this octonary lattice is not* 4*-universal.*

Proof. (a) Denote by L the given octonary lattice, and by ℓ a quaternary $\mathbb{Z}_2$-lattice. First, assume that ℓ is orthogonally split by $2^a\mathbb{H}$. Since $2^a\mathbb{H}$ is primitively represented by $\mathbb{H} \perp \langle 1, -1 \rangle$, ℓ is primitively represented by L.

Now, assume that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2, 2^{a_3}\epsilon_3, 2^{a_4}\epsilon_4 \rangle$. Since $\langle 1, -1 \rangle$ primitively represents $\mathbb{Z}_2^\times \cup 8\mathbb{Z}_2$, we may assume that $a_1, a_2, a_3, a_4 \in \{1, 2\}$. It is easily seen that, up to rearrangement, at least one among

$$2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2, \quad 2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2 + 2^{a_3}\epsilon_3, \quad 2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2 + 2^{a_3}\epsilon_3 + 2^{a_4}\epsilon_4$$

is a multiple of 8, so that it is primitively represented by the $\mathbb{Z}_2$-lattice $\langle 1, -1\rangle$. Hence, one may easily verify that ℓ is primitively represented by L.

Next, assume that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle \perp 2^{a_3}\mathbb{A}$. We may suppose $a_1, a_2 \in \{1, 2\}$ and $a_3 \in \{0, 1\}$. It is easily seen that, up to rearrangement, at least one among

$$2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2, \quad 2^{a_1}\epsilon_1 + 2^{a_3+1}, \quad 2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2 + 2^{a_3+1}$$

is a multiple of 8, or we have $a_1 = a_2 = 1$ and $a_3 = 0$. In the former, one may easily verify that ℓ is primitively represented by L. In the latter, observe that

$$\mathbb{Z}_2[x_7e_7 + x_8e_8 + e_5 - 3e_6, e_1 + \epsilon_2 e_2, e_3 + 3e_4, e_3 + e_5 + 3e_6]$$

is a primitive sublattice of $\mathbb{H}^3 \perp \langle 1, -1\rangle$ isometric to $\langle 2\epsilon_1, 2\epsilon_2\rangle \perp \mathbb{A}$, where $x_7, x_8 \in \mathbb{Z}_2^\times$ with $x_7^2 - x_8^2 = 2^{a_1}\epsilon_1 + 6$.

Finally, assume that $\ell \cong 2^{a_1}\mathbb{A} \perp 2^{a_3}\mathbb{A}$. We may suppose $a_1, a_3 \in \{0, 1\}$. If $a_1 = a_3 = 0$, then clearly $L \cong \ell \perp \mathbb{H} \perp \langle 1, -1\rangle$. If $a_1 = a_3 = 1$, then

$$\mathbb{Z}_2[e_1 + 2e_2, e_1 + e_3 + 2e_4, e_5 + 2e_6, e_3 - 2e_4 + e_5 + 3e_7 - e_8]$$

is a primitive sublattice of $\mathbb{H}^3 \perp \langle 1, -1\rangle$ isometric to $2\mathbb{A} \perp 2\mathbb{A}$.

Now we prove that $\mathbb{A} \perp 2\mathbb{A}$ is not primitively represented by L. It suffices to show that $2\mathbb{A}$ is not primitively represented by $\langle 1, 3, 1, 3\rangle \perp \mathbb{A}$.

Suppose that $M \cong \langle 1, 3, 1, 3\rangle \perp \mathbb{A}$ in $e_1, \ldots, e_6$. Let $z = \sum_1^6 z_i e_i$ be a typical primitive vector in M such that $4 = Q(z) = z_1^2 + 3z_2^2 + z_3^2 + 3z_4^2 + 2(z_5^2 + z_5z_6 + z_6^2)$. Since it is impossible that $z_1, \ldots, z_4$ are all even, we may assume that z_i is odd for some $1 \le i \le 4$. Since $N := \mathbb{Z}_2[e_i, z - z_ie_i]$ is unimodular, it splits L.

Note that

$$N \cong \langle 1, 3\rangle \quad \text{and} \quad N^{\perp} \cong \langle 1, 3\rangle \perp \mathbb{A}.$$

This observation allows us to assume that $z_3 = z_4 = z_5 = z_6 = 0$.

Now suppose to the contrary that we have two primitive vectors $z = z_1e_1 + z_2e_2$, $w = \sum_1^6 w_ie_i$ in M such that $Q(z) = 4 = Q(w)$ and $B(z, w) = 2$. Since $z_1z_2 \equiv 1 \pmod 2$ and $2 = B(z, w) = z_1w_1 + 3z_2w_2$, one may easily show that $Q(w_1e_1 + w_2e_2) \equiv 4 \pmod 8$ and $\sum_3^6 w_ie_i$ is primitive. However, any integer in $8\mathbb{Z}_2$ is not primitively represented by $\langle 1, 3\rangle \perp \mathbb{A}$, which is a contradiction. Hence, $2\mathbb{A}$ is not primitively represented by M.

(b) Denote by L the given octonary lattice, and by ℓ a quaternary $\mathbb{Z}_2$-lattice. Suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2, 2^{a_3}\epsilon_3, 2^{a_4}\epsilon_4\rangle$ and assume that $a_1 \le a_2 \le a_3 \le a_4$. Since $\langle 1, -1\rangle \perp \langle 2, -2\rangle$ primitively represents all binary lattices of the form $\langle \epsilon\rangle \perp \langle \delta\rangle$, $\langle \epsilon\rangle \perp \langle 2\delta\rangle$, $\langle \epsilon\rangle \perp \langle 2^4\alpha\rangle$, $\langle 2\epsilon\rangle \perp \langle 2^3\alpha\rangle$, $\langle 2^3\alpha\rangle \perp \langle 2^3\beta\rangle$, we may suppose (a_1, a_2, a_3, a_4) to be one among the following quadruples:

$$\begin{gathered}(0,0,0,2),\ (0,0,0,3),\ (0,0,2,2),\ (0,0,2,3),\ (0,2,2,2),\ (0,2,2,3),\ (1,1,1,1),\\ (1,1,1,2),\ (1,1,2,2),\ (1,2,2,2),\ (2,2,2,2),\ (2,2,2,3).\end{gathered}$$

First, assume that $a_1 = 0$. Then ϵ_1 is primitively represented by $\langle 1, -1\rangle$. If $a_4 = 3$, then either $2^2\epsilon_2 + 2^2\epsilon_3 \equiv 0 \pmod{16}$ or $2^2\epsilon_2 + 2^2\epsilon_3 + 2^3\epsilon_4 \equiv 0 \pmod{16}$. Hence, for instance, if $(a_2, a_3) = (0, 2)$, then $\langle \epsilon_2, -\epsilon_2\rangle \perp \mathbb{H} \perp \langle 2, -2\rangle$ primitively represents $\langle \epsilon_2, 2^2\epsilon_3, 2^3\epsilon_4\rangle$, for either the primitive sublattice

$$\mathbb{Z}_2\left[e_1, 2e_2 + \left(\frac{\epsilon_2 + \epsilon_3}{2} + 1\right)e_5 + \left(\frac{\epsilon_2 + \epsilon_3}{2} - 1\right)e_6, e_3 + 2^2\epsilon_4e_4\right]$$

or the primitive sublattice

$$\mathbb{Z}_2\left[e_1, 2e_2 + \left(\frac{\epsilon_2+\epsilon_3}{2}+1\right)e_5 + \left(\frac{\epsilon_2+\epsilon_3}{2}-1\right)e_6 + e_3 - 2^2\epsilon_4 e_4, e_3 + 2^2\epsilon_4 e_4\right]$$

is isometric to $\langle \epsilon_2, 2^2\epsilon_3, 2^3\epsilon_4\rangle$. The primitive representability for cases when $(a_2, a_3) = (0,0)$ or $(2,2)$ can be proved in a similar manner. If $a_4 = 2$, then at least one among $2^2\epsilon_3 + 2^2\epsilon_4$, $2^2\epsilon_2 + 2^2\epsilon_3$, $2^2\epsilon_1 + 2^2\epsilon_2$, $2^2(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$ is congruent to 0 modulo 16, and hence, is primitively represented by $\langle 2, -2\rangle$. Hence the primitive representability for this case can be proved in a similar manner.

Next, assume that $a_1 = 1$. Then $2\epsilon_1$ is primitively represented by $\langle 2, -2\rangle$, and at least one among the following integers is congruent to 0 modulo 8:

$$2^{a_3}\epsilon_3 + 2^{a_4}\epsilon_4,\ 2^{a_2}\epsilon_2 + 2^{a_3}\epsilon_3,\ 2^{a_2}\epsilon_2 + 2^{a_3}\epsilon_3 + 2^{a_4}\epsilon_4,\ 2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2,$$
$$2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2 + 2^{a_3}\epsilon_3 + 2^{a_4}\epsilon_4.$$

Hence, such an integer is primitively represented by $\langle 1, -1\rangle$, which leads to the primitive representability for cases when $a_1 = 1$.

Finally, assume that $a_1 = 2$. If $(a_1, a_2, a_3, a_4) = (2,2,2,3)$, then $2^3\epsilon_4$ is primitively represented by $\langle 1, -1\rangle$, and either $2^2\epsilon_2 + 2^2\epsilon_3 \equiv 0 \pmod{16}$ or $2^2\epsilon_2 + 2^2\epsilon_3 + 2^3\epsilon_4 \equiv 0 \pmod{16}$. If $(a_1, a_2, a_3, a_4) = (2,2,2,2)$, then there exists a pair among $2^2\epsilon_1$, $2^2\epsilon_2$, $2^2\epsilon_3$, $2^2\epsilon_4$, which adds up to be congruent to 8 modulo 16 (for if $\epsilon_1+\epsilon_3 \equiv \epsilon_2+\epsilon_3 \equiv 0 \pmod 4$, then $\epsilon_1+\epsilon_2 \equiv 2 \pmod 4$), say $2^2\epsilon_1$ and $2^2\epsilon_2$. Then $2^2\epsilon_1 + 2^2\epsilon_2$ is primitively represented by $\langle 1, -1\rangle$, and either $2^2\epsilon_3 + 2^2\epsilon_4$ or $2^2\epsilon_1 + 2^2\epsilon_2 + 2^2\epsilon_3 + 2^2\epsilon_4$ is primitively represented by $\langle 2, -2\rangle$, and we are done.

Next, suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle \perp 2^{a_3}\mathbb{H}$. If $a_1 = 0$ or $a_1 \geq 3$ (up to rearrangement), then $\langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle$, and $2^{a_3}\mathbb{H}$ is primitively represented by $\mathbb{H} \perp \langle 2, -2\rangle$. If $a_1 = 1$, then the former is primitively represented by $\mathbb{H} \perp \langle 2, -2\rangle$, and the latter by $\mathbb{H} \perp \langle 1, -1\rangle$. If $a_1 = a_2 = 2$, then $2^{a_1}\epsilon_1 + 2^{a_2}\epsilon_2 \equiv 0 \pmod 8$. Hence, in this case, the former is primitively represented by $A(0,0) \perp \langle 1, -1\rangle$, and the latter by $\mathbb{H} \perp \langle 2, -2\rangle$.

Now, suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle \perp 2^{a_3}\mathbb{A}$. First, assume that $a_3 \geq 3$. If $a_1 = 0$ or $a_1 \geq 3$ (up to rearrangement), then $\langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\mathbb{H}\perp\langle 1, -1\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H}\perp\langle 2, -2\rangle$. If $a_1 = 1$, then $\langle 2\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\mathbb{H} \perp \langle 2, -2\rangle$ or $\langle 1, -1\rangle \perp \langle 2, -2\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H}\perp\langle 1, -1\rangle$. If $a_1 = a_2 = 2$ then $2^2\epsilon_1$ is primitively represented by $\mathbb{H}$, $2^2\epsilon_1+2^2\epsilon_2$ by $\langle 1, -1\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H} \perp \langle 2, -2\rangle$.

Next, assume that $a_3 = 2$. We may assume that a_1, $a_2 \neq 2$ or 3. If $a_1 = 1$ or $a_1 \geq 4$ (up to rearrangement), then $\langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\mathbb{H} \perp \langle 2, -2\rangle$ or $\langle 1, -1\rangle \perp \langle 2, -2\rangle$, and $2^2\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle$. Now, suppose that $a_1 = a_2 = 0$. If $\epsilon_1 + \epsilon_2 \not\equiv 4 \pmod 8$ then $\langle \epsilon_1, \epsilon_2\rangle$ is primitively represented by $\langle 1, -1\rangle \perp \langle 2, -2\rangle$. We are left with the case when $\epsilon_1 + \epsilon_2 \equiv 4 \pmod 8$.

Now, assume that $a_3 = 1$. We may assume that a_1, $a_2 \neq 1$ or 2. If $a_1 \geq 3$ and $a_2 \geq 4$, then $\langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\langle 1, -1\rangle \perp \langle 2, -2\rangle$, and $2\mathbb{A}$ by $\mathbb{H} \perp \mathbb{H}$. If $a_1 = a_2 = 3$, then $2^3\epsilon_1$ is primitively represented by $\langle 1, -1\rangle$, $2^3\epsilon_1 + 2^3\epsilon_2$ by $\langle 2, -2\rangle$, and $2\mathbb{A}$ by $\mathbb{H} \perp \mathbb{H}$. If $a_1 = 0$, then $2^{a_2}\epsilon_2$ is primitively represented by $\langle 1, -1\rangle$, and

$$\mathbb{Z}_2[e_1 - 2\epsilon_1 e_2, e_1 + e_3 + e_4 + 2e_5] \cong -2\epsilon_1\mathbb{A}$$

is a primitive sublattice in $\mathbb{H} \perp \langle 2, -2\rangle \perp \langle -\epsilon_1\rangle$ which is isometric to $2\mathbb{A}$.

Finally, assume that $a_3 = 0$. We may assume that a_1, $a_2 \neq 0$ or 1. It suffices to show that $\ell' \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle$ is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle \perp \langle 2, 6\rangle$. If $a_1 \geq 3$ (up to rearrangement), then ℓ' is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle$. If $a_1 = a_2 = 2$, then $2^2\epsilon_1$ is primitively represented by $\mathbb{H}$, and $2^2\epsilon_1 + 2^2\epsilon_2$ by $\langle 1, -1\rangle$.

If $\ell \cong 2^{a_1}\mathbb{H} \perp 2^{a_3}\mathbb{H}$, then it is clear that ℓ is primitively represented by L.

Suppose that $2^{a_1}\mathbb{H} \perp 2^{a_3}\mathbb{A}$. If $a_3 = 0$ or $a_3 \geq 3$, then $2^{a_1}\mathbb{H}$ is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H} \perp \langle 2, -2\rangle$. If $a_3 = 2$, then $2^{a_1}\mathbb{H}$ is primitively represented by $\mathbb{H} \perp \langle 2, -\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H} \perp \langle 1, -1\rangle$. Now assume that $a_3 = 1$. If $a_1 \geq 1$, then

$$\mathbb{Z}_2[e_1 + e_2, 2^{a_1 - 1}e_1 - 2^{a_1 - 1}e_2 + e_3 + e_4]$$

is a primitive sublattice of $\langle 1, -1\rangle \perp \langle 2, -2\rangle$ that is isometric to $2^{a_1}\mathbb{H}$, and $2\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \mathbb{H}$. If $a_1 = 0$, then

$$\mathbb{Z}_2[e_1 + 2e_2, e_1 + 2e_3 + e_5 + e_6]$$

is a primitive sublattice of $\mathbb{H} \perp \langle 1, -1\rangle \perp \langle 2, -2\rangle$ that is isometric to $2\mathbb{A}$.

Finally, suppose that $2^{a_1}\mathbb{A} \perp 2^{a_3}\mathbb{A}$. We may assume that $a_1 < a_3$. If $a_1 \geq 2$, then $2^{a_1}\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \langle 1, -1\rangle$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H} \perp \langle 2, -2\rangle$. Assume that $a_1 = 0$. It suffices to show that $2^{a_3}\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \langle 1,-1\rangle \perp \langle 2,6\rangle$. If $a_3 \geq 2$, then $2^{a_3}\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \langle 1,-1\rangle$. If $a_3 = 1$, then

$$\mathbb{Z}_2[a_1 + 2a_2, a_1 + 2a_4 + a_5 + a_6]$$

is a primitive sublattice of $\mathbb{H} \perp \langle 1,-1\rangle \perp \langle 2,6\rangle$ that is isometric to $2\mathbb{A}$. Now, assume that $a_1 = 1$. Then $2\mathbb{A}$ is primitively represented by $\mathbb{H} \perp \mathbb{H}$. If $a_3 \geq 3$, then

$$\mathbb{Z}_2\left[(2^{a_3-1}+1)e_1 + (2^{a_3-1}-1)e_2, e_1 - e_2 + (2^{a_3-2}+1)e_3 + (2^{a_3-2}-1)e_4\right]$$

is a primitive sublattice of $\langle 1,-1\rangle \perp \langle 2,-2\rangle$ that is isometric to $2^{a_3}\mathbb{A}$. We are left with the case $a_3 = 2$.

Now we prove that $\langle 1,3\rangle \perp 4\mathbb{A}$ is not primitively represented by L. It suffices to show that $4\mathbb{A}$ is not primitively represented by $\langle 1,-1,1,3,2,-2\rangle$.

Suppose that $M \cong \langle 1,-1,1,3,2,-2\rangle$ in $e_1,\dots,e_6$. Let $z = \sum_1^6 z_ie_i$ be a typical primitive vector in M such that $8 = Q(z) = z_1^2 - z_2^2 + z_3^2 + 3z_4^2 + 2z_5^2 - 2z_6^2$. Since it is impossible that $z_1, \dots, z_4$ are all even, we may assume that z_i is odd for some $1 \leq i \leq 4$. Since $N := \mathbb{Z}_2[e_i, z - z_ie_i]$ is unimodular, it splits M. Note that

$$N \cong \langle 1,-1\rangle \quad \text{and} \quad N^\perp \cong \langle 1,3,2,-2\rangle.$$

This observation allows us to assume that $z_3 = z_4 = z_5 = z_6 = 0$.

Now suppose to the contrary that we have two primitive vectors $z = z_1e_1 + z_2e_2$, $w = \sum_1^6 w_ie_i$ in M such that $Q(z) = 8 = Q(w)$ and $B(z,w) = 4$. Since $z_1z_2 \equiv \pm 3 \pmod 8$ and $4 = B(z,w) = z_1w_1 - z_2w_2$, one may easily show that

$Q(w_1e_1+w_2e_2)\equiv 0 \pmod{16}$ and $\sum_3^6 w_ie_i$ is primitive. However, any integer that is congruent to 8 modulo 16 is not primitively represented by $\langle 1,3,2,-2\rangle$, which is a contradiction. Hence, $4\mathbb{A}$ is not primitively represented by M.

Finally, we prove that $2\mathbb{A}\perp 4\mathbb{A}$ is not primitively represented by L. Suppose to the contrary that there is a primitive sublattice ℓ of L such that

$$\ell\cong 2\mathbb{A}\perp 4\mathbb{A} \quad \text{in} \quad x,y,z,w.$$

Suppose that $L\cong I_6\perp\langle 2,2\rangle$ in $e_1,\dots,e_8$ and write $z=\sum_1^8 z_ie_i$ and $w=\sum_1^8 w_ie_i$. If $z_i\equiv w_i\equiv 0 \pmod 2$ for all $1\le i\le 6$, then we must have $z_7z_8w_7w_8\equiv 1 \pmod 2$ which contradicts the fact that $\mathbb{Z}_2[z,w]$ is a primitive sublattice of L. Hence, by exchanging the role of z and w if necessary, we may assume that z_i is odd for some $1\le i\le 6$. Since $M:=\mathbb{Z}_2[e_i,z-z_ie_i]$ is unimodular, it splits L. Write $w=w_M+w'$ for some $w_M\in M$ and $w'\in L':=M^\perp$. Since $M\cong\langle 1,-1\rangle$, we have

$$B(z,w_M)\equiv B(z,w)\equiv 4 \pmod 8 \quad \text{implies} \quad Q(w_M)\equiv 0 \pmod{16}.$$

Note that if α, β are integers such that $\alpha^2-\beta^2\equiv 0 \pmod{16}$, then $\alpha\equiv\beta \pmod 2$. Hence, z,w_M is not a primitive sequence of vectors in L. Therefore, w' is a primitive vector in L' such that $Q(w')\equiv 8 \pmod{16}$.

First, assume that $\mathfrak{n}L'=\mathbb{Z}_2$. Then $L'\cong\langle 1,-1,1,-1,2,-2\rangle$ in some basis, say $e_1',\dots,e_6'$. Write $w'=\sum_1^6 w_i'e_i'$. If $w_i'\equiv 0 \pmod 2$ for all $1\le i\le 4$, then we must have $w_5'w_6'\equiv 1 \pmod 2$. In this case, since $N:=\mathbb{Z}_2[e_5',w'-w_5'e_5]$ is 2-modular, N splits L', and we have

$$N\cong\langle 2,6\rangle \quad \text{and} \quad N^\perp\cong\langle 1,1,1,5\rangle.$$

Otherwise, $w_i' \equiv 1 \pmod 2$ for some $1 \le i \le 4$. In this case, since $N := \mathbb{Z}_2[e_i', w' - w_i'e_i]$ is unimodular, N splits L', and we have

$$N \cong \langle 1, -2 \rangle \quad \text{and} \quad N^\perp \cong \langle 1, -1, 2, -2 \rangle.$$

Now, assume that $\mathfrak{n}L' \subseteq 2\mathbb{Z}_2$. Then $L' \cong \mathbb{H} \perp \mathbb{H} \perp \langle 2, -2 \rangle$ in some basis, say $e_1', \dots, e_6'$. Write $w' = \sum_1^6 w_i' e_i$. If $w_i' \equiv 0 \pmod 2$ for all $1 \le i \le 4$, then we must have $w_5'w_6' \equiv 1 \pmod 2$. In this case, since $N := \mathbb{Z}_2[e_5', w' - w_5'e_5]$ is 2-modular, N splits L', and we have

$$N \cong \langle 2, 6 \rangle \quad \text{and} \quad N^\perp \cong \mathbb{H} \perp \mathbb{A}.$$

Otherwise, $w_i' \equiv 1 \pmod 2$ for some $1 \le i \le 4$. In this case, let $j = 3 - i$ if $i = 1, 2$ and $j = 7 - i$ if $i = 3, 4$. Since $N := \mathbb{Z}_2[w', e_j]$ is unimodular, N splits L', and we have

$$N \cong \mathbb{H} \quad \text{and} \quad N^\perp \cong \mathbb{H} \perp \langle 2, -2 \rangle.$$

So far we have divided the quadruple (x, y, z, w) into four possibly overlapping cases. In the first case, we may assume that $L \cong \langle 1, -1, 1, 1, 1, 5, 2, 6 \rangle$ in $e_1, \dots, e_8$, $z = z_1e_1 + z_2e_2$, $w = w_1e_1 + w_2e_2 + w_7e_7 + w_8e_8$, and $z_1z_2w_7w_8 \equiv 1 \pmod 2$. Since

$$\left.\begin{aligned} B(z_1e_1 + z_2e_2, x_1e_1 + x_2e_2) &\equiv 0 \\ B(w_1e_1 + w_2e_2, x_1e_1 + x_2e_2) &\equiv 0 \\ B(w_7e_7 + w_8e_8, x_7e_7 + x_8e_8) &\equiv 0 \end{aligned}\right\} \pmod 4,$$

we have $Q(x_1e_1 + x_2e_2) \equiv Q(x_7e_7 + x_8e_8) \equiv 0 \pmod 8$. Hence, $Q(\sum_3^6 x_ie_i) \equiv 4 \pmod 8$. However, this is a contradiction, for no integer that is congruent to 4 modulo 8 is primitively represented by the $\mathbb{Z}_2$-lattice $\langle 1, 1, 1, 5 \rangle$.

In the next case, we may assume that $L \cong \langle 1,-1,1,-1,1,-1,2,-2\rangle$ in $e_1,\dots,e_8$, $z = z_1e_1+z_2e_2$, $w = w_1e_1+w_2e_2+w_3e_3+w_4e_4$, and $z_1z_2w_3w_4 \equiv 1 \pmod 2$. Since

$$\left.\begin{aligned} B(z_1e_1+z_2e_2, x_1e_1+x_2e_2) &\equiv 0 \\ B(w_1e_1+w_2e_2, x_1e_1+x_2e_2) &\equiv 0 \\ B(w_3e_3+w_4e_4, x_3e_3+x_4e_4) &\equiv 0 \end{aligned}\right\} \pmod 4, \tag{*}$$

we have $Q(x_1e_1+x_2e_2) \equiv Q(x_3e_3+x_4e_4) \equiv 0 \pmod 8$. Moreover, the equations (*) and similar equations containing y_i's instead of x_i's lead to the conclusion $Q(y_1e_1+y_2e_2) \equiv Q(y_3e_3+y_4e_4) \equiv 0 \pmod 8$ and

$$B(x_1e_1+x_2e_2, y_1e_1+y_2e_2) \equiv B(x_3e_3+x_4e_4, y_3e_3+y_4e_4) \equiv 0 \pmod 4.$$

Hence, $Q(\sum_5^8 x_ie_i) \equiv Q(\sum_5^8 y_ie_i) \equiv 4 \pmod 8$. However, this implies that $x_5 \equiv x_6 \equiv y_5 \equiv y_6 \equiv 0 \pmod 2$ and $x_7x_8y_7y_8 \equiv 1 \pmod 2$, which is a contradiction.

In the third case, we may assume that $L \cong \langle 1,-2\rangle \perp \mathbb{H} \perp \mathbb{A} \perp \langle 2,6\rangle$ in $e_1,\dots,e_8$, $z = z_1e_1+z_2e_2$, $w = w_1e_1+w_2e_2+w_7e_7+w_8e_8$, and $z_1z_2w_7w_8 \equiv 1 \pmod 2$. A similar reasoning to the first two cases leads to a conclusion that $Q(\sum_3^6 x_ie_i) \equiv Q(\sum_3^6 y_ie_i) \equiv 4 \pmod 8$ and $B(\sum_3^6 x_ie_i, \sum_3^6 y_ie_i) \equiv 2 \pmod 4$. However, one may easily show that $2\mathbb{A}$ is not primitively represented by $\mathbb{H}\perp\mathbb{A}$, which is a contradiction.

In the final case, we may assume that $L \cong \langle 1,-1\rangle \perp \mathbb{H} \perp \mathbb{H} \perp \langle 2,-2\rangle$ in $e_1,\dots,e_8$, $z = z_1e_1+z_2e_2$, $w = w_1e_1+w_2e_2+w_3e_3+w_4e_4$, and $z_1z_2w_3w_4 \equiv 1 \pmod 2$. A similar reasoning to the first two cases leads to a conclusion that

$Q(\sum_5^8 x_ie_i) \equiv Q(\sum_5^8 y_ie_i) \equiv 4 \pmod 8$ and $B(\sum_5^8 x_ie_i, \sum_5^8 y_ie_i) \equiv 2 \pmod 4$. However, one may easily show that $2\mathbb{A}$ is not primitively represented by $\mathbb{H} \perp \langle 2, -2\rangle$, which is a contradiction. Hence, $2\mathbb{A}\perp 4\mathbb{A}$ is not primitively represented by L.

(c) Denote by L the given octonary lattice, and by ℓ a quaternary $\mathbb{Z}_2$-lattice. Suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2, 2^{a_3}\epsilon_3, 2^{a_4}\epsilon_4\rangle$ and assume that $a_1 \le a_2 \le a_3 \le a_4$. Since $\langle 2^2\epsilon, 2^2\delta\rangle$ is primitively represented by $\langle 1, -1\rangle \perp 2\mathbb{H}$, we may assume that the exponents satisfy either

$$\text{(i) } 0 = a_1 \text{ and } a_4 \le 1 \quad \text{or} \quad \text{(ii) } a_1 = a_2 = a_3 = 1.$$

First, assume that case (i) holds. If $a_4 = 0$ then, either there is a pair of $\epsilon_1, \ldots, \epsilon_4$ that adds up to be a multiple of 4, or the sum of all four is a multiple of 4. If $a_3 = 0$ and $a_4 = 1$, then either $\epsilon_2 + \epsilon_3 \equiv 0 \pmod 4$ or $\epsilon_2 + \epsilon_3 + 2\epsilon_4 0 \pmod 4$. If $a_3 = 1$, then $2\epsilon_3 + 2\epsilon_4 \equiv 0 \pmod 4$. Now, assume that case (ii) holds. If $a_4 \ge 3$, then $2^{a_4}\epsilon_4$ is primitively represented by $\langle 1, -1\rangle$, and $2\epsilon_2 + 2\epsilon_3 \equiv 0 \pmod 4$. If $a_4 = 2$, then $2^2\epsilon_4$ is primitively represented by $2\mathbb{H}$, and either $2\epsilon_2 + 2\epsilon_3 \equiv 0 \pmod 8$ or $2\epsilon_2 + 2\epsilon_3 + 2^2\epsilon_4 \equiv 0 \pmod 8$. Finally, suppose that $a_4 = 1$. If there is a pair of $2\epsilon_1, \ldots, 2\epsilon_4$ that adds up to be a multiple of 8, then the sum of the rest two is a multiple of 4. If no such pair exists, then the sum of all four is a multiple of 8, and the sum of any two is a multiple of 4.

Next, suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle \perp 2^{a_3}\mathbb{H}$ and assume that $a_1 \le a_2$. Then we may suppose that $a_1 \le 1$. If $a_1 = 0$, then $2^{a_3}\mathbb{H}$ is primitively represented by $\mathbb{H}\perp 2\mathbb{H}$. Now, suppose that $a_1 = 1$. If $a_3 = 0$, then $\langle 2\epsilon_1, 2^{a_2}\epsilon_2\rangle$

is primitively represented by $\mathbb{H} \perp 2\mathbb{H}$. If $a_3 \geq 1$, then $2^{a_3}\mathbb{H}$ is primitively represented by $\langle 1, -1\rangle \perp 2\mathbb{H}$.

Now, suppose that $\ell \cong \langle 2^{a_1}\epsilon_1, 2^{a_2}\epsilon_2\rangle \perp 2^{a_3}\mathbb{A}$ and assume that $a_1 \leq a_2$. Then we may suppose that $a_1 \leq 1$. Moreover, we may assume that $a_3 \notin \{a_1, a_1-1, a_2, a_2-1\}$. If $a_1 = 0$, then $2^{a_3}\mathbb{A}$ is primitively represented by $\mathbb{H}\perp 2\mathbb{H}$ for $a_3 \geq 1$. If $a_1 = 1$, then $2^{a_3}\mathbb{A}$ is primitively represented by $\langle 1, -1\rangle \perp 2\mathbb{H}$ for $a_3 \geq 2$.

Clearly, $2^{a_1}\mathbb{H} \perp 2^{a_3}\mathbb{H}$ is primitively represented by L. Suppose that $\ell \cong 2^{a_1}\mathbb{H} \perp 2^{a_3}\mathbb{A}$. If $a_1 \geq 1$, then $2^{a_1}\mathbb{H}$ is primitively represented by $2\mathbb{H} \perp \langle 1, -1\rangle$. If $a_3 \geq 1$, then $2^{a_3}\mathbb{A}$ is primitively represented by $\mathbb{H} \perp 2\mathbb{H}$.

Finally, suppose that $\ell \cong 2^{a_1}\mathbb{A} \perp 2^{a_3}\mathbb{A}$. We may assume $a_1 < a_3$. If $a_1 \geq 1$, then $2^{a_1}\mathbb{A}$ is primitively represented by $\mathbb{H}\perp 2\mathbb{H}$, and $2^{a_3}\mathbb{A}$ by $\mathbb{H}\perp\langle 1, -1\rangle$. Now, suppse that $a_1 = 0$. then $a_3 \geq 1$ and it suffices to show that $2^{a_3}\mathbb{A}$ is primitively represented by $\mathbb{A}\perp\langle 1, -1\rangle\perp 2\mathbb{H}$. If $a_3 = 1$, observe that $\langle -1\rangle \perp 2\mathbb{H} \cong \langle 3\rangle \perp 2\mathbb{A}$. If $a_3 \geq 2$, then $2^{a_3}\mathbb{A}$ is primitively represented by $\langle 1, -1\rangle \perp 2\mathbb{H}$.

Now, we prove that L cannot primitively represent $\mathbb{H} \perp \mathbb{A}$. It suffices to show $\mathbb{H}\perp\langle 1, -1\rangle\perp 2\mathbb{H}$ cannot primitively represent $\mathbb{A}$. Pick any primitive vector $z = \sum_1^6 x_i e_i \in M \cong \mathbb{H}\perp\langle 1, -1\rangle\perp 2\mathbb{H}$ with $2 = Q(z) = 2x_1x_2 + x_3^2 - x_4^2 + 4x_5x_6$. $Q(z) \equiv 2 \pmod 4$ implies that $x_1x_2 \equiv 1 \pmod 2$ and $x_3 \equiv x_4 \pmod 2$. However, the B value of two such vectors must be even. $\square$

3.3.4 The minimal rank of primitively n-universal $\mathbb{Z}_2$-lattices for $n \geq 5$

We prove that $u_2^*(n) = 2n$ for $n \geq 5$ by the lemma below.

Lemma 3.3.5. *For $n \geq 5$, the following $\mathbb{Z}_2$-lattices are primitively n-universal.*

(a) $\mathbb{H}^{n-1} \perp \langle 1, -1 \rangle$

(b) $\mathbb{H}^{n-2} \perp \langle 1, -1 \rangle \perp \langle 2, -2 \rangle$

(c) $\mathbb{H}^{n-2} \perp \langle 1, -1 \rangle \perp 2\mathbb{H}$

Proof. (a) Suppose that $n = 5$. Any quinary $\mathbb{Z}_2$-lattice is split by a unary lattice. Hence, by Lemma 3.3.4, it suffices to show that any $\mathbb{Z}_2$-lattice ℓ of the form $\mathbb{A} \perp 2\mathbb{A} \perp \langle 2^a\epsilon \rangle$ is primitively represented by $\mathbb{H}^4 \perp \langle 1, -1 \rangle$. If $a \geq 3$, then $\langle 2^a\epsilon \rangle$ is primitively represented by $\langle 1, -1 \rangle$. If $a \leq 2$, then ℓ has a splitting other than the form $\mathbb{A} \perp 2\mathbb{A} \perp \langle 2^a\epsilon \rangle$.

Suppose that $n = 6$. Any senary $\mathbb{Z}_2$-lattice either is split by a unary lattice or is an orthogonal sum of binary lattices. For the former case, the primitive representability follows from the case $n = 5$. For the latter case, consider a $\mathbb{Z}_2$-lattice of the form $L_1 \perp L_2 \perp L_3$ where L_i are binary lattices. It is clear that $L_1 \perp L_2 \cong L_1 \perp L_3 \cong L_2 \perp L_3 \cong \mathbb{A} \perp 2\mathbb{A}$ is impossible. Hence, we may assume that $L_1 \perp L_2 \not\cong \mathbb{A} \perp 2\mathbb{A}$. Therefore, $L_1 \perp L_2$ is primitively represented by $\mathbb{H}^3 \perp \langle 1, -1 \rangle$, and L_3 by $\mathbb{H}^2$.

The case when $n \geq 7$ follows by induction on n. It follows from the case of $n - 1$ for n odd, from the cases of $n - 1$ and $n - 2$ for n even.

(b) Suppose that $n = 5$. Any quinary $\mathbb{Z}_2$-lattice is split by some unary lattice. Hence, by Lemma 3.3.4, it suffices to show that any $\mathbb{Z}_2$-lattice of the form $\langle 1, 3\rangle \perp 4\mathbb{A} \perp \langle 2^a\epsilon\rangle$ or $2\mathbb{A} \perp 4\mathbb{A} \perp \langle 2^a\epsilon\rangle$ is primitively represented by $\mathbb{H}^3 \perp \langle 1, -1\rangle \perp \langle 2, -2\rangle$. The former case is trivial. For the latter case, we may assume that $a = 0$ or $a \geq 4$. If $a = 0$, then

$$\mathbb{Z}_2[e_1 - 2\epsilon e_2, e_1 + e_7 + e_8 + 2e_9, e_3 + 4e_4, e_3 + e_5 + 4e_6]$$

is a primitive sublattice of $\mathbb{H}^3 \perp \langle 2, -2, -\epsilon\rangle$ that is isometric to $2\mathbb{A} \perp 4\mathbb{A}$. If $a \geq 4$, then $\langle 2^a\epsilon\rangle$ is primitively represented by $\langle 2, -2\rangle$, and $2\mathbb{A} \perp 4\mathbb{A}$ by $\mathbb{H}^3 \perp \langle 1, -1\rangle$ according to Lemma 3.3.4.

Suppose $n = 6$. Any senary $\mathbb{Z}_2$-lattice either is split by some unary lattice, or is an orthogonal sum of binary lattices. For the former case, the primitive representability follows from the case $n = 5$. For the latter case, consider a $\mathbb{Z}_2$-lattice of the form $L_1 \perp L_2 \perp L_3$ where L_i are binary lattices. It is clear that $L_1 \perp L_2 \cong L_1 \perp L_3 \cong L_2 \perp L_3 \cong 2\mathbb{A} \perp 4\mathbb{A}$ is impossible. Hence, we may assume that $L_1 \perp L_2 \ncong 2\mathbb{A} \perp 4\mathbb{A}$. Therefore, $L_1 \perp L_2$ is primitively represented by $\mathbb{H}^2 \perp \langle 1, -1\rangle \perp \langle 2, -2\rangle$, and L_3 by $\mathbb{H}^2$.

The case when $n \geq 7$ follows by induction on n.

(c) Suppose that $n = 5$. Any quinary $\mathbb{Z}_2$-lattice is split by a unary lattice. Hence, by Lemma 3.3.4, it suffices to show that any $\mathbb{Z}_2$-lattice of the form $\mathbb{H} \perp \mathbb{A} \perp \langle 2^a\epsilon\rangle$ is primitively represented by $\mathbb{H}^3 \perp \langle 1, -1\rangle \perp 2\mathbb{H}$. We may assume that $a \geq 2$, and in this case $\langle 2^a\epsilon\rangle$ is primitively represented by $2\mathbb{H}$.

Suppose that $n = 6$. Any senary $\mathbb{Z}_2$-lattice either is split by some unary lattice, or is an orthogonal sum of binary lattices. For the former case, the primitive representability follows from the case $n = 5$. For the latter case, consider a $\mathbb{Z}_2$-lattice of the form $L_1 \perp L_2 \perp L_3$ where L_i are binary lattices. It is clear that $L_1 \perp L_2 \cong L_1 \perp L_3 \cong L_2 \perp L_3 \cong \mathbb{H} \perp \mathbb{A}$ is impossible. Hence, we may assume that $L_1 \perp L_2 \ncong \mathbb{H} \perp \mathbb{A}$. Therefore, $L_1 \perp L_2$ is primitively represented by $\mathbb{H}^2 \perp \langle 1, -1 \rangle \perp 2\mathbb{H}$, and L_3 by $\mathbb{H}^2$.

The case when $n \geq 7$ follows by induction on n. $\square$

Chapter 4

Primitively 2-universal $\mathbb{Z}$-lattices of rank six

4.1 The minimal rank of primitively 2-universal $\mathbb{Z}$-lattices

It is well known in [13] that the minimal rank of 2-universal $\mathbb{Z}$-lattices is five, which implies that the minimal rank of primitively 2-universal $\mathbb{Z}$-lattices is at least five. The aim of this section is to show that the minimal rank of primitively 2-universal $\mathbb{Z}$-lattices is six.

Lemma 4.1.1. (a) *Let V be a quinary quadratic space over a local field F at $\mathfrak{p}$. Then* $\operatorname{ind} V = 2$ *if and only if* $S_{\mathfrak{p}}V = (-1, -dV)$.

(b) *For any quinary $\mathbb{Z}$-lattice L, there are infinitely many binary $\mathbb{Z}$-lattices* *that are not primitively represented by L.*

Proof. (a) Since V is isotropic, $V \cong \mathbb{H} \perp U$ for some ternary quadratic space U over F. Then $\operatorname{ind} V = 2$ if and only if U is isotropic, if and only if $S_{\mathfrak{p}}U = (-1,-1)$, if and only if $S_{\mathfrak{p}}V = (-1,-dV)$.

(b) Since $S_\infty(\mathbb{Q}L) = 1 \neq (-1,-dL)$, there is a prime q such that $S_q(\mathbb{Q}L) = (-1,-dL)$ by Hilbert Reciprocity Law. Then $\operatorname{ind}(\mathbb{Q}_qL) = 1$ by (a). Hence L_q is not primitively 2-universal by Corollary 3.1.6. Therefore there are infinitely many binary $\mathbb{Z}$-lattices that are not primitively represented by L. $\square$

Theorem 4.1.2. *The minimal rank of primitively* 2*-universal* $\mathbb{Z}$*-lattices is* 6.

Proof. By the above lemma, the minimal rank of primitively 2-universal $\mathbb{Z}$-lattices is at least six. Furthermore, one may easily verify that I_6 is primitively 2-universal, for it is primitively 2-universal over $\mathbb{Z}_p$ for any prime p, and it is of class number one (see Theorem 4.3.1). The theorem follows from this. $\square$

Definition 4.1.3. The prime q in the proof of the above lemma such that $\operatorname{ind}\mathbb{Q}_qL = 1$ is called the **core prime** of L (see also [17, Lemma 2.4]).

The existence of the core prime will play a significant role when we determine binary $\mathbb{Z}$-lattices that are primitively represented by a quinary $\mathbb{Z}$-lattice in Sections 4.3 and 4.4.

4.2 Candidates of primitively 2-universal senary $\mathbb{Z}$-lattices

In this section, we find all candidates of primitively 2-universal senary $\mathbb{Z}$-lattices.

A $\mathbb{Z}$-sublattice $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$ of a $\mathbb{Z}$-lattice L is called a k-**section** of L if there are vectors $e_{k+1}, \ldots, e_n$ such that $\{e_1, \ldots, e_n\}$ is a Minkowski reduced basis for L. Recall that a k-section of L is not unique in general.

Let L be a primitively 2-universal senary $\mathbb{Z}$-lattice. We find all possible k-sections of L inductively on $k = 1, \ldots, 6$. To obtain all possible candidates of $(k+1)$-sections containing a specific k-section, we recurrently turn to Lemma 4.2.1, which is quite well known (see [15, Lemma 2.1]).

Lemma 4.2.1. *Let M and N be positive definite $\mathbb{Z}$-lattices of rank m and n respectively. Suppose that $N = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ is Minkowski reduced. Suppose further that M is represented by N, but not by the k-section $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$.*

(a) *We have $Q(e_{k+1}) \leq C_4(k+1)\max\{\mu_m(M), Q(e_k)\}$ where the constant $C_4(j)$ is defined in* [4] *that depends only on j and $\mu_m(M)$ is the m-th successive minimum of M (see* [4, Theorem 3.1]*).*

(b) *Suppose further that $n \leq m+4$ and N is m-universal. Then for any $x = x_1e_1 + \cdots + x_ne_n \in L$, we have $Q(x) \geq Q(e_j)$ whenever $x_j \neq 0$. Also, we have $Q(e_{k+1}) \leq \mu_m(M)$.*

Proof. (1) Suppose to the contrary that $Q(e_{k+1}) > C_4(k+1)\max\{\mu_m(M), Q(e_k)\}$.

Since $Q(e_{k+1}) \leq C_4(k+1)\mu_{k+1}(N)$, this implies

$$\mu_{k+1}(N) > \max\{\mu_m(M), Q(e_k)\}. \tag{*}$$

The hypothesis that N represents M and the inequality (*) together implies that

$$M \to \operatorname{span}_{\mathbb{Q}}\{x \in L : Q(x) < \mu_{k+1}(N)\} \cap N = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k,$$

which is a contradiction.

(2) Since N represents I_m, we have $N = I_m \perp N'$ for $N' = \mathbb{Z}e_{m+1} \oplus \cdots \oplus \mathbb{Z}e_n$, and N' also is Minkowski reduced. Since $\operatorname{rank} N' \leq 4$, for any

$$x = x_{m+1}e_{m+1} + \cdots + x_n e_n \in N',$$

we have $Q(x) \geq Q(e_j)$ whenever $x_j \neq 0$ (see [4, Lemma 1.2 of Ch. 12]). Hence the former assertion holds. Moreover, the successive minima of N' must appear on the diagonal, hence the same for N itself. In particular, $Q(e_{k+1}) = \mu_{k+1}(N)$. Now suppose to the contrary that $Q(e_{k+1}) > \mu_m(M)$, then $\mu_{k+1}(N) > \mu_m(M)$. This inequality and the hypothesis that N represents M together implies that

$$M \to \operatorname{span}_{\mathbb{Q}}\{x \in L : Q(x) < \mu_{k+1}(N)\} \cap N \subseteq \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k,$$

which is a contradiction. $\square$

If a $\mathbb{Z}$-lattice M is not primitively 2-universal, we define the **truant** of M to be, among all the binary $\mathbb{Z}$-lattices that is not primitively represented by

M, the least one up to isometry with respect to the following total order in terms of Gram matrices:

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \prec \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} c_1 < c_2, & \text{or} \\ c_1 = c_2 \text{ and } a_1 < a_2, & \text{or} \\ c_1 = c_2,\ a_1 = a_2 \text{ and } b_1 < b_2, \end{cases}$$

where we assume that all lattices are Minkowski reduced, that is, $0 \le 2b_i \le a_i \le c_i$ for any $i = 1, 2$.

Now we find all candidates of k-sections for each $k = 1, \ldots, 6$ inductively. Since I_2 is the truant of any lattice of rank less than two, clearly the 2-section must be I_2. Since I_2 cannot primitively represent $\langle 1, 2\rangle$, we must have $1 \le Q(e_3) \le 2$ by the last lemma. By repeating this and removing duplicates as well as any candidates that have the truant, we finally obtain the following list of 201 candidates of primitively 2-universal senary $\mathbb{Z}$-lattices:

Table 4.1: The 201 candidates of P2U senary $\mathbb{Z}$-lattices

Type	5-section	Candidates	Possible k's
A	I_5	$I_5 \perp \langle k \rangle$	$k = 1,\ 2$
B	$I_4 \perp \langle 2 \rangle$	$I_4 \perp \langle 2, k \rangle$	$2 \le k \le 5$
		$I_4 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & k \end{smallmatrix}\right)$	$k = 2,\ 3,\ 5,\ 6$
C	$I_4 \perp \langle 3 \rangle$	$I_4 \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & k \end{smallmatrix}\right)$	$k = 3$

Table 4.1: The 201 candidates of P2U senary $\mathbb{Z}$-lattices

Type	5-section	Candidates	Possible k's
D	$I_3 \perp \langle 2, 2\rangle$	$I_3 \perp \langle 2, 2, k\rangle$	$2 \leq k \leq 6$
		$I_3 \perp \langle 2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & k \end{pmatrix}$	$3 \leq k \leq 8$
		$I_3 \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & k \end{pmatrix}$	$k = 3,\ 5 \leq k \leq 7$
E	$I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle k\rangle$	$2 \leq k \leq 5,\ k = 7,\ 8$
		$I_3 \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & k \end{pmatrix}$	$2 \leq k \leq 9$
F	$I_3 \perp \langle 2, 3\rangle$	$I_3 \perp \langle 2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & k \end{pmatrix}$	$k = 3$
G	$I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	$I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle k\rangle$	$k = 3,\ 4,\ 6 \leq k \leq 18$
		$I_3 \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & k \end{pmatrix}$	$3 \leq k \leq 19$
		$I_3 \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & k \end{pmatrix}$	$3 \leq k \leq 19$
H	$I_2 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle 2\rangle$	$I_2 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle 2, k\rangle$	$3 \leq k \leq 5$
		$I_2 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & k \end{pmatrix}$	$k = 2,\ 4,\ 5$
		$I_2 \perp \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & k \end{pmatrix}$	$3 \leq k \leq 6$
		$I_2 \perp \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & k \end{pmatrix}$	$k = 3,\ 4,\ 6$

Table 4.1: The 201 candidates of P2U senary $\mathbb{Z}$-lattices

Type	5-section	Candidates	Possible k's
I	$I_2 \perp \begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix}$	$I_2 \perp \begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix} \perp \langle k \rangle$	$k = 2$
		$I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&2&0\\1&1&0&k \end{pmatrix}$	$2 \le k \le 5$
		$I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&k \end{pmatrix}$	$2 \le k \le 4$
J	$I_2 \perp \begin{pmatrix} 2&1\\1&2 \end{pmatrix} \perp \langle 3 \rangle$	$I_2 \perp \begin{pmatrix} 2&1\\1&2 \end{pmatrix} \perp \begin{pmatrix} 3&1\\1&k \end{pmatrix}$	$k = 3$
K	$I_2 \perp \begin{pmatrix} 2&1&1\\1&2&1\\1&1&3 \end{pmatrix}$	$I_2 \perp \begin{pmatrix} 2&1&1\\1&2&1\\1&1&3 \end{pmatrix} \perp \langle k \rangle$	$3 \le k \le 20$, $22 \le k \le 24$
		$I_2 \perp \begin{pmatrix} 2&1&1&0\\1&2&1&0\\1&1&3&1\\0&0&1&k \end{pmatrix}$	$3 \le k \le 24$
		$I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&3&0\\1&1&0&k \end{pmatrix}$	$3 \le k \le 25$
		$I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&3&1\\1&1&1&k \end{pmatrix}$	$3 \le k \le 25$

We refer to any of the above candidates by the expression such as (type) or $(\text{type})_k$. For instance, by type K we mean each of 89 candidates in the last four rows. Among them, by types K^{i}, K^{ii}, $\mathrm{K}^{\mathrm{iii}}$ and K^{iv} we mean 21, 22, 23 and 23 candidates in each of four rows, respectively. For instance, by type K^{iv} we mean any $\mathbb{Z}$-lattice of the form $I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&3&1\\1&1&1&k \end{pmatrix}$. Finally, by $\mathrm{K}^{\mathrm{iv}}_3$ we mean the lattice $I_2 \perp \begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&3&1\\1&1&1&3 \end{pmatrix}$.

Moreover, when we refer to each of the candidates in Table 4.1, we always assume that the basis $e_1, \ldots, e_6$ corresponds to the Gram matrix given in the table. For instance, when we consider the $\mathbb{Z}$-lattice $\mathrm{K}_3^{\mathrm{iv}}$, we have

$$(B(e_i, e_j)) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&2&1&1&1\\ 0&0&1&2&1&1\\ 0&0&1&1&3&1\\ 0&0&1&1&1&3 \end{pmatrix}.$$

From now on, by abusing of terminology, the s-section of $\mathbb{Z}e_1+\cdots+\mathbb{Z}e_6$ always denotes $\mathbb{Z}e_1+\cdots+\mathbb{Z}e_s$ for any $s = 1, 2, \ldots, 6$.

4.3 The proof of primitive 2-universality (ordinary cases)

In this section, we prove that some candidates given in Table 4.1 are, in fact, primitively 2-universal. Let L be one of candidates of a primitively 2-universal $\mathbb{Z}$-lattices given in Table 4.1. We prove the primitive 2-universality of L for cases when L itself or the 5-section of L is of class number one.

4.3.1 Class number one case

As explained in Chapter 2, if a $\mathbb{Z}$-lattice O is of class number one, then O primitively represents any $\mathbb{Z}$-lattice that is primitively represented over $\mathbb{Z}_p$ for any prime p. Hence, if L is of class number one, then L is primitively 2-universal if and only if L_p primitively represents all binary $\mathbb{Z}_p$-lattices for any prime p.

Theorem 4.3.1. *If L is of class number one, then L is primitively 2-universal. In fact, there are 10 such cases.*

Proof. One may easily show that

$$L \cong \mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{B}_2^{\mathrm{i}},\ \mathrm{B}_2^{\mathrm{ii}},\ \mathrm{B}_3^{\mathrm{ii}},\ \mathrm{D}_3^{\mathrm{iii}},\ \mathrm{E}_2^{\mathrm{ii}},\ \mathrm{E}_3^{\mathrm{ii}} \text{ or } \mathrm{I}_3^{\mathrm{ii}}.$$

Note that $dL = 1,\ 2,\ 4,\ 3,\ 5,\ 8,\ 4,\ 7,\ 4,$ and 8, respectively. Hence L_p is primitively 2-universal for all odd prime p by Lemma 3.2.1. Also L_2 is primitively 2-universal by Lemma 3.3.1 for the following eight cases.

$$\begin{array}{ll}
\mathrm{A}_1 \cong \mathbb{H}^2 \perp \langle -1,-1\rangle & \mathrm{A}_2 \cong \mathbb{H}^2 \perp \langle 5,10\rangle \\
\mathrm{B}_2^{\mathrm{ii}} \cong \mathbb{H}^2 \perp \langle 1,3\rangle & \mathrm{B}_3^{\mathrm{ii}} \cong \mathbb{H}^2 \perp \langle 1,5\rangle \\
\mathrm{D}_3^{\mathrm{iii}} \cong \mathbb{H}^2 \perp \langle 5,40\rangle & \mathrm{E}_2^{\mathrm{ii}} \cong \mathbb{H}^2 \perp \langle 3,12\rangle \\
\mathrm{E}_3^{\mathrm{ii}} \cong \mathbb{H}^2 \perp \langle 1,-1\rangle & \mathrm{I}_3^{\mathrm{ii}} \cong \mathbb{H}^2 \perp \langle -1,-8\rangle
\end{array}$$

Now, assume that $L \cong \mathrm{B}_2^{\mathrm{i}}$. Then $L_2 \cong \mathbb{H} \perp M \cong \langle 1,-1\rangle \perp M$ where $M \cong \langle 1,5,2,2\cdot 3\rangle$ is primitively 1-universal by [7, Theorem 5.2]. Hence any diagonal binary $\mathbb{Z}_2$-lattice is primitively represented by L_2. Now, denote by e, f the hyperbolic basis of $\mathbb{H}$, and pick any primitive vector x, $y \in M$ with $Q(x) = 0$ and $Q(y) = 2^{a+1}$ for a nonnegative integer a. Then $\mathbb{Z}_2[e, 2^a f + x]$ and $\mathbb{Z}_2[e + 2^a f, e + y]$ are primitive sublattices of $\mathbb{H} \perp M$ that is isometric to $2^a\mathbb{H}$ and $2^a\mathbb{A}$, respectively. Hence, L_2 is primitively 2-universal.

Finally, assume that $L \cong \mathrm{I}_2^{\mathrm{ii}}$. Then $L_2 \cong \mathbb{H} \perp M \cong \langle 1,-1\rangle \perp M$ where $M \cong \langle 3,7\rangle \perp \left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. One may easily show that $Q^*(M) = \{3,7\}(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2$. Hence, by Lemma 3.3.1, any diagonal binary $\mathbb{Z}_2$-lattice is primitively represented by L_2 except $\langle 1,1\rangle$ and $\langle 1,5\rangle$, which also are primitively represented by L_2. The proof of the fact that any binary improper modular lattice is primitively represented by $\mathbb{H} \perp M$ is quite similar to the previous case. Hence, L_2 is primitively 2-universal. $\square$

Remark 4.3.2. In fact, for senary lattices, Lemma 4.3.1 generalizes Budarina's result [3], where L is required to be of class number one and to be of squarefree odd discriminant. One may verify from the proof that there are four such cases out of our 201 candidates.

4.3.2 Class number one 5-section case

In the remaining of this section, we consider the case when the 5-section of L, say M, has class number one. Since M is a primitive sublattice of L, L primitively represents any $\mathbb{Z}$-lattice that is primitively represented by M. Hence, if M is of class number one, then L primitively represents any $\mathbb{Z}$-lattice that is locally primitively represented by M. Note that by Lemma 4.1.1, M_q is not primitively 2-universal for some prime q, and hence there are infinitely many $\mathbb{Z}$-lattices that are not primitively represented by M.

Recall that any core prime q of M satisfies $S_qU \neq (-1,-1)$, where $\mathbb{Q}M \cong U \perp \langle dM \rangle$ (see Definition 4.1.3). One may easily check that the 5-section of L whose type is not of H has class number one, and the 5-section $I_2 \perp \mathbb{A} \perp \langle 2 \rangle$ of L with type H has class number two. Note that the genus mate of this lattice is $I_4 \perp \langle 6 \rangle$. Hereafter α, β denote integers in $\mathbb{Z}_p$, and ϵ, δ denote units in $\mathbb{Z}_p$, unless stated otherwise, where the prime p could be easily verified from the context.

Lemma 4.3.3. *For the* 5*-section* M *of each type given below, the core prime* q *of* M *and local structures over* $\mathbb{Z}_q$ *of any binary* $\mathbb{Z}$*-lattice* ℓ *that is not primitively represented by* M *are given as follows:*

Table 4.2: The core prime and the local structures

Type	M	q	Local structures
A	I_5	2	$\ell_2 \cong \langle 1, 8\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 4\mathbb{Z}_2$
B	$I_4 \perp \langle 2\rangle$	2	$\ell_2 \cong \langle 2, 16\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 8\mathbb{Z}_2$
D	$I_3 \perp \langle 2, 2\rangle$	2	$\ell_2 \cong \langle 1, 16\alpha\rangle$, $\langle 4, 16\alpha\rangle$, $\langle 20, 16\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2$
E	$I_3 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$	3	$\ell_3 \cong \langle 3, 9\alpha\rangle$ or $\mathfrak{s}(\ell_3) \subseteq 9\mathbb{Z}_3$
G	$I_3 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$	5	$\ell_5 \cong \langle 5, 25\alpha\rangle$ or $\mathfrak{s}(\ell_5) \subseteq 25\mathbb{Z}_5$
I	$I_2 \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right)$	2	$\ell_2 \cong \langle 1, 32\alpha\rangle$, $\langle 5, 16\epsilon\rangle$, $\langle 4, 32\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2$
K	$I_2 \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{smallmatrix}\right)$	7	$\ell_7 \cong \langle 7, 49\alpha\rangle$ or $\mathfrak{s}(\ell_7) \subseteq 49\mathbb{Z}_7$

Proof. One may easily verify that the prime q given in Table 4.2 is the only core prime for each 5-section M, and M_p primitively represents ℓ_p for any prime $p \neq q$. Hence, ℓ is primitively represented by M if and only if ℓ_q is primitively represented by M_q.

Assume that L is of type A. Note that $M_2 \cong \mathbb{H} \perp N$, where $N \cong \langle -1, -1, -1\rangle$

and

$$Q^*(N) = \{3, 5, 7\}(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2^\times.$$

Hence M_2 primitively represents all binary lattices of the form $\langle \alpha, \theta \rangle$, where $\theta \in Q^*(N))$, $\mathbb{H}$, and $\mathbb{A}$. Moreover, M_2 primitively represents $\langle 1, 1 \rangle$ and $\langle 1, 4\epsilon \rangle$, for $\mathbb{Z}_2[e_1, e_2+e_3+e_4+\sqrt{4\epsilon - 3}e_5]$ is a primitive sublattice of I_5 that is isometric to $\langle 1, 4\epsilon \rangle$.

Assume that L is of type B. Note that $M_2 \cong \mathbb{H} \perp N$, where $N \cong \langle 1, 3, 10 \rangle$ and

$$Q^*(N) = \mathbb{Z}_2^\times \cup \{6, 10, 14\}(\mathbb{Z}_2^\times)^2 \cup 4\mathbb{Z}_2^\times.$$

Hence M_2 primitively represents all binary lattices of the form $\langle \alpha, \theta \rangle$, where $\theta \in Q^*(N)$, $2^a\mathbb{H}$, and $2^a\mathbb{A}$ $(0 \le a \le 1)$. Moreover, M_2 primitively represents $\langle 2, 2 \rangle$ and $\langle 2, 8\epsilon \rangle$, for $\mathbb{Z}_2[e_1+e_2, e_1-e_2+2e_4+\sqrt{4\epsilon - 3}e_5]$ is a primitive sublattice of $I_4 \perp \langle 2 \rangle$ that is isometric to $\langle 2, 8\epsilon \rangle$.

Assume that L is of type D. Note that $M_2 \cong \mathbb{H} \perp N$, where $N \cong \langle 5, 2, 6 \rangle$ and

$$Q^*(J) = \{3, 5, 7\}(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2^\times \cup \{12, 28\}(\mathbb{Z}_2^\times)^2 \cup 8\mathbb{Z}_2^\times.$$

Hence, M_2 primitively represents all binary lattices of the form $\langle \alpha, \theta \rangle$, where $\theta \in Q^*(N)$, $2^a\mathbb{H}$, and $2^a\mathbb{A}$ $(0 \le a \le 2)$. Moreover, M_2 primitively represents $\langle \theta, \theta' \rangle$, where θ, $\theta' \in \{1, 4, 20\}$.

Assume that L is of type E. Note that $M_3 \cong \mathbb{H} \perp N$, where $N \cong \langle 1, 1, 3 \cdot \Delta_3 \rangle$ and $Q^*(N) = \{1, \Delta_3, 3 \cdot \Delta_3\}(\mathbb{Z}_3^\times)^2$. Hence, M_3 primitively represents any binary $\mathbb{Z}_3$-lattice that represents 1, Δ_3, or $3 \cdot \Delta_3$.

Assume that L is of type G. Note that $M_5 \cong \mathbb{H} \perp N$, where $N \cong \langle 1, 1, 5\cdot\Delta_5 \rangle$ and $Q^*(N) = \{1, \Delta_5, 5 \cdot \Delta_5\}(\mathbb{Z}_5^\times)^2$. Hence, M_5 primitively represents any binary $\mathbb{Z}_5$-lattice that represents 1, Δ_5, or $5 \cdot \Delta_5$.

Assume that L is of type I. Note that $M_2 \cong \mathbb{H} \perp N$, where $N \cong \langle 3, 7, 12 \rangle$ and

$$Q^*(N) = \{3, 7\}(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2^\times \cup \{12, 20, 28\}(\mathbb{Z}_2^\times)^2 \cup 8\mathbb{Z}_2^\times.$$

Hence, M_2 primitively represents all binary lattices of the form $\langle \alpha, \theta \rangle$, where $\theta \in Q^*(N)$, $2^a\mathbb{H}$, and $2^a\mathbb{A}$ $(0 \le a \le 2)$. Moreover, M_2 primitively represents $\langle \theta, \theta' \rangle$, where θ, $\theta' \in \{1, 5, 4\}$, $\langle 1, 16\epsilon \rangle$, $\langle 5, 32\alpha \rangle$, and $\langle 4, 16\epsilon \rangle$.

Assume that L is of type K. Note that $M_7 \cong \mathbb{H} \perp N$, where $N \cong \langle 1, 1, 7\cdot\Delta_7 \rangle$ and $Q^*(N) = \{1, \Delta_7, 7 \cdot \Delta_7\}(\mathbb{Z}_7^\times)^2$. Hence, M_7 primitively represents any binary $\mathbb{Z}_7$-lattice that represents 1, Δ_7, or $7 \cdot \Delta_7$. This completes the proof.

□

We first complete the proof of the case when the 5-section splits L orthogonally.

Theorem 4.3.4. *Suppose that L is of class number at least two. If M is of class number one and orthogonally splits L, then L is primitively 2-universal. In fact, there are 51 such cases.*

Proof. By the above lemma, L is of type B^{i}(except $\mathrm{B}^{\mathrm{i}}_2$), D^{i}, E^{i}, G^{i}, I^{i}, or K^{i}. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a $\mathbb{Z}$-lattice which is not primitively represented by M. We assume that ℓ is Minkowski reduced, that is, $0 \le 2b \le a \le c$. If we show that ℓ is primitively represented by L, then we are done. To do this, we may consider two $\mathbb{Z}$-lattices

$$\ell' \cong \begin{pmatrix} a-k & b \\ b & c \end{pmatrix}, \qquad \ell'' \cong \begin{pmatrix} a & b \\ b & c-k \end{pmatrix}.$$

If either ℓ' or ℓ'' is primitively represented by M, then clearly ℓ is primitively represented by $L \cong M \perp \langle k\rangle$. Moreover, ℓ' (ℓ'') is primitively represented by M if and only if ℓ'_q (ℓ''_q, repsectively) is primitively represent by M_q for the core prime q of M.

First, assume that L is of type B^{i}. By Lemma 4.3.3 we may assume that

$$\ell_2 \cong \langle 2, 16\alpha\rangle \quad \text{or} \quad \mathfrak{n}(\ell_2) \subseteq 8\mathbb{Z}_2.$$

Suppose that $k = 3$ or 5. Since $\mathfrak{s}\ell''_2 = \mathbb{Z}_2$, ℓ'_2 is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 7$. One may directly check that ℓ is primitively represented by L if $c \leq 6$.

Now, suppose that $k = 4$. If $a \not\equiv 0 \pmod 4$, then $d\ell''_2 \not\equiv 0 \pmod{16}$, and thus ℓ''_2 is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 6$. If $c \not\equiv 0 \pmod 4$, then $c \equiv 2 \pmod{16}$, and thus ℓ''_2 is split by $\langle c-4\rangle$. Furthermore, since $c - 4 \equiv -2 \pmod{16}$, ℓ'' is primitively represented by M if $c \geq 6$. If $a \equiv c \equiv 0 \pmod 4$, then $\mathfrak{s}(\ell_2) \subseteq 4\mathbb{Z}_2$ since ℓ_2 is not unimodular, which implies $\mathfrak{s}(\ell_2) \subseteq 8\mathbb{Z}_2$. Hence ℓ''_2 is split by $\langle c-4\rangle$. Since $c-4 \equiv 4 \pmod 8$, ℓ'' is primitively represented by M since $c \geq 8$. One may directly check that ℓ is primitively represented by L if $c \leq 6$.

Assume that L is of type D^{i}. By Lemma 4.3.3 we may assume that

$$\ell_2 \cong \langle 1, 16\alpha\rangle, \quad \langle 4, 16\alpha\rangle, \quad \langle 20, 16\alpha\rangle, \quad \text{or} \quad \mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2.$$

Suppose that $k = 3$ or 5. If $a \not\equiv 0 \pmod{16}$, then $d\ell_2'' \not\equiv 0 \pmod{16}$, and thus ℓ_2'' is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 7$. Assume $a \equiv 0 \pmod{16}$. Since $a-k$ is odd, ℓ_2' is split by $\langle a-k\rangle$. Furthermore, since $a-k \not\equiv 1 \pmod 8$, ℓ_2' is primitively represented by M_2. Hence, ℓ' is primitively represented by M. One may directly check that ℓ is primitively represented by L if $c \leq 6$.

Now, suppose that $k = 2$ or 6. If $a \not\equiv 0 \pmod 8$, then $d\ell_2'' \not\equiv 0 \pmod{16}$, and thus ℓ_2'' is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 9$. If c is odd, then $c \equiv 1 \pmod 8$, and thus ℓ_2'' is split by $\langle c-k\rangle$. Furthermore, since $c-k \not\equiv 1 \pmod 8$, ℓ'' is primitively represented by M if $c \geq 9$. If $a \equiv 0 \pmod 8$ and c is even, then $\mathfrak{s}(\ell_2) \subseteq 2\mathbb{Z}_2$ since ℓ_2 is not unimodular, which implies $\mathfrak{s}(\ell_2) \subseteq 4\mathbb{Z}_2$, and then $c-k \equiv 2 \pmod 4$ and $\mathfrak{s}(\ell_2'') = 2\mathbb{Z}_2$. Hence, ℓ'' is primitively represented by M if $c \geq 9$. One may directly check that ℓ is primitively represented by L if $c \leq 8$.

Finally, suppose that $k = 4$. If $a \not\equiv 0 \pmod 4$, then $d\ell_2'' \not\equiv 0 \pmod{16}$, and thus ℓ_2'' is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 6$. If c is odd, then $c \equiv 1 \pmod 8$, and thus ℓ_2'' is split by $\langle c-4\rangle$. Furthermore, since $c-4 \not\equiv 1 \pmod 8$, ℓ'' is primitively represented by M if $c \geq 6$. If $a \equiv 0 \pmod 4$ and c is even, then $\mathfrak{s}(\ell_2) \subseteq 2\mathbb{Z}_2$ since ℓ_2 is not unimodular, which implies $\mathfrak{s}(\ell_2) \subseteq 4\mathbb{Z}_2$ and

$d\ell_2 \equiv 0 \pmod{64}$. If $a \not\equiv 0 \pmod{16}$ then $d\ell_2'' \not\equiv 0 \pmod{64}$, and hence again ℓ'' is primitively represented by M if $c \geq 6$. If $a \equiv 0 \pmod{16}$ then $\mathfrak{s}(\ell_2') = (4)$, and thus ℓ_2' is split by $\langle a-4 \rangle$. Furthermore, since $a-4 \equiv -4$ (16), ℓ_2' is primitively represented by M_2. Hence, ℓ' is primitively represented by M since $a \geq 16$. One may directly check that ℓ is primitively represented by L if $c \leq 5$.

Next, assume that L is of type E^{i}. By Lemma 4.3.3 we may assume that

$$\ell_3 \cong \langle 3, 9\alpha \rangle \quad \text{or} \quad \mathfrak{s}(\ell_3) \subseteq 9\mathbb{Z}_3.$$

Suppose that $k \neq 3$. Then $\mathfrak{s}(\ell_3'') = \mathbb{Z}_3$, and thus ℓ_3'' is primitively represented by M_3. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 11$. One may directly check that ℓ is primitively represented by L if $c \leq 10$.

Now, suppose that $k = 3$. If $a \not\equiv 0 \pmod 9$ then $d\ell_3'' \not\equiv 0 \pmod{27}$, and thus ℓ_3'' is primitively represented by M_3. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 5$. If $a \equiv 0 \pmod 9$ then $\mathfrak{s}(\ell_3') = 3\mathbb{Z}_3$, and thus ℓ_3' is split by $\langle a-k \rangle$ where $a - k \equiv 6 \pmod 9$, and then ℓ_3' is primitively represented by M_3. Hence, ℓ' is primitively represented by M since $a \geq 9$. One may directly check that ℓ is primitively represented by L if $c \leq 4$.

Now, assume that L is of type G^{i}. By Lemma 4.3.3 we may assume that

$$\ell_5 \cong \langle 5, 25\alpha \rangle \quad \text{or} \quad \mathfrak{s}(\ell_5) \subseteq 25\mathbb{Z}_5.$$

Suppose that $k \neq 10$ or 15. Then $\mathfrak{s}(\ell_5'') = \mathbb{Z}_5$, and thus ℓ_5'' is primitively represented by M_5. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 25$. One may directly check that ℓ is primitively represented by L if $c \leq 24$.

Now, suppose that $k = 10$ or 15. If $a \not\equiv 0 \pmod{25}$ then $d\ell''_5 \not\equiv 0 \pmod{125}$, and thus ℓ''_5 is primitively represented by M_5. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 21$. If $a \equiv 0 \pmod{25}$ then $\mathfrak{s}(\ell'_5) = 5\mathbb{Z}_5$, and thus ℓ'_5 is split by $\langle a-k\rangle$ where $a-k \equiv 15$ or $10 \pmod{25}$, and then ℓ'_5 is primitively represented by M_5. Hence, ℓ' is primitively represented by M since $a \geq 25$. One may directly check that ℓ is primitively represented by L if $c \leq 20$.

Next, assume that L is of type I^{i}. By Lemma 4.3.3 we may assume that

$$\ell_2 \cong \langle 1, 2^5\alpha\rangle,$$

$$\ell_2 \cong \langle 1, 32\alpha\rangle, \quad \langle 5, 16\epsilon\rangle, \quad \langle 4, 32\alpha\rangle, \quad \text{or} \quad \mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2.$$

If $a \not\equiv 0 \pmod 8$, then $d\ell''_2 \not\equiv 0 \pmod{16}$, and thus ℓ''_2 is primitively represented by M_2. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 3$. If c is odd, then $c \equiv 1 \pmod 4$, and thus ℓ''_2 is split by $\langle c-2\rangle$. Furthermore, since $c - 2 \equiv 3 \pmod 4$, ℓ'' is primitively represented by M if $c \geq 3$. If $a \equiv 0 \pmod 8$ and c is even, then $\mathfrak{s}(\ell_2) \subseteq 2\mathbb{Z}_2$ since ℓ_2 is not unimodular, which implies $\mathfrak{s}(\ell_2) \subseteq 4\mathbb{Z}_2$, and then $c - k \equiv 2 \pmod 4$ and $\mathfrak{s}(\ell''_2) = 2\mathbb{Z}_2$. Hence, ℓ'' is primitively represented by M since $c \geq 8$. One may directly check that ℓ is primitively represented by L if $c \leq 2$.

Finally, assume that L is of type K^{i}. By Lemma 4.3.3 we may assume that

$$\ell_7 \cong \langle 7, 49\alpha\rangle \quad \text{or} \quad \mathfrak{s}(\ell_7) \subseteq 49\mathbb{Z}_7.$$

Suppose that $k \neq 7$ or 14. Then $\mathfrak{s}(\ell''_7) = \mathbb{Z}_7$, and thus ℓ''_7 is primitively represented by M_7. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 33$. One may directly check that ℓ is primitively represented by L if $c \leq 32$.

Now, suppose that $k = 7$ or 14. If $a \not\equiv 0 \pmod{49}$ then $d\ell''_7 \not\equiv 0 \pmod{343}$, and thus ℓ''_7 is primitively represented by M_7. Hence, ℓ'' is primitively represented by M if it is positive definite, that is, $c \geq 19$. If $a \equiv 0 \pmod{49}$ then $\mathfrak{s}(\ell'_7) = 7\mathbb{Z}_7$, and thus ℓ'_7 is split by $\langle a-k \rangle$ where $a-k \equiv 35$ or $42 \pmod{49}$, and then ℓ'_7 is primitively represented by M_7. Hence, ℓ' is primitively represented by M since $a \geq 49$. One may directly check that ℓ is primitively represented by L if $c \leq 18$. This completes the proof. $\square$

Remark 4.3.5. If $L \cong \mathrm{C}_3$ or F_3, then we may take $M \cong I_3 \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, which is a primitive sublattice of L. If $L \cong \mathrm{H}_3^{\mathrm{iii}}$, then we may take $M \cong I_2 \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, which is a primitive sublattice of L. Since the class number of M is one and M splits L orthogonally, the proofs of these three candidates are quite similar to the above theorem.

Let $R = \mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p. Let O be an R-lattice and let $\mathfrak{B} = \{e_1, \ldots, e_n\}$ be the fixed (ordered) basis for the R-lattice O. When only the corresponding symmetric matrix M_O is given instead of the basis for O, we assume that $\mathfrak{B}$ is the basis for O such that $(B(e_i, e_j)) = M_O$. We define

$$O' = R \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

by the R-sublattice of O generated by m vectors $a_{11}e_1 + \cdots + a_{1n}e_n, \dots,$ $a_{m1}e_1 + \cdots + a_{mn}e_n$. Note that if the rank of O' is m, then the symmetric matrix corresponding to O' is that

$$M_{O'} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} M_O \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix}.$$

4.3.3 A class number one superlattice of the 5-section case

Recall that we are assuming that L is one of 201 candidates of primitively 2-universal senary $\mathbb{Z}$-lattices given in Table 4.1, and $\mathfrak{B} = \{e_1, \dots, e_6\}$ is the basis for L such that $(B(e_i, e_j))$ is the symmetric matrix given in Table 4.1 corresponding to L. Furthermore, $M = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_5$ is the 5-section of L.

Lemma 4.3.6. *Let $d = dL$ and let q be the core prime of M. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a binary $\mathbb{Z}$-lattice. If $\ell' \cong \left(\begin{smallmatrix} qa-ds^2 & qb-dst \\ qb-dst & qc-dt^2 \end{smallmatrix}\right)$ is positive definite and is primitively represented by N for some integers s and t, then ℓ is primitively represented by L, where L and N are given as follows:*

1. *The $\mathbb{Z}$-lattice L is of type* E^{ii}, *and $N \cong qI_3 \perp \mathbb{A}$.*

2. The $\mathbb{Z}$-lattice L is of type G^{ii} *or* $\mathrm{G}^{\mathrm{iii}}$*, and* $N \cong qI_3 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$.

3. The $\mathbb{Z}$-lattice L is of type K^{ii}, $\mathrm{K}^{\mathrm{iii}}$ *or* K^{iv}*, and* $N \cong qI_2 \perp \left(\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & -2 & 5 \end{smallmatrix}\right)$.

Proof. One may easily show that there is a representation $\psi : L^q \to N \perp \langle d \rangle$ such that $\psi(L^q) \cap N = M^q$.

Since all the other cases can be proved in similar manners, we only provide the proof of the case when L is of type K^{iv}. Since N primitively represents ℓ', there are integers c_i's and d_i's for $i = 1, \dots, 5$ such that the primitive sublattice of N

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix} \cong \ell' \cong \begin{pmatrix} 7a-(7k-5)s^2 & 7b-(7k-5)st \\ 7b-(7k-5)st & 7c-(7k-5)t^2 \end{pmatrix},$$

where $\left(\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{smallmatrix}\right)$ is a primitive matrix. Then we have

$$\left.\begin{aligned} 3(c_3-2c_4-2c_5)^2 &\equiv 5s^2 \\ 3(c_3-2c_4-2c_5)(d_3-2d_4-2d_5) &\equiv 5st \\ 3(d_3-2d_4-2d_5)^2 &\equiv 5t^2 \end{aligned}\right\} \pmod 7,$$

for $\left(\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & -2 & 5 \end{smallmatrix}\right) \equiv 3\left(\begin{smallmatrix} 1 \\ -2 \\ -2 \end{smallmatrix}\right)\begin{pmatrix} 1 & -2 & -2 \end{pmatrix} \pmod 7$. Hence, after replacing (s,t) by $(-s,-t)$, if necessary, we may assume that

$$c_3-2c_4-2c_5+2s \equiv d_3-2d_4-2d_5+2t \equiv 0 \pmod 7.$$

Therefore, there are integers a_3, a_4, a_5, b_3, b_4 and b_5 satisfying

$$\begin{pmatrix} c_3 & d_3 \\ c_4 & d_4 \\ c_5 & d_5 \\ s & t \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \\ s & t \end{pmatrix}.$$

Now, consider the sublattice

$$O = \mathbb{Z}\begin{bmatrix} c_1 & c_2 & a_3 & a_4 & a_5 & s \\ d_1 & d_2 & b_3 & b_4 & b_5 & t \end{bmatrix}$$

of L. Since

$$\begin{aligned} &\begin{pmatrix} a_3 & a_4 & a_5 & s \\ b_3 & b_4 & b_5 & t \end{pmatrix} \begin{pmatrix} 14 & 7 & 7 & 7 \\ 7 & 14 & 7 & 7 \\ 7 & 7 & 21 & 7 \\ 7 & 7 & 7 & 7q \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \\ s & t \end{pmatrix} \\ &\qquad = \begin{pmatrix} c_3 & c_4 & c_5 & s \\ d_3 & d_4 & d_5 & t \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 5 & -2 & 0 \\ 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 7q-5 \end{pmatrix} \begin{pmatrix} c_3 & d_3 \\ c_4 & d_4 \\ c_5 & d_5 \\ s & t \end{pmatrix}, \end{aligned}$$

the $\mathbb{Z}$-lattice O is isometric to ℓ. Now, since

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & -a_3+2a_5 & 2a_3+a_4+a_5+s & a_3-a_4 \\ d_1 & d_2 & -b_3+2b_5 & 2b_3+b_4+b_5+t & b_3-b_4 \end{pmatrix}$$

is a primitive matrix, so is

$$\begin{pmatrix} c_1 & c_2 & -a_3+2a_5 & 2a_3+a_4+a_5+s & a_3-a_4 & a_5 \\ d_1 & d_2 & -b_3+2b_5 & 2b_3+b_4+b_5+t & b_3-b_4 & b_5 \end{pmatrix}.$$

Therefore, the matrix

$$\begin{pmatrix} c_1 & c_2 & a_3 & a_4 & a_5 & s \\ d_1 & d_2 & b_3 & b_4 & b_5 & t \end{pmatrix}$$

is primitive, which implies that O is a primitive sublattice of L. This completes the proof. $\square$

Theorem 4.3.7. *If L is of type*

$$\mathrm{G}^{\mathrm{ii}},\ \mathrm{G}^{\mathrm{iii}},\ \mathrm{K}^{\mathrm{ii}},\ \mathrm{K}^{\mathrm{iii}},\ \text{or } \mathrm{K}^{\mathrm{iv}},$$

then L is primitively 2-universal. There are exactly 110 such $\mathbb{Z}$-lattices, and among them, only $\mathrm{E}_2^{\mathrm{ii}}$ and $\mathrm{E}_3^{\mathrm{ii}}$ have class number one.

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a $\mathbb{Z}$-lattice which is not primitively represented by M.

Suppose that L is of type E. Note that $\det(\mathrm{E}^{\mathrm{ii}}) = 3k-2$. By Lemma 4.3.3, we may assume that

$$\ell_3 \cong \langle 3, 9\alpha \rangle \quad \text{or} \quad \mathfrak{s}(\ell_3) \subseteq 9\mathbb{Z}_3.$$

Observe that $N = 3I_3 \perp \mathbb{A}$ is of class number one, and that 3 is the only core prime of N. Furthermore, $N_7 \cong 3\mathbb{H} \perp O$, where $O = \langle \Delta_3, 3, 3 \rangle$. Thus

N_3 primitively represents all binary $\mathbb{Z}_3$-lattices of the form $\langle 3\alpha, \theta\rangle$, where $\theta \in Q^*(O) = \{\Delta_3, 3, 3\cdot\Delta_3\}(\mathbb{Z}_3^\times)^2$. Hence, N_3 primitively represents $\ell' \cong \left(\begin{smallmatrix} 3a & 3b \\ 3b & 3c-(3k-2)\end{smallmatrix}\right)$. Therefore N primitively represents ℓ' if it is positive definite. Hence, by Lemma 4.3.6, ℓ is primitively represented by L if $c \geq 12$. One may directly check that ℓ is primitively represented by L if $c \leq 11$.

Now, suppose that L is of type G. Note that

$$\det(\mathrm{G}^{\mathrm{ii}}) = 5k-2 \quad \text{and} \quad \det(\mathrm{G}^{\mathrm{iii}}) = 5k-3.$$

By Lemma 4.3.3, we may assume that

$$\ell_5 \cong \langle 5, 25\alpha\rangle \quad \text{or} \quad \mathfrak{s}(\ell_5) \subseteq 25\mathbb{Z}_5.$$

Observe that $N = 5I_3 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3\end{smallmatrix}\right)$ is of class number one, and that 5 is the only core prime of N. Furthermore, $N_5 \cong 5\mathbb{H} \perp O$, where $O = \langle \Delta_5, 5, 5\cdot\Delta_5\rangle$. Thus N_5 primitively represents all binary $\mathbb{Z}_5$-lattices of the form $\langle 5\alpha, \theta\rangle$, where $\theta \in Q^*(O) = \{\Delta_5, 5, 5\cdot\Delta_5\}(\mathbb{Z}_5^\times)^2$. Hence, N_5 primitively represents

$$\ell' \cong \left(\begin{smallmatrix} 5a & 5b \\ 5b & 5c-(5k-2)\end{smallmatrix}\right) \quad \text{and} \quad \left(\begin{smallmatrix} 5a & 5b \\ 5b & 5c-(5k-3)\end{smallmatrix}\right).$$

Therefore N primitively represents ℓ' if it is positive definite. Hence, by Lemma 4.3.6, ℓ is primitively represented by L if $c \geq 25$. One may directly check that ℓ is primitively represented by L if $c \leq 24$.

Finally, suppose that L is of type K. Note that

$$\det(\mathrm{K}^{\mathrm{ii}}) = 7k-3, \quad \det(\mathrm{K}^{\mathrm{iii}}) = 7k-6, \quad \text{and} \quad \det(\mathrm{K}^{\mathrm{iv}}) = 7k-5.$$

By Lemma 4.3.3, we may assume that

$$\ell_7 \cong \langle 7, 49\alpha \rangle \quad \text{or} \quad \mathfrak{s}(\ell_7) \subseteq 49\mathbb{Z}_7.$$

Observe that $N = 7I_2 \perp \left(\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & -2 & 5 \end{smallmatrix}\right)$ is of class number one, and that 7 is the only core prime of N. Furthermore, $N_7 \cong 7\mathbb{H} \perp O$, where $O = \langle \Delta_7, 7, 7 \rangle$. Thus N_7 primitively represents all binary $\mathbb{Z}_7$-lattices of the form $\langle 7\alpha, \theta \rangle$, where $\theta \in Q^*(O) = \{\Delta_7, 7, 7 \cdot \Delta_7\}(\mathbb{Z}_7^\times)^2$. Hence, N_7 primitively represents

$$\ell' \cong \left(\begin{smallmatrix} 7a & 7b \\ 7b & 7c-(7k-3) \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 7a & 7b \\ 7b & 7c-(7k-6) \end{smallmatrix}\right), \quad \text{and} \quad \left(\begin{smallmatrix} 7a & 7b \\ 7b & 7c-(7k-5) \end{smallmatrix}\right).$$

Therefore N primitively represents ℓ' if it is positive definite. Hence, by Lemma 4.3.6, ℓ is primitively represented by L if $c \geq 33$. One may directly check that ℓ is primitively represented by L if $c \leq 32$. $\square$

Remark 4.3.8. In fact, we do not use the fact that M is the 5-section of L in the above theorem. If $L \cong \mathrm{D}_3^{\mathrm{ii}}$, $\mathrm{H}_3^{\mathrm{iv}}$, or $\mathrm{I}_3^{\mathrm{iii}}$, then we may take a primitive sublattice M of L as in Table 4.3. Then q is the only core prime of M, and one may apply Lemma 4.3.6 for each pair of L and N in the table. Therefore, the proofs of primitive 2-universalities of these three candidates are quite similar to Theorem 4.3.7.

Summing up all, we have proved the primitive 2-universalities of 175 $\mathbb{Z}$-lattices among 201 candidates, and the primitive 2-universalities of 26 candidates remain unproven.

Table 4.3: The core prime of M and the choice of N

L	M	q	N
$\mathrm{D}_3^{\mathrm{ii}}$	$I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	5	$5I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$
$\mathrm{H}_3^{\mathrm{iv}}$ or $\mathrm{I}_3^{\mathrm{iii}}$	$I_2 \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$	7	$7I_2 \perp \begin{pmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & -2 & 5 \end{pmatrix}$

4.4 The proof of primitive 2-universality (exceptional cases)

Let L be one of the remaining 26 candidates of primitively 2-universal $\mathbb{Z}$-lattices which we do not consider in Section 4.3. We try to find quinary or quaternary primitive sublattices of L which have class number one or two to prove primitive 2-universality of L, in each exceptional case. The next two lemmas summarize some data needed for computations which will be used throughout this section.

Lemma 4.4.1. *For each given quaternary $\mathbb{Z}_2$-lattice N, the binary $\mathbb{Z}_2$-lattice ℓ that is not primitively represented by N satisfies one of the conditions given in Table 4.4.*

Proof. Since one may prove the lemma by direct computations, the proof is left to the readers. □

Lemma 4.4.2. *For the 5-section M and its core prime q of each type given in Table 4.5, if a binary $\mathbb{Z}$-lattice ℓ is not primitively represented by M, then ℓ satisfies one of the conditions given in the table.*

Table 4.4: The local structures

N	Binary $\mathbb{Z}_2$-lattices that are not primitively represented by N
I_4	$\ell_2 \cong \langle \epsilon, 4\alpha \rangle$, $\langle 2, 6 \rangle$, $\langle 2\epsilon, 8\alpha \rangle$, $\mathfrak{n}(\ell_2) \subseteq 4\mathbb{Z}_2$, or $\mathbb{Q}_2\ell \cong \mathbb{Q}_2\mathbb{H}$
$\langle 1, 1, 2, 2 \rangle$	$\ell_2 \cong \langle \epsilon, 4\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle \epsilon, 16\alpha \rangle$, $\langle 2\epsilon, 8\alpha \rangle$, $\langle 4\epsilon, 4\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle 4\epsilon, 16\alpha \rangle$, $\mathbb{A}$, $\mathfrak{n}(\ell_2) \subseteq 8\mathbb{Z}_2$, or $\mathbb{Q}_2\ell \cong \mathbb{Q}_2\mathbb{H}$
$\langle 1, 2, 2, 4 \rangle$	ℓ_2 is unimodular, $\ell_2 \cong \langle \epsilon, 16\alpha \rangle$, $\langle 2\epsilon, 8\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle 2\epsilon, 32\alpha \rangle$, $\langle 4\epsilon, 16\alpha \rangle$, $\langle 8\epsilon, 8\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle 8\epsilon, 32\alpha \rangle$, $\mathbb{A}^2$, $\mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2$, or $\mathbb{Q}_2\ell \cong \mathbb{Q}_2\mathbb{H}$
$\langle 1, 2 \rangle \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$	$\ell_2 \cong \langle 1, 1 \rangle$, $\langle 3, -1 \rangle$, $\langle 1, 20 \rangle$, $\langle -1, -4 \rangle$, $\langle \epsilon, 16\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle \epsilon, 64\alpha \rangle$, $\langle 2, 2 \rangle$, $\langle 2, 6 \rangle$, $\langle 2, 10 \rangle$, $\langle 2\epsilon, 16\alpha \rangle$ with $\epsilon \equiv 1 \pmod 4$, $\langle 2\epsilon, 32\alpha \rangle$ with $\epsilon \equiv -1 \pmod 4$, $\langle 12, 12 \rangle$, $\langle 4, 20 \rangle$, $\langle 4\epsilon, 16\delta \rangle$ with $\epsilon\delta \equiv 3 \pmod 8$, $\langle 4\epsilon, 64\alpha \rangle$, $\mathfrak{n}(\ell_2) \subseteq 8\mathbb{Z}_2$, or $\mathbb{Q}_2\ell \cong \mathbb{Q}_2\mathbb{H}$

Proof. Since one may prove the lemma by direct computations, the proof is left to the readers. $\square$

Recall that a finite sequence of vectors $v_1, \ldots, v_m$ in $\mathbb{Z}^n$ $(m \leq n)$ is primitive if and only if the greatest common divisor g of the determinants of all $m \times m$ submatrices of the coefficient matrix of $v_1, \ldots, v_m$, which is defined by the $m \times n$ matrix whose rows are $v_1, \ldots, v_m$, is a unit. Also, we say that $v_1, \ldots, v_m$ is

Table 4.5: The core prime and the local structures

Type	M	q	Local structures
C	$I_4 \perp \langle 3\rangle$	2	$\ell_2 \cong \langle 3, 8\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 4\mathbb{Z}_2$
F	$I_3 \perp \langle 2, 3\rangle$	3	$\ell_2 \cong \langle 4\epsilon, 4\delta\rangle$ or $\mathbb{A}$, or $\ell_3 \cong \langle 6, 9\alpha\rangle$ or $\mathfrak{s}(\ell_3) \subseteq 9\mathbb{Z}_3$
J	$I_2 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \perp \langle 3\rangle$	2	$\ell_2 \cong \langle 1, 8\alpha\rangle$ or $\mathfrak{n}(\ell_2) \subseteq 4\mathbb{Z}_2$
-	$I_3 \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$	2	$\ell_2 \cong \langle 3, 7\rangle$, $\langle -1, 4\rangle$, $\langle 2, 2\rangle$, $\langle 2, 64\alpha\rangle$, $\langle 10, 32\epsilon\rangle$, $\langle 8, 64\alpha\rangle$, or $\mathfrak{s}(\ell_2) \subseteq 16\mathbb{Z}_2$

p-primitive for a prime p if g is prime to p. Then it is clear that $v_1, \ldots, v_m$ is primitive if and only if it is p-primitive for any prime p.

Lemma 4.4.3. *Let $L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n+1}$ be a free $\mathbb{Z}$-module of rank $n+1$, and let $M = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$. Suppose that $v_1, \ldots, v_m$ are vectors in M for some $1 \le m \le n$.*

(a) *Suppose that $v_1, \ldots, v_m$ is p-primitive for some prime p. Then $v_1, \ldots, v_{m-1}, v_m + pw$ also is p-primitive for any $w \in M$.*

(b) *Suppose that $v_1, \ldots, v_m$ is p-primitive for some odd prime p. Then for any $w \in M$, either $v_1, \ldots, v_{m-1}, v_m + w$ or $v_1, \ldots, v_{m-1}, v_m - w$ also is p-primitive.*

(c) *Suppose that $v_1, \ldots, v_m$ is primitive. For a vector $y = y_1e_1 + \cdots + y_ne_n +$*

$y_{n+1}e_{n+1} \in L$, put

$$\mathcal{P}(y) := \{p : \gcd(y_1, \ldots, y_n, y_{n+1}) \text{ is divisible by a prime } p\},$$
$$\mathcal{P}(y_{n+1}) := \{p : y_{n+1} \text{ is divisible by a prime } p\}.$$

If $\mathcal{P}(y_{n+1}) \backslash \mathcal{P}(y) = \varnothing$, then $v_1, \ldots, v_{m-1}, v_m+y$ is primitive. If $\mathcal{P}(y_{n+1}) \backslash \mathcal{P}(y) = \{p\}$ for an odd prime p, then either $v_1, \ldots, v_{m-1}, v_m + y$ or $v_1, \ldots, v_{m-1}, v_m - y$ is primitive.

Proof. (a) The lemma follows from the fact that the determinant of any $m \times m$ submatrix of the $m \times n$ coefficient matrix of $v_1, \ldots, v_m$ is congruent modulo p to the determinant of the corresponding $m \times m$ submatrix of the $m \times n$ coefficient matrix of $v_1, \ldots, v_{m-1}, v_m + pw$.
(b) Suppose to the contrary that both

$$v_1, \ldots, v_{m-1}, v_m + w \quad \text{and} \quad v_1, \ldots, v_{m-1}, v_m - w$$

are not p-primitive. This implies that the determinant of any $m \times m$ submatrix of the $m \times n$ coefficient matrices C^η of $v_1, \ldots, v_{m-1}, v_m + \eta w$ is a multiple of p for any $\eta \in \{1, -1\}$. Observe that, by multilinearity of the determinant, the determinant of any $m \times m$ submatrix of C^η is equal to

$$\begin{aligned}&\det(\text{the corresponding } m \times m \text{ submatrix of } C)\\ &\qquad\qquad + \eta \det(\text{the corresponding } m \times m \text{ submatrix of } C'),\end{aligned}$$

where C is the $m \times n$ coefficient matrix of $v_1, \ldots, v_m$, and C' is that of $v_1, \ldots, v_{m-1}, w$. Since p is odd, if the determinants of any $m \times m$ submatrix of C^1 and the corresponding $m \times m$ submatrix of C^{-1} are multiples of p

simultaneously, then so are the determinants of the corresponding $m \times m$ submatrices of C and C'. This implies that $v_1, \dots, v_m$ is not p-primitive, which is a contradiction.

(c) We have to show that the greatest common divisor of the determinants of $m \times m$ submatrices of $m \times (n+1)$ coefficient matrix of $v_1, \dots, v_{m-1}, v_m + \eta y$ is 1 for some $\eta \in \{1, -1\}$. Let g_1 be the greatest common divisor of the determinants of all $m \times m$ submatrices containing the $(n+1)$-th column, and g_2 be the greatest common divisor of the determinants of those not containing the column. Then what we have to show is $(g_1, g_2) = 1$. Let

$$w = y_1 e_1 + \cdots + y_n e_n \in M.$$

Then g_2 is equal to the greatest common divisor of the determinants of all $m \times m$ submatrices of $m \times n$ coefficient matrix of $v_1, \dots, v_{m-1}, v_m + \eta w$. Note that $g_1 = |y_{n+1}|$. Hence it suffices to show that $v_1, \dots, v_{m-1}, v_m + \eta w$ is q-primitive for any $q \in \mathcal{P}(y_{n+1})$. If $\mathcal{P}(y_{n+1}) \setminus \mathcal{P}(y) = \varnothing$, it follows from (1). If $\mathcal{P}(y_{n+1}) \setminus \mathcal{P}(y) = \{p\}$, it follows from (1) that both $v_1, \dots, v_{k-1}, v_k + w$ and $v_1, \dots, v_{k-1}, v_k - w$ are q-primitive for any $q \in \mathcal{P}(y_{n+1})$ such that $q \neq p$ (equivalently, for any $q \in \mathcal{P}(y)$), and it follows from (2) that at least one of the two is p-primitive. $\square$

4.4.1 Type B^{ii}

Theorem 4.4.4. *The lattice* $\mathrm{B}^{\mathrm{ii}}_q \cong I_4 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right)$ $(q = 5,\ 6)$ *is primitively* 2-*universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice which $I_4 \perp \langle 2 \rangle$ does not primitively represent. Then $\mathfrak{s}\ell_2 = (2)$ and $d\ell_2 \subseteq (2^5)$ or $\mathfrak{n}\ell_2 \subseteq (8)$ by Lemma 4.3.3. In particular, a, b and c are all even. If we show that L primitively represents ℓ, then we are done.

Denote by N the 4-section of L. Then N primitively represent a binary $\mathbb{Z}$-lattice ℓ' if and only if ℓ' is positive definite and N_2 primitively represent ℓ'_2, by the last lemma. Now suppose that

$$\ell' \cong \begin{pmatrix} a-A & b-B \\ b-B & c-C \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$$

for some integers s_1, s_2, t_1 and t_2. If $\ell' = \ell'(s_1, s_2; t_1, t_2)$ is primitively represented by N then evidently ℓ is primitively represented by $L \cong N \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right)$. Finally, put

$$l^{(1)} = \ell'(1,0;0,1) \cong \begin{pmatrix} a-2 & b-1 \\ b-1 & c-q \end{pmatrix}, \quad l^{(2)} = \ell'(0,1;1,0) \cong \begin{pmatrix} a-q & b-1 \\ b-1 & c-2 \end{pmatrix},$$

$$l^{(3)} = \ell'(-1,1;0,1) \cong \begin{pmatrix} a-q & b-(q-1) \\ b-(q-1) & c-q \end{pmatrix},$$

$$l^{(4)} = \ell'(0,0;0,1) \cong \begin{pmatrix} a & b \\ b & c-q \end{pmatrix}, \quad l^{(5)} = \ell'(0,1;0,0) \cong \begin{pmatrix} a-q & b \\ b & c \end{pmatrix}.$$

Let $q = 5$. If $a \in (4)$ then $a \in (8)$. Then $d\ell^{(1)} = dl - 5a + 2b - 2c + 9 \equiv 1\ (4)$, hence N_2 primitively represents $l_2^{(1)}$. Hence N primitively represents $l^{(1)}$ if $a \ge 10$, or if $a = 8$ and $c \ge 7$. Now suppose $a \equiv 2\ (4)$ then $a \equiv 2\ (16)$. If $c \in (4)$ then $d\ell^{(2)} = dl - 2a + 2b - 5c + 9 \equiv 1\ (4)$, hence I_4 primitively represents $l^{(2)}$ if $a \geq 10$. If $a = 2$, then we must have $b = 0$ and $c \in (16)$, hence N primitively represents $c - 18$ if $c \geq 32$, then L primitively represents $\mathbb{Z}[e_5, v - e_5 + 2e_6] \cong \langle 2, c\rangle$ where v is a primitive vector in N such that $Q(v) = c - 18$. Finally suppose $a \equiv c \equiv 2$ (4), then $a \equiv c \equiv 2$ (16). Then $d\ell^{(3)} = dl - 5a + 8b - 5c + 9 \equiv 5$ (8), hence N primitively represents $l^{(3)}$ since $a \geq 18$. A direct calculation shows that the only remnant $\langle 2, 16\rangle$ also is primitively represented by L.

Let $q = 6$. If $\mathfrak{n}\ell_2 \subseteq (8)$ then $d\ell^{(1)} = dl - 6a + 2b - 2c + 11 \equiv 3$ (8), hence N_2 primitively represents $l_2^{(1)}$. Hence N primitively represents $l^{(1)}$ if $a \geq 11$, or if $a = 8$ and $c \geq 8$. Now suppose $\mathfrak{s}\ell_2 = (2)$ and $d\ell_2 \subseteq (2^5)$. If $a \equiv 2$ (4) then $a \equiv 2$ (16), hence $d\ell^{(4)} = dl - 6a \equiv 4$ (16), hence N_2 primitively represents $l_2^{(4)}$. Hence N primitively represents $l^{(4)}$ if $c \geq 9$. If $a \in (4)$ then $c \equiv 2$ (16), hence $d\ell^{(5)} = dl - 6c \equiv 4$ (16), then N primitively represents $l^{(5)}$ if $a \geq 9$, or if $a \geq 8$ and $c \geq 9$. Finitely many remnants with $c \leq 8$ can be verified directly. $\square$

4.4.2 Type D$^{\text{ii}}$

Lemma 4.4.5. *Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a $\mathbb{Z}$-lattice and let m and $k \geq 2$ be positive integers. Suppose that*

$$\begin{pmatrix} 2a - (2k-1)s^2 & 2b - (2k-1)st \\ 2b - (2k-1)st & 2c - (2k-1)t^2 \end{pmatrix}$$

is positive definite and is primitively represented by $2I_m \perp \langle 4, 1\rangle$ for some integers s and t. Then the binary $\mathbb{Z}$-lattice ℓ is primitively represented by

$I_m \perp \langle 2\rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & k \end{smallmatrix}\right)$.

Proof. Suppose that $2I_m \perp \langle 4, 1\rangle$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & \cdots & c_m & c_{m+1} & c_{m+2} \\ d_1 & \cdots & d_m & d_{m+1} & d_{m+2} \end{bmatrix} \cong \begin{pmatrix} 2a-(2k-1)s^2 & 2b-(2k-1)st \\ 2b-(2k-1)st & 2c-(2k-1)t^2 \end{pmatrix}.$$

Then

$$\left.\begin{aligned} c_{m+2} \equiv c_{m+2}^2 \equiv s^2 \equiv s \\ d_{m+2} \equiv d_{m+2}^2 \equiv t^2 \equiv t \end{aligned}\right\} \pmod 2,$$

hence there exist integers a_2 and b_2 such that $c_{m+2} = 2a_2 + s$ and $d_{m+2} = 2b_2 + t$. Now we claim that $I_m \perp \langle 2\rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right)$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & \cdots & c_m & c_{m+1} & a_2 & s \\ d_1 & \cdots & d_m & d_{m+1} & b_2 & t \end{bmatrix} \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For the representation, we must verify the identity

$$\begin{pmatrix} a_2 & s \\ b_2 & t \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 2q \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ s & t \end{pmatrix} = \begin{pmatrix} c_{m+2} & s \\ d_{m+2} & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2q-1 \end{pmatrix}\begin{pmatrix} c_{m+2} & d_{m+2} \\ s & t \end{pmatrix},$$

which is evident. For the primitivity, observe that

$$\begin{pmatrix} c_1 & \cdots & c_{m+2} \\ d_1 & \cdots & d_{m+2} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_{m+1} & 2a_2+s \\ d_1 & \cdots & d_{m+1} & 2b_2+t \end{pmatrix}$$

is primitive, hence so is

$$\begin{pmatrix} c_1 & \cdots & c_{m+1} & 2a_2+s & a_2 \\ d_1 & \cdots & d_{m+1} & 2b_2+t & b_2 \end{pmatrix},$$

hence so is

$$\begin{pmatrix} c_1 & \cdots & c_{m+1} & a_2 & s \\ d_1 & \cdots & d_{m+1} & b_2 & t \end{pmatrix}.$$

This completes the proof. $\square$

Theorem 4.4.6. *The lattice* $\mathrm{D}^{\mathrm{ii}}_k \cong I_3 \perp \langle 2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right)$ $(4 \le k \le 8)$ *is primitively 2-universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice. To apply the last lemma, observe that $N = 2I_3 \perp \langle 4, 1 \rangle$ is of class number one, the only core prime of N is 2, $N_2 \cong 2\mathbb{H} \perp \langle 1, 10, 12 \rangle \cong 2\mathbb{H} \perp \langle 3, 10, 4 \rangle$ hence N_2 primitively represents all binary $\mathbb{Z}_2$-lattices of the form $\langle 2\alpha, \epsilon \rangle$. Consider a $\mathbb{Z}$-lattice $\ell' \cong \begin{pmatrix} 2a & 2b \\ 2b & 2c-(2k-1) \end{pmatrix}$, then $\mathfrak{s}\ell'_2 = \mathbb{Z}_2$ and $d\ell' \subseteq (2)$, hence N_2 primitively represents ℓ'_2. Then N primitively represents ℓ' if $c > 2(2k-1)/3$, hence L primitively represents ℓ if $c \ge 11$. Finitely many remnants with $c \le 10$ can be verified directly. $\square$

4.4.3 Type $\mathrm{D}^{\mathrm{iii}}$

We reserve the case of lattice $\mathrm{D}^{\mathrm{iii}}_5$ to the end of this section.

Lemma 4.4.7. *Let* $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ *be a positive definite binary* $\mathbb{Z}$*-lattice and let* m *and* $q \ge 2$ *be positive integers. Suppose that* I_{m+2} *primitively represents*

$$\mathbb{Z}\begin{bmatrix} c_1 & \cdots & c_m & c_{m+1} & c_{m+2} \\ d_1 & \cdots & d_m & d_{m+1} & d_{m+2} \end{bmatrix} \cong \begin{pmatrix} a-(k-1)s^2 & b-(k-1)st \\ b-(k-1)st & c-(k-1)t^2 \end{pmatrix}$$

for some integers s and t that satisfy $c_{m+1}+c_{m+2}+s \equiv d_{m+1}+d_{m+2}+t \equiv 0\ (2)$. Then ℓ is primitively represented by $I_m \perp \left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & k \end{smallmatrix}\right)$.

Proof. Define integers a_1, a_2, b_1 and b_2 by equations

$$\begin{aligned} a_1 &:= a_2 + c_{m+2}, & a_2 &:= \tfrac{c_{m+1}-c_{m+2}-s}{2}, \\ b_1 &:= b_2 + d_{m+2}, & b_2 &:= \tfrac{d_{m+1}-d_{m+2}-t}{2}, \end{aligned}$$

so that they satisfy $c_{m+1} = a_1 + a_2 + s$, $c_{m+2} = a_1 - a_2$, $d_{m+1} = b_1 + b_2 + t$ and $d_{m+2} = b_1 - b_2$. Now we claim that $I_m \perp \left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & k \end{smallmatrix}\right)$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & \cdots & c_m & a_1 & a_2 & s \\ d_1 & \cdots & d_m & b_1 & b_2 & t \end{bmatrix} \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For the representation, we must verify the identity

$$\begin{aligned} &\begin{pmatrix} a_1 & a_2 & s \\ b_1 & b_2 & t \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & k \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ s & t \end{pmatrix} \\ &\qquad = \begin{pmatrix} c_{m+1} & c_{m+2} & s \\ d_{m+1} & d_{m+2} & t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k-1 \end{pmatrix} \begin{pmatrix} c_{m+1} & d_{m+1} \\ c_{m+2} & d_{m+2} \\ s & t \end{pmatrix}, \end{aligned}$$

which is evident. For the primitivity, observe that

$$\begin{pmatrix} c_1 & \cdots & c_{m+2} \\ d_1 & \cdots & d_{m+2} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_m & a_1+a_2+s & a_1-a_2 \\ d_1 & \cdots & d_m & b_1+b_2+t & b_1-b_2 \end{pmatrix}$$

is primitive, hence so is

$$\begin{pmatrix} c_1 & \cdots & c_m & a_1+a_2+s & a_1-a_2 & a_1 \\ d_1 & \cdots & d_m & b_1+b_2+t & b_1-b_2 & b_1 \end{pmatrix},$$

hence so is

$$\begin{pmatrix} c_1 & \cdots & c_m & a_1 & a_2 & s \\ d_1 & \cdots & d_m & a_1 & b_2 & t \end{pmatrix}.$$

This completes the proof. $\square$

Lemma 4.4.8. *Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a $\mathbb{Z}$-lattice. Suppose that there exists a primitive sublattice $\mathbb{Z}\left[\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{smallmatrix}\right]$ of the $\mathbb{Z}$-lattice I_5, which is isometric to ℓ, for some integers c_i and d_i such that the set*

$$\{(\overline{c_i+c_j}, \overline{d_i+d_j}) \mid 1 \le i < j \le 5\}$$

is a proper subset of $(\mathbb{Z}/2\mathbb{Z})^2$, where $\overline{n}$ is the residue class of n modulo 2 for any integer n. Then ℓ satisfies one of the followings:

(i) *a or $c \equiv 1 \pmod 4$ and $d\ell \equiv 0 \pmod 4$;*

(ii) *$a \not\equiv 1 \pmod 8$, $c \not\equiv 1 \pmod 8$, and $d\ell \equiv 2 \pmod 4$.*

Proof. Consider the set $C = \{(\overline{c_i}, \overline{d_i}) \mid 1 \le i \le 5\}$. Since $\mathbb{Z}\left[\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{smallmatrix}\right]$ is a primitive sublattice of I_5, the set C contains at least two nonzero vectors in $(\mathbb{Z}/2\mathbb{Z})^2$. Furthermore, one may easily see from the assumption that C is one of

$$\{(\overline{1},\overline{0}),(\overline{0},\overline{1})\}, \quad \{(\overline{1},\overline{0}),(\overline{1},\overline{1})\}, \quad \{(\overline{0},\overline{1}),(\overline{1},\overline{1})\},$$

which respectively corresponds to each of the followings:

(a) $a+c \equiv 1 \pmod 4$ and b is even;

(b) $a \equiv 5 \pmod 8$ and $b \equiv c \pmod 2$;

(c) $a \equiv b \pmod 2$ and $c \equiv 5 \pmod 8$.

The lemma follows directly from this. $\square$

Theorem 4.4.9. *The $\mathbb{Z}$-lattice $\mathrm{D}_k^{\mathrm{iii}} \cong I_3 \perp \left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & k \end{smallmatrix}\right)$ for $k=6$ or 7 is primitively 2-universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a $\mathbb{Z}$-lattice such that $0 \le 2b \le a \le c$. Note that the 5-section $M \cong I_3 \perp \langle 2,2\rangle$ in this case has class number one. Hence, we may assume that ℓ is not primitively represented by M locally, that is, one of the following conditions holds:

(i) $\ell_2 \cong \langle 1, 16\alpha\rangle$ for some $\alpha \in \mathbb{Z}_2$;

(ii) $\ell_2 \cong \langle 4, 16\alpha\rangle$ or $\langle 20, 16\alpha\rangle$ for some $\alpha \in \mathbb{Z}_2$;

(iii) $\mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2$.

Note that we have $a \equiv 1 \pmod 8$, $a \equiv 4 \pmod{16}$, or $a \equiv 0 \pmod{16}$. Assume that both of the $\mathbb{Z}$-lattices

$$\ell(1) \cong \begin{pmatrix} a-(k-1) & b \\ b & c \end{pmatrix}, \qquad \ell(2) \cong \begin{pmatrix} a & b \\ b & c-(k-1) \end{pmatrix}$$

are positive definite. Then one may easily show that $\ell(s)$ is primitively represented by I_5 for some $s = 1, 2$. Let $N = \mathbb{Z}\left[\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{smallmatrix}\right]$ be a primitive binary $\mathbb{Z}$-sublattice of I_5 which is isometric to $\ell(s)$. If there is an (i, j) with $1 \le i < j \le 5$ such that

$$c_i + c_j + (2 - s) \equiv d_i + d_j + (s - 1) \equiv 0 \pmod 2,$$

then ℓ is primitively represented by $\mathrm{D}_k^{\mathrm{iii}}$ by Lemma 4.4.7. If there does not exist such an (i, j), then by Lemma 4.4.8, one of the followings must hold:

(a) $a - (2 - s)(k - 1) \equiv 1 \pmod 4$ or $c - (s - 1)(k - 1) \equiv 1 \pmod 4$, and $d\ell(s) \equiv 0 \pmod 4$;

(b) $a - (2 - s)(k - 1) \not\equiv 1 \pmod 8$, $c - (s - 1)(k - 1) \not\equiv 1 \pmod 8$, and $d\ell(s) \equiv 2 \pmod 4$.

However, one may easily verify that none of (a) and (b) holds in each case. For instance, consider case (i) when $k = 6$. If a is odd, then $a \equiv 1 \pmod 8$. Since $d\ell(2) \equiv 3 \pmod 8$, $\ell(2)$ is primitively represented by I_5 so that we may take $s = 2$. Since $d\ell(2)$ is odd, $\ell(2)$ satisfies neither (a) nor (b). Now, suppose that a is even. Then $a \equiv 0 \pmod 4$ and $c \equiv 1 \pmod 8$. Therefore, similarly to the above, $\ell(1)$ is primitively represented by I_5 and hence $\ell(1)$ does not satisfy any of (a) and (b).

Now, we have to consider the case when neither $\ell(1)$ nor $\ell(2)$ is positive definite. Note that if $a \ge 9$, then both $\ell(1)$ and $\ell(2)$ are positive definite. Hence, we may assume that $a = 1$ or $a = 4$. If $a = 1$, then $b = 0$ and $c \equiv 0 \pmod 8$ by (i). Since $\ell(2)$ is positive definite if $c \ge 9$, one may apply the same method as the above to prove the theorem. One may directly check that

ℓ is primitively represented by L if $c \le 8$. Now, assume that $a = 4$. Then we have either $b = 0$ or $b = 2$. If $b = 0$, then $c \equiv 0 \pmod{16}$ by (ii). Hence, $\ell(2)$ is positive definite, and we may apply the same method to the above to prove the theorem. If $b = 2$, then $c \equiv 1 \pmod 8$ by (i). Note that I_3 represents $c - k \equiv 2, 3 \pmod 8$ by Legendre's three-square theorem. If we choose a vector v in the 3-section of L such that $Q(v) = c - k$, then clearly, $\mathbb{Z}[e_4 + e_5, v + e_6]$ is a primitive sublattice of L isometric to $\left(\begin{smallmatrix} 4 & 2 \\ 2 & c \end{smallmatrix}\right)$. $\square$

4.4.4 Type $\mathbf{H^i}$

The main obstacle for type H lattices is that the 5-section $I_2 \perp \mathbb{A} \perp \langle 2 \rangle$ of L is of class number two, and the genus mate $I_4 \perp \langle 6 \rangle$ is not represented by L. Lemma 4.4.5 gives some information on binary $\mathbb{Z}$-lattices that are primitively represented by the 5-section of L, though it has class number two.

Lemma 4.4.10. *If a binary $\mathbb{Z}$-lattice ℓ is not primitively represented by the $\mathbb{Z}$-lattice $M = I_2 \perp \langle 2 \rangle \perp \mathbb{A}$, then either $\ell_2 \cong \langle 6, 16\alpha \rangle$ or $\mathfrak{n}(\ell_2) \subseteq 8\mathbb{Z}_2$.*

Proof. Fix a basis for M corresponding to the Gram matrix in the statement of the lemma. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a binary $\mathbb{Z}$-lattice which does not satisfy the conclusion. Since M primitively represents $\langle 1, 1, 2, 2 \rangle$, we may assume that

$$\ell_2 \cong \begin{cases} \langle 1,-1\rangle,\ \langle \epsilon, 4\delta\rangle,\ \langle \epsilon, 16\alpha\rangle, \\ \langle 2,-2\rangle,\ \langle 2\epsilon, 8\alpha\rangle, \\ \langle 4\epsilon, 4\delta\rangle \text{ with } \epsilon\delta \equiv -1 \pmod 4),\ \langle 4\epsilon, 16\alpha\rangle, \\ \mathbb{H}^2, \\ \mathbb{H} \text{ or } \mathbb{A}. \end{cases}$$

We define the binary $\mathbb{Z}$-lattices

$$\ell'(u,t) \cong \begin{pmatrix} 2a-3t^2 & 2ua+2b \\ 2ua+2b & 2u^2a+4ub+2c \end{pmatrix} \quad \text{and} \quad \ell''(u,t) \cong \begin{pmatrix} 2a+4ub+2u^2c & 2b+2uc \\ 2b+2uc & 2c-3t^2 \end{pmatrix}.$$

Note that

$$\ell \cong \begin{pmatrix} a & ua+b \\ ua+b & u^2a+2ub+c \end{pmatrix} \cong \begin{pmatrix} a+2ub+u^2c & b+uc \\ b+uc & c \end{pmatrix}$$

for any integer u. Hence, if $\ell'(u,t)$ or $\ell''(u,t)$ is primitively represented by $N \cong \langle 1,2,2,4\rangle$ for some integers u and t, then ℓ is primitively represented by M by Lemma 4.4.5. Since $N_p \cong \mathbb{H} \perp \mathbb{H}$ for any odd prime p, N_p is primitively 2-universal over $\mathbb{Z}_p$ for any odd prime p. Note that

$$\mathfrak{s}(\ell'(u,1)_2) = \mathfrak{s}(\ell''(u,1)_2) = \mathbb{Z}_2.$$

First, assume that $\mathfrak{s}(\ell_2) = \mathbb{Z}_2$. Assume that a is odd. Since $d\ell''(0,1) \equiv 2 \pmod 4$, $\ell''(0,1)_2$ is primitively represented by N_2. Note that $\ell''(0,1)$ is positive definite. Hence, $\ell''(0,1)$ is primitively represented by N. Now, assume that a is even. Then $a \equiv 0 \pmod 4$ and c is odd. In this case,

$d\ell'(0,1) \equiv 2 \pmod 4$ and $\ell'(0,1)$ is positive definite. Hence, $\ell'(0,1)$ is primitively represented by N.

Now, assume that $\ell_2 \cong \langle 2,-2\rangle$. Note that $a \equiv b \equiv c \equiv 0 \pmod 2$. First, suppose that $a \equiv 2 \pmod 4$. If $a \not\equiv -2 \pmod{16}$, then

$$d\ell''(0,1) \equiv 4 \pmod 8 \quad \text{and} \quad d\ell''(0,1) \not\equiv -4 \pmod{32}.$$

Hence, $\ell''(0,1)$ is primitively represented by N. If $a \equiv -2 \pmod{16}$, then

$$\mathfrak{s}(\ell''(0,2)_2) = 4\mathbb{Z}_2 \quad \text{and} \quad d\ell''(0,2) \equiv 32 \pmod{64}.$$

Hence, $\ell''(0,2)$ is primitively represented by N. Now, suppose that $a \equiv 0 \pmod 4$. Then $a \equiv 0 \pmod{16}$ and $c \equiv 2 \pmod 4$. If $c \not\equiv -2 \pmod{16}$, then $\ell'(0,1)$ is primitively represented by N, and if $c \equiv -2 \pmod{16}$, then $\ell'(0,2)$ is primitively represented by N.

Next, assume that $\ell_2 \cong \langle 2\epsilon, 8\alpha\rangle$. Note that $a \equiv b \equiv c \equiv 0 \pmod 2$. If $a \equiv 2 \pmod 4$ and $a \not\equiv 6 \pmod{16}$, then

$$d\ell''(0,1) \equiv 4 \pmod 8 \quad \text{and} \quad d\ell''(0,1) \not\equiv -4 \pmod{32}.$$

Hence, $\ell''(0,1)$ is primitively represented by N. Similarly, if $c \equiv 2 \pmod 4$ and $c \not\equiv 6 \pmod{16}$, then $\ell'(0,1)$ is primitively represented by N. Now, assume that $a \equiv 6 \pmod{16}$ or $c \equiv 6 \pmod{16}$. Since we are assuming that

$$\ell_2 \not\cong \langle 6, 16\alpha\rangle,$$

we have $d\ell \equiv 16 \pmod{32}$. Assume that $a \equiv c \equiv 6 \pmod{16}$. Then, $b \equiv 2 \pmod 4$. Hence, there is an $\eta \in \{1,-1\}$ such that

$$a + 2\eta b + c \equiv 6 \pmod 8.$$

Since $d\ell''(\eta, 1) \equiv 8 \pmod{16}$, we have

$$\ell''(\eta, 1)_2 \cong \langle \epsilon, 8\delta \rangle,$$

which is primitively represented by $N_2 \cong \langle 1, 2, 2, 4 \rangle$. Furthermore, since $\ell''(\eta, 1)$ is positive definite, it is primitively represented by N. Next, assume that $a \equiv 6 \pmod{16}$ and $c \not\equiv 6 \pmod{16}$. Then either $c \equiv 8 \pmod{16}$ and $b \equiv 0 \pmod 8$, or $c \equiv 0 \pmod{16}$ and $b \equiv 4 \pmod 8$. In any case,

$$a - 2b + c \equiv -2 \pmod{16}.$$

Since $d\ell''(-1, 1) \equiv 12 \pmod{32}$, we have

$$\ell''(-1, 1)_2 \cong \langle -3, -4 \rangle,$$

which is primitively represented by $N_2 \cong \langle 1, 2, 2, 4 \rangle$. Furthermore, since $\ell''(-1, 1)$ is positive definite, it is primitively represented by N. Finally, assume that $a \not\equiv 6 \pmod{16}$ and $c \equiv 6 \pmod{16}$. Then, similarly to the above, $\ell'(-1, 1)$ is primitively represented by N in this case.

Now, assume that $\mathfrak{s}(\ell_2) = 4\mathbb{Z}_2$. Note that $a \equiv b \equiv c \equiv 0 \pmod 4$. If $a \equiv 4 \pmod 8$, then $d\ell''(0, 1) \equiv 8 \pmod{16}$. Since $\ell''(0, 1)$ is positive definite, $\ell''(0, 1)$ is primitively represented by N. Similarly, if $c \equiv 4 \pmod 8$, then $\ell'(0, 1)$ is primitively represented by N in this case.

Next, assume that $\ell_2 \cong \mathbb{H}^2$. Note that $a \equiv b - 2 \equiv c \equiv 0 \pmod 4$. Assume that $a \equiv 4 \pmod 8$. Since $d\ell''(0, 1) \equiv 8 \pmod{16}$, we have

$$\ell''(0, 1)_2 \cong \langle \epsilon, 8\delta \rangle,$$

which is primitively represented by $N_2 \cong \langle 1,2,2,4\rangle$. Since $\ell''(0,1)$ is positive definite, $\ell''(0,1)$ is primitively represented by N. Next, assume that $c \equiv 4 \pmod 8$. Then, similarly to the above, $\ell'(0,1)$ is primitively represented by N in this case. Finally, assume that $a \equiv c \equiv 0 \pmod 8$. Since

$$a - 2b + c \equiv 4 \pmod 8,$$

we have $d\ell''(-1,1) \equiv 8 \pmod{16}$. Since $\ell''(-1,1)$ is positive definite, $\ell''(-1,1)$ is primitively represented by N.

Finally, assume that $\ell_2 \cong \mathbb{H}$ or $\mathbb{A}$. Note that $a \equiv b-1 \equiv c \equiv 0 \pmod 2$. Assume that $a \equiv 6 \pmod 8$. Since $d\ell''(0,1) \equiv 8 \pmod{16}$, we have

$$\ell''(0,1)_2 \cong \langle \epsilon, 8\delta\rangle,$$

which is primitively represented by $N_2 \cong \langle 1,2,2,4\rangle$. Hence, $\ell''(0,1)$ is primitively represented by N. Assume that $c \equiv 6 \pmod 8$. Then, similarly to the above, $\ell'(0,1)$ is primitively represented by N in this case. Now, suppose that neither a nor c is congruent to 6 modulo 8. Then we have

$$a \equiv 2 \pmod 8 \quad \text{or} \quad a \equiv 0 \pmod 4,$$

and the same with c. First, assume that $a \equiv c \equiv 2 \pmod 8$. Then, there is an $\eta \in \{1,-1\}$ such that

$$a + 2\eta b + c \equiv 6 \pmod 8.$$

Since $d\ell''(\eta,1) \equiv 8 \pmod{16}$, $\ell''(\eta,1)$ is primitively represented by $N_2 \cong \langle 1,2,2,4\rangle$. Hence, $\ell''(\eta,1)$ is primitively represented by N if it is positive definite, that is, if $a \geq 7$, or if $a = 2$ and $c \geq 15$. Clearly, $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 10 \end{smallmatrix}\right)$ are primitively represented by M. Next, assume that $a \equiv 0 \pmod 4$ and $c \equiv 2 \pmod 8$. Then

$$4a - 4b + c \equiv 6 \pmod 8.$$

Since $d\ell'(-2,1) \equiv 8 \pmod{16}$, we have $\ell'(-2,1)_2 \cong \langle \epsilon, 8\delta \rangle$, which is primitively represented by N_2. Since

$$\ell'(-2,1) \cong \left(\begin{smallmatrix} 2a-3 & -4a+2b \\ -4a+2b & 8a-8b+2c \end{smallmatrix}\right)$$

is positive definite, it is primitively represented by N. Now, assume that $a \equiv 2 \pmod 8$ and $c \equiv 0 \pmod 4$. Then, similarly to the above, $\ell''(-2,1)$ is primitively represented by N if it is positive definite, that is, if $a \geq 11$, or if $a = 10$ and $c \geq 4$. The case when $a = 2$ will be postponed to the end of this proof. Finally, assume that $a \equiv c \equiv 0 \pmod 4$. Then, there is an $\eta \in \{1, -1\}$ such that

$$a + 2\eta b + c \equiv 6 \pmod 8.$$

Hence, $\ell''(\eta, 1)$ is primitively represented by N if it is positive definite, that is, if $a \geq 7$, or if $a = 4$ and $c \geq 5$. Clearly $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ is primitively represented by M.

Now, suppose that $a = 2$, $b = 1$, and $c \equiv 0 \pmod 4$. If $c \not\equiv 0 \pmod{16}$, then $\langle 1,1,2 \rangle$ represents $c-2$. If we choose a vector v in the 3-section of M such that $Q(v) = c-2$, then clearly, $\mathbb{Z}[e_4, v+e_5]$ is a primitive sublattice of M isometric to $\left(\begin{smallmatrix} 2 & 1 \\ 1 & c \end{smallmatrix}\right)$. If $c \equiv 0 \pmod{16}$, then $\langle 1,1,2 \rangle$ primitively represents $c-14$. If we choose a vector w in the 3-section of M such that $Q(v) = c-14$, then clearly,

$$\mathbb{Z}[e_4, w - 2e_4 + 3e_5]$$

is a primitive sublattice of M isometric to $\left(\begin{smallmatrix} 2 & 1 \\ 1 & c \end{smallmatrix}\right)$. $\square$

Theorem 4.4.11. *The* $\mathbb{Z}$*-lattice* $\mathrm{H}^{\mathrm{i}}_k \cong I_2 \perp \mathbb{A} \perp \langle 2, k \rangle$ $(3 \le k \le 5)$ *is primitively 2-universal.*

Proof. Denote by M the 5-section of L and let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice which M does not primitively represent. Then we may assume that ℓ satisfies either (i) a or $c \equiv 6$ (2^4) and $d\ell \subseteq (2^5)$; or (ii) $a \equiv c \equiv 0$ (8) and $b \in (4)$, by the last lemma. If M primitively represents a $\mathbb{Z}$-lattice $\ell' \cong \left(\begin{smallmatrix} a & b \\ b & c-k \end{smallmatrix}\right)$ then evidently $L \cong M \perp \langle k \rangle$ primitively represents ℓ. Moreover, M primitively represents ℓ' if ℓ' is positive definite and neither $l_2 \cong \langle 6, 2^4\alpha \rangle$ nor $\mathfrak{n}\ell'_2 \subseteq (8)$, by the same lemma.

Let $q = 3$ or 5, then ℓ'_2 is unimodular, hence M_2 primitively represents ℓ'_2. Hence M primitively represents ℓ' if $c \ge 7$, or if $\ell \cong \left(\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right)$. Let $q = 4$. (i) Note that $a \equiv b \equiv c \equiv 0$ (2). If $a \equiv 2$ (4) then $d\ell' \equiv 8$ (16), hence M primitively represents ℓ' since $c \ge 6$. If $a \in (4)$ then $c \equiv 6$ (2^4), hence ℓ'_2 is split by $c - 4 \cong 2$ (2^4), thus M primitively represents ℓ' since $c \ge 6$. (ii) Note that $\mathfrak{s}\ell'_2 = (4)$, hence M primitively represents ℓ' since $c \ge 6$. $\square$

4.4.5 Type H^{ii}

Lemma 4.4.12. *Let* $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ *be a positive definite binary* $\mathbb{Z}$*-lattice and let* q *and* r *be positive integers. Suppose that* $\langle 2, 2, 1, 1 \rangle$ *primitively represents*

$\left(\begin{smallmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C \end{smallmatrix}\right)$, *where*

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} s_4 & s_6 \\ t_4 & t_6 \end{pmatrix} \begin{pmatrix} 2q-1 & 0 \\ 0 & 2r-1 \end{pmatrix} \begin{pmatrix} s_4 & t_4 \\ s_6 & t_6 \end{pmatrix}$$

for some integers s_4, s_6, t_4 and t_6 such that at least one of A and C is odd. Then ℓ is primitively represented by $I_2 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right) \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & r \end{smallmatrix}\right)$.

Proof. Suppose that $\langle 2,2,1,1\rangle$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_5 \\ d_1 & d_2 & d_3 & d_5 \end{bmatrix} \cong \begin{pmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C \end{pmatrix}.$$

Then

$$\left.\begin{aligned} c_3+c_5 \equiv c_3^2+c_5^2 \equiv A \equiv s_4^2+s_6^2 \equiv s_4+s_6 \\ c_3d_3+c_5d_5 \equiv B \equiv s_4t_4+s_6t_6 \\ d_3+d_5 \equiv d_3^2+d_5^2 \equiv C \equiv t_4^2+t_6^2 \equiv t_4+t_6 \end{aligned}\right\} \pmod 2.$$

One may easily observe that by hypothesis we may assume $c_3-s_4 \equiv c_5-s_6 \equiv d_3-t_4 \equiv d_5-t_6 \equiv 0 \pmod 2$, after exchanging indices 3 and 5 if necessary. Hence there exist integers a_3, a_5, b_3 and b_5 such that $c_3 = 2a_3+s_4$, $c_5 = 2a_5+s_6$, $d_3 = 2b_3+t_4$ and $d_5 = 2b_5+t_6$. Now we claim that $I_2 \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right) \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & r \end{smallmatrix}\right)$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & a_3 & s_4 & a_5 & s_6 \\ d_1 & d_2 & b_3 & t_4 & b_5 & t_6 \end{bmatrix} \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For the representation, we must verify the identities

$$\begin{pmatrix} a_3 & s_4 \\ b_3 & t_4 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 2q \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ s_4 & t_4 \end{pmatrix} = \begin{pmatrix} c_3 & s_4 \\ d_3 & t_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2q-1 \end{pmatrix} \begin{pmatrix} c_3 & d_3 \\ s_4 & t_4 \end{pmatrix}$$

and

$$\begin{pmatrix} a_5 & s_6 \\ b_5 & t_6 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 2r \end{pmatrix} \begin{pmatrix} a_5 & b_5 \\ s_6 & t_6 \end{pmatrix} = \begin{pmatrix} c_5 & s_6 \\ d_5 & t_6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2r-1 \end{pmatrix} \begin{pmatrix} c_5 & d_5 \\ s_6 & t_6 \end{pmatrix},$$

which are evident. For the primitivity, observe that

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_5 \\ d_1 & d_2 & d_3 & d_5 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & 2a_3+s_4 & 2a_5+s_6 \\ d_1 & d_2 & 2b_3+t_4 & 2b_5+t_6 \end{pmatrix}$$

is primitive, hence so is

$$\begin{pmatrix} c_1 & c_2 & a_3 & 2a_3+s_4 & a_5 & 2a_5+s_6 \\ d_1 & d_2 & b_3 & 2b_3+t_4 & b_5 & 2b_5+t_6 \end{pmatrix},$$

hence so is

$$\begin{pmatrix} c_1 & c_2 & a_3 & s_4 & a_5 & s_6 \\ d_1 & d_2 & b_3 & t_4 & b_5 & t_6 \end{pmatrix}.$$

This completes the proof. $\square$

Theorem 4.4.13. *The lattice* $\mathrm{H}^{\mathrm{ii}}_k \cong I_2 \perp \mathbb{A} \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & q \end{smallmatrix}\right)$ $(k = 2,\, 4,\, 5)$ *is primitively 2-universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice not primitively represented by the 5-section of L. Then we may assume that ℓ satisfies either (i) a or $c \equiv 6\ (2^4)$ and $d\ell \subseteq (2^5)$; or (ii) $a \equiv c \equiv 0\ (8)$ and $b \in (4)$, by Lemma 4.4.10. According to the last lemma, if $\langle 2,2,1,1\rangle$

primitively represents a $\mathbb{Z}$-lattice

$$\ell' \cong \begin{pmatrix} 2a-3 & 2b \\ 2b & 2c-(2k-1) \end{pmatrix},$$

then L primitively represents ℓ. Also, $\langle 2,2,1,1\rangle$ primitively represents ℓ' if ℓ' is positive definite and does not satisfy the conditions in Lemma 4.4.1.

Let $k = 2$ or 4. Note that $a \equiv b \equiv c \equiv 0$ (2), hence $d\ell' \equiv 1$ (4), thus $\langle 2,2,1,1\rangle_2$ primitively represents ℓ'_2. Hence $\langle 2,2,1,1\rangle$ primitively represents ℓ' if $a \geq 7$, or if $a = 6$ and $c \geq 5$, which is true. Let $k = 5$ and consider two more $\mathbb{Z}$-lattices

$$\ell'' \cong \begin{pmatrix} 2a-(2k-1) & 2b \\ 2b & 2c \end{pmatrix}, \qquad \ell''' \cong \begin{pmatrix} 2a & 2b \\ 2b & 2c-(2k-1) \end{pmatrix}.$$

According to Lemma 4.4.5, if $\langle 2,2,4,1\rangle$ primitively represents ℓ'' or ℓ''' then ℓ is primitively represented by $I_2 \perp \langle 2\rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 5 \end{smallmatrix}\right)$, and hence by L. (i) Note that $a \equiv b \equiv c \equiv 0$ (2). If $a \equiv c \equiv 2$ (4), then $d\ell' \equiv 3$ (8), and thus $\langle 2,2,1,1\rangle_2$ primitively represents ℓ'_2. Hence $\langle 2,2,1,1\rangle$ primitively represents ℓ' if $a \geq 9$, or if $a = 6$ and $c \geq 6$, which is true. If $a - 2 \equiv c \equiv 0$ (4), then $\mathfrak{s}\ell'''_2 = \mathbb{Z}_2$ and $d\ell''' \equiv 20$ (2^5), and thus $\langle 2,2,4,1\rangle_2$ primitively represents ℓ'''_2 by Lemma 4.4.1. Hence $\langle 2,2,4,1\rangle$ primitively represents ℓ''' if $c \geq 9$, which is true. Similarly if $a \equiv c-2 \equiv 0$ (4), then $a \in (8)$ and $c \equiv 6$ (2^4), and hence $\langle 2,2,4,1\rangle$ primitively represents ℓ'' if $a \geq 9$, or if $a = 8$ and $c \geq 5$, which is true. (ii) Observe that $d\ell' \equiv 3$ (8), and hence $\langle 2,2,1,1\rangle$ primitively represents ℓ' if $a \geq 9$, or if $a = 8$ and $c \geq 7$, which is true. $\square$

4.4.6 Type $\mathrm{H^{iii}}$

Lemma 4.4.14. *Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a positive definite binary $\mathbb{Z}$-lattice and let q and r be positive integers. Suppose that $\langle 2,2,4,1\rangle$ primitively represents $\left(\begin{smallmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C \end{smallmatrix}\right)$ where*

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} s_5 & s_6 \\ t_5 & t_6 \end{pmatrix} \begin{pmatrix} 2q-1 & 1 \\ 1 & 2r-1 \end{pmatrix} \begin{pmatrix} s_5 & t_5 \\ s_6 & t_6 \end{pmatrix}$$

for some integers s_5, s_6, t_5 and t_6. Then ℓ is primitively represented by $I_2 \perp \langle 2\rangle \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{smallmatrix}\right)$.

Proof. Suppose that $\langle 2,2,4,1\rangle$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \cong \begin{pmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C \end{pmatrix}.$$

Then

$$\left.\begin{aligned} c_4 \equiv c_4^2 \equiv A \equiv s_5^2 + s_6^2 \equiv s_5 + s_6 \\ d_4 \equiv d_4^2 \equiv C \equiv t_5^2 + t_6^2 \equiv t_5 + t_6 \end{aligned}\right\} \pmod 2.$$

Hence there exist integers a_4 and b_4 such that $c_4 = 2a_4 + s_5 + s_6$ and $d_4 = 2b_4 + t_5 + t_6$. Now we claim that $I_2 \perp \langle 2\rangle \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{smallmatrix}\right)$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & a_4 & s_5 & s_6 \\ d_1 & d_2 & d_3 & b_4 & t_5 & t_6 \end{bmatrix} \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For the representation, we must verify the identity

$$\begin{pmatrix} a_4 & s_5 & s_6 \\ b_4 & t_5 & t_6 \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2q & 2 \\ 2 & 2 & 2r \end{pmatrix} \begin{pmatrix} a_4 & b_4 \\ s_5 & t_5 \\ s_6 & t_6 \end{pmatrix}$$
$$= \begin{pmatrix} c_4 & s_5 & s_6 \\ d_4 & t_5 & t_6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2q-1 & 1 \\ 0 & 1 & 2r-1 \end{pmatrix} \begin{pmatrix} c_4 & d_4 \\ s_5 & t_5 \\ s_6 & t_6 \end{pmatrix},$$

which is evident. For the primitivity, observe that

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & 2a_4+s_5+s_6 \\ d_1 & d_2 & d_3 & 2b_4+t_5+t_6 \end{pmatrix}$$

is primitive, hence so is

$$\begin{pmatrix} c_1 & c_2 & c_3 & a_4 & s_5 & 2a_4+s_5+s_6 \\ d_1 & d_2 & d_3 & b_4 & t_5 & 2b_4+t_5+t_6 \end{pmatrix},$$

hence so is

$$\begin{pmatrix} c_1 & c_2 & c_3 & a_4 & s_5 & s_6 \\ d_1 & d_2 & d_3 & b_4 & t_5 & t_6 \end{pmatrix}.$$

This completes the proof. □

Theorem 4.4.15. *The lattice* $\mathrm{H}_k^{\mathrm{iii}} \cong I_2 \perp \langle 2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & k \end{smallmatrix}\right)$ $(4 \le k \le 6)$ *is primitively* 2*-universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice which the 5-section of L does not primitively represent. Then we may assume that ℓ satisfies either (i) a or $c \equiv 6$ (2^4) and $d\ell \subseteq (2^5)$; or (ii) $a \equiv c \equiv 0$ (8) and $b \in (4)$, by Lemma 4.4.10.

Let $k = 4$ or 5 and consider $\mathbb{Z}$-lattices

$$\ell' \cong \begin{pmatrix} 2a-(2k-1) & 2b \\ 2b & 2c \end{pmatrix}, \qquad \ell'' \cong \begin{pmatrix} 2a & 2b \\ 2b & 2c-(2k-1) \end{pmatrix},$$

and

$$\ell''' \cong \begin{pmatrix} 2a-3 & 2b-1 \\ 2b-1 & 2c-(2k-1) \end{pmatrix}.$$

(i) According to Lemma 4.4.5, if $\langle 2,2,4,1 \rangle$ primitively represents ℓ' or ℓ'' then ℓ is primitively represented by $I_2 \perp \langle 2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & k \end{smallmatrix}\right)$, hence by L. Note that $a \equiv b \equiv c \equiv 0$ (2). If $a \equiv 2$ (4) then $a \equiv 6$ (2^4), hence $d\ell''$ is $\equiv 12$ (2^5) if $k = 4$, and $\equiv 20$ (2^5) if $k = 5$, respectively, thus $\langle 2,2,4,1 \rangle_2$ primitively represents ℓ''_2 by Lemma 4.4.1. Hence $\langle 2,2,4,1 \rangle$ primitively represents ℓ'' if $c \ge 7$, or if $\ell \cong \left(\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right)$. If $a \in (4)$ then $a \in (8)$ and $c \equiv 6$ (2^4), hence similarly $\langle 2,2,4,1 \rangle$ primitively represents ℓ' since $a \ge 8$. (ii) According to the last lemma, if $\langle 2,2,1,1 \rangle$ primitively represents ℓ''' then then L primitively represents ℓ. Note that $d\ell'''$ is $\equiv 4$ (2^4) if $k = 4$, and $\equiv 10$ (2^4) if $k = 5$, respectively. Hence $\langle 2,2,1,1 \rangle$ primitively represents ℓ''' if $a \ge 9$, or if $a = 8$ and $c \ge 7$, which is true.

Let $k = 6$ and consider a primitive sublattice $M := \mathbb{Z}[e_1+e_2, e_3, e_4, e_5, e_6] \cong \langle 2,2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 6 \end{smallmatrix}\right)$ of L. Note that M is of class number one, $dM = 2^6$, and the only core prime of M is 2, hence M primitively represents a binary $\mathbb{Z}$-lattice $l^{(4)}$ if and only if $l^{(4)}$ is positive definite and M_2 primitively represents $l_2^{(4)}$. Also observe that $M \cong \mathbb{A} \perp \langle 2,2,48\rangle \cong \mathbb{H} \perp \langle 2,10,48\rangle$,

$$Q^*(\langle 2,2,48\rangle) = \{2,10,4,20,24,56,48,96,112,64\alpha\},$$

and

$$Q^*(\langle 2,10,48\rangle) = \{2,10,12,28,24,56,48,96,112,32\epsilon\}.$$

(i) By previous inspection, M_2 primitively represents l_2, hence M primitively represents ℓ. (ii) By previous inspection, M_2 primitively represents all binary $\mathbb{Z}_2$-lattices $l^{(4)}$ with $\mathfrak{n}\ell^{(4)} = (8)$, hence we may assume $a \equiv c \equiv 0\ (2^4)$ and $b \in (8)$. Observe that $M^{\perp} = \mathbb{Z}[e_1 - e_2] \cong \langle 2\rangle$ and $e_1 - e_2 = -(e_1+e_2)+2e_1$, hence if M primitively represents $l^{(4)} \cong \left(\begin{smallmatrix} a & b \\ b & c-2\cdot 2^2\end{smallmatrix}\right)$, then L primitively represents ℓ by Lemma 4.4.3. Actually M primitively represents $l^{(4)}$ since $c \geq 16$ and $\mathfrak{s}\ell_2^{(4)} = (8)$. $\square$

4.4.7 Type $\mathbf{H}^{\mathbf{iv}}$

Lemma 4.4.16. *Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)$ be a positive definite binary $\mathbb{Z}$-lattice and let q and r be positive integers. Suppose that $\langle 2,2,1,1\rangle$ primitively represents $\left(\begin{smallmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C\end{smallmatrix}\right)$ where*

$$\begin{pmatrix} A & B \\ B & C\end{pmatrix} = \begin{pmatrix} s_5 & s_6 \\ t_5 & t_6\end{pmatrix}\begin{pmatrix} 2q-1 & 1 \\ 1 & 2r-2\end{pmatrix}\begin{pmatrix} s_5 & t_5 \\ s_6 & t_6\end{pmatrix}$$

for some integers s_5, s_6, t_5 and t_6 such that at least one of A and C is odd. Then ℓ is primitively represented by $I_2 \perp \left(\begin{smallmatrix} 2&0&0&1\\0&2&1&1\\0&1&q&1\\1&1&1&r \end{smallmatrix}\right)$.

Proof. Suppose that $\langle 2,2,1,1\rangle$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \cong \begin{pmatrix} 2a-A & 2b-B \\ 2b-B & 2c-C \end{pmatrix}.$$

Then

$$\left.\begin{aligned} c_3+c_4 \equiv c_3^2+c_4^2 \equiv A \equiv s_5^2 \equiv s_5 \equiv (s_5+s_6)+s_6 \\ c_3d_3+c_4d_4 \equiv B \equiv s_5t_5+s_5t_6+s_6t_5 \\ \equiv (s_5+s_6)(t_5+t_6)+s_6t_6 \\ d_3+d_4 \equiv d_3^2+d_4^2 \equiv C \equiv t_5^2 \equiv t_5 \equiv (t_5+t_6)+t_6 \end{aligned}\right\} \pmod 2.$$

One may easily observe that by hypothesis we may assume $c_3 - s_6 \equiv c_4 - s_5 - s_6 \equiv d_3 - t_6 \equiv d_4 - t_5 - t_6 \equiv 0 \pmod 2$, after exchanging indices 3 and 4 if necessary. Hence there exist integers a_3, a_4, b_3 and b_4 such that $c_3 = 2a_3 + s_6$, $c_4 = 2a_4 + s_5 + s_6$, $d_3 = 2b_3 + t_6$ and $d_4 = 2b_4 + t_5 + t_6$. Now we claim that $I_2 \perp \left(\begin{smallmatrix} 2&0&0&1\\0&2&1&1\\0&1&q&1\\1&1&1&r \end{smallmatrix}\right)$ primitively represents

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & a_3 & a_4 & s_5 & s_6 \\ d_1 & d_2 & b_3 & b_4 & t_5 & t_6 \end{bmatrix} \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

For the representation, we must verify the identity

$$\begin{pmatrix} a_3 & a_4 & s_5 & s_6 \\ b_3 & b_4 & t_5 & t_6 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 4 & 2 & 2 \\ 0 & 2 & 2q & 2 \\ 2 & 2 & 2 & 2r \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \\ s_5 & t_5 \\ s_6 & t_6 \end{pmatrix}$$
$$= \begin{pmatrix} c_3 & c_4 & s_5 & s_6 \\ d_3 & d_4 & t_5 & t_6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2q-1 & 1 \\ 0 & 0 & 1 & 2r-2 \end{pmatrix} \begin{pmatrix} c_3 & d_3 \\ c_4 & d_4 \\ s_5 & t_5 \\ s_6 & t_6 \end{pmatrix},$$

which is evident. For the primitivity, observe that

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & 2a_3 + s_6 & 2a_4 + s_5 + s_6 \\ d_1 & d_2 & 2b_3 + t_6 & 2b_4 + t_5 + t_6 \end{pmatrix}$$

is primitive, hence so is

$$\begin{pmatrix} c_1 & c_2 & a_3 & a_4 & 2a_3 + s_6 & 2a_4 + s_5 + s_6 \\ d_1 & d_2 & b_3 & b_4 & 2b_3 + t_6 & 2b_4 + t_5 + t_6 \end{pmatrix},$$

hence so is

$$\begin{pmatrix} c_1 & c_2 & a_3 & a_4 & s_5 & s_6 \\ d_1 & d_2 & b_3 & b_4 & t_5 & t_6 \end{pmatrix}.$$

This completes the proof. $\square$

Theorem 4.4.17. *The lattice* $\mathrm{H}_k^{\mathrm{iv}} \cong I_2 \perp \left(\begin{smallmatrix} 2&0&0&1\\ 0&2&1&1\\ 0&1&2&1\\ 1&1&1&k \end{smallmatrix}\right)$ $(k = 4,\, 6)$ *is primitively 2-universal.*

Proof. Let $\ell \cong \left(\begin{smallmatrix} a&b\\ b&c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice which the 5-section of L does not primitively represent. Then we may assume that ℓ satisfies either (i) a or $c \equiv 6$ (2^4) and $d\ell \subseteq (2^5)$; or (ii) $a \equiv c \equiv 0$ (8) and $b \in (4)$, by Lemma 4.4.10. According to the last lemma, if $\langle 2,2,1,1\rangle$ primitively represents a $\mathbb{Z}$-lattice

$$\ell' \cong \begin{pmatrix} 2a-3 & 2b-1 \\ 2b-1 & 2c-(2k-2) \end{pmatrix},$$

then L primitively represents ℓ. Also, $\langle 2,2,1,1\rangle$ primitively represents ℓ' if ℓ' is positive definite and does not satisfy the conditions in Lemma 4.4.1. Note that $a \equiv b \equiv c \equiv 0$ (2), hence $d\ell' \equiv 1$ (4), thus $\langle 2,2,1,1\rangle_2$ primitively represents ℓ'_2. Hence $\langle 2,2,1,1\rangle$ primitively represents ℓ' if $a \ge 9$, if $a = 8$ and $c \ge 7$, or if $a = 6$ and $c \ge 6$, which is true. $\square$

4.4.8 Type I^{ii}

We reserve the lattice $\mathrm{I}_5^{\mathrm{ii}}$ to the end of this section. For lattices $\mathrm{I}_4^{\mathrm{ii}}$ and $\mathrm{I}_4^{\mathrm{iii}}$ in the next subsection, note that the quaternary orthogonal summand $\mathbb{Z}[e_3, e_4, e_5, e_6]$ of L has nonsquare discriminant. Hence, such a quaternary $\mathbb{Z}$-lattice is not primitively 2-universal over $\mathbb{Z}_p$ for infinitely many primes p.

Theorem 4.4.18. (a) *The* $\mathbb{Z}$*-lattice* $N \cong \left(\begin{smallmatrix} 2&1&1&1\\ 1&2&1&1\\ 1&1&2&0\\ 1&1&0&4 \end{smallmatrix}\right)$ *primitively represents any binary* $\mathbb{Z}$*-lattice* ℓ' *satisfying all of the following three conditions:*

(1) $\mathfrak{n}(\ell'_2) \subseteq 2\mathbb{Z}_2$ *and* ℓ'_2 *represents some element in* $\{6, -2, 4, 20\}$*;*

(2) ℓ'_3 *represents* Δ_3 *or* 3*;*

(3) ℓ'_p *represents a unit in* $\mathbb{Z}_p$ *for any odd prime* p *with* $\left(\frac{3}{p}\right) = -1$*, where* $\left(\frac{\cdot}{\cdot}\right)$ *is the Legendre symbol.*

(b) *If the quinary* $\mathbb{Z}$*-lattice* $M \cong \langle 2 \rangle \perp N$ *does not primitively represent a positive definite binary* $\mathbb{Z}$*-lattice* ℓ*, then* ℓ *satisfies* $\mathfrak{n}(\ell_2) = \mathbb{Z}_2$*,* $\ell_3 \cong \langle 3 \cdot \Delta_3, 9\alpha \rangle$ *for some* $\alpha \in \mathbb{Z}_3$*, or* $\mathfrak{s}(\ell_3) \subseteq 9\mathbb{Z}_3$*.*

(c) *The* $\mathbb{Z}$*-lattice* $\mathrm{I}_4^{\mathrm{ii}} \cong I_2 \perp N$ *is primitively* 2*-universal.*

Proof. (a) Note that N is of class number one and $dN = 12$. Hence, a binary $\mathbb{Z}$-lattice ℓ' is primitively represented by N if and only if ℓ'_p is primitively represented by N_p for any prime p. Since $N_2 \cong \mathbb{A} \perp \langle -2, -2 \rangle \cong \mathbb{H} \perp \langle 6, -2 \rangle$, ℓ'_2 is primitively represented by N_2 if $\mathfrak{n}(\ell'_2) \subseteq (2)$ and ℓ'_2 represents some element in $\{6, -2, 4, 20\} \subset \mathbb{Z}_2$. Since $N_3 \cong \mathbb{H} \perp \langle \Delta_3, 3 \rangle$, ℓ'_3 is primitively represented by N_3 if ℓ'_3 represents Δ_3 or 3. Now, suppose $p \neq 2, 3$. If $\left(\frac{3}{p}\right) = 1$ then $N_p \cong \mathbb{H} \perp \mathbb{H}$, and hence N_p is primitively 2-universal. If $\left(\frac{3}{p}\right) = -1$, that is, $dN_p = \Delta_p$, then $N_p \cong \mathbb{H} \perp \langle 1, -\Delta_p \rangle$. Hence, ℓ'_p is primitively represented by N_p if it represents a unit in $\mathbb{Z}_p$. Note that for any odd prime p, $\left(\frac{3}{p}\right) = -1$ if and only if

$$p \equiv 5,\, 7 \pmod{12}.$$

(b) Note that M is of class number one, $dM = 24$, and 3 is the only core prime of M. Hence, a binary lattice ℓ is primitively represented by M if and only if ℓ_p is primitively represented by M_p for $p = 2, 3$. Note that

$M_2 \cong \mathbb{H} \perp \mathbb{H}^2 \perp \langle 6\rangle$ primitively represents any binary lattice ℓ_2 satisfying $\mathfrak{n}(\ell_2) \subseteq 2\mathbb{Z}_2$, and $M_3 \cong \mathbb{H} \perp \langle 1,1,3\rangle$ primitively represents all binary lattices representing 1, Δ_3, or 3.

(c) Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a $\mathbb{Z}$-lattice which is primitively represented by neither the 5-section of L nor the primitive sublattice

$$M := \mathbb{Z}[e_1+e_2, e_3, e_4, e_5, e_6] \cong \langle 2\rangle \perp N$$

of L. Then by Lemma 4.3.3 and by (1) given above, we may assume that ℓ satisfies one of the following conditions:

(i) $\ell_2 \cong \langle 1, 32\alpha\rangle$ for some $\alpha \in \mathbb{Z}_2$;

(ii) $\ell_2 \cong \langle 5, 16\epsilon\rangle$ for some $\epsilon \in \mathbb{Z}_2^\times$;

(iii) $a \equiv b \equiv c \equiv 0 \pmod{12}$.

First, assume that case (iii) holds. Observe that $M^\perp = \mathbb{Z}[e_1 - e_2] \cong \langle 2\rangle$ and $e_1 - e_2 = -(e_1+e_2) + 2e_1$. Hence, if $\ell' \cong \left(\begin{smallmatrix} a & b \\ b & c-2\cdot 2^2 \end{smallmatrix}\right)$ is primitively represented by M, then ℓ is primitively represented by L by Lemma 4.4.3. In fact, ℓ' is primitively represented by M for $\mathfrak{s}(\ell'_2) \subseteq 4\mathbb{Z}_2$ and $\mathfrak{s}(\ell'_3) = \mathbb{Z}_3$.

Denote by O the 5-section of L. Then $O^\perp = \mathbb{Z}(-e_3 - e_4 + e_5 + 2e_6) \cong \langle 12\rangle$. Hence, if

$$\ell'' \cong \begin{pmatrix} a - 12\cdot 2^2 & b \\ b & c \end{pmatrix} \quad \text{or} \quad \ell''' \cong \begin{pmatrix} a & b \\ b & c - 12\cdot 2^2 \end{pmatrix}$$

is primitively represented by O, then ℓ is primitively represented by L by Lemma 4.4.3. Moreover, ℓ'' (ℓ''') is primitively represented by O if and only if

ℓ'' (ℓ''', respectively) is positive definite and ℓ''_2 (ℓ'''_2, respectively) is primitively represented by O_2.

Now, assume that case (i) holds. If $c \le 64$, then one may directly check that ℓ is primitively represented by L. Now, we assume that $c \ge 65$. First, suppose that a is odd. Since $d\ell''' \equiv 16 \pmod{32}$, ℓ'''_2 is primitively represented by O_2. Furthermore, since ℓ''' is positive definite, it is primitively represented by O. Now, suppose that a is even. Since c is odd, similarly to the above, ℓ'' is primitively represented by O if $a \ge 64$ so that ℓ'' is positive definite. Hence, we may assume that $a < 64$. Note that $c \equiv 1 \pmod 8$ and one of the following conditions holds:

(α) $a \equiv 4 \pmod{32}$ and $b \equiv 2 \pmod 4$;

(β) $a \equiv 16 \pmod{32}$ and $b \equiv 4 \pmod 8$;

(γ) $a \equiv 0 \pmod{32}$ and $b \equiv 0 \pmod 8$.

We define

$$\ell^{(4)} \cong \begin{cases} \left(\begin{smallmatrix} a & b \\ b & c-1-4 \end{smallmatrix}\right) & \text{if } a = 48, \\ \left(\begin{smallmatrix} a-4 & b \\ b & c-1 \end{smallmatrix}\right) & \text{if } a = 36, \\ \left(\begin{smallmatrix} a & b \\ b & c-9-4 \end{smallmatrix}\right) & \text{if } a = 32 \text{ and } c \equiv 1 \pmod{16}, \\ \left(\begin{smallmatrix} a & b \\ b & c-1-4 \end{smallmatrix}\right) & \text{if } a = 32 \text{ and } c \equiv 9 \pmod{16}, \\ \left(\begin{smallmatrix} a & b \\ b & c-9-4 \end{smallmatrix}\right) & \text{if } a = 16 \text{ and } c \equiv 0 \pmod 3, \\ \left(\begin{smallmatrix} a & b \\ b & c-1-4 \end{smallmatrix}\right) & \text{if } a = 16 \text{ and } c \not\equiv 0 \pmod 3, \\ \left(\begin{smallmatrix} a & b \\ b & c-1 \end{smallmatrix}\right) & \text{if } a = 4 \text{ and } c \equiv 0, 1 \pmod 3, \\ \left(\begin{smallmatrix} a & b \\ b & c-9 \end{smallmatrix}\right) & \text{if } a = 4 \text{ and } c \equiv 2 \pmod 3. \end{cases}$$

Then by (a), $\ell^{(4)}$ is primitively represented by N. Hence, ℓ is primitively represented by L in each case. The proof of case (ii) is quite similar to this. $\square$

4.4.9 Type $\mathrm{I}^{\mathrm{iii}}$

Theorem 4.4.19. (a) *If a $\mathbb{Z}$-lattice $\langle 2\rangle \perp \begin{pmatrix} 2&1&1&1\\ 1&2&1&1\\ 1&1&2&1\\ 1&1&1&2 \end{pmatrix}$ does not primitively represent at positive definite binary $\mathbb{Z}$-lattice ℓ then ℓ satisfies one of the following: $\mathfrak{n}\ell_2 = \mathbb{Z}_2$; $l_5 \cong \langle 10, 25\alpha\rangle$; or $\mathfrak{s}\ell_5 \subseteq (25)$.*

(b) *The lattice $\mathrm{I}_2^{\mathrm{iii}} \cong I_2 \perp \begin{pmatrix} 2&1&1&1\\ 1&2&1&1\\ 1&1&2&1\\ 1&1&1&2 \end{pmatrix}$ is primitively 2-universal.*

Proof. (a) Denote by M the given lattice. Note that M is of class number one, $dM = 10$ and the only core prime of M is 5. Hence by Lemma 3.2.1, M primitively represents a binary lattice ℓ if and only if ℓ is positive definite, M_2 primitively represents l_2 and M_5 primitively represents l_5. Note that $M_2 \cong \mathbb{H} \perp \mathbb{H} \perp \langle 10\rangle$ primitively represents all binary lattices l_2 satisfying $\mathfrak{n}\ell_2 \subseteq (2)$ by Lemma 3.3.1, and $M_5 \cong \mathbb{H} \perp \langle 1, 2, 5\rangle$ primitively represents all binary lattices of the form $\langle \alpha, \theta\rangle$ ($\theta = 1,\ 2,\ 5$) including $\langle 10, 10\rangle \cong \langle 5, 5\rangle$.

(b) Let $\ell \cong \begin{pmatrix} a&b\\ b&c \end{pmatrix}$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice. We may assume that ℓ is primitively represented by none of the following primitive sublattices of L: (i) the 5-section, (ii) $\mathbb{Z}[e_1, e_2, e_3 - e_4, e_5, e_6] \cong I_2 \perp \langle 2\rangle \perp \mathbb{A}$ and (iii) $M := \mathbb{Z}[e_1 + e_2, e_3, e_4, e_5, e_6] \cong \langle 2\rangle \perp \begin{pmatrix} 2&1&1&1\\ 1&2&1&1\\ 1&1&2&1\\ 1&1&1&2 \end{pmatrix}$. Then $a \equiv c \equiv 0\ (2^4 \cdot 5)$ and $b \in (8 \cdot 5)$, by Lemmata 4.3.3 and 4.4.10, and by (a). Now observe that $M^{\perp} = \mathbb{Z}[e_1 - e_2] \cong \langle 2\rangle$ and $e_1 - e_2 = -(e_1 + e_2) + 2e_1$, hence if M primitively represents $\ell' \cong \begin{pmatrix} a & b \\ b & c-2\cdot 2^2 \end{pmatrix}$, then L primitively represents ℓ by Lemma 4.4.3. Actually M primitively represents ℓ' since $c \geq 80$, $\mathfrak{s}\ell'_2 \subseteq (8)$ and $\mathfrak{s}\ell'_5 = \mathbb{Z}_5$. $\square$

Theorem 4.4.20. (a) *The $\mathbb{Z}$-lattice $N \cong \begin{pmatrix} 2&1&1&1 \\ 1&2&1&1 \\ 1&1&2&1 \\ 1&1&1&4 \end{pmatrix}$ primitively represents a positive definite binary $\mathbb{Z}$-lattice ℓ' if all of the following three conditions hold:*

(1) *$\mathfrak{n}\ell'_2 \subseteq 2\mathbb{Z}_2$ and ℓ'_2 represents twice an odd integer;*

(2) *ℓ'_{13} represents 1 or 13;*

(3) *ℓ'_p represents a unit in $\mathbb{Z}_p$ for any odd prime p with $\left(\frac{13}{p}\right) = \left(\frac{p}{13}\right) = -1$, where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol.*

(b) *The lattice $\mathrm{I}_4^{\mathrm{iii}} \cong I_2 \perp N$ is primitively 2-universal.*

Proof. (a) Note that N is of class number one. Hence N primitively represents a positive definite binary $\mathbb{Z}$-lattice ℓ' if and only if N_p primitively represents ℓ'_p for all prime p. For $p = 2$, $N_2 \cong \mathbb{H} \perp \mathbb{A}$, hence N_2 primitively represents ℓ'_2 if ℓ'_2 (primitively) represents twice a 2-adic unit, by Lemma 3.3.1. For $p = 13$, $N_{13} \cong \mathbb{H} \perp \langle 1, 13\rangle$, hence N_{13} primitively represents ℓ'_{13} if ℓ'_{13} represents 1 or 13, by Lemma 3.2.1. Now suppose $p \neq 2, 13$. If Legendre symbol $\left(\frac{13}{p}\right) = \left(\frac{p}{13}\right) = 1$, then $N_p \cong \mathbb{H} \perp \mathbb{H}$ hence N_p is primitively 2-universal. Otherwise dN_p is a non-square unit, say Δ_p, hence $N_p \cong \mathbb{H} \perp \langle 1, -\Delta_p\rangle$, then N_p primitively represents ℓ'_p if ℓ'_p represents a p-adic unit.

(b) Let $\ell \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $(0 \leq 2b \leq a \leq c)$ be a positive definite $\mathbb{Z}$-lattice which the 5-section of L does not primitively represent. Then we may assume that

ℓ satisfies one of (i) a or $c \equiv 1$ (4) and $d\ell \subseteq (2^4)$; (ii) $a \equiv b \equiv c \equiv 0$ (4), a or $c \equiv 4$ (2^4) and $d\ell \subseteq (2^6)$; or (iii) $a \equiv c \equiv 0$ (2^4) and $b \in (8)$, by Lemma 4.3.3.

For cases (i) and (ii), observe that L primitively represents $\mathbb{Z}[e_1, e_2, e_3 - e_4, e_5, e_6] \cong I_2 \perp \langle 2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$. Hence according to the Lemma 4.4.5, if $\langle 2, 2, 4, 1 \rangle$ primitively represents a $\mathbb{Z}$-lattice

$$l^{(1)} \cong \begin{pmatrix} 2a-7 & 2b \\ 2b & 2c \end{pmatrix} \quad \text{or} \quad l^{(2)} \cong \begin{pmatrix} 2a & 2b \\ 2b & 2c-7 \end{pmatrix},$$

then ℓ is primitively represented by $I_2 \perp \langle 2 \rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$, hence by L. Also, $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(1)}$ if $l^{(1)}$ is positive definite and does not satisfy the conditions in Lemma 4.4.1, and similarly for $l^{(2)}$. (i) If a is odd then $d\ell^{(2)} \equiv 2$ (4), hence $\langle 2, 2, 4, 1 \rangle_2$ primitively represents $l_2^{(2)}$. Hence $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(2)}$ if $c \geq 10$. If a is even then $a \in (4)$ and $c \equiv 1$ (4), hence similarly $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(1)}$ if $a \geq 10$ and we cannot have $a = 8$. If $a = 4$ then $b = 2$ and $d\ell^{(2)} \equiv 8$ (2^4), hence $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(2)}$ if $c \geq 10$. (ii) If $a \equiv 4$ (8) then $d\ell^{(2)} \equiv 8$ (2^4), hence $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(2)}$ if $c \geq 10$. If $a \in (8)$ then $a \in (2^4)$ and $c \equiv 4$ (2^4), hence similarly $\langle 2, 2, 4, 1 \rangle$ primitively represents $l^{(1)}$ since $a \geq 16$. Finitely many remnants can be verified directly.

Now assume case (iii). Denote by $\mathcal{P}$ the set of odd primes p such that Legendre symbol $\left(\frac{13}{p}\right) = \left(\frac{p}{13}\right) = -1$. Then a prime p is $\in \mathcal{P}$ if and only if $p \equiv 5,\ 7,\ 11,\ 15,\ 19,\ 21$ (mod 26), hence

$$\mathcal{P} = \{5, 7, 11, 19, 31, 37, 41, 47, \dots\}.$$

First assume $a \notin (13)$. Define

$$l(r) \cong \begin{pmatrix} a-w & ra+b \\ ra+b & r^2a+2rb+c \end{pmatrix} \text{ where } w = \begin{cases} 2 & \text{if } a \equiv 1,3,5,6,11,12 \ (13), \\ 18 & \text{if } a \equiv 2,4,8,9 \ (13), \\ 26 & \text{if } a \equiv 10 \ (13), \\ 34 & \text{if } a \equiv 7 \ (13). \end{cases}$$

Clearly L primitively represents ℓ if N primitively represents $l(r)$ for some integer r. Note that for all r, N_2 primitively represents $l(r)_2$ and N_{13} primitively represents $l(r)_{13}$, hence N_p primitively represents $l(r)_p$ for all $p \notin \mathcal{P}$. Also, $l(r)$ is positive definite if $a > \frac{4}{3}w(r^2 + \max\{0, r\} + 1)$. Denote by $p_1, \ldots, p_t$ the distinct prime factors of $a-w$ in $\mathcal{P}$. If $t = 0$ then N primitively represents $l(0)$ if and only if $l(0)$ is positive definite, hence if $a \geq 46$. If $t = 1$ then either $-a+b$ or b is prime to p_1, hence N primitively represents either $l(0)$ or $l(-1)$ if $a \geq 91$. If $t = 2$ then at least one of $-a+b$, b, $a+b$ is prime to p_1p_2, hence N primitively represents $l(r)$ for some $r \in \{-1, 0, 1\}$ if $a \geq 137$. Similarly if $t = 3$ then N primitively represents $l(r)$ for some $r \in \{-2, -1, 0, 1\}$ if $a \geq 227$. Now assume $t \geq 4$ then we have $a \geq 2 \cdot 5 \cdot 7 \cdot 19 \cdot 31^{t-3} > \frac{4}{3} \cdot 34\big((t+4)^2 2^{2(t-3)} + 1\big)$, hence $l(r)$ is positive definite for all $r \in \mathbb{Z} \cap [-(t+4)2^{t-3}, (t+4)2^{t-3} - 1]$. On the other hand, there exists some $s \in \mathbb{Z} \cap [-(t+4)2^{t-3}, (t+4)2^{t-3} - 1]$ such that $ra + b$ is prime to $p_1 \cdots p_t$ by [13, Lemma 3], hence N primitively represents $l(r)$ for such an r. Hence we may assume $a \leq 226$. For each $a = 16$, 32, $\ldots$, 192 and 224, we have $t = 1$, and clearly both $l(-1)$ and $l(0)$ are positive definite, hence N primitively represents either.

Finally assume $a \in (2^4 \cdot 13)$. Define

$$\ell'(r) \cong \begin{cases} \begin{pmatrix} a-26 & ra+b \\ ra+b & r^2a+2rb+c \end{pmatrix} & \text{if } b \notin (13), \\ \begin{pmatrix} a-2 & ra+b \\ ra+b & r^2a+2rb+c \end{pmatrix} & \text{if } b \in (13) \text{ and } c \notin (13), \\ \begin{pmatrix} a-2 & ra+b \\ ra+b & r^2a+2rb+c-2 \end{pmatrix} & \text{if } b \equiv c \equiv 0 \ (13). \end{cases}$$

Again L primitively represents ℓ if N primitively represents $\ell'(r)$ for some integer r, and for all r, N_2 primitively represents $\ell'(r)_2$ and N_{13} primitively represents $\ell'(r)_{13}$, hence N_p primitively represents $\ell'(r)_p$ for all $p \notin \mathcal{P}$. Note that $\ell'(r)$ is positive definite if $a > \frac{4}{3} \cdot 26(r^2 + \max\{0, r\} + 2)$. By an argument similar to the above (defining t for respectively $a-26$ or $a-2$, instead of $a-w$), we conclude that we are done if $a \geq 209$. If $a = 208$ then $208 - 26 = 182$ and $208 - 2 = 206$ has 1 and 0 prime factors in $\mathcal{P}$, respectively, hence again we are done. $\square$

4.4.10 Type J

Theorem 4.4.21. (a) *Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ be a binary $\mathbb{Z}$-lattice such that $a \equiv 1 \pmod 2$ and $c \equiv 0 \pmod{16}$. If $\left(\begin{smallmatrix} a & b \\ b & c-6 \end{smallmatrix}\right)$ is primitively represented by the $\mathbb{Z}$-lattice $\langle 1, 2\rangle \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, then ℓ is primitively represented by the $\mathbb{Z}$-lattice $\langle 1\rangle \perp \mathbb{A} \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$.*

(b) *The $\mathbb{Z}$-lattice $L \cong \mathrm{J}_3 \cong I_2 \perp \mathbb{A} \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ is primitively 2-universal.*

Proof. (a) We fix bases for $\mathbb{Z}$-lattices $\langle 1, 2\rangle \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $\langle 1\rangle \perp \mathbb{A} \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ corresponding to the given Gram matrices. With respect to such bases, if there exists a primitive sublattice $\mathbb{Z}\left[\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{smallmatrix}\right] \cong \left(\begin{smallmatrix} a & b \\ b & c-6 \end{smallmatrix}\right)$ of $\langle 1,2\rangle \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ for c_i, $d_i \in \mathbb{Z}$, then clearly, the sublattice

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & 0 & c_3 & c_4 \\ d_1 & d_2-1 & 2 & d_3 & d_4 \end{bmatrix}$$

of $\langle 1\rangle \perp \mathbb{A} \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ is isometric to ℓ. Since the greatest common divisor of the determinants of all 2×2 submatrices containing the third column is 2, it suffices to show that $\left(\begin{smallmatrix} c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{smallmatrix}\right)$ is 2-primitive. Since a is odd, exactly one or three of c_1, c_3, c_4 are odd. Furthermore, since

$$d_1^2 + 2d_2^2 + 3d_3^2 + 2d_3d_4 + 3d_4^2 \equiv 10 \pmod{16},$$

d_1 is even and d_3, d_4 are odd. Therefore, $\left(\begin{smallmatrix} c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{smallmatrix}\right)$ is 2-primitive.
(b) Denote by M the 5-section of L. Let $M' := \mathbb{Z}[e_1,e_3,e_4,e_5,e_6]$ and $N := \mathbb{Z}[e_1,e_3,e_5,e_6]$ be primitive sublattices of L. Note that

$$M' \cong \langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad N \cong \langle 1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a $\mathbb{Z}$-lattice which is primitively represented by neither M nor N. Then ℓ satisfies either

(i) a or $c \equiv 1 \pmod 8$ and $d\ell \equiv 0 \pmod{16}$, or

(ii) $a \equiv c \equiv 0 \pmod 4$ and b is even.

If $c \le 32$, then one may directly check that ℓ is primitively represented by L. Hence, we may assume that $c \ge 33$. Also, we assume that $a \ne 4$ for the moment. Consider the $\mathbb{Z}$-lattices

$$\ell' = \ell'_u(s,t;s') \cong \begin{pmatrix} a+2ub+u^2c-s^2-6s'^2 & b+uc-st \\ b+uc-st & c-t^2 \end{pmatrix},$$
$$\ell'' = \ell''_u(s,t;t') \cong \begin{pmatrix} a-s^2 & ua+b-st \\ ua+b-st & u^2a+2ub+c-t^2-6t'^2 \end{pmatrix},$$

where u, s, t, s' and t' are integers. Observe that the orthogonal complement of N in M' is $N^\perp = \mathbb{Z}[-e_3+2e_4] \cong \langle 6\rangle$. Hence, by Lemma 4.4.3, if $\ell'_u(s,t;s')$ ($\ell''_u(s,t;t')$) is primitively represented by N for some s' (t', respectively) even, then ℓ is primitively represented by L. Moreover, by (1) given above, if all of the following three conditions hold, then ℓ is primitively represented by L:

(a) $\ell'_u(s,t;1)$ ($\ell''_u(s,t;1)$) is primitively represented by N;

(b) $a+2ub+u^2c-s^2$ $(u^2a+2ub+c-t^2) \equiv 0 \pmod{16}$;

(c) $c-t^2$ ($a-s^2$, respectively) is odd.

Assume that case (i) holds. First, suppose that a is odd. We consider the $\mathbb{Z}$-lattice

$$\ell''_u(0,0;1) \cong \left(\begin{smallmatrix} a & ua+b \\ ua+b & u^2a+2ub+c-6 \end{smallmatrix}\right).$$

Since $d\ell''_u(0,0;1) \equiv 2 \pmod 4$, $\ell''_u(0,0;1)_2$ is primitively represented by $N_2 \cong \langle 1,2\rangle \perp \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$. Hence, for any integer u, $\ell''_u(0,0;1)$ is primitively represented by N. Since a is odd, there is a $u \in \{-2,-1,0,1\}$ such that $ua+b \equiv 0 \pmod 4$, which implies that $u^2a+2ub+c \equiv 0 \pmod{16}$. Hence, we are done by (1). Now, suppose that a is even. Then $a \equiv 0,\ 4 \pmod{16}$ and c is odd.

Hence, similarly to the above, if we take an integer $u \in \{-2,-1,0,1\}$ such that $u^2a+2ub+c \equiv 0 \pmod{16}$, then $\ell'_u(0,0;1)$ is primitively represented by N.

Now, assume that case (ii) holds. First, suppose that either $a \equiv 0$, $4 \pmod{16}$ or $c \equiv 0$, $4 \pmod{16}$. If we define a $\mathbb{Z}$-lattice ℓ''' by

$$\ell''' = \begin{cases} \ell''_0(1,0;1) & \text{if } a \equiv 0 \pmod{16}, \\ \ell''_0(1,2;1) & \text{if } a \equiv 4 \pmod{16}, \\ \ell'_0(0,1;1) & \text{if } c \equiv 0 \pmod{16}, \\ \ell'_0(2,1;1) & \text{if } c \equiv 4 \pmod{16}, \end{cases}$$

then $d\ell''' \equiv 2 \pmod 4$. Hence, ℓ''' is primitively represented by N. Therefore, by (1), ℓ is primitively represented by L in each case. Now, suppose that a, $c \equiv 8$ or $12 \pmod{16}$. First, assume that $b \equiv 2 \pmod 4$. One may easily show that there is an $\eta \in \{1,-1\}$ such that

$$a+2\eta b+c \equiv 0 \text{ or } 4 \pmod{16}.$$

Hence, one of $\ell''_\eta(1,0;1)$ or $\ell''_\eta(1,2;1)$ is primitively represented by N. Therefore, by (1), ℓ is primitively represented by L. Now, assume that $b \equiv 0 \pmod 4$. If we define a $\mathbb{Z}$-lattice $\ell^{(4)}$ by

$$\ell^{(4)} = \begin{cases} \ell''_0(0,1;0) & \text{if } a \equiv 8 \pmod{16}, \\ \ell'_0(1,0;0) & \text{if } c \equiv 8 \pmod{16}, \\ \ell''_0(0,4;0) & \text{if } a \equiv c \equiv 12 \pmod{16} \text{ and } b \equiv 0 \pmod 8, \\ \ell''_0(0,0;2) & \text{if } a \equiv c \equiv 12 \pmod{16} \text{ and } b \equiv 4 \pmod 8, \end{cases}$$

then one may easily show that

$$d\ell^{(4)} \equiv 8 \pmod{16}, \quad \ell_2^{(4)} \cong \langle 12, -4\rangle, \quad \text{or} \quad d\ell^{(4)} \cong 32 \pmod{64}.$$

Hence, $\ell^{(4)}$ is primitively represented by N, which implies that ℓ is primitively represented by L in each case.

Finally, we consider the remaining case when $a = 4$. Note that $b = 0$ or $b = 2$. It is well known that the 4-section $N' \cong I_2 \perp \mathbb{A}$ of L is primitively 1-universal (see [2]). If we choose a primitive vector v in N' such that $Q(v) = c$, then clearly, $\mathbb{Z}[e_5 - e_6, v]$ is a primitive sublattice of L which is isometric to $\langle 4, c\rangle$. If we choose a vector w in N' such that $Q(w) = c - 3$, then clearly,

$$\mathbb{Z}[e_5 - e_6, w + e_5]$$

is a primitive sublattice of L which is isometric to $\left(\begin{smallmatrix} 4 & 2 \\ 2 & c \end{smallmatrix}\right)$. $\square$

4.4.11 Lattices $\mathrm{D}_5^{\mathrm{iii}}$ and $\mathrm{I}_5^{\mathrm{ii}}$

Finally, we consider the case when $L \cong \mathrm{D}_5^{\mathrm{iii}}$ or $\mathrm{I}_5^{\mathrm{ii}}$. The quaternary primitive $\mathbb{Z}$-sublattice N of L given in Table 4.6 has class number two and its genus mate N' is also primitively represented by L. Hence, any binary $\mathbb{Z}$-lattice that is represented by the genus of N is primitively represented by L. Note that $dN = 16$ and the sublattice $\mathbb{Z}[e_3, e_4, e_5, e_6]$ of L is isometric to N for both cases.

Table 4.6: The $\mathbb{Z}$-lattice N and its genus mate N'

L	N	N'
$\mathrm{D}_5^{\mathrm{iii}}$	$\langle 1\rangle \perp \begin{pmatrix} 2&0&1\\0&2&1\\1&1&5 \end{pmatrix}$	$I_3 \perp \langle 16\rangle$
$\mathrm{I}_5^{\mathrm{ii}}$	$\begin{pmatrix} 2&1&1&1\\1&2&1&1\\1&1&2&0\\1&1&0&5 \end{pmatrix}$	$I_2 \perp \begin{pmatrix} 4&2\\2&5 \end{pmatrix}$

Lemma 4.4.22. *If a quaternary $\mathbb{Z}_2$-lattice $I_3 \perp \langle 16\rangle$ does not primitively represent a binary $\mathbb{Z}_2$-lattice ℓ_2, then ℓ_2 satisfies one of the following conditions: Given each quaternary $\mathbb{Z}_2$-lattice N_2 below, if N_2 does not primitively represent a binary $\mathbb{Z}_2$-lattice ℓ_2, then ℓ_2 satisfies one of the conditions given in Table 4.7.*

Proof. The necessary conditions for one N may be verified easily by a direct calculation. Then the other can be obtained by a scaling by 5. $\square$

Theorem 4.4.23. (a) *Let $\ell \cong \begin{pmatrix} a&b\\b&c \end{pmatrix}$ be a binary $\mathbb{Z}$-lattice. Suppose that, for some integers s, t, and t',*

$$\ell' \cong \begin{pmatrix} a-2s^2 & b-2st \\ b-2st & c-2t^2-8t'^2 \end{pmatrix}$$

is positive definite and ℓ'_2 is primitively represented by the quaternary $\mathbb{Z}_2$-lattice $I_3 \perp \langle 16\rangle$. Then ℓ is primitively represented by the $\mathbb{Z}$-lattice $L \cong \mathrm{D}_5^{\mathrm{iii}}$.

(b) *The $\mathbb{Z}$-lattice $L \cong \mathrm{D}_5^{\mathrm{iii}} \cong I_3 \perp \begin{pmatrix} 2&0&1\\0&2&1\\1&1&5 \end{pmatrix}$ is primitively 2-universal.*

Table 4.7: The local structures

N	Binary $\mathbb{Z}_2$-lattices that are not primitively represented by N
$I_3 \perp \langle 16\rangle$	$l_2 \cong \langle -1,\alpha\rangle$, $\langle 1,12\rangle \cong \langle 5,-4\rangle$, $\langle 1,8\rangle$, $\langle 1,40\rangle$, $\langle 5,24\rangle$, $\langle 5,-8\rangle$, $\langle 1,2^6\alpha\rangle$, $\langle 3,2^5\alpha\rangle$, $\langle 5,2^6\alpha\rangle$, $\langle 2,10\rangle$, $\langle 6,6\rangle$, $\langle 2\epsilon,12\rangle$, $\langle 2,8\rangle \cong \langle 10,40\rangle$, $\langle 6,-8\rangle \cong \langle -2,24\rangle$, $\langle 2\epsilon,2^5\delta\rangle$ with $\epsilon\delta \equiv 3\ (8)$, $\langle 2\epsilon,2^7\alpha\rangle$, $\mathfrak{n}\ell_2 \subseteq (4)$ or $\mathbb{Q}_2 l \cong \mathbb{Q}_2 H$
$5I_3 \perp \langle 80\rangle$	$l_2 \cong \langle 3,\alpha\rangle$, $\langle 1,12\rangle \cong \langle 5,-4\rangle$, $\langle 1,24\rangle$, $\langle 1,-8\rangle$, $\langle 5,8\rangle$, $\langle 5,40\rangle$, $\langle 1,2^6\alpha\rangle$, $\langle 5,2^6\alpha\rangle$, $\langle -1,2^5\alpha\rangle$, $\langle 2,10\rangle$, $\langle -2,-2\rangle$, $\langle 2\epsilon,-4\rangle$, $\langle 2,8\rangle \cong \langle 10,40\rangle$, $\langle 6,-8\rangle \cong \langle -2,24\rangle$, $\langle 2\epsilon,2^5\delta\rangle$ with $\epsilon\delta \equiv 3\ (8)$, $\langle 2\epsilon,2^7\alpha\rangle$, $\mathfrak{n}\ell_2 \subseteq (4)$ or $\mathbb{Q}_2 l \cong \mathbb{Q}_2 H$

Proof. (a) Consider two primitive sublattices of L,

$$N := \mathbb{Z}[e_3, e_4, e_5, e_6] \quad \text{and} \quad N' := \mathbb{Z}[e_1, e_2, e_3, e_4 + e_5 - 2e_6] \cong I_3 \perp \langle 16\rangle,$$

where the ordered basis $\{e_i\}_{i=1}^6$ for L corresponds to the Gram matrix given in the statement. Moreover, we fix an ordered basis for N' corresponding to the Gram matrix given in the defining equation above. Note that the class number of N is two and N' is the other class is the genus of N. Hence, if ℓ' satisfies all conditions given above, then ℓ' is primitively represented by either N or N'.

If ℓ' is primitively represented by N, then one may directly check that ℓ is primitively represented by L. Now, suppose that the primitive sublattice

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

of N' is isometric to ℓ'. Then clearly, the sublattice

$$\mathbb{Z}\begin{bmatrix} c_1 & c_2 & c_3 & c_4+s & c_4 & -2c_4 \\ d_1 & d_2 & d_3 & d_4+t & d_4+2t' & -2d_4 \end{bmatrix}$$

of L is isometric to ℓ. To see that such a sublattice is primitive, note that the greatest common divisor of all 2×2 submatrices of the above coefficient matrix divides (g_1, g_2), where g_1 and g_2 are the greatest common divisor of all 2×2 submatrices of

$$\begin{pmatrix} c_1 & c_2 & c_3 & -2c_4 \\ d_1 & d_2 & d_3 & -2d_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4+2t' \end{pmatrix},$$

respectively. Since $\left(\begin{smallmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{smallmatrix}\right)$ is primitive, g_1 divides 2 and g_2 is odd. Hence, $(g_1, g_2) = 1$.

(b) Let $\ell \cong \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ $(0 \le 2b \le a \le c)$ be a $\mathbb{Z}$-lattice which is primitively represented by neither the genus of N nor the 5-section of L. Then by Lemma 4.3.3 and by (1) given above, we may assume that ℓ satisfies one of the following conditions:

(i) $\ell_2 \cong \langle 1, -16\rangle$;

(ii) $\ell_2 \cong \langle 1, 64\alpha\rangle$ for some $\alpha \in \mathbb{Z}_2$;

(iii) $\ell_2 \cong \langle 4, 16\alpha\rangle$ or $\langle 20, 16\alpha\rangle$ for some $\alpha \in \mathbb{Z}_2$;

(iv) $\mathfrak{n}(\ell_2) \subseteq 16\mathbb{Z}_2$.

If $c \le 21$, then one may directly check that ℓ is primitively represented by L. Hence, we may assume that $c \ge 22$. If either

$$\ell'(s,t;s') \cong \begin{pmatrix} a-2s^2-8s'^2 & b-2st \\ b-2st & c-2t^2 \end{pmatrix} \quad \text{or} \quad \ell''(s,t;t') \cong \begin{pmatrix} a-2s^2 & b-2st \\ b-2st & c-2t^2-8t'^2 \end{pmatrix}$$

satisfies all conditions given in (2), then ℓ is primitively represented by L.

Assume that case (i) holds. First, suppose that a is odd. Note that $a \equiv 1 \pmod 8$. Since $d\ell''(0,2;1) \equiv 32 \pmod{64}$, $\ell''(0,2;1)_2 \cong \langle 1, 32\epsilon\rangle$ is primitively represented by N_2. Since $\ell''(0,2;1)$ is positive definite, ℓ is primitively represented by L. Now, suppose that a is even. Note that $c \equiv 1 \pmod 8$, and we have

$$a \equiv 20 \pmod{32}, \quad a \equiv -16 \pmod{64}, \quad \text{or} \quad a \equiv 0 \pmod{128}.$$

Since $\ell'(2,0;1)$ is positive definite, ℓ is primitively represented by L, by the similar reasoning.

Next, assume that case (ii) holds. Suppose that a is odd. Note that $a \equiv 1 \pmod 8$. Since $d\ell''(0,0;1) \equiv -8 \pmod{64}$, $\ell''(0,0;1)_2 \cong \langle 1, -8\rangle$ is primitively represented by N_2. Furthermore, since $\ell''(0,0;1)$ is positive definite, ℓ is primitively represented by L. Now, suppose that a is even. Note that $c \equiv 1 \pmod 8$, and we have

$$a \equiv 4 \pmod{32}, \quad a \equiv 16 \pmod{64}, \quad \text{or} \quad a \equiv 0 \pmod{64}.$$

If $a \geq 16$, then $\ell'(0,0;1)$ is positive definite. Hence, ℓ is primitively represented by L in this case. If $a = 4$, then $\ell''(0,0;1)_2 \cong \langle 1, 32\epsilon \rangle$ and $\ell''(0,0;1)$ is positive definite. Hence, ℓ is primitively represented by L.

Now, assume that case (iii) holds. First, suppose that $a \equiv c \equiv 4 \pmod{16}$. Note that $b \equiv 4 \pmod 8$. One may easily show that there is an $\eta \in \{1, -1\}$ such that $d\ell''(1,\eta;0) \equiv 32 \pmod{64}$. Then, $\ell''(1,\eta;0)_2 \cong \langle 2, 16\epsilon \rangle$ is primitively represented by N_2. Since $\ell''(1,\eta;0)$ is positive definite, ℓ is primitively represented by L. Next, suppose that $a \equiv 4 \pmod{16}$, $b \equiv 0 \pmod 8$, and $c \equiv 0 \pmod{16}$. In this case, either $\ell''(1,0;0)$ or $\ell''(1,2;0)$ is isometric to $\langle 2, 16\epsilon \rangle$ over $\mathbb{Z}_2$. Furthermore, since it is positive definite, ℓ is primitively represented by L. Similarly to the above, if $a \equiv 0 \pmod{16}$, $b \equiv 0 \pmod 8$, and $c \equiv 4 \pmod{16}$, then ℓ is primitively represented by L.

Finally, assume that case (iv) holds. If we define a $\mathbb{Z}$-lattice ℓ''' by

$$\ell''' = \begin{cases} \ell''(0,1;0) & \text{if } a \equiv 16 \pmod{32}, \\ \ell'(0,1;0) & \text{if } c \equiv 16 \pmod{32}, \\ \ell'(1,1;0) & \text{if } a \equiv c \equiv 0 \pmod{32} \text{ and } b \equiv 8 \pmod{16}, \\ \ell''(1,0;1) & \text{if } a \equiv 0 \pmod{32},\ b \equiv 0 \pmod{16}, \text{ and } c \equiv 0 \pmod{64}, \\ \ell'(0,1;1) & \text{if } a \equiv 0 \pmod{64},\ b \equiv 0 \pmod{16}, \text{ and } c \equiv 0 \pmod{32}, \\ \ell'(1,1;1) & \text{if } a \equiv c \equiv 16 \pmod{32} \text{ and } b \equiv 0 \pmod{32}, \\ \ell'(1,0;0) & \text{if } a \equiv 32 \pmod{64},\ b \equiv 16 \pmod{32},\ c \equiv -32 \pmod{128}, \\ \ell'(0,1;0) & \text{if } a \equiv -32 \pmod{128},\ b \equiv 16 \pmod{32},\ c \equiv 32 \pmod{64}, \\ \ell'(1,1;0) & \text{if } a \equiv c \equiv 32 \pmod{128} \text{ and } b \equiv -16 \pmod{64}, \\ \ell'(1,3;0) & \text{if } a \equiv c \equiv 32 \pmod{128} \text{ and } b \equiv 16 \pmod{64}, \end{cases}$$

then one may easily show that

$$d\ell''' \equiv 16 \pmod{128}, \quad d\ell''' \equiv 32 \pmod{64}, \quad \text{or} \quad d\ell''' \equiv 64 \pmod{256}.$$

Hence, ℓ is primitively represented by L in each case. $\square$

Theorem 4.4.24. *The followings hold.*

(a) *Let $\ell \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a positive definite binary $\mathbb{Z}$-lattice. Suppose that*

$$\ell' \cong \begin{pmatrix} a-2s^2 & b-2st \\ b-2st & c-2t^2-8t'^2 \end{pmatrix}$$

is positive definite and a $\mathbb{Z}_2$-lattice $5I_3 \perp \langle 80 \rangle$ primitively represents ℓ'_2. Then ℓ is primitively represented by the lattice $\mathrm{I}_5^{\mathrm{ii}}$.

(b) *The lattice $\mathrm{I}_5^{\mathrm{ii}} \cong I_2 \perp \begin{pmatrix} 2&1&1&1 \\ 1&2&1&1 \\ 1&1&2&0 \\ 1&1&0&5 \end{pmatrix}$ is primitively 2-universal.*

Proof. Consider two primitive sublattices of L:

$$N := \mathbb{Z}[e_3, e_4, e_5, e_6] \quad \text{and} \quad N' := \mathbb{Z}[e_1, e_2, e_3+e_4-e_5, e_6] \cong I_2 \perp \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}.$$

(a) Note that $\mathbb{Q}N \cong \mathbb{Q}N'$ and $N_2 \cong N'_2 \cong (5I_3 \perp \langle 80 \rangle)_2$, hence they are locally isometric to each other, thus the genus of N is identical to the genus of N'. Moreover N (hence N' also) is of class number two, hence the genus of N consists of the classes of N and N'. Therefore, if ℓ' satisfies the hypotheses, then either N or N' primitively represents ℓ'.

If N primitively represents ℓ' then the conclusion is clear. Suppose N' primitively represents ℓ', then clearly a primitive sublattice

$$M := N' \perp \mathbb{Z}e_5 \cong I_2 \perp \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \perp \langle 2 \rangle$$

of L primitively represents $\ell'' \cong \begin{pmatrix} a & b \\ b & c-8t'^2 \end{pmatrix}$. Observe that $M^\perp = \mathbb{Z}[e_3 - e_4] \cong \langle 2 \rangle$ and $e_3 - e_4 = -(e_3 + e_4 - e_5) - e_5 + 2e_3$, hence if M primitively represents ℓ'', then L primitively represents ℓ by Lemma 4.4.3.

(b) Let $\ell \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $(0 \le 2b \le a \le c)$ be a positive definite $\mathbb{Z}$-lattice which none of the 5-section of L, N or N' primitively represents. Then we may assume that ℓ satisfies one of (i) a or $c \equiv 1$ (8) and $d\ell \subseteq (2^6)$; (ii) a or $c \equiv 5$ (8) and $d\ell \equiv -2^4$ (2^7); (iii) $a \equiv b \equiv c \equiv 0$ (4), a or $c \equiv 4$ (2^5) and $d\ell \subseteq (2^6)$; (iv) $a \equiv c \equiv 0$ (2^4) and $b \in (8)$, by Lemma 4.3.3 and by the last lemma. According to (a), if either of $\mathbb{Z}$-lattices

$$\ell'(s,t;s') \cong \begin{pmatrix} 2a - 2s^2 - 8s'^2 & 2b - 2st \\ 2b - 2st & 2c - 2t^2 \end{pmatrix}$$

and

$$\ell''(s,t;t') \cong \begin{pmatrix} 2a - 2s^2 & 2b - 2st \\ 2b - 2st & 2c - 2t^2 - 8t'^2 \end{pmatrix}$$

satisfies the hypotheses of (a) then L primitively represents ℓ.

(i) If a is odd then $a \equiv 1$ (8), hence $d\ell''(0,1;1) \equiv 6$ (2^4),thus N_2 primitively represents $\ell''(0,1;1)_2$. Hence L primitively represents ℓ if $c \ge 14$. If a is even then $c \equiv 1$ (8), and a is $\equiv 4$ (2^5), $\equiv 2^4$ (2^6) or $\in (2^6)$. Hence by a similar argument using $\ell'(1,0;1)$, L primitively represents ℓ if $a \ge 14$. If $a = 4$ then $d\ell''(0,0;1) \equiv 2^5$ (2^6), hence L primitively represents ℓ if $c \ge 11$. (ii) If a is odd then $a \equiv 5$ (8), hence $d\ell''(0,2;1) \equiv 2^5$ (2^6), thus L primitively represents ℓ if $c \ge 22$. If a is even then $c \equiv 5$ (8), and a is $\equiv 4$ (2^5), $\equiv -2^4$ (2^6) or

$\in (2^7)$. Hence by a similar argument using $\ell'(2,0;1)$, L primitively represents ℓ if $a \geq 22$. If $a = 4$ then $d\ell'(0,0;1) \equiv 80\ (2^7)$, hence L primitively represents ℓ since $c \geq 29$. The proof for (iii) and (iv) is identical to the case $\mathrm{D}_5^{\mathrm{iii}}$. $\square$

Bibliography

[1] M. Bhargava, *On the Conway–Schneeberger fifteen theorem*, Quadratic Forms and Their Applications (Dublin, 1999), Contemp. Math., vol.272, pp. 27–37, Amer. Math. Soc., Providence, RI, 2000.

[2] N. V. Budarina, *On primitively universal quadratic forms*, Lith. Math. J. **50**(2010), 140–163.

[3] N. V. Budarina, *On primitively 2-universal quadratic forms* (Russian. Russian summary), Algebra i Analiz 23, 3 (2011), 31–62; translation in St. Petersburg Math. J. **23**(2012), 435–458.

[4] J. W. S. Cassels, *Rational Quadratic Forms*, Academic Press, 1978.

[5] K. Conrad, *A multivariable Hensel's lemma*, online at `https://kconrad.math.uconn.edu/blurbs/gradnumthy/multivarhensel.pdf`.

[6] L. E. Dickson, *Quaternary Quadratic Forms Representing all Integers*, Amer. J. Math. **49**(1927), 39–56.

[7] A. G. Earnest and B. L. K. Gunawardana, *Local criteria for universal and primitively universal quadratic forms*, J. Number Theory **225**(2021), 260–280.

[8] A. G. Earnest and B. L. K. Gunawardana, *On locally primitively universal quadratic forms*, Ramanujan J. **55**(2021), 1145–1163.

[9] Z. He and Y. Hu, *On k-universal quadratic lattices over unramified dyadic local fields*, Journal of Pure and Applied Algebra **227**(2023), Article 107334.

[10] Z. He, Y. Hu, and F. Xu, *On indefinite k-universal quadratic forms over number fields*, Math. Z. **304**(2023), Article 20.

[11] J. Ju, D. Kim, K. Kim, M. Kim and B.-K. Oh, *Primitively universal quaternary quadratic forms*, J. Number Theory **242**(2023), 181–207.

[12] M. Kim, *Recent developments on universal forms*, Algebraic and arithmetic theory of quadratic forms, Contemp. Math., vol. **344**, pp. 215–228, Amer. Math. Soc., Providence, RI, 2004.

[13] B. M. Kim, M.-H. Kim and B.-K. Oh, 2-*universal positive definite integral quinary quadratic forms*, Integral Quadratic Forms and Lattices (Seoul, 1998), Contemp. Math., vol. **249**, pp. 51–62, Amer. Math. Soc., Providence, RI, 1999.

[14] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge University Press, 1993.

[15] I. Lee, B.-K. Oh and H. Yu, *A finiteness theorem for positive definite almost n-regular quadratic forms*, J. Ramanujan Math. Soc. **35**(2020), 81–94.

[16] B.-K. Oh, *Universal Z-lattices of minimal rank*, Proc. Amer. Math. Soc. **128**(2000), 683–689.

[17] B.-K. Oh, *The representation of quadratic forms by almost universal forms of higher rank*, Math. Z. **244**(2003), 399–413.

[18] O. T. O'Meara, *The integral representations of quadratic forms over local fields*, Amer. J. Math. **80**(1958), 843–878.

[19] O. T. O'Meara, *Introduction to quadratic forms*, Springer, 1963.

[20] S. Ramanujan, *On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$*, Proc. Camb. Philol. Soc. **19**(1917), 11–21.

국문초록

소수 p 및 양의 정수 n에 대하여, 환 $\mathbb{Z}_p$ 위의 랭크가 n인 정수계수 이차형식을 모두 원시적으로 표현하는 $\mathbb{Z}_p$ 위의 정수계수 이차형식을 원시 n보편 형식이라 한다. Earnest와 Gunawardana는 [7]에서 어떤 $\mathbb{Z}_p$ 위의 정수계수 이차형식이 원시 1보편 형식인지 판단할 수 있는 기준을 제시하였다. 이 논문에서는, p가 홀수인 소수이거나 n이 5 이상이면, $\mathbb{Z}_p$ 위의 원시 n보편 형식의 최소 랭크가 $2n$임을 증명하였다. 나아가, $\mathbb{Z}_p$ 위의 최소 랭크의 원시 2보편 형식을 완전히 분류하였다.

양의 정수 n에 대하여, 랭크가 n인 양의 정부호 정수계수 이차형식을 모두 원시적으로 표현하는 양의 정부호 정수계수 이차형식을 원시 n보편 형식이라 한다. [11]에서는 사변수 원시 1보편 정수계수 이차형식은 등장동형인 것을 같게 볼 때 정확히 107개 있음을 증명하였다. 이 논문에서는, 원시 2보편 정수계수 이차형식의 최소 랭크가 6임을 증명하고, 또 육변수 원시 2보편 정수계수 이차형식은 등장동형인 것을 같게 볼 때 정확히 201개 있음을 증명하였다.

주요어휘: 원시n보편성

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