이학박사 학위논문

Super duality for quantum affine superalgebras of type A
(A형 양자 아핀 초대수에 대한 초 쌍대성)

2023년 8월

서울대학교 대학원
수리과학부
이 신 명

Super duality for quantum affine superalgebras of type A
(A형 양자 아핀 초대수에 대한 초 쌍대성)

지도교수 권 재 훈

이 논문을 이학박사 학위논문으로 제출함

2023년 4월

서울대학교 대학원
수리과학부
이 신 명

이 신 명의 이학박사 학위논문을 인준함

2023년 6월

위 원 장	서 의 린	(인)
부 위 원 장	권 재 훈	(인)
위 원	Masato Okado	(인)
위 원	박 의 용	(인)
위 원	유 필 상	(인)

Super duality for quantum affine superalgebras of type A

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Sin-Myung Lee

Dissertation Director : Professor Jae-Hoon Kwon

Department of Mathematical Sciences
Seoul National University

August 2023

Abstract

We propose a new approach to the study of representations of quantum affine (super)algebras, motivated from super duality. Namely, we study a category of interest by finding its bosonic or fermionic counterpart, and then construct supersymmetric analogues and functors to interpolate bosons and fermions. A key role is played by R-matrices and their spectral decompositions, which enables a uniform treatment for super and non-super cases.

In this thesis, we consider two module categories of quantum affine (super)algebras of type A. First, the category of polynomial representations is studied, where a uniform approach is possible thanks to the powerful Schur–Weyl-type duality. We construct a functor that directly relates the category for quantum affine algebras to the one for superalgebras, and lift it to an equivalence between inverse limits of categories.

Second, we introduce a category of infinite-dimensional representations called q-oscillator representations, whose irreducible objects naturally correspond to finite-dimensional irreducible representations. Since the former can be seen as a bosonic counterpart of the latter, we explain the correspondence by introducing an analogous category for quantum affine superalgebras. In the spirit of super duality, the connection provided by the super analogue is expected to give rise to an equivalence of categories.

Key words: Super duality, quantum affine algebra, general linear Lie superalgebra, R-matrix, Schur–Weyl duality, oscillator representation
Student Number: 2017-22587

Contents

Chapter 1

Introduction

Quantum groups, more specifically quantizations of universal enveloping algebras of Kac-Moody algebras, have arguably been one of the most significant and interesting objects in modern mathematics. They appear as hidden yet fundamental algebraic structures in various branches of mathematics, such as mathematical physics, combinatorics, number theory, harmonic analysis, algebraic geometry and noncommutative geometry.

Among Kac-Moody algebras, the best-understood class is formed by affine Lie algebras. Their importance stems from an interplay between two completely different descriptions: as an infinite-dimensional analogue of complex semisimple Lie algebras, and as a central extension of loop algebras. While the same applies to *quantum affine algebras*, an extra structure called a universal R-matrix arises as a result of quantization, which is a characteristic of quantum groups.

The original motivation of Drinfeld [28] and Jimbo [47] to introduce quantum groups was to find a systematic method to obtain solutions, *R-matrices*, of the celebrated Yang–Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$

in pursuit of integrability in (1+1)D quantum field theory. A specific model is realized by a tensor product of finite-dimensional representations of a quantum affine algebra, to which the universal R-matrix applies to produce an R-matrix in a uniform manner. Since the ubiquity of the Yang–Baxter equation is a source of the wide occurrence and utility of quantum groups, it is no surprise that R-matrices have played an essential role in the representation theory of quantum affine algebras.

1.1 Quantum affine superalgebras

Lie superalgebras are $\mathbb{Z}/2\mathbb{Z}$-graded generalizations of Kac-Moody algebras, introduced by Kac [48] as a uniform approach to bosons and fermions. Accordingly, *quantum affine superalgebras* arise from supersymmetric integrable systems and related R-matrices [87]. Moreover, it has recently been recognized that their variants associated with $\mathfrak{psl}(2,2)$ realize the S-matrix of string worldsheet in the context of AdS/CFT correspondence [6] or the R-matrix of deformed Hubbard model [90].

Despite rising interests, finite-dimensional representations of quantum affine superalgebras are much less understood than those in non-super cases. Since introduced, they have been studied mainly in connection with integrable models [3, 6, 87], hence limited to specific representations. It is only recently that a systematic study has begun: for type A quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}_{M|N})$, obtained are a classification of the finite-dimensional irreducibles [96]; fundamental representations and simple tensor products [98, 99]; asymptotic limit of Kirillov-Reshetikhin modules, Q-operators and generalized Baxter's TQ relations [100, 101]. This success is due to the existence of a Drinfeld realization [94] and an RTT presentation [98] for $U_q(\widehat{\mathfrak{gl}}_{M|N})$, which are not known in general.

In non-super cases, two different presentations of quantum affine algebras provide us two different perspectives on finite-dimensional representations. The Drinfeld realization [29] is a quantum version of the loop algebra realization of affine Lie algebras, and the associated highest weight theory is a suitable framework for finite-dimensional representations. Especially the corresponding character theory, called *q-characters* [34], has been studied by various methods and leads to a number of significant developments, which include the T-system [37, 68, 82] and a generalization of Bethe ansatz equations and Baxter's TQ relations [32]. We remark that the aforementioned works on quantum affine superalgebras by Zhang are also in this vein.

More familiar Drinfeld–Jimbo presentation allows one to utilize powerful tools from the representation theory of quantum groups, such as crystals and canonical bases. The pioneering work is [58], establishing fundamental results on the structure of tensor products of fundamental representations in terms of *singularities of normalized R-matrices* (see Section 2.2.2). Although it made use of heavy tools most of which are yet unavailable for superalgebras, the idea has been refined and developed to yield remarkable results, most notably Schur–Weyl-type duality functors [51] and a monoidal categorification of cluster algebras [60, 62].

Under both approaches lies the R-matrix, and the absence of it is one of critical difficulties in the representation theory for quantum affine superalgebras. Indeed, Zhang's works relied to a degree on explicit computations and specific modules, and a uniform approach is still desirable. Moreover, no direct relation to non-super theory has been known, other than that they arise as special cases $M = 0$ or $N = 0$.

1.2 Super duality

Several difficulties in representation theory of quantum affine superalgebras already appear in the classical theory. For instance, the Weyl group of a Lie superalgebra is too small to control the representation theory, and so many results for integrable representations of Lie algebras cannot be directly generalized to super cases. Already a fundamental problem of finding finite-dimensional irreducible characters took 20 years to be solved, in a rather non-elementary way through the Kazhdan–Lusztig theory involving geometry of super Grassmannians [89] or categorification of quantum group representations [10].

Super duality [20, 21] is a novel and powerful approach to the representation theory of Lie superalgebras. Roughly speaking, super duality views a representation of $\overline{\mathfrak{g}}$ and the corresponding one of $\mathfrak{g}$ as the fermionic and the bosonic aspects of the same representation of $\widetilde{\mathfrak{g}}$, respectively. This gives an explicit connection between representations of $\overline{\mathfrak{g}}$ and $\mathfrak{g}$ via the ones of $\widetilde{\mathfrak{g}}$, and hence provides a useful viewpoint to super representation theory.

Typically, a super duality is based on a triple of parabolic Bernstein-Gelfand-Gelfand categories: $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_n}$ of a Lie (super)algebra $\overline{\mathfrak{g}}_n$, $\mathcal{O}_{\mathfrak{g}_m}$ of the corresponding Lie (super)algebra $\mathfrak{g}_m$, and the one $\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{g}}_{m+n}}$ of the intermediating Lie superalgebra $\widetilde{\mathfrak{g}}_{m+n}$. They are connected by *truncation* functors T and $\overline{T}$ from $\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{g}}_{m+n}}$ to $\mathcal{O}_{\mathfrak{g}_m}$ and $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_n}$, which are given explicitly by picking out only bosons and fermions, respectively. Super duality asserts that at infinite rank limit $m, n \to \infty$, T and $\overline{T}$ become equivalences of highest weight categories:

$$\begin{array}{ccc} & \widetilde{\mathcal{O}}_{\widetilde{\mathfrak{g}}_\infty} & \\ {}^{T}\swarrow\,\simeq & & \simeq\,\searrow^{\overline{T}} \\ \mathcal{O}_{\mathfrak{g}_\infty} & & \overline{\mathcal{O}}_{\overline{\mathfrak{g}}_\infty} \\ {\scriptstyle T_m}\downarrow & & \downarrow{\scriptstyle \overline{T}_n} \\ \mathcal{O}_{\mathfrak{g}_m} & & \overline{\mathcal{O}}_{\overline{\mathfrak{g}}_n} \end{array} \tag{1.2.1}$$

where T_m, $\overline{T}_n$ are truncations to finite ranks.

To illustrate it, let us explain briefly how super duality solves the irreducible character problem for $\overline{\mathfrak{g}}_N := \mathfrak{gl}_{M|N}$, the general linear Lie superalgebra. Recall that in a parabolic BGG category $\mathcal{O}_{\mathfrak{g}_N}$ of $\mathfrak{g}_N := \mathfrak{gl}_{M+N}$, the problem is solved by Kazhdan–Lusztig theory: once we compute a transition matrix between the basis of irreducible modules and the one of parabolic Verma modules in $\mathcal{O}_{\mathfrak{g}_N}$, an irreducible character can be written as a linear combination of parabolic Verma characters, which are easy to compute.

Let $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_N}$ be a category of finite-dimensional $\mathfrak{gl}_{M|N}$-modules, and $\widetilde{\mathcal{O}}_{\widetilde{\mathfrak{g}}_{2N}}$ a certain module category over $\mathfrak{gl}_{M|2N}$ associated with a non-standard choice of a Borel subalgebra. Identifying $\mathfrak{gl}_{M+N}$ and $\mathfrak{gl}_{M|N}$ with subalgebras of $\mathfrak{gl}_{M|2N}$, the truncations T and $\overline{T}$ are defined in a natural way and induce functorial connections between the categories which become equivalences after an infinite rank limit $N \to \infty$. This completes the above diagram.

Since the transition matrix gets stabilized as $N \to \infty$ and truncations preserve simples and Vermas, the same matrix solves the problem in the limit category $\mathcal{O}_{\mathfrak{g}_\infty}$, and hence in $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_\infty}$ through $\overline{T} \circ T^{-1}$. Applying $\overline{T}_N$, we obtain a solution for $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_N}$, that is the finite-dimensional irreducible characters of $\mathfrak{gl}_{M|N}$ in terms of KL polynomials (from $\mathcal{O}_{\mathfrak{g}_N}$) and Verma characters (of $\mathfrak{gl}_{M|N}$).

The same strategy works to lift various properties of $\mathcal{O}_{\mathfrak{g}_N}$ to $\overline{\mathcal{O}}_{\overline{\mathfrak{g}}_N}$, or vice versa. Consequently, we understand that finite-dimensional representations of $\mathfrak{gl}_{M|N}$ behave as infinite-dimensional representations of $\mathfrak{gl}_{M+N}$ in a BGG category, rather than as finite-dimensional's as naively expected. This gives a conceptual explanation on difficulties in super theory, and tells us how to overcome them (see the introduction of [21]).

1.3 Main results

Throughout this thesis, q is assumed to be an indeterminate. Let $\widehat{\mathfrak{gl}}_{M|N} = \mathfrak{gl}_{M|N} \otimes \mathbb{C}[t, t^{-1}]$ be the affine Lie superalgebra (or the loop superalgebra) associated with $\mathfrak{gl}_{M|N}$. The aim of this thesis is to understand representations of the quantum affine superalgebra $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ of type A. The approach we take here, motivated from super duality, is rather new and gives a way to understand representations of quantum affine superalgebras in connection with those of quantum affine algebras.

More precisely, let $\mathcal{U}(\epsilon)$ be the *generalized quantum group* of affine type A associated with a (01)-sequence ϵ with M 0's and N 1's, which recovers the usual quantum affine algebra of type A when $M = 0$ or $N = 0$. We study representations of the generalized quantum group $\mathcal{U}(\epsilon)$ when $M \neq N$, and relate module categories of $\mathcal{U}(\epsilon)$ and $\mathcal{U}(\epsilon')$ for a subsequence ϵ' of ϵ, including non-super cases. Since there exists an algebra isomorphism between $\mathcal{U}(\epsilon)$ and $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ (up to mild extension), our study is naturally related to the one of quantum affine superalgebras.

Two module categories are to be considered. First, the category of *polynomial representations* is especially parallel to the non-super cases thanks to a Schur–Weyl-type duality, and we are able to establish a super-duality-type equivalence between the category for generalized quantum groups and the one for quantum affine algebras. The other is the category of *q-oscillator representations* of $U_q(\widehat{\mathfrak{gl}}_n)$, whose irreducible objects naturally correspond to irreducible finite-dimensional representations through super analogues. Since q-oscillators can be seen as bosonic counterparts of finite-dimensional representations, it is expected that there exists a super duality for the correspondence. Below we explain the results in more details.

1.3.1 Generalized quantum groups

Generalized quantum groups $\mathcal{U}(\epsilon)$ are Hopf algebras over $\mathbb{Q}(q)$, which are not super but the parities are implicitly encoded by ϵ. When ϵ is homogeneous, that is $\epsilon_{M|0} = (0, \ldots, 0)$ or $\epsilon_{0|N} = (1, \ldots, 1)$, $\mathcal{U}(\epsilon)$ recovers $U_q'(\widehat{\mathfrak{gl}}_M)$ or $U'_{-q^{-1}}(\widehat{\mathfrak{gl}}_N)$, respectively. In general generalized quantum groups do not quantize $U(\widehat{\mathfrak{gl}}_{M|N})$, but still arise as symmetry algebras of certain R-matrices. Those R-matrices are obtained by 2D reductions of solutions of a tetrahedron equation (a 3D analogue of the Yang–Baxter equation) [70].

Although $\mathcal{U}(\epsilon)$ is not really the same as the quantum affine superalgebra, it is closely related to. Indeed, we provide an algebra isomorphism between $\mathcal{U}(\epsilon)$ and $U_q'(\widehat{\mathfrak{gl}}_{M|N})$[1], after a mild extension, which gives rise to an equivalence between module categories. This equivalence is not a priori monoidal since the isomorphism does not respect comultiplications. Nevertheless, we expect that representation theories of $\mathcal{U}(\epsilon)$ and $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ are intimately linked. For example, weight space decomposition is preserved under the equivalence, and so is the usual character.

Using the algebra isomorphism, we prove the existence of the universal R-matrix for $\mathcal{U}(\epsilon)$ by defining a nondegenerate Hopf pairing (*cf.* [78]). This is a main advantage of $\mathcal{U}(\epsilon)$ over $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ that allows us to adopt the methods in the representation theory of quantum affine algebras to the one of $\mathcal{U}(\epsilon)$.

[1]The definition of $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ implicitly depends on ϵ, see Definition 3.1.4.

1.3.2 Super duality for polynomial representations

Let $\mathcal{C}(\epsilon)$ be the category of finite-dimensional $\mathcal{U}(\epsilon)$-modules with polynomial weights. For usual quantum affine algebras $U_q'(\widehat{\mathfrak{sl}}_n)$, every irreducible object in this category can be obtained as a quotient of a tensor product of fundamental representations $V(\varpi_i)_x$ ($i = 1, \ldots, n-1$, $x \in \mathbb{Q}(q)^\times$). Moreover, $V(\varpi_i)_x$ can be realized as an affinization of the i-th q-exterior power of $V(\varpi_1)$, the natural representation of $U_q(\mathfrak{sl}_n)$.

More precisely, the *fusion construction* [56] of an irreducible representation is implemented by taking the image of a composition of normalized R-matrices

$$R^{\mathrm{norm}}_{V(\varpi_l),V(\varpi_m)}(z_1/z_2) : V(\varpi_l)_{z_1} \otimes V(\varpi_m)_{z_2} \longrightarrow \mathbb{Q}(q)(z_1/z_2) \otimes_{\mathbb{Q}(q)[z_1^{\pm 1}, z_2^{\pm 1}]} V(\varpi_m)_{z_2} \otimes V(\varpi_l)_{z_1},$$

a $U_q'(\widehat{\mathfrak{sl}}_n)$-linear map that satisfies the Yang–Baxter equation. As explained, it is well-known that the poles of $R^{\mathrm{norm}}_{V(\varpi_l),V(\varpi_m)}(z_1/z_2)$ in z_1/z_2 contain much information on the structure of $V(\varpi_l)_{z_1} \otimes V(\varpi_m)_{z_2}$. Therefore, roughly speaking, the category $\mathcal{C}(\epsilon_{0|n})$ is generated by fundamental representations $\{V(\varpi_i)_x\}$ and its monoidal structure is determined by their normalized R-matrices.

Accordingly, our study of $\mathcal{C}(\epsilon)$ for general ϵ begins from the construction of fundamental representations and R-matrices for $\mathcal{U}(\epsilon)$. We introduce the l-th fundamental representation $\mathcal{W}_{l,\epsilon}(x)$ of $\mathcal{U}(\epsilon)$ ($l \in \mathbb{Z}_{\geq 0}$) from the l-th q-supersymmetric (encoded by ϵ) power of the natural representation $\mathcal{W}_{1,\epsilon}$.

The universal R-matrix for $\mathcal{U}(\epsilon)$ gives rise to the normalized R-matrix

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(z_1/z_2) : \mathcal{W}_{l,\epsilon}(z_1) \otimes \mathcal{W}_{m,\epsilon}(z_2) \longrightarrow \mathbb{Q}(q)(z_1/z_2) \otimes_{\mathbb{Q}(q)[z_1^{\pm 1}, z_2^{\pm 1}]} \mathcal{W}_{m,\epsilon}(z_2) \otimes \mathcal{W}_{l,\epsilon}(z_1)$$

and we compute its spectral decomposition, that is the formula

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(z) = \sum_{t \in H_\epsilon(l,m)} \prod_{i=1}^{t} \frac{1 - q^{|l-m|+2i} z}{z - q^{|l-m|+2i}} \mathcal{P}^{l,m}_t$$

where $\mathcal{P}^{l,m}_t$ is a projection to the irreducible component $V_\epsilon((l+m-t, t))$ of $\mathcal{W}_{l,\epsilon}(z_1) \otimes \mathcal{W}_{m,\epsilon}(z_2) \cong \mathcal{W}_{m,\epsilon}(z_2) \otimes \mathcal{W}_{l,\epsilon}(z_1)$ over the finite type subalgebra $\mathring{\mathcal{U}}(\epsilon)$.

The key observation is that the coefficients in the spectral decomposition of $\mathcal{R}^{\mathrm{norm}}_{l,m}$ for $\mathcal{U}(\epsilon)$ is independent of ϵ. In particular, the set of poles of $\mathcal{R}^{\mathrm{norm}}_{l,m}$ remains the same for any choice of sufficiently large ϵ (so that $H_\epsilon(l,m)$ is stabilized), including $\epsilon = \epsilon_{0|N}$. This suggests that the monoidal category $\mathcal{C}(\epsilon)$ has a similar structure for any ϵ. For example, following the argument of [60] we prove that the composition of specializations (whenever defined) of normalized R-matrices

$$R_\epsilon(\boldsymbol{l}, \boldsymbol{c}) : \mathcal{W}_{l_1,\epsilon}(c_1) \otimes \cdots \otimes \mathcal{W}_{l_t,\epsilon}(c_t) \longrightarrow \mathcal{W}_{l_t,\epsilon}(c_t) \otimes \cdots \otimes \mathcal{W}_{l_1}(c_1)$$

has simple image or zero, which is a super analogue of the fusion construction.

Indeed, such a heuristic can be made more precise, by introducing a super analogue of a generalized quantum affine Schur–Weyl duality functor [51], an exact monoidal functor

$$\mathcal{F}_\epsilon : R\text{-gmod} \longrightarrow \mathcal{C}_{\mathbb{Z}}(\epsilon) \subset \mathcal{C}(\epsilon)$$

for a quiver Hecke algebra R of type A_∞. Here $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ is a monoidal Serre subcategory of $\mathcal{C}(\epsilon)$ that is an analogue of the Hernandez–Leclerc subcategory for quantum affine algebras [41]. The functor allows us to analyze $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ uniformly in ϵ, in terms of the representation theory of quiver Hecke algebras.

Now we explain this similarity in the context of super duality. We introduce a truncation functor $\mathfrak{tr}^\epsilon_{\epsilon'} : \mathcal{C}(\epsilon) \to \mathcal{C}(\epsilon')$ for a subsequence ϵ' of ϵ. The truncation preserves all the ingredients above, namely fundamental representations, R-matrices and their spectral decompositions. In particular, $\mathfrak{tr}^\epsilon_{\epsilon'}$ is compatible with the duality functor $\mathcal{F}_\epsilon$, in the sense that $\mathfrak{tr}^\epsilon_{\epsilon'} \circ \mathcal{F}_\epsilon \cong \mathcal{F}_{\epsilon'}$ naturally.

Decomposing $\mathcal{C}_{\mathbb{Z}}(\epsilon) = \bigoplus_{\ell \geq 0} \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon)$ by degree and $\mathcal{F}_\epsilon = \bigoplus_{\ell \geq 0} \mathcal{F}^\ell_\epsilon$, we can identify $\mathcal{F}^\ell_\epsilon$ with a super analogue of the quantum affine Schur–Weyl duality functor in [17]. As in non-super cases, this functor can be shown to be an equivalence whenever $\ell < M+N$, and hence $\mathfrak{tr}^\epsilon_{\epsilon'}$ is also an equivalence on the degree ℓ components for every ϵ, ϵ' whose lengths are larger than ℓ.

It can be interpreted as a super-duality-type equivalence as follows. Let $\epsilon^\infty = (\epsilon_1, \epsilon_2, \dots)$ be an infinite (01)-sequence with infinitely many 0's and 1's. Taking an ascending chain of finite subsequences $\epsilon^{(k)}$ of ϵ^∞, we can define $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$ as the limit of an inverse system $\left(\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^{(k)}), \mathfrak{tr}^{\epsilon^{(k)}}_{\epsilon^{(k-1)}}\right)_{k \geq 1}$ and $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty) = \bigoplus \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$. Similarly we obtain $\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^\infty)$ and $\mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^\infty)$ with respect to $\overline{\epsilon}^\infty$ and $\underline{\epsilon}^\infty$, the subsequences of ϵ^∞ consisting of 1's and 0's from ϵ^∞ respectively. Then truncations induce equivalences of categories $\mathcal{S}_{0|\infty}$ and $\mathcal{S}_{\infty|0}$ that fit into the following diagram, and hence a super duality:

$$\begin{array}{ccccc} & & \mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}) & & \\ & \overset{\mathcal{S}_{\infty|0}}{\swarrow} \simeq & & \simeq \overset{\mathcal{S}_{0|\infty}}{\searrow} & \\ \mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty}) & & & & \mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty}) \\ \downarrow & & & & \downarrow \\ \mathcal{C}_{\mathbb{Z}}(\epsilon_{n|0}) & & & & \mathcal{C}_{\mathbb{Z}}(\epsilon_{0|n}). \end{array} \tag{1.3.1}$$

Since any (01)-sequence ϵ is a subsequence of ϵ^{∞}, we have a truncation $\mathfrak{tr}_{\epsilon} : \mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}) \to \mathcal{C}_{\mathbb{Z}}(\epsilon)$ as well. Therefore, we can lift known properties of the category $\mathcal{C}_{\mathbb{Z}}(\epsilon_{0|n})$ (of polynomial representations of $U_q'(\widehat{\mathfrak{gl}}_n)$) to the one $\mathcal{C}(\epsilon)$ for quantum affine superalgebras. Two examples are given: the T-system for Kirillov–Reshetikhin-type modules and a description of the Grothendieck ring of $\mathcal{C}_{\mathbb{Z}}(\epsilon)$.

1.3.3 Oscillator representations and super duality

As another manifestation of the super duality philosophy in the representation theory of quantum affine algebras, we introduce a category $\widehat{\mathcal{O}}_{\rm osc}$ of q-oscillator representations[2] of $U_q'(\widehat{\mathfrak{gl}}_n)$. They are infinite-dimensional in general and so not much of structures have been studied so far, except through a general study [36, 80] (which is at its earliest stage) on the affinization $\widehat{\mathcal{O}}$ of a BGG category, containing $\widehat{\mathcal{O}}_{\rm osc}$.

On the other hand, $\widehat{\mathcal{O}}_{\rm osc}$ is generated by fundamental q-oscillator representations $\mathcal{W}_l^{\rm osc}(x)$, which can be seen as another bosonic analogue of finite-dimensional fundamental representations $V(\varpi_i)_x$. In the virtue of super duality, this alludes that they would show similar behaviors with finite-dimensional representations, and such an analogy can be explained by considering super analogues of q-oscillator representations and truncations.

Let us first explain the classical picture. Take $n \geq 4$ and fix $2 \leq r \leq n-2$. There exists a $(\mathfrak{gl}_{r+(n-r)}, GL_\ell)$-duality on a tensor power of a bosonic Fock space $S(\mathbb{C}^{r*} \oplus \mathbb{C}^{n-r})^{\otimes \ell}$. The irreducible $\mathfrak{gl}_n$-modules occurring in this space are infinite-dimensional highest weight representations, and called oscillator representations [44]. Thanks to the duality, they form a semisimple monoidal category $O_{\rm osc}$ whose Grothendieck ring structure is determined by

[2] We remark that they are different from the q-oscillator representations [2] related to Baxter's Q-operators, which are infinite-dimensional representations of the Borel subalgebra $U_q(\mathfrak{b})$ and also referred as prefundamental representations [40]. See also Remark 5.3.3.

the branching rule of GL_ℓ.

If we consider instead a fermionic counterpart $\Lambda(\mathbb{C}^{r*}\oplus\mathbb{C}^{n-r})^{\otimes\ell}$, it is actually isomorphic to the ℓ-th power of the usual exterior power $\Lambda(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ of $\mathfrak{gl}_n$. It still carries a duality between finite-dimensional representations of $\mathfrak{gl}_n$ and GL_ℓ, and hence more familiar category $\mathcal{F}$ of finite-dimensional polynomial representations of $\mathfrak{gl}_n$ can be understood from the representations of GL_ℓ in the same way as above.

In summary, we have two $(\mathfrak{gl}_n, GL_\ell)$-dualities with the same one ends, and hence the category $O_{\rm osc}$ and $\mathcal{F}$ are simultaneously controlled by the same representation theory. Moreover, one is obtained from the other by switching bosons and fermions. This strongly indicates the existence of super duality, which is indeed true but intricate.

The second goal of this thesis is to establish a quantum affine version of such a correspondence. For this, we begin from constructing a q-analogue of oscillator representations. This is performed by investigating a decomposition of tensor powers of q-deformed bosonic Fock space $\mathcal{W}^{\rm osc}$. Indeed, they are semisimple and their irreducible components are infinite-dimensional highest weight $U_q(\mathfrak{gl}_n)$-modules, which recover the irreducible oscillator representations of $\mathfrak{gl}_n$ under the classical limit $q \to 1$.

To introduce an affine version, the idea is to replace polynomial representations of $U_q(\mathfrak{gl}_n)$ in the study of polynomial representations of $U'_q(\widehat{\mathfrak{gl}}_n)$, by q-oscillator representations of $U_q(\mathfrak{gl}_n)$. Let $\widehat{\mathcal{O}}_{\rm osc}$ be the category of $U'_q(\widehat{\mathfrak{gl}}_n)$-modules that are direct sums of irreducible q-oscillator representations over $U_q(\mathfrak{gl}_n)$, and we call the objects in $\widehat{\mathcal{O}}_{\rm osc}$ the q-oscillator representations of $U'_q(\widehat{\mathfrak{gl}}_n)$.

Then we can adapt the methods above for polynomial representations to q-oscillators. We construct fundamental q-oscillator representations of $U'_q(\widehat{\mathfrak{gl}}_n)$ and normalized R-matrices on their tensor products, and compute the spectral decomposition. Since the classical decomposition of tensor products of two fundamental q-oscillator representations are not of finite length, there is no well-defined notion of denominators. Still, it is possible to consider poles of coefficients in the spectral decomposition, and the fusion construction can be justified.

By fusion we obtain a family of irreducible objects in $\widehat{\mathcal{O}}_{\rm osc}$ that naturally corresponds to the one of finite-dimensional irreducible $U'_q(\widehat{\mathfrak{gl}}_n)$-modules. This correspondence can be made more direct by introducing a super analogue of q-oscillator representations.

Motivated by super duality, we take an alternating (01)-sequence $\boldsymbol{\epsilon} = (0101\ldots10)$, and repeat the above constructions over $\mathcal{U}(\boldsymbol{\epsilon})$ to obtain a category $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$ of q-oscillator representations of $\mathcal{U}(\boldsymbol{\epsilon})$. Then truncation functors from $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$ to $\widehat{\mathcal{O}}_{\rm osc}$, and to a category

$\widehat{\mathcal{O}}_{\mathrm{osc}.\overline{\boldsymbol{\epsilon}}}$ of finite-dimensional representations of $U_q'(\widehat{\mathfrak{gl}}_n)$ connect irreducible q-oscillators and finite-dimensional irreducible representations of $U_q'(\widehat{\mathfrak{gl}}_n)$, through interpolating irreducible q-oscillator representations of $\mathcal{U}(\boldsymbol{\epsilon})$. In conclusion, we obtain the following diagram

$$\begin{array}{ccccc} & & \widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}} & & \\ & \swarrow^{\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}} & & \searrow^{\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}} & \\ \widehat{\mathcal{O}}_{\mathrm{osc}} & & & & \widehat{\mathcal{O}}_{\mathrm{osc}.\overline{\boldsymbol{\epsilon}}} \end{array}$$

which resembles the super duality diagram (1.2.1). We strongly expect that the truncation functors become equivalences after taking a proper infinite rank limit, and we propose several evidences towards a desired *quantum affine super duality.*

Finally, we remark that oscillator representations arise more naturally in other types. For example, the spin representations of $\mathfrak{so}_{2n}$ can be constructed using a Clifford algebra. If one repeats it with a Weyl algebra, a bosonic counterpart of the Clifford algebra, one obtains the oscillator representations of $\mathfrak{sp}_{2n}$ [44]. As above, one can establish a pair of Howe dualities for this spin-oscillator correspondence, which is again nicely explained by a super duality [21]. Their quantum affine versions are studied in [74, 76].

1.4 Organization

This thesis is organized as follows.

- Chapter 2 provides a pragmatic review on Lie superalgebras, quantum affine algebras and quiver Hecke algebras, focusing on type A case.

- In Chapter 3, we introduce generalized quantum groups $\mathcal{U}(\epsilon)$ of affine type A. We give an algebra isomorphism between $\mathcal{U}(\epsilon)$ and the quantum affine superalgebra $U_q'(\widehat{\mathfrak{gl}}_{M|N})$ in Section 3.1.2. In Section 3.1.3 we use this isomorphism to prove the existence of a nondegenerate Hopf pairing on $\mathcal{U}(\epsilon)$, and hence of a universal R-matrix. We also recall in Section 3.2 basic facts on polynomial representations of a subalgebra $\mathring{\mathcal{U}}(\epsilon)$ of finite type A from [75, 77].

- In Chapter 4 we study the category $\mathcal{C}(\epsilon)$ of polynomial representations of $\mathcal{U}(\epsilon)$. In Section 4.1 we give a quick review on the super duality for polynomial representations of $\mathfrak{gl}_{M|N}$. Section 4.2 is devoted to a supersymmetric generalization of important constructions in the theory of finite-dimensional representations of quantum affine algebras $U_q'(\widehat{\mathfrak{sl}}_n)$: fundamental representations (Section 4.2.1), normalized R-matrices with the spectral decomposition (Section 4.2.2), the fusion construction of irreducible representations (Section 4.2.3) and a generalized quantum affine Schur–Weyl duality functor(Section 4.2.4). Then we introduce truncation functors to relate polynomial representations of $\mathcal{U}(\epsilon)$ and $\mathcal{U}(\epsilon')$ for a subsequence ϵ' of ϵ (Section 4.3.1). The highlight is a super-duality-type equivalence (Theorem 4.3.28), which is established from the equivalence of duality functors at high ranks (Section 4.3.2), and an interpretation of the infinite rank limit by inverse limits of categories (Section 4.3.3).

- Chapter 5 begins with a brief account on Howe dualities, to motivate oscillator representations and associated super duality. We define and study q-oscillator representations of $U_q(\mathfrak{gl}_n)$ in Section 5.2 with their super analogues, aiming to reproduce the following two results on polynomial representations: the tensor product decomposition (5.2.2) and the compatibility with truncations (Theorem 5.2.14). Then we define the category $\widehat{\mathcal{O}}_{\rm osc}$ of q-oscillator representations of $U_q'(\widehat{\mathfrak{gl}}_n)$, and its super version, connected by truncations (Section 5.3.1). Our study of $\widehat{\mathcal{O}}_{\rm osc}$ is parallel with the case of polynomial representations. We introduce fundamental q-oscillator representations (Section 5.2.1) and normalized R-matrices with a computation of their spectral decomposition (Section 5.3.2). Again we apply the fusion construction to obtain a family of irreducible objects in $\widehat{\mathcal{O}}_{\rm osc}$ (Section 5.3.3). We conclude the chapter with discussions towards a super duality explaining the correspondence.

- Chapter 6 consists of detailed proofs of several results in this thesis.

Chapter 2

Preliminaries

In this chapter, we review necessary backgrounds on Lie superalgebras, quantum affine algebras and quiver Hecke algebras. Since this thesis deals only with type A cases, we focus on type A and make the exposition as concrete as possible.

In Section 2.1, we recall basic facts on the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ and its finite-dimensional representations. Section 2.2 is devoted to (untwisted) quantum affine algebras and their finite-dimensional representations. Specifically, we concentrate on fundamental representations, normalized R-matrices and the fusion construction, which will be reproduced for quantum affine superalgebras in later chapters. The final Section 2.3 is on quiver Hecke algebras, as they will play a crucial role in the investigation of polynomial representations via Schur–Weyl-type duality in Chapter 4.

The following notations will be used throughout the thesis:

- $\mathbb{C}^{M|N}$: a vector superspace whose even part is $\mathbb{C}^M$ and odd part is $\mathbb{C}^N$.
- $|v|$: the parity of a homogeneous vector v in a superspace V.
- $\mathfrak{S}_n$: the symmetric group on n letters.
- $\mathscr{P}$: the set of partitions.
- $\ell(\lambda)$: the length of a partition λ.
- $K(\mathcal{A})$: the Grothendieck group of an abelian category $\mathcal{A}$.
- $\Bbbk = \mathbb{Q}(q)$, for indeterminate q.

- For $n \in \mathbb{Z}$ and a symbol x,

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad [n]_x! = [n]_x[n-1]_x \cdots [1]_x, \quad \begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x!}{[n]_x![m-n]_x!}.$$

  When we put $x = q$, we omit q and just write $[n] = [n]_q$.

- For a $\Bbbk$-algebra A, $x, y \in A$ and $t \in \Bbbk$, we define $[x, y]_t = xy - tyx$.

- For a statement P, $\delta(P) = 1$ if P is true and $\delta(P) = 0$ if not. As a special case, we also write $\delta_{ij} = \delta(i = j)$.

2.1 General linear Lie superalgebra $\mathfrak{gl}_{M|N}$

In this section, we recall several basic facts on Lie superalgebras, focusing on the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ in order to keep the presentation explicit. Up to a slight modification, most of statements here remain true for basic Lie superalgebras. See [24, 81] for general introductions on basic Lie superalgebras.

Definition 2.1.1. For a $\mathbb{C}$-vector superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$, the endomorphism superalgebra $\mathrm{End}(V)$ has a structure of Lie superalgebra by the supercommutator

$$[x, y] = xy - (-1)^{|x||y|} yx,$$

called the *general linear Lie superalgebra* $\mathfrak{gl}(V)$.

When $V = \mathbb{C}^{M|N}$, $\mathfrak{gl}(V)$ is also denoted by $\mathfrak{gl}_{M|N}$.

As always, it is convenient to consider a matrix representation of $\mathfrak{gl}_{M|N}$. Take a homogeneous ordered basis $\{\mathbf{e}_i\}_{i \in \mathbb{I}(M|N)}$ of $\mathbb{C}^{M|N}$, indexed by

$$\mathbb{I}(M|N) = \{1 < \cdots < M < M+1 < \cdots < M+N\}$$

with the parity

$$|\mathbf{e}_1| = \cdots = |\mathbf{e}_M| = \overline{0}, \quad |\mathbf{e}_{M+1}| = \cdots = |\mathbf{e}_{M+N}| = \overline{1}.$$

Then each element of $\mathfrak{gl}_{M|N}$ can be written with respect to this basis as an $(M + N) \times$

$(M+N)$-matrix of the block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and D are $M \times M$- and $N \times N$-matrices, respectively. The even part $(\mathfrak{gl}_{M|N})_{\bar{0}}$ consists of the matrices with $B = C = 0$, while the odd part $(\mathfrak{gl}_{M|N})_{\bar{1}}$ of those with $A = D = 0$. Note that as a Lie algebra, $(\mathfrak{gl}_{M|N})_{\bar{0}} \cong \mathfrak{gl}_M \oplus \mathfrak{gl}_N$. We also define the supertrace

$$\mathrm{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr} A - \mathrm{tr} D$$

and the *special linear Lie superalgebra* $\mathfrak{sl}_{M|N}$ is defined as the kernel of str.

For $i, j \in \mathbb{I}(M|N)$, let E_{ij} denote the element of $\mathfrak{gl}_{M|N}$ with the matrix representation $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ with respect to the above basis. Then $\{E_{ij}\}_{i,j\in\mathbb{I}(M|N)}$ is a homogeneous basis of $\mathfrak{gl}_{M|N}$.

The Lie subalgebra $\mathfrak{h}$ of diagonal matrices is a Cartan subalgebra of $\mathfrak{gl}_{M|N}$, which is by definition a Cartan subalgebra of the even subalgebra $(\mathfrak{gl}_{M|N})_{\bar{0}}$. Let us take the dual basis $\{\delta_i\}_{i\in\mathbb{I}(M|N)}$ for $\mathfrak{h}^*$ of the $\{E_{ii}\}_{i\in\mathbb{I}(M|N)}$.

As a prototype of *basic* Lie superalgebras, $\mathfrak{gl}_{M|N}$ has the following structures analogous to those of finite-dimensional semisimple Lie algebras.

- With respect to the adjoint action, $\mathfrak{gl}_{M|N}$ has the root space decomposition

  $$\mathfrak{gl}_{M|N} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

  for some $\Phi \subset \mathfrak{h}^*$, where for $\alpha \in \mathfrak{h}^*$

  $$\mathfrak{g}_\alpha = \{x \in \mathfrak{gl}_{M|N} \,|\, [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

- The set of roots $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

  $$\begin{aligned} \Phi_{\bar{0}} &= \{\delta_i - \delta_j \,|\, i \neq j;\ i, j \leq M \text{ or } i, j > M\}, \\ \Phi_{\bar{1}} &= \{\pm(\delta_i - \delta_j) \,|\, i \leq M < j\}, \end{aligned}$$

  and $\mathfrak{g}_{\delta_i-\delta_j} = \mathbb{C}E_{ij}$. Here the parity of a root α is determined by whether $\mathfrak{g}_\alpha \subset (\mathfrak{gl}_{M|N})_{\bar{0}}$ or $\mathfrak{g}_\alpha \subset (\mathfrak{gl}_{M|N})_{\bar{1}}$.

- The supertrace form

$$\begin{aligned}\mathfrak{gl}_{M|N} \times \mathfrak{gl}_{M|N} &\longrightarrow \mathbb{C}\\ (A, B) &\longmapsto \mathrm{str}(AB)\end{aligned}$$

defines a nondegenerate invariant supersymmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{gl}_{M|N}$. It induces a nondegenerate bilinear form $\mathfrak{h}^*$ as well, given by

$$(\delta_i, \delta_j) = \begin{cases} 1 & \text{if } i = j \leq M \\ -1 & \text{if } i = j > M \\ 0 & \text{if } i \neq j. \end{cases}$$

Then we also have

$$(\alpha, \alpha) = \begin{cases} \pm 2 & \text{if } \alpha \in \Phi_{\bar{0}} \\ 0 & \text{if } \alpha \in \Phi_{\bar{1}}. \end{cases}$$

In contrast, there exist several substantial differences with non-super ones. Especially, the Weyl group of $\mathfrak{gl}_{M|N}$ is defined to be the one $\mathfrak{S}_M \times \mathfrak{S}_N$ of even part, which is much smaller than $\mathfrak{S}_{M+N}$ as expected from its non-super counterpart $\mathfrak{gl}_{M+N}$. This results in critical obstructions of the study of representations of Lie superalgebras.

For example, Borel subalgebras are not necessarily conjugate under the Weyl group action. Recall that from the matrix representation of $\mathfrak{gl}_{M|N}$ above, we get the standard Borel $\mathfrak{b}_{\mathrm{std}}$ of upper-triangular matrices. However, if we take another ordered basis $\{v_i\}_{i\in\mathbb{I}}$ of $\mathbb{C}^{M|N}$ parametrized by $\mathbb{I} = \{1 < 2 < \cdots < M+N\}$ with a different $\mathbb{Z}/2\mathbb{Z}$-grading, then the corresponding matrix representation of $\mathfrak{gl}_{M|N}$ yields another Borel subalgebra (of upper-triangular matrices) that is not conjugate to $\mathfrak{b}_{\mathrm{std}}$.

Indeed, if we have two $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{I}$, then the associated Borel subalgebras are conjugate to each other if and only if two gradings are the same. Thus, the conjugacy classes of Borel subalgebras are classified by (01)-*sequences*: a sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_{M+N})$ of M 0's and N 1's, which assigns $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{I}$ by $|i| = \epsilon_i$. In particular, the standard Borel $\mathfrak{b}_{\mathrm{std}}$ corresponds to $\epsilon_{M|N} \coloneqq (0^M, 1^N)$. We will use the notation $\mathfrak{gl}_\epsilon$ when we want to stress the choice of a Borel subalgebra of $\mathfrak{gl}_{M|N}$ associated with ϵ.

Now let us consider finite-dimensional representations of $\mathfrak{gl}_{M|N}$ and see what is different from the ones of $\mathfrak{gl}_{M+N}$. Upon a choice of a Borel subalgebra $\mathfrak{b}$, one can do the highest weight theory for $\mathfrak{gl}_{M|N}$ with respect to $\mathfrak{b}$: for each $\lambda \in \mathfrak{h}^*$, we construct a Verma module of highest weight λ, which has a unique irreducible quotient $L(\mathfrak{b}, \lambda)$. It is not a priori clear how much a choice of Borel affects the theory, but the following facts are known (see [24, Section 1.5, 2.1]):

- For any Borel $\mathfrak{b}$, every finite-dimensional irreducible $\mathfrak{gl}_{M|N}$-module is of highest weight with respect to $\mathfrak{b}$.
- For another Borel $\mathfrak{b}'$, $L(\mathfrak{b}, \lambda)$ is also of highest weight with respect to $\mathfrak{b}'$, and the $\mathfrak{b}'$-highest weight vector (and so its highest weight) can be found by means of *odd reflections.*
- $L(\mathfrak{b}_{\text{std}}, \lambda)$ is finite-dimensional if and only if $\lambda \in \mathfrak{h}^*$ is dominant with respect to the even subalgebra $\mathfrak{gl}_M \oplus \mathfrak{gl}_N$, that is,

  $$\lambda_i - \lambda_{i+1}, \lambda_j - \lambda_{j+1} \in \mathbb{Z}_{\geq 0} \text{ for any } i = 1, \ldots, M,\ j = M+1, \ldots, M+N,$$

  where $\lambda = \sum_{i \in \mathbb{I}(M|N)} \lambda_i \delta_i$.

On the other hand, the following difficulties are much harder to overcome:

- Linkage is not entirely controlled by its Weyl group $\mathfrak{S}_M \times \mathfrak{S}_N$.
- Finite-dimensional $\mathfrak{gl}_{M|N}$-modules are not semisimple in general.
- There is no uniform formula (such as Weyl character formula) for finite-dimensional irreducible characters.

It is now understood that these originate from a BGG category of $\mathfrak{gl}_\infty$, which is equivalent to the (limit of) category of finite-dimensional representations of $\mathfrak{gl}_{M|N}$ by super duality. We do not give here a general account on super duality, and content ourselves with the one in the introduction. Rather, specific examples that are related to the main results of this thesis will be given at the beginning of Chapter 4 and 5. We refer interested readers to [24], and also to [19] for generalizations beyond classical Lie superalgebras.

Remark 2.1.2. There are two classes of finite-dimensional $\mathfrak{gl}_{M|N}$-modules that exhibit as nice behaviors as in non-super case. The first one is formed by typical representations (see [24, Section 2.2]): their highest weights are so special that this small Weyl group

$\mathfrak{S}_M \times \mathfrak{S}_N$ fully controls the linkages between them, and hence semisimplicity and Weyl-type character formula can be obtained as in non-super cases.

The second class consists of polynomial representations. They are in resonance with the representations of symmetric groups in virtue of the Schur–Weyl duality (see [24, Section 3.2]), and so can be treated uniformly with the ones of $\mathfrak{gl}_{M+N}$. When we study quantum affine analogues of polynomial representations in Chapter 4, the corresponding Schur–Weyl-type duality will be again used in a crucial way.

2.2 Quantum affine algebra

In this section, we first recall the definition of (untwisted) affine Lie algebras and quantum affine algebras. Then we explain several basic constructions [1, 58] which will be reproduced for super cases in later chapters. For a detailed account on quantum affine algebras, we refer the reader to recent surveys [14, 39].

2.2.1 Affine Lie algebras and quantum affine algebras

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $n-1$. The corresponding Cartan matrix $(a_{ij})_{i,j=1,\dots,n-1}$ is determined by

$$a_{ij} = \langle h_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

where α_i is the simple root, h_i the simple coroot and $(\,\cdot\,,\,\cdot\,)$ the (normalized) Killing form on $\mathfrak{g}$.

We define the *affine Lie algebra* $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$ by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}C \oplus \mathbb{C}d,$$

whose Lie bracket is given by

$$[x \otimes t^n + ad, y \otimes t^m + bd] = [x, y] \otimes t^{m+n} + n\delta_{n,-m}(x, y)C + amy \otimes t^m - bnx \otimes t^n$$

for $x, y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$ and $a, b \in \mathbb{C}$, and C is central. Then $\widehat{\mathfrak{g}}$ is an example of the Kac-Moody algebra of untwisted affine type [49]. Namely, $\widehat{\mathfrak{g}}$ is the Kac-Moody algebra associated with the generalized Cartan matrix $A = (a_{ij})_{i,j=0,\dots,n-1}$ obtained from the

Cartan matrix for $\mathfrak{g}$ by adjoining the 0-th row and column:

$$a_{0j} = \langle -\theta^\vee, \alpha_j \rangle, \quad a_{j0} = \langle h_j, -\theta \rangle \ (j = 1, \dots, n-1), \quad a_{00} = \langle \theta^\vee, \theta \rangle$$

where θ (resp. $\theta^\vee$) is the maximal root (resp. coroot) of $\mathfrak{g}$. It is known that A is symmetrizable, that is, there exist positive integers[1] d_i ($i = 0, 1, \dots, n-1$) such that $d_i a_{ij} = d_j a_{ji}$ for all i, j.

As a Kac-Moody algebra, $\widehat{\mathfrak{g}}$ has the weight lattice

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_{n-1} \oplus \mathbb{Z}\delta$$

and the dual weight lattice

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d$$

paired with P by

$$\langle h_i, \Lambda_j \rangle = \delta_{ij}, \quad \langle h_i, \delta \rangle = 0 = \langle d, \Lambda_j \rangle, \quad \langle d, \delta \rangle = 1.$$

Take simple roots

$$\alpha_i = \sum_j a_{ji}\Lambda_j + \delta_{i,0}\delta \in P,$$

and let $\Pi^\vee = \{h_0, h_1, \dots, h_{n-1}\}$ and $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be the set of simple coroots and simple roots, respectively. Note that we have $\delta = \theta + \alpha_0$.

The tuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ is called a Cartan datum associated with A.

Definition 2.2.1. The quantum group $U_q(\widehat{\mathfrak{g}})$ associated with the above Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the $\mathbb{Q}(q)$-algebra generated by e_i,f_i ($i = 0, 1, \dots, n-1$) and q^h ($h \in P^\vee$) subject to the following defining relations:

$$\begin{aligned}
&q^0 = 1, \quad q^{h+h'} = q^h q^{h'}, \\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \\
&e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},
\end{aligned}$$

[1] we always choose d_i so that the greatest common divisor of $d_1, \dots, d_{n-1}$ is 1.

$$\begin{aligned}
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0
\end{aligned} \qquad \text{for } i \neq j,$$

where $q_i = q^{d_i}$ and $k_i = q_i^{h_i}$.

The *quantum affine algebra* $U'_q(\widehat{\mathfrak{g}})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by e_i, f_i and $k_i^{\pm 1}$ $(i = 0, 1, \ldots, n-1)$.

The quantum group $U_q(\widehat{\mathfrak{g}})$ and the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$ have a Hopf algebra structure given by

$$\begin{aligned}
&\Delta : q^h \mapsto q^h \otimes q^h, \quad e_i \mapsto e_i \otimes k_i^{-1} + 1 \otimes e_i, \quad f_i \mapsto f_i \otimes 1 + k_i \otimes f_i, \\
&S : q^h \mapsto q^{-h}, \quad e_i \mapsto -e_i k_i, \quad f_i \mapsto -k_i^{-1} f_i.
\end{aligned}$$

Therefore, the category of finite-dimensional modules has the structure of a rigid monoidal category (see [15, Section 5.1] for definition).

Remark 2.2.2. Let $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$ be the *classical weight lattice*, whose dual is given by

$$P_{\mathrm{cl}}^\vee = \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{cl}}, \mathbb{Z}) = \{h \in P^\vee \mid \langle h, \delta \rangle = 0\} = \bigoplus_{i=0}^{n-1} \mathbb{Z} h_i.$$

Then the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$ can be seen as the quantum group associated with a Cartan datum $(A, P_{\mathrm{cl}}, \{\alpha_i\}, P_{\mathrm{cl}}^\vee, \{h_i\})$, where we are abusing notations $\alpha_i \in P_{\mathrm{cl}}$. Note that $\{\alpha_i\}$ is not linearly independent in P_{cl}, as $\alpha_0 + \theta = 0$.

In the study of finite-dimensional representations, it is more natural to use quantum affine algebras $U'_q(\widehat{\mathfrak{g}})$, rather than $U_q(\widehat{\mathfrak{g}})$. Indeed, every nontrivial integrable representation of $U_q(\widehat{\mathfrak{g}})$ is infinite-dimensional [49, Chapter 12]. Responsible is the imaginary root δ, and so we have to reduce the weight lattice P to P_{cl} to consider finite-dimensional representations.

There is another presentation of $U'_q(\widehat{\mathfrak{g}})$ called the Drinfeld realization [4, 29], which is a quantum analogue of the realization of $\widehat{\mathfrak{g}}$ as a (central extension of) loop algebra $\mathfrak{g}[t, t^{-1}]$. This presentation also possesses a triangular decomposition, and the corresponding highest weight theory is suitable to study finite-dimensional representations of $U'_q(\widehat{\mathfrak{g}})$.

The weights with respect to the diagonal subalgebra in the Drinfeld realization are called ℓ-weights. Then the finite-dimensional irreducible representations are ℓ-highest weight modules and classified by their ℓ-highest weights. Furthermore, the character theory in ℓ-weights, called the q-character [34], plays a fundamental role in the development of the theory of finite-dimensional representations for quantum affine algebras. Since we will not pursue this direction in the sequel, we refer the reader to [14] for further explanation.

2.2.2 Finite-dimensional representations of quantum affine algebras

In this section, we take the algebraic closure of $\mathbb{Q}(q)$ in $\bigcup_{m>0}\mathbb{C}((q^{1/m}))$ as a base field $\mathbf{k}$, and define $U_q'(\widehat{\mathfrak{g}})$ over $\mathbf{k}$ with the same presentation as above.

A $U_q'(\widehat{\mathfrak{g}})$-module V is said to be *integrable* if V has the weight space decomposition

$$V = \bigoplus_{\lambda\in P_{\mathrm{cl}}} V_\lambda, \quad V_\lambda = \left\{ v \in V \mid k_i v = q_i^{\langle h_i, \lambda\rangle} u \text{ for all } i \right\}$$

and the actions of e_i, f_i $(i = 0, 1, \ldots, n-1)$ on V are locally nilpotent. Note that the second condition follows automatically when we consider finite-dimensional representations with the weight space decomposition. Every module in this thesis is assumed to be integrable.

Recall that for $\lambda \in P$, the integer $\langle C, \lambda\rangle$ is called the level of λ. Then the image P^0_{cl} of the set P^0 of level 0 weights under the projection $\mathrm{cl} : P \to P_{\mathrm{cl}}$ can be identified with the weight lattice of $\mathfrak{g}$ by

$$\varpi_i := \mathrm{cl}(\Lambda_i - a_i^\vee \Lambda_0),$$

where $a_i^\vee$ is the coefficient of h_i in $\theta^\vee$. We call $\varpi_i \in P^0_{\mathrm{cl}}$ the i-th level 0 fundamental weight. It is known that any finite-dimensional integrable $U_q'(\widehat{\mathfrak{g}})$-module has weights in P^0_{cl}.

Let $V(\varpi_i)_x$ denote the *i-th fundamental representation* of *spectral parameter* x, for $i = 1, 2, \ldots, n-1$ and $x \in \mathbf{k}^\times$ [58], which plays the role of a fundamental representation in the theory of integrable representations of Kac-Moody algebras. For example, every finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module can be obtained as a quotient of a submodule of a tensor product of fundamental representations, where the submodule is generated by the tensor product of dominant *extremal* weight vectors (or ℓ-highest weight vectors [16]).

Fundamental representations have various nice properties, such as the existence of canonical bases with simple crystals. We do not give here a general construction by [58]. Instead, we focus on the type A case [1, Appendix B], where they can be explicitly realized on the q-exterior powers of the natural representation $V(\varpi_1)$.

Example 2.2.3. Let us consider the fundamental representations of type A quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_n)$.

For $1 \le i \le n-1$ and $x \in \mathbf{k}^\times$, $V(\varpi_i)_x$ has a basis $\{b_J\}$ labeled by subsets J of $\mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$ of i elements. The action of the generators with respect to this basis is given by

$$e_i b_J = \begin{cases} x^{\delta_{i0}} b_{(J \setminus \{i+1\}) \cup \{i\}} & \text{if } i+1 \in J,\, i \notin J \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i b_J = \begin{cases} x^{-\delta_{i0}} b_{(J \setminus \{i\}) \cup \{i+1\}} & \text{if } i \in J,\, i+1 \notin J \\ 0 & \text{otherwise,} \end{cases}$$

$$k_i b_J = \begin{cases} q b_J & \text{if } i \in J,\, i+1 \notin J \\ q^{-1} b_J & \text{if } i \notin J,\, i+1 \in J \\ b_J & \text{otherwise.} \end{cases}$$

If we set $\delta_1 = \varpi_1$ and $\delta_k = \varpi_k - \varpi_{k-1}$ for $2 \le k \le n-1$, then $\mathrm{wt}(b_J) = \sum_{j \in J} \delta_j$. In particular, a weight space $V(\varpi_i)_{\varpi_i}$ is spanned by a single vector $b_{\{1,2,\ldots,i\}}$ that is an ℓ-highest weight vector. It is also called a *dominant extremal weight vector* of $V(\varpi_i)$, an analogue of a highest weight vector.

Note that the above description is valid even for $i = 0, n$, resulting in the trivial representation. Hence, by convention we also set $V(\varpi_0)_x = V(\varpi_n)_x = \mathbf{k}$.

Next, we explain the *fusion construction*, which produces the finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-modules as quotients of tensor products of fundamental representations in a proper order. The key ingredient is the normalized R-matrix, whose construction is to be recalled now.

For an integrable $U_q'(\widehat{\mathfrak{g}})$-module V, define the affinization V_{aff} to be a P-graded $U_q'(\widehat{\mathfrak{g}})$-module

$$V_{\mathrm{aff}} = \mathbf{k}[z^{\pm 1}] \otimes V, \quad (V_{\mathrm{aff}})_\lambda = z^k \otimes V_{\mathrm{cl}(\lambda)} \text{ for } \lambda = \iota \circ \mathrm{cl}(\lambda) + k\delta \in P$$

where $\iota : P_{\mathrm{cl}} \longrightarrow P$ is defined by $\iota(\mathrm{cl}(\Lambda_i)) = \Lambda_i$. The $U_q'(\widehat{\mathfrak{g}})$-action is given by

$$e_i = z^{\delta_{i0}} \otimes e_i, \quad f_i = z^{-\delta_{i0}} \otimes f_i, \quad k_i = 1 \otimes k_i.$$

For any $x \in \mathbf{k}^\times$, we set $V_x = V_{\mathrm{aff}}/(z-x)V_{\mathrm{aff}}$. In particular, $V_1 \cong V$ and $(V_x)_y \cong V_{xy}$. For example, we have $V(\varpi_i)_x \cong (V(\varpi_i)_1)_x$ and hence compatible with the notation. We also write $V_z \coloneqq V_{\mathrm{aff}}$ for indeterminate z.

There exists a $\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U_q'(\widehat{\mathfrak{g}})$-linear map

$$\mathcal{R}^{\mathrm{univ}}_{V,W}(z_1, z_2) : V_{z_1} \otimes W_{z_2} \longrightarrow \mathbf{k}(z_1, z_2) \otimes_{\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}]} (W_{z_2} \otimes V_{z_1})$$

called a *universal R-matrix* (see section 4.2.2 for construction). When $V = V(\varpi_i)$ and $W = V(\varpi_j)$, we have distinguished vectors $u_{\varpi_i} \in V$ and $u_{\varpi_j} \in W$, the dominant extremal weight vectors. We normalize $\mathcal{R}^{\mathrm{univ}}_{V(\varpi_i),V(\varpi_j)}$ to

$$\begin{aligned} \mathcal{R}^{\mathrm{norm}}_{i,j}(z_1, z_2) : V(\varpi_i)_{z_1} \otimes V(\varpi_j)_{z_2} &\longrightarrow \mathbf{k}(z_1, z_2) \otimes_{\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}]} (V(\varpi_j)_{z_2} \otimes V(\varpi_i)_{z_1}) \\ u_{\varpi_i} \otimes u_{\varpi_j} &\longmapsto u_{\varpi_j} \otimes u_{\varpi_i}, \end{aligned}$$

and call it a *normalized R-matrix.*

It is known that $\mathcal{R}^{\mathrm{norm}}_{i,j}(z_1, z_2)$ only depends on z_1/z_2, and its image is contained in $\mathbf{k}(z_1/z_2) \otimes_{\mathbf{k}[(z_1/z_2)^{\pm 1}]} (V(\varpi_j)_{z_2} \otimes V(\varpi_i)_{z_1})$. Let $d_{i,j}(z) \in \mathbf{k}[z]$ be the *denominator* of $\mathcal{R}^{\mathrm{norm}}_{i,j}$, namely the monic polynomial of minimal degree such that the image of $d_{ij}(z_1/z_2) \cdot \mathcal{R}^{\mathrm{norm}}_{i,j}$ is in $V(\varpi_j)_{z_2} \otimes V(\varpi_i)_{z_1}$. If $x, y \in \mathbf{k}^\times$ are such that $d_{i,j}(x/y) \neq 0$, then we can specialize $\mathcal{R}^{\mathrm{norm}}_{i,j}$ to obtain a $U_q'(\widehat{\mathfrak{g}})$-module homomorphism

$$R_{i,j}(x/y) : V(\varpi_i)_x \otimes V(\varpi_j)_y \longrightarrow V(\varpi_j)_y \otimes V(\varpi_i)_x.$$

We refer to [85, Section 4] for denominator formulas for various quantum affine algebras.

Above all, normalized R-matrices solve the Yang-Baxter equation:

$$(\mathcal{R}^{\mathrm{norm}}_{j,k} \otimes 1) \circ (1 \otimes \mathcal{R}^{\mathrm{norm}}_{i,k}) \circ (\mathcal{R}^{\mathrm{norm}}_{i,j} \otimes 1) = (1 \otimes \mathcal{R}^{\mathrm{norm}}_{i,j}) \circ (\mathcal{R}^{\mathrm{norm}}_{i,k} \otimes 1) \circ (1 \otimes \mathcal{R}^{\mathrm{norm}}_{j,k})$$

holds as a map from $V(\varpi_i)_{z_1} \otimes V(\varpi_j)_{z_2} \otimes V(\varpi_k)_{z_3}$. Therefore, for any $w \in \mathfrak{S}_t$, we can define without ambiguity the composition

$$\mathcal{R}^{w}_{i_1,\dots,i_t} : V(\varpi_{i_1})_{z_1}\otimes\cdots\otimes V(\varpi_{i_t})_{z_t} \longrightarrow \mathbf{k}(z_1,\dots,z_t)\otimes\Big(V(\varpi_{i_{w(1)}})_{z_{w(1)}}\otimes\cdots\otimes V(\varpi_{i_{w(t)}})_{z_{w(t)}}\Big)$$

with respect to any reduced expression of w. If $c_1,\dots,c_t \in \mathbf{k}^\times$ are given such that $d_{i_j,i_k}(c_j/c_k) \neq 0$ for all $j<k$ satisfying $w(j)>w(k)$, then we also obtain its specialization at $z_i = c_i$,

$$R^{w}_{i_1,\dots,i_t}(c_1,\dots,c_t) : V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t} \longrightarrow V(\varpi_{i_{w(1)}})_{c_{w(1)}}\otimes\cdots\otimes V(\varpi_{i_{w(t)}})_{c_{w(t)}}.$$

Theorem 2.2.4 ([58]). *Suppose that $i_1,\dots,i_t \in \{1,\dots,n-1\}$ and $c_1,\dots,c_t \in \mathbf{k}^\times$ are such that $d_{i_j,i_k}(c_j/c_k)\neq 0$ for all $j<k$. Then the following statements hold.*

(1) *$V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t}$ is generated by $u_{\varpi_{i_1}}\otimes\cdots\otimes u_{\varpi_{i_t}}$, and $V(\varpi_{i_t})_{c_t}\otimes\cdots\otimes V(\varpi_{i_1})_{c_1}$ is cogenerated by $u_{\varpi_{i_t}}\otimes\cdots\otimes u_{\varpi_{i_1}}$.*

(2) *The head of $V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t}$ and the socle of $V(\varpi_{i_t})_{c_t}\otimes\cdots\otimes V(\varpi_{i_1})_{c_1}$ are simple.*

(3) *For the longest element w_0 of $\mathfrak{S}_t$, the image of*

$$R^{w_0}_{i_1,\dots,i_t}(c_1,\dots,c_t) : V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t} \longrightarrow V(\varpi_{i_t})_{c_t}\otimes\cdots\otimes V(\varpi_{i_1})_{c_1}$$

is isomorphic to the head of $V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t}$ and the socle of $V(\varpi_{i_t})_{c_t}\otimes\cdots\otimes V(\varpi_{i_1})_{c_1}$. In particular, the image is simple.

Conversely, for any finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module V, there exists a pair of sequences $(i_1,\dots,i_t)\in\{1,\dots,n-1\}^t$, $(c_1,\dots,c_t)\in(\mathbf{k}^\times)^t$, unique up to permutation, such that $d_{i_j,i_k}(c_j/c_k)\neq 0$ for all $j<k$, and V is isomorphic to the head of $V(\varpi_{i_1})_{c_1}\otimes\cdots\otimes V(\varpi_{i_t})_{c_t}$, and so to the image of $R^{w_0}_{i_1,\dots,i_t}(c_1,\dots,c_t)$.

Therefore, any finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module can be obtained as the image of a composition of normalized R-matrices on a tensor product of fundamental representations. This method is called a fusion construction [56], and originates from the fusion of solvable lattice models in mathematical physics.

Recall from the last part of Section 2.2 that the finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-modules are classified by their ℓ-highest weights. More precisely, finite-dimensional irreducibles are in bijection with *dominant* ℓ-weights, which are by definition $(n-1)$-tuples

$(\Psi_i(z))_{i=1,\dots,n-1}$ of rational functions in z such that

$$\Psi_i(z) = q_i^{\deg P_i} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}, \qquad \text{for some polynomial } P_i \in \mathbb{C}[z] \text{ with constant term 1.}$$

For example, the fundamental representation $V(\varpi_i)_x$ corresponds to

$$P_j(z) = \begin{cases} 1 + o(i)(-1)^h q^{-h^\vee} xz & \text{if } j = i \\ 1 & \text{otherwise,} \end{cases}$$

where h (resp. $h^\vee$) is the Coxeter (resp. dual Coxeter) number of $\widehat{\mathfrak{g}}$, and $o(i) = \pm 1$ is chosen so that $o(i) = -o(j)$ whenever $a_{ij} < 0$ [84, Remark 3.3].

If V and W are ℓ-highest weight $U_q'(\widehat{\mathfrak{g}})$-modules with the ℓ-highest weight vector v and w respectively, then $v \otimes w \in V \otimes W$ is of ℓ-highest weight, whose ℓ-weight is equal to the (componentwise) product of the ones of v and w. Hence, if one knows the ℓ-highest weight of the given finite-dimensional irreducible $U_q'(\widehat{\mathfrak{g}})$-module, then one can easily find the pair of sequences in the above theorem.

Example 2.2.5. We continue to consider the type A example. Observe that $V(\varpi_i)$ is already irreducible over $U_q(\mathfrak{sl}_n)$ with highest weight ϖ_i. Then as a $U_q(\mathfrak{sl}_n)$-module, the tensor product decomposition is given by

$$V(\varpi_l) \otimes V(\varpi_m) \cong \bigoplus_{t=\max\{l+m-n,0\}}^{\min\{l,m\}} V(\varpi_{l+m-t} + \varpi_t),$$

where $V(\lambda)$ denotes the irreducible highest weight representation of $U_q(\mathfrak{sl}_n)$ of highest weight λ (in the right hand side, we understand $\varpi_0 = \varpi_n = 0$).

Since the normalized R-matrix $\mathcal{R}^{\mathrm{norm}}_{l,m}(z_1/z_2)$ is also $U_q(\mathfrak{sl}_n)$-linear, by Schur's lemma we can write it as

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(z_1/z_2) = \sum_{t=\max\{l+m-n,0\}}^{\min\{l,m\}} \rho_t^{l,m}(z_1/z_2) \mathcal{P}_t^{l,m}$$

for some $\rho_t^{l,m}(z) \in \mathbf{k}(z)$, where $\mathcal{P}_t^{l,m}$ is a projection from $V(\varpi_l) \otimes V(\varpi_m)$ to the direct summand $V(\varpi_{l+m-t} + \varpi_t)$ of $V(\varpi_m) \otimes V(\varpi_l)$. This expression is called the *spectral decomposition* of the normalized R-matrix, and known to contain much information on the structure of $V(\varpi_l)_x \otimes V(\varpi_m)_y$.

The coefficients $\rho_t^{l,m}(z)$ are given in [27]:

$$\rho_t^{l,m}(z) = \prod_{i=t+1}^{\min\{l,m\}} \frac{z-(-q)^{l+m-2i+2}}{1-(-q)^{l+m-2i+2}z}, \tag{2.2.1}$$

from which one obtain the denominator formula

$$\begin{aligned} d_{l,m}(z) &= \prod_{i=\max\{l+m-n+1,1\}}^{\min\{l,m\}} (1-(-q)^{l+m-2i+2}z) \\ &= \begin{cases} (1-(-q)^{|l-m|+2}z)(1-(-q)^{|l-m|+4}z)\cdots(1-(-q)^{l+m}z) & \text{if } l+m \le n \\ (1-(-q)^{|l-m|+2}z)(1-(-q)^{|l-m|+4}z)\cdots(1-(-q)^{2n-l-m}z) & \text{if } l+m > n. \end{cases} \end{aligned}$$

In general, the denominator is much easier to compute than the spectral decomposition.

2.3 Quiver Hecke algebra

Quiver Hecke algebras or *Khovanov-Lauda-Rouquier algebras* are vast generalizations of the affine Hecke algebra of type A. They were first introduced by Khovanov-Lauda [65] and Rouquier [88] independently to categorify the negative half of quantum groups. Since then, there have been growing interests and studies on quiver Hecke algebras, their cyclotomic quotients (which categorify integrable highest weight modules) and representations.

More recently, another aspect of quiver Hecke algebras was discovered in [51], as a partner of quantum affine algebras in spirit of the celebrated Schur–Weyl duality. Motivated from a duality between finite-dimensional representations of $U_q'(\widehat{\mathfrak{sl}}_n)$ and those of affine Hecke algebras of type A [17, 26, 35], they introduce a general construction of a quiver Hecke algebra action on a (completion of) tensor product of representations of quantum affine algebras, and hence a functorial relation between two representation theory.

The purpose of this section is to provide a background on quiver Hecke algebras, needed for constructing and making use of the duality functor in Section 4.2.4. After a quick review on quiver Hecke algebras, we consider a quiver Hecke algebra of type A_∞ and its finite-dimensional simple modules, following the approach of [51, 57]. One can observe a similarity with the story of Section 2.2.2, striking enough to motivate the existence of the duality.

Throughout this section, we fix a base field $\mathbf{k}$.

2.3.1 Quiver Hecke algebra

Let $A = (a_{ij})_{i,j\in J}$ be a symmetrizable generalized Cartan matrix with positive integers d_i ($i \in J$) such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in J$. Set

$$\mathbb{N}[J] = \left\{ \sum_{i\in J} c_i \cdot i \in \mathbb{Z}[J] \mid c_i \in \mathbb{Z}_{\geq 0} \right\},$$

where $\mathbb{Z}[J]$ is a free abelian group generated by J. Note that $\mathbb{N}[J]$ is naturally identified with the positive cone Q_+ of the root lattice of the corresponding Kac-Moody algebra $\mathfrak{g}(A)$, by $i \leftrightarrow \alpha_i$. For $\beta = \sum_{i\in J} c_i \cdot i \in \mathbb{N}[J]$ with $\mathrm{ht}(\beta) \coloneqq \sum c_i = \ell$, we put

$$J^\beta = \left\{ \nu = (\nu_1, \dots, \nu_\ell) \in J^\ell \mid \nu_1 + \cdots + \nu_\ell = \beta \right\}.$$

Suppose that we are given a matrix $(Q_{ij}(u,v))_{i,j\in J}$ with entries $Q_{ij}(u,v) \in \mathbf{k}[u,v]$ satisfying

(1) $Q_{ij}(u,v) = Q_{ji}(v,u)$ for $i \neq j$ and $Q_{ii}(u,v) = 0$,

(2) the coefficient of $u^p v^q$ ($p, q \in \mathbb{Z}_{\geq 0}$) in $Q_{ij}(u,v)$ is zero unless $d_i a_{ii} p + d_j a_{jj} q = -2 d_i a_{ij}$,

(3) the coefficient of $u^{-a_{ij}}$ in $Q_{ij}(u,v)$ is nonzero.

Definition 2.3.1. The *quiver Hecke algebra* $R(\beta)$ at $\beta \in \mathbb{N}[J]$ associated with $(Q_{ij})_{i,j\in J}$ is the $\mathbb{Z}$-graded $\mathbf{k}$-algebra generated by

$$e(\nu)\ (\nu \in J^\beta), \quad x_k\ (k = 1, \dots, \mathrm{ht}(\beta)), \quad \tau_m\ (m = 1, \dots, \mathrm{ht}(\beta) - 1),$$

subject to the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu\nu'} e(\nu), \quad \sum_{\nu\in J^\beta} e(\nu) = 1,$$
$$x_k x_{k'} = x_{k'} x_k, \quad x_k e(\nu) = e(\nu) x_k, \quad \tau_m e(\nu) = e(s_m(\nu)) \tau_m,$$

$$(\tau_m x_k - x_{s_m(k)}\tau_m)e(\nu) = \begin{cases} -e(\nu) & \text{if } k = m \text{ and } \nu_m = \nu_{m+1} \\ e(\nu) & \text{if } k = m+1 \text{ and } \nu_m = \nu_{m+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_m^2 e(\nu) = Q_{\nu_m,\nu_{m+1}}(x_m,x_{m+1})e(\nu), \quad \tau_m\tau_{m'} = \tau_{m'}\tau_m \quad \text{if } |m-m'|>1,$$

$$(\tau_{m+1}\tau_m\tau_{m+1} - \tau_m\tau_{m+1}\tau_m)e(\nu) = \begin{cases} \frac{Q_{\nu_m,\nu_{m+1}}(x_m,x_{m+1}) - Q_{\nu_m,\nu_{m+1}}(x_{m+2},x_{m+1})}{x_m - x_{m+2}}e(\nu) & \text{if } \nu_m = \nu_{m+2} \\ 0 & \text{otherwise,} \end{cases}$$

and the grading

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = a_{\nu_k,\nu_k}, \quad \deg \tau_m e(\nu) = -a_{\nu_m,\nu_{m+1}}.$$

We also set

$$R(\ell) = \bigoplus_{\mathrm{ht}(\beta)=\ell} R(\beta) \quad (\ell \ge 1), \quad R = \bigoplus_{\ell \ge 0} R(\ell)$$

where $R(0) = \mathbf{k}$. By the standard argument in the theory of Hecke algebras, we have

$$R(\beta) = \bigoplus_{\nu \in J^\beta, w \in \mathfrak{S}_\ell} \mathbf{k}[x_1, \dots, x_\ell]e(\nu)\tau_w,$$

as a vector space, when $\mathrm{ht}(\beta) = \ell$. Here $\tau_w = \tau_{i_1}\cdots\tau_{i_l}$ is defined after fixing a reduced expression[2] $w = s_{i_1}\cdots s_{i_l}$ for each $w \in \mathfrak{S}_\ell$.

For $\beta_1, \beta_2 \in \mathbb{N}[J]$ with $\mathrm{ht}(\beta_1) = \ell_1$, $\mathrm{ht}(\beta_2) = \ell_2$, let

$$e(\beta_1, \beta_2) = \sum_{\substack{(\nu_1,\dots,\nu_{\ell_1}) \in J^{\beta_1} \\ (\nu_{\ell_1+1},\dots,\nu_{\ell_1+\ell_2}) \in J^{\beta_2}}} e(\nu_1, \dots, \nu_{\ell_1+\ell_2}) \in R(\beta_1 + \beta_2).$$

Then we have a $\mathbf{k}$-algebra homomorphism

$$\begin{aligned} R(\beta_1) \otimes R(\beta_2) &\longrightarrow e(\beta_1,\beta_2)R(\beta_1+\beta_2)e(\beta_1,\beta_2) \\ e(\nu_1,\dots,\nu_{\ell_1}) \otimes e(\nu'_1,\dots,\nu'_{\ell_2}) &\longmapsto e(\nu_1,\dots,\nu_{\ell_1},\nu'_1,\dots,\nu'_{\ell_2}), \end{aligned}$$

[2]Note however that τ_w does depend on the choice of a reduced expression, as the braid relation in τ_m does not hold in the quiver Hecke algebras.

$$x_k \otimes 1 \mapsto x_k e(\beta_1, \beta_2), \quad 1 \otimes x_{k'} \mapsto x_{\ell_1+k'} e(\beta_1, \beta_2),$$
$$\tau_m \otimes 1 \mapsto \tau_m e(\beta_1, \beta_2), \quad 1 \otimes \tau_{m'} \mapsto \tau_{\ell_1+m'} e(\beta_1, \beta_2),$$

so that $R(\beta_1 + \beta_2)$ has a right $R(\beta_1) \otimes R(\beta_2)$-module structure. Hence one can define the convolution product of an $R(\beta_1)$-module M_1 and an $R(\beta_2)$-module M_2 by

$$M_1 \circ M_2 := R(\beta_1 + \beta_2) \otimes_{R(\beta_1) \otimes R(\beta_2)} (M_1 \otimes M_2)$$

which is an $R(\beta_1 + \beta_2)$-module.

Consider the category $R(\beta)$-gmod of finite-dimensional graded $R(\beta)$-modules, and

$$R\text{-gmod} = \bigoplus_{\beta \in \mathbb{N}[J]} R(\beta)\text{-gmod}.$$

Then the category R-gmod is equipped with a monoidal structure by the convolution product, and the degree shift q defined by $(qM)_k = M_{k-1}$ for $M = \bigoplus_{k \in \mathbb{Z}} M_k \in R$-gmod. Consequently, the Grothendieck group

$$K(R\text{-gmod}) = \bigoplus_{\beta \in \mathbb{N}[J]} K(R(\beta)\text{-gmod})$$

possesses a $\mathbb{Z}[q^{\pm 1}]$-algebra structure. Similar construction works for the category $R(\beta)$-gproj of finitely generated projective graded $R(\beta)$-modules, yielding another $\mathbb{Z}[q^{\pm 1}]$-algebra $K(R\text{-gproj})$.

Now let $U_{\mathcal{A}}^-(\mathfrak{g})$ be the integral form of the negative half of the quantum group associated with the given generalized Cartan matrix, that is, the subalgebra over $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$ generated by divided powers $f_i^m/[m]!$ ($i \in J$, $m \in \mathbb{Z}_{\geq 0}$). The original motivation of introducing quiver Hecke algebras is the following categorification theorem.

Theorem 2.3.2 ([65, 88]). *For a symmetrizable generalized Cartan matrix A and a parameter matrix $(Q_{ij})_{i,j \in J}$, let $R(\beta)$ be the quiver Hecke algebra associated with (Q_{ij}). Then there exists an $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism*

$$U_{\mathcal{A}}^-(\mathfrak{g}) \cong K(R\text{-gproj}) = \bigoplus_{\beta} K(R(\beta)\text{-gproj}), \quad U_{\mathcal{A}}^-(\mathfrak{g})^\vee \cong K(R-\text{gmod})$$

where $U_{\mathcal{A}}^-(\mathfrak{g})^\vee$ is the graded dual over $\mathbb{Z}[q^{\pm 1}]$ with respect to the $-Q_+$-grading.

Moreover, the first (resp. latter) isomorphism matches the set of indecomposable projective (resp. simple) modules with the (resp. dual) canonical basis when A is symmetric and char$\mathbf{k} = 0$ [91], and even gives a monoidal categorification of the quantum cluster algebra structure on $U^-_{\mathcal{A}}(\mathfrak{g})^\vee$ [54]. In addition, there exist certain quotients called the cyclotomic quiver Hecke algebras known to categorify the integrable highest weight representations of $U_q(\mathfrak{g})$ [50], while a large part of structures and representations of cyclotomic quiver Hecke algebras are still unknown.

Next, we recall the notion of renormalized R-matrices for modules over symmetric quiver Hecke algebras [51, Section 1.3].

Definition 2.3.3. A quiver Hecke algebra $R(\beta)$ associated with A and (Q_ij) is said to be *symmetric* if A is symmetric and $Q_{ij}(u,v)$ is a polynomial in $u-v$ for all $i,j \in J$.

Every quiver Hecke algebra that will appear in this thesis is symmetric. Let us introduce elements

$$\varphi_m e(\nu) = \begin{cases} (\tau_m x_m - x_m \tau_m)e(\nu) & \text{if } \nu_m = \nu_{m+1} \\ \tau_m e(\nu) & \text{if } \nu_m \neq \nu_{m+1} \end{cases}$$

of $R(\beta)$, where $\nu \in J^\beta$ and $1 \le m \le \ell - 1$ ($\ell = \mathrm{ht}(\beta)$). Unlike the generators τ_m, the family $\{\varphi_m\}_{1 \le m \le \ell-1}$ satisfies the braid relation, and so we obtain a well-defined element φ_w for $w \in \mathfrak{S}_\ell$ by taking any reduced expression of w. Moreover, one can check the following properties:

(1) For $w \in \mathfrak{S}_\ell$ and $1 \le k \le \ell$, $\varphi_w x_k = x_{w(k)} \varphi_w$,

(2) For $w \in \mathfrak{S}_\ell$ and $1 \le m \le \ell - 1$, if $w(k+1) = w(k)+1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.

Hence given an $R(\beta_i)$-module M_i $(i = 1,2)$, we obtain an $R(\beta_1+\beta_2)$-module homomorphism

$$\begin{aligned} R_{M,N} : M \circ N &\longrightarrow N \circ M \\ u_1 \otimes u_2 &\longmapsto \varphi_{w[\mathrm{ht}(\beta_1),\mathrm{ht}(\beta_2)]}(u_2 \otimes u_1) \end{aligned}$$

where $w[\ell_1, \ell_2] \in \mathfrak{S}_{\ell_1+\ell_2}$ is defined by

$$w[\ell_1,\ell_2](k) = \begin{cases} k + \ell_2 & \text{if } k \le \ell_1 \\ k - \ell_1 & \text{if } k > \ell_1. \end{cases}$$

For an $R(\beta)$-module M, define the affinization of M to be the $\mathbf{k}[z] \otimes R(\beta)$-module $M_z = \mathbf{k}[z] \otimes M$ on which the generators act by

$$\begin{aligned} e(\nu) &= 1 \otimes e(\nu), \quad \tau_m = 1 \otimes \tau_m, \\ x_k &= z \otimes 1 + 1 \otimes x_k, \end{aligned}$$

where z is an indeterminate (of degree 2). Then we define a *renormalized R-matrix* by

$$\mathbf{r}_{M_1,M_2} = z^{-s} R_{(M_1)_z,M_2}|_{z=0} : M_1 \circ M_2 \longrightarrow M_2 \circ M_1$$

where s is the largest nonnegative integer such that $\mathrm{im} R_{(M_1)_z,M_2} \subset z^s R_{M_2,(M_1)_z}$ [51, Proposition 1.10]. In particular, $\mathbf{r}_{M_1,M_2}$ never vanishes.

Since the braid relation is satisfied by φ_m, renormalized R-matrices solve the Yang-Baxter equation

$$\mathbf{r}_{M,N}\mathbf{r}_{L,N}\mathbf{r}_{L,M} = \mathbf{r}_{L,M}\mathbf{r}_{L,N}\mathbf{r}_{M,N} : L \circ M \circ N \longrightarrow N \circ M \circ L.$$

Again this allows us to define without ambiguity a module homomorphism

$$\mathbf{r}^w_{M_1,\dots,M_t} : M_1 \circ \cdots \circ M_t \longrightarrow M_{w(1)} \circ \cdots \circ M_{w(t)}$$

for any $w \in \mathfrak{S}_t$ and $R(\beta_i)$-module M_i. We remark that although $\mathbf{r}_{M_i,M_j} \neq 0$ for any i, j, it may happen that $\mathbf{r}^w_{M_1,\dots,M_t}$ vanishes for some w (see [51, Proposition 1.15], [59, Corollary 2.9]).

2.3.2 Quiver Hecke algebra of type A and their simple modules

We fix the following Dynkin quiver Γ of type A_∞:

$$\cdots \longrightarrow \underset{-2}{\circ} \longrightarrow \underset{-1}{\circ} \longrightarrow \underset{0}{\circ} \longrightarrow \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \cdots \tag{2.3.1}$$

with the associated Cartan matrix $(a_{ij})_{i,j\in\mathbb{Z}}$ defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $J = \mathbb{Z}$, and put

$$\begin{aligned} P_{ij}(u, v) &= (u - v)^{d_{ij}} \quad \text{where } d_{ij} \text{ is the number of arrows from } i \text{ to } j \text{ in } \Gamma, \\ Q_{ij}(u, v) &= P_{ij}(u, v)P_{ji}(v, u) \quad \text{for } i \neq j. \end{aligned}$$

Then we obtain a symmetric quiver Hecke algebra $R(\beta)$ associated with the quiver Γ.

Let us recall the classification of the finite-dimensional simple $R(\beta)$-modules following [51, 57]. Recall that the positive roots of the Kac-Moody algebra of type A_∞ are of the form

$$\beta_{(a,b)} := \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$$

for pairs (a, b) of integers such that $a \leq b$. We call such a pair a segment of length $\ell = b - a + 1$. Thus, the positive roots are in bijection with the segments. We also assign to the latter the lexicographic order

$$(a, b) \leq (a', b') \iff a < a' \text{ or } (a = a',\ b \leq b').$$

A finite sequence of segments $((a_1, b_1), \ldots, (a_t, b_t))$ is called a multisegment, and is said to be ordered if $(a_k, b_k) \geq (a_{k+1}, b_{k+1})$ for all $1 \leq k \leq t - 1$.

For each segment (a, b) of length ℓ, we define a 1-dimensional $R(\beta_{(a,b)})$-module $L(a, b) = \mathbf{k}u(a, b)$ defined by

$$x_k u(a, b) = \tau_m u(a, b) = 0, \quad e(\nu)u(a, b) = \begin{cases} u(a, b) & \text{if } \nu = (a, a + 1, \ldots, b) \\ 0 & \text{otherwise.} \end{cases}$$

As we have seen in the last subsection, for an ordered multisegment $((a_1, b_1), \ldots, (a_t, b_t))$, there is an $R(\beta)$-module homomorphism ($\beta = \sum_{i=1}^{t} \beta_{(a_i,b_i)}$)

$$\mathbf{r}^{w_0} : L(a_1, b_1) \circ \cdots \circ L(a_t, b_t) \longrightarrow L(a_t, b_t) \circ \cdots \circ L(a_1, b_1),$$

associated with the longest element $w_0 \in \mathfrak{S}_t$.

Proposition 2.3.4 ([51, 57]). *There exists a one-to-one correspondence between the ordered multisegments and the finite-dimensional simple graded R-modules (up to isomorphisms and grading shifts), given by*

$$((a_1, b_1), \dots, (a_t, b_t)) \longmapsto \mathrm{hd}\left(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)\right).$$

Moreover, $\mathbf{r}^{w_0}$ *has a simple image which is isomorphic to* $\mathrm{hd}\left(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)\right)$.

As an example, one has the following result for ordered couples of segments.

Proposition 2.3.5 ([51, Proposition 4.3]). *Let two segments $(a, b) \geq (a', b')$ be given.*

(1) *If any of the following holds: $a' < a \leq b \leq b'$, $a > b' + 1$, or $a = a' \leq b' \leq b$, then $L(a, b) \circ L(a', b')$ is irreducible and*

$$L(a, b) \circ L(a', b') \xrightarrow{\mathbf{r}} L(a', b') \circ L(a, b)$$

is an isomorphism.

(2) *If $a' < a \leq b' < b$, then we have an exact sequence*

$$\begin{aligned} 0 \longrightarrow L(a', b) \circ L(a, b') \longrightarrow\ & L(a, b) \circ L(a', b') \longrightarrow \\ & L(a', b') \circ L(a, b) \xrightarrow{\mathbf{r}} L(a', b) \circ L(a, b') \longrightarrow 0. \end{aligned}$$

(3) *If $a = b' + 1$, then we have an exact sequence*

$$0 \longrightarrow L(a', b) \longrightarrow L(a, b) \circ L(a', b') \xrightarrow{\mathbf{r}} L(a', b') \circ L(a, b) \longrightarrow L(a', b) \longrightarrow 0.$$

Here we ignore the grading and $\mathbf{r}$ *denotes the corresponding renormalized R-matrix.*

Remark 2.3.6. Observe that the above theory of modules over type A_∞ quiver Hecke algebras resembles that of type A affine Hecke algebras [9, 92, 95]. This can be understood through an algebra isomorphism between them (after a completion) [11, 88], which will be explained and used to prove the equivalence of generalized quantum affine Schur-Weyl duality functor later in Chapter 4.

Chapter 3

Generalized Quantum Groups of type A

In this chapter, we introduce the generalized quantum group $\mathcal{U}(\epsilon)$ of affine type A [70]. Since it is a variant of the usual quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}_{M|N})$, we first seek for a relation between them, namely an algebra isomorphism up to a mild extension. Then we use the isomorphism to construct a nondegenerate Hopf pairing on $\mathcal{U}(\epsilon)$, which leads to a universal R-matrix by the standard argument of [78]. We also recall some basic facts on polynomial representations of the finite type subalgebra $\mathring{\mathcal{U}}(\epsilon)$.

Let us fix here notations which will be used throughout this thesis.

- $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$: a (01)-sequence of length $n \geq 4$
- $M = |\{i \,|\, \epsilon_i = 0\}|$ and $N = |\{i \,|\, \epsilon_i = 1\}|$.
- $\epsilon_{M|N}$: a (01)-sequence with $\epsilon_1 = \cdots = \epsilon_M = 0$, $\epsilon_{M+1} = \cdots = \epsilon_{M+N} = 1$.
- $\mathbb{I} = \{1 < 2 < \cdots < n\}$.
- $P_{\rm fin} = \bigoplus_{i \in \mathbb{I}} \mathbb{Z}\delta_i$: a free abelian group of rank n.
- $P_{\geq 0} = \bigoplus_{i \in \mathbb{I}} \mathbb{Z}_{\geq 0}\delta_i \subset P_{\rm fin}$.
- $\deg \lambda = \sum \lambda_i$ for $\lambda = \sum \lambda_i \delta_i \in P_{\geq 0}$.
- $q_i = (-1)^{\epsilon_i} q^{(-1)^{\epsilon_i}} = \begin{cases} q & \text{if } \epsilon_i = 0 \\ -q^{-1} & \text{if } \epsilon_i = 1 \end{cases} \quad (i \in \mathbb{I})$.

- $\mathbf{q}(\,\cdot\,,\,\cdot\,)$: a $\Bbbk$-valued symmetric biadditive form on $P_{\rm fin}$ defined by

$$\mathbf{q}(\lambda,\mu)=\prod q_i^{c_id_i}\qquad \text{for } \lambda=\sum c_i\delta_i,\ \mu=\sum d_i\delta_i\in P_{\rm fin}.$$

- $(\,\cdot\,|\,\cdot\,)$: a symmetric bilinear form on $P_{\rm fin}$ such that $(\delta_i|\delta_j)=(-1)^{\epsilon_i}\delta_{ij}$.
- $I=\{0,1,\ldots,n-1\}$.
- $\alpha_i=\delta_i-\delta_{i+1}\in P_{\rm fin}\quad (i\in I)$.
- $I_{\rm even}=\{i\in I\,|\,(\alpha_i|\alpha_i)=\pm 2\}$, $I_{\rm odd}=\{i\in I\,|\,(\alpha_i|\alpha_i)=0\}$.
- $p(i)=\begin{cases}0 & \text{if } i\in I_{\rm even}\\ 1 & \text{if } i\in I_{\rm odd}.\end{cases}$
- $P^0=\bigoplus_{i\in\mathbb{I}}\mathbb{Z}\boldsymbol{\delta}_i\oplus\mathbb{Z}\boldsymbol{\delta}$: a free abelian group of rank $n+1$.
- $\boldsymbol{\alpha}_i=\boldsymbol{\delta}_i-\boldsymbol{\delta}_{i+1}+\delta_{i,0}\boldsymbol{\delta}\in P^0$.
- $Q=\bigoplus_{i\in I}\mathbb{Z}\boldsymbol{\alpha}_i\subset P^0,\quad Q_+=\sum\mathbb{Z}_{\geq 0}\boldsymbol{\alpha}_i$.
- $\mathrm{ht}(\beta)=\sum d_i$ for $\beta=\sum d_i\boldsymbol{\alpha}_i\in Q_+$.
- $\mathrm{cl}:P^0\to P_{\rm fin}$: the linear map defined by $\mathrm{cl}(\boldsymbol{\delta}_i)=\delta_i$, $\mathrm{cl}(\boldsymbol{\delta})=0$.

A subscript $i\in I$ is always understood modulo n.

Note that $P_{\rm fin}$ is the weight lattice for $\mathfrak{gl}_{M|N}$ (equivalently, for $\mathfrak{gl}_{M+N}$), and $P_{\geq 0}$ is the set of polynomial weights. Moreover, P^0 is the set of level zero weights of the affine Lie superalgebra $\widehat{\mathfrak{gl}}_{M|N}$ (see Section 2.2.2).

3.1 Generalized quantum group of affine type A

3.1.1 Definition

Definition 3.1.1 ([70, 79]). Given a (01)-sequence ϵ of length n, the *generalized quantum group of affine type A associated with ϵ* is defined to be the $\Bbbk$-algebra $\mathcal{U}(\epsilon)$, generated by e_i, f_i $(i\in I)$ and k_μ $(\mu\in P_{\rm fin})$ subject to the following defining relations:

$$k_0=1,\quad k_{\mu+\mu'}=k_\mu k_{\mu'},$$

$$
\begin{aligned}
&k_\mu e_i k_\mu^{-1} = \mathbf{q}(\mu, \alpha_i) e_i, \quad k_\mu f_i k_\mu^{-1} = \mathbf{q}(\mu, -\alpha_i) f_i,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},\\
&e_i^2 = f_i^2 = 0 \qquad (i \in I_{\mathrm{odd}}),\\
&e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \qquad (i - j \not\equiv \pm 1 \ (\mathrm{mod}\, n)),\\
&\left.\begin{aligned} e_i^2 e_j - (-1)^{\epsilon_i}[2] e_i e_j e_i + e_j e_i^2 = 0\\ f_i^2 f_j - (-1)^{\epsilon_i}[2] f_i f_j f_i + f_j f_i^2 = 0 \end{aligned}\right. \quad (i \in I_{\mathrm{even}} \text{ and } i - j \equiv \pm 1 \ (\mathrm{mod}\, n)),\\
&\left.\begin{aligned} &e_i e_{i-1} e_i e_{i+1} - e_i e_{i+1} e_i e_{i-1} + e_{i+1} e_i e_{i-1} e_i\\ &\qquad - e_{i-1} e_i e_{i+1} e_i + (-1)^{\epsilon_i}[2] e_i e_{i-1} e_{i+1} e_i = 0\\ &f_i f_{i-1} f_i f_{i+1} - f_i f_{i+1} f_i f_{i-1} + f_{i+1} f_i f_{i-1} f_i\\ &\qquad - f_{i-1} f_i f_{i+1} f_i + (-1)^{\epsilon_i}[2] f_i f_{i-1} f_{i+1} f_i = 0 \end{aligned}\right. \quad (i \in I_{\mathrm{odd}}),
\end{aligned}
$$

where we put $k_i := k_{\alpha_i}$.

Moreover, $\mathcal{U}(\epsilon)$ is endowed with a Hopf algebra structure given by

$$
\begin{aligned}
&\Delta : k_\mu \mapsto k_\mu \otimes k_\mu, \quad e_i \mapsto e_i \otimes k_i^{-1} + 1 \otimes e_i, \quad f_i \mapsto f_i \otimes 1 + k_i \otimes f_i,\\
&S : k_\mu \mapsto k_\mu^{-1}, \quad e_i \mapsto -e_i k_i, \quad f_i \mapsto -k_i^{-1} f_i.
\end{aligned}
$$

We let $\mathcal{U}(\epsilon)^+$ (resp. $\mathcal{U}(\epsilon)^-$) be the subalgebra generated by e_i (resp. f_i) for $i \in I$, and $\mathcal{U}(\epsilon)^0$ the one generated by k_μ for $\mu \in P_{\mathrm{fin}}$. Then the proof of [43, Theorem 3.1.5] applies here to prove the following triangular decomposition.

Proposition 3.1.2. *The multiplication*

$$
\mathcal{U}(\epsilon)^+ \otimes \mathcal{U}(\epsilon)^0 \otimes \mathcal{U}(\epsilon)^- \longrightarrow \mathcal{U}(\epsilon)
$$

is an isomorphism of $\Bbbk$*-vector spaces.*

Observe that if $\epsilon_j = 0$ (resp. $\epsilon_j = 1$) for all $j \in \mathbb{I}$, then $\mathcal{U}(\epsilon)$ recovers the quantum affine algebra of type A. More precisely, its subalgebra generated by $e_i, f_i, k_i^{\pm 1}$ ($i \in I$) is isomorphic to $U_q'(\widehat{\mathfrak{sl}}_n)$ (resp. $U'_{-q^{-1}}(\widehat{\mathfrak{sl}}_n)$) or its quotient by $q^C - 1$.

When $\widetilde{\epsilon}$ is obtained from ϵ by permuting entries, $\mathcal{U}(\widetilde{\epsilon})$ is related to $\mathcal{U}(\epsilon)$ by the following algebra isomorphism. This can be seen as a super analogue of the Lusztig's braid group symmetry on quantum groups [78, Chapter 37] (*cf.* [94, Proposition 8.2.1]).

Proposition 3.1.3 ([79]). *Suppose that $\widetilde{\epsilon}$ is obtained from ϵ by permuting the entries ϵ_i and ϵ_{i+1} for some $i \in I$. There exists an algebra isomorphism $T_i : \mathcal{U}(\epsilon) \longrightarrow \mathcal{U}(\widetilde{\epsilon})$ given by*

$$\begin{aligned}
&T_i(k_{\delta_i}) = k_{\delta_{i+1}}, \quad T_i(k_{\delta_{i+1}}) = k_{\delta_i}, \quad T_i(k_{\delta_j}) = k_{\delta_j} \quad (j \neq i, i+1),\\
&T_i(e_i) = -f_i k_i, \quad T_i(f_i) = -k_i^{-1} e_i,\\
&T_i(e_j) = [e_i, e_j]_{\mathbf{q}_{\widetilde{\epsilon}}(\alpha_i,\alpha_j)}, \quad T_i(f_j) = [f_j, f_i]_{\mathbf{q}_{\widetilde{\epsilon}}(\alpha_i,\alpha_j)^{-1}} \quad (|i-j| = 1),\\
&T_i(e_j) = e_j, \quad T_i(f_j) = f_j \quad (|i-j| > 1)
\end{aligned}$$

where $\mathbf{q}_{\widetilde{\epsilon}}(\,\cdot\,,\,\cdot\,)$ denotes the biadditive function associated to $\widetilde{\epsilon}$.

Finally, we introduce a bar involution, the $\mathbb{Q}$-algebra involution on $\mathcal{U}(\epsilon)$ given by

$$\overline{q} = q^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{k_\mu} = k_\mu^{-1}.$$

3.1.2 Quantum affine superalgebra and algebra isomorphism

Let us recall the definition [94] of a quantum affine superalgebra, namely a quantized universal enveloping algebra of an affine Lie superalgebra $\widehat{\mathfrak{gl}}_{M|N}$.

Definition 3.1.4. Let $U(\epsilon)$ be the $\Bbbk$-superalgebra generated by E_i, F_i ($i \in I$) and K_μ ($\mu \in P_{\mathrm{fin}}$) with parities

$$p(E_i) = p(F_i) = p(i), \quad p(K_\mu) = 0,$$

subject to the following defining relations:

$$\begin{aligned}
&K_0 = 1, \quad K_{\mu+\mu'} = K_\mu K_{\mu'},\\
&K_\mu E_i K_\mu^{-1} = q^{(\mu|\alpha_i)} E_i, \quad K_\mu F_i K_\mu^{-1} = q^{-(\mu|\alpha_i)} F_i,\\
&E_i F_j - (-1)^{p(i)p(j)} F_j E_i = (-1)^{\epsilon_i} \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q - q^{-1}},\\
&E_i^2 = F_i^2 = 0 \qquad (i \in I_{\mathrm{odd}}),\\
&E_i E_j - (-1)^{p(i)p(j)} E_j E_i = F_i F_j - (-1)^{p(i)p(j)} F_j F_i = 0 \qquad (i - j \not\equiv \pm 1 \pmod n),\\
&\left.\begin{aligned} E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0\\ F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 \end{aligned}\right. \qquad (i \in I_{\mathrm{even}} \text{ and } i - j \equiv \pm 1 \pmod n),
\end{aligned}$$

$$\begin{aligned}[E_i,[[E_{i-1},E_i]_{(-1)^{p(i-1)}q},E_{i+1}]_{(-1)^{(p(i-1)+1)p(i+1)}q^{-1}}]_{(-1)^{p(i-1)+p(i+1)+1}} &= 0 \\ [F_i,[[F_{i-1},F_i]_{(-1)^{p(i-1)}q},F_{i+1}]_{(-1)^{(p(i-1)+1)p(i+1)}q^{-1}}]_{(-1)^{p(i-1)+p(i+1)+1}} &= 0\end{aligned} \qquad (i \in I_{\mathrm{odd}}).$$

There is a Hopf (super)algebra structure on $U(\epsilon)$ by the same formula as above. Note that when $M \neq N$, the subalgebra of $U(\epsilon)$ generated by $E_i, F_i, K_{\alpha_i}^{\pm 1}$ is isomorphic to the quantum affine superalgebra $U_q'(\widehat{\mathfrak{sl}}_{M|N})$ [94, Theorem 6.8.2], more precisely its quotient by the canonical central element.

Remark 3.1.5. A Drinfeld realization for quantum affine superalgebras $U_q'(\widehat{\mathfrak{sl}}_{M|N})$ is also established in [94, Section 8], using a braid group symmetry (*cf.* [4]).

Observe that the defining relations of $\mathcal{U}(\epsilon)$ and $U(\epsilon)$ differ only by signs. We resolve this discrepancy by adjoining sign operators σ_i to the algebras, as follows.

Introduce a commutative bialgebra Σ over $\Bbbk$ generated by σ_j ($j \in \mathbb{I}$) satisfying $\sigma_j^2 = 1$, with the comultiplication $\Delta(\sigma_j) = \sigma_j \otimes \sigma_j$. Then $U(\epsilon)$ carries a Σ-module algebra structure given by

$$\sigma_j K_\mu = K_\mu, \quad \sigma_j E_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} E_i, \quad \sigma_j F_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)} F_i,$$

that is, the multiplication and the unit morphism are Σ-module homomorphisms. Thus we can form a semidirect product $U(\epsilon)[\sigma]$ of $U(\epsilon)$ and Σ, and similarly we obtain $\mathcal{U}(\epsilon)[\sigma]$.

Now let us assume $M \neq 0$. Given ϵ, there exists a unique sequence $1 \leq i_1 < i_2 < \cdots < i_l \leq n$ such that

$$\epsilon_{i_k - 1} \neq \epsilon_{i_k} = \epsilon_{i_k+1} = \cdots = \epsilon_{i_{k+1}-1} \neq \epsilon_{i_{k+1}} \quad \text{for } 1 \leq k \leq l.$$

Here we understand the subscripts modulo n, and $i_{l+1} := i_1$. For example, to $\epsilon = (001011)$ corresponds the sequence $1 < 3 < 4 < 5$. Put

$$\sigma_{\leq j} = \sigma_1 \sigma_2 \cdots \sigma_j \ (j = \mathbb{I}).$$

We assign to each generators E_i, F_i, K_{δ_j} ($i \in I$, $j \in \mathbb{I}$) of $U(\epsilon)$ certain elements $\tau(E_i)$, $\tau(F_i)$, $\tau(K_{\delta_j})$ of $\mathcal{U}(\epsilon)[\sigma]$ respectively. First, we define

$$\tau(K_{\delta_j}) = k_{\delta_j} \sigma_j.$$

(1) If $i \in I_{\text{even}}$ with $(\epsilon_i, \epsilon_{i+1}) = (0,0)$, then we set

$$\tau(E_i) = e_i, \quad \tau(F_i) = f_i.$$

(2) If $i \in I_{\text{odd}}$ with $(\epsilon_i, \epsilon_{i+1}) = (0,1)$, then we set

$$\tau(E_i) = e_i\sigma_{\leq i}, \quad \tau(F_i) = f_i\sigma_{\leq i}\sigma_i\sigma_{i+1}.$$

(3) If $i \in I_{\text{even}}$ with $(\epsilon_i, \epsilon_{i+1}) = (1,1)$, then $i \in \{i_k, i_k+1, \dots, i_{k+1}-2\}$ for some k. When $k < l$, we set

$$\tau(E_i) = -e_i(\sigma_i\sigma_{i+1})^{i-i_k+1}, \quad \tau(F_i) = f_i(\sigma_i\sigma_{i+1})^{i-i_k},$$

and when $k = l$,

$$\tau(E_i) = \begin{cases} -e_i(\sigma_i\sigma_{i+1})^{i-i_l+1} \\ -e_i(\sigma_i\sigma_{i+1})^{i-(i_l-n)+1}, \end{cases} \qquad \tau(F_i) = \begin{cases} -f_i(\sigma_i\sigma_{i+1})^{i-i_l} & \text{if } i_l \leq i \leq n \\ -f_i(\sigma_i\sigma_{i+1})^{i-(i_l-n)} & \text{if } 1 \leq i \leq i_1 - 2. \end{cases}$$

(4) If $i \in I_{\text{odd}}$ with $(\epsilon_i, \epsilon_{i+1}) = (1,0)$, then $i = i_{k+1} - 1$ for some k. When $k < l$, we set

$$\tau(E_i) = e_i\sigma_{\leq i}(-\sigma_i\sigma_{i+1})^{i_{k+1}-i_k}, \quad \tau(F_i) = f_i\sigma_{\leq i}(\sigma_i\sigma_{i+1})^{i_{k+1}-i_k-1},$$

and when $k = l$,

$$\tau(E_i) = e_i\sigma_{\leq i}(-\sigma_i\sigma_{i+1})^{i_1-(i_l-n)}, \quad \tau(F_i) = f_i\sigma_{\leq i}(\sigma_i\sigma_{i+1})^{i_1-(i_l-n)-1}.$$

The following table illustrates the image of E_i and F_i under τ for $\epsilon = (001011)$:

$i \in I$	0	1	2	3	4	5
E_i	$e_0\sigma_{\leq 6}$	e_1	$e_2\sigma_{\leq 2}$	$-e_3\sigma_{\leq 3}\sigma_3\sigma_4$	$e_4\sigma_{\leq 4}$	$-e_5\sigma_5\sigma_6$
F_i	$f_0\sigma_{\leq 6}\sigma_6\sigma_1$	f_1	$f_2\sigma_{\leq 2}\sigma_2\sigma_3$	$f_3\sigma_{\leq 3}$	$f_4\sigma_{\leq 4}\sigma_4\sigma_5$	f_5

Theorem 3.1.6 (*cf.* [75, Proposition 4.4]). *Suppose that $M \neq 0$. Then τ extends to a $\Bbbk$-algebra isomorphism $\tau : U(\epsilon)[\sigma] \to \mathcal{U}(\epsilon)[\sigma]$ with $\tau(\sigma_j) = \sigma_j$.*

Proof. It is straightforward to check that τ maps the defining relations of $U(\epsilon)$ to zero. For example, take $\epsilon = (001011)$ and let us verify

$$\tau(K_{\alpha_2})\tau(E_3)\tau(K_{-\alpha_2}) - \tau(q^{(\alpha_2|\alpha_3)}E_3) = 0.$$

Indeed, from the above table we have

$$\begin{aligned}
\tau(K_{\alpha_2})\tau(E_3)\tau(K_{-\alpha_2}) &= -k_2\sigma_2\sigma_3 e_3\sigma_1\sigma_2\sigma_4 k_2^{-1}\sigma_2\sigma_3 \\
&= k_2 e_3 k_2^{-1}\sigma_1\sigma_2\sigma_4 \\
&= \mathbf{q}(\alpha_2, \alpha_3)e_3\sigma_1\sigma_2\sigma_4 \\
&= -q^{-1}e_3\sigma_1\sigma_2\sigma_4 \\
&= \tau(q^{(\alpha_2|\alpha_3)}E_3).
\end{aligned}$$

Since one can define the inverse map in the same manner, τ is invertible. □

Remark 3.1.7. The isomorphism τ induces an equivalence between module categories of $\mathcal{U}(\epsilon)$ and $U(\epsilon)$, which preserves the notions of weight (see Remark 3.2.2). However, since τ does not respect the comultiplication, this equivalence is not monoidal a priori.

3.1.3 Universal R-matrix

In the theory of quantum groups, a standard way to construct an intertwiner on a tensor product of two modules is to apply to the tensor product a distinguished element Θ, called a *universal R-matrix*, in the (often completed) tensor square of the quantum group. Drinfeld [28] provided a systematic method to construct a (quantum group with) universal R-matrix as the Casimir element of a nondegenerate Hopf pairing (see for example [64, Chapter XI]).

In this section, we reformulate a half of the generalized quantum group $\mathcal{U}(\epsilon)^-$ following [78]. The key result is Theorem 3.1.10, which asserts that the Serre relations for $\mathcal{U}(\epsilon)$ generate the radical of a symmetric bilinear form on a free associative algebra $'\mathbf{f}(\epsilon)$. This implies the nondegeneracy of a Hopf pairing on $\mathcal{U}(\epsilon)$, and hence a universal R-matrix.

Let $'\mathbf{f}(\epsilon)$ be the free associative $\Bbbk$-algebra with unity generated by θ_i for $i \in I$, which is Q_+-graded with $|\theta_i| = \boldsymbol{\alpha}_i$. We define an algebra homomorphism $r : {'\mathbf{f}(\epsilon)} \to {'\mathbf{f}(\epsilon)} \otimes {'\mathbf{f}(\epsilon)}$

by $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$, where the multiplication on $'\mathbf{f}(\epsilon) \otimes {'\mathbf{f}}(\epsilon)$ is twisted by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \mathbf{q}(|x_2|, |y_1|)^{-1}(x_1 y_1) \otimes (x_2 y_2)$$

for homogeneous x_2, y_1.

For later use, let us also introduce a $\Bbbk$-linear map ${}_ir : {'\mathbf{f}}(\epsilon) \longrightarrow {'\mathbf{f}}(\epsilon)$ $(i \in I)$ defined by

(1) ${}_ir(1) = 0$, ${}_ir(\theta_j) = \delta_{ij}$ for $j \in I$,

(2) ${}_ir(xy) = {}_ir(x)y + \mathbf{q}(|x|, \alpha_i)^{-1} x \cdot {}_ir(y)$ for homogeneous x,

and $r_i : {'\mathbf{f}}(\epsilon) \longrightarrow {'\mathbf{f}}(\epsilon)$ by

(1) $r_i(1) = 0$, $r_i(\theta_j) = \delta_{ij}$ for $j \in I$,

(2) $r_i(xy) = x r_i(y) + \mathbf{q}(|y|, \alpha_i)^{-1} r_i(x) y$ for homogeneous y.

Proposition 3.1.8. *There exists a unique $\Bbbk$-valued symmetric bilinear form $(\ ,\ )$ on ${'\mathbf{f}}(\epsilon)$ satisfying*

(1) $(1, 1) = 1$, $(\theta_i, \theta_j) = \delta_{ij}$,

(2) $(x, yy') = (r(x), y \otimes y')$ *for* x, y, $y' \in {'\mathbf{f}}(\epsilon)$,

(3) $(xx', y) = (x \otimes x', r(y))$ *for* x, x', $y \in {'\mathbf{f}}(\epsilon)$,

where we set $(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2)$. *Moreover, the following property holds:*

$$(\theta_i x, y) = (y, {}_ir(x)), \quad (x\theta_i, y) = (x, r_i(y)).$$

Proof. The proof is a straightforward induction on the height with respect to the Q_+-grading on ${'\mathbf{f}}(\epsilon)$, see [78, Chapter 1]. □

Let $\mathcal{I}$ denote the radical of this bilinear form. Consider a $\Bbbk$-algebra $\mathrm{U}(\epsilon)$ generated by E_i, F_i $(i \in I)$ and K_μ $(\mu \in P_{\mathrm{fin}})$ subject to the relations

$$\begin{aligned}
\mathrm{K}_0 = 1, \quad \mathrm{K}_{\mu+\mu'} &= \mathrm{K}_\mu \mathrm{K}_{\mu'}, \\
\mathrm{K}_\mu \mathrm{E}_i \mathrm{K}_\mu^{-1} = \mathbf{q}(\mu, \alpha_i)\mathrm{E}_i, &\quad \mathrm{K}_\mu \mathrm{F}_i \mathrm{K}_\mu^{-1} = \mathbf{q}(\mu, \alpha_i)^{-1}\mathrm{F}_i, \\
\mathrm{E}_i \mathrm{F}_j - \mathrm{F}_j \mathrm{E}_i &= \delta_{ij} \frac{\mathrm{K}_{\alpha_i} - \mathrm{K}_{\alpha_i}^{-1}}{q - q^{-1}}, \\
h(\mathrm{E}_0, \ldots, \mathrm{E}_n) = h(\mathrm{F}_0, \ldots, \mathrm{F}_n) = 0 &\ \text{ whenever } h(\theta_0, \ldots, \theta_n) \in \mathcal{I}, \text{ for } h \in \Bbbk \langle x_0, \ldots, x_n \rangle.
\end{aligned}$$

Lemma 3.1.9. *The following elements are contained in the radical $\mathcal{I}$:*

(1) θ_i^2 *for* $i \in I_{\mathrm{odd}}$,

(2) $\theta_i\theta_j - \theta_j\theta_i$ *if* $i - j \not\equiv \pm 1 \pmod n$,

(3) $\theta_i^2\theta_j - (-1)^{\epsilon_i}[2]\theta_i\theta_j\theta_i + \theta_j\theta_i^2$ *if* $i \in I_{\mathrm{even}}$ *and* $i - j \equiv \pm 1 \pmod n$,

(4) $\theta_i\theta_{i-1}\theta_i\theta_{i+1} - \theta_i\theta_{i+1}\theta_i\theta_{i-1} + \theta_{i+1}\theta_i\theta_{i-1}\theta_i - \theta_{i-1}\theta_i\theta_{i+1}\theta_i + (-1)^{\epsilon_i}[2]\theta_i\theta_{i-1}\theta_{i+1}\theta_i$ *if* $i \in I_{\mathrm{odd}}$.

Proof. It can be checked by a direct calculation (see [78, Proposition 1.4.3] for (2),(3)). For example, for $i \in I_{\mathrm{odd}}$,

$$(\theta_i^2, \theta_i^2) = (\theta_i, {}_ir(\theta_i^2)) = (\theta_i, (1 + \mathbf{q}(\alpha_i, \alpha_i)^{-1})\theta_i) = 0$$

as $\mathbf{q}(\alpha_i, \alpha_i) = q_i q_{i+1} = -1$. □

By the lemma, we obtain a surjective algebra map $\pi : \mathcal{U}(\epsilon) \longrightarrow \mathrm{U}(\epsilon)$. The following theorem is our first main result, which enables us to reproduce the Lusztig's construction of a universal R-matrix.

Theorem 3.1.10. *When $M \neq N$, the map $\pi : \mathcal{U}(\epsilon) \longrightarrow \mathrm{U}(\epsilon)$ is an isomorphism.*

Proof. As above, we define $\mathrm{U}(\epsilon)[\sigma]$ and consider the algebra maps

$$U(\epsilon)[\sigma] \xrightarrow{\tau} \mathcal{U}(\epsilon)[\sigma] \xrightarrow{\pi} \mathrm{U}(\epsilon)[\sigma]$$

where we use the same symbol π to denote the obvious extension of π.

As in Proposition 3.1.2, $\mathcal{U}(\epsilon)[\sigma]$ also has the triangular decomposition

$$\mathcal{U}(\epsilon)[\sigma] \cong \mathcal{U}(\epsilon)^+ \otimes \mathcal{U}(\epsilon)^0[\sigma] \otimes \mathcal{U}(\epsilon)^-$$

where $\mathcal{U}(\epsilon)^0[\sigma]$ is the (semi)direct product of $\mathcal{U}(\epsilon)^0$ and Σ. One can show that $U(\epsilon)$ and $\mathrm{U}(\epsilon)$ possess similar decompositions, and π respects the decomposition. Hence, it is sufficient to prove that the restrictions

$$\pi|_{\mathcal{U}(\epsilon)^\pm} : \mathcal{U}(\epsilon)^\pm \longrightarrow \mathrm{U}(\epsilon)^\pm$$

are injective.

Suppose we are given $y \in \mathcal{U}(\epsilon)^+ \cap \ker \pi$, which can be assumed to be homogeneous as $\pi|_{\mathcal{U}(\epsilon)^+}$ preserves the Q_+-grading. Let us also assume $\beta := |y|$ has the minimal height such that $\mathcal{U}(\epsilon)^+_\beta \cap \ker \pi \neq 0$.

Since τ is an isomorphism, there exists a unique $x \in U(\epsilon)^+_\beta$ and a monomial ς in σ_j such that $\tau(x\varsigma) = y$. We claim that $x = 0$, which follows by [94, Proposition 6.5.1] once we check that

$$xF_i - (-1)^{p(i)p(\beta)} F_i x = 0 \quad \text{for all } i \in I.$$

Indeed, using the defining relations of $U(\epsilon)$ we can express

$$xF_i - (-1)^{p(i)p(\beta)} F_i x = x' K_{\alpha_i} + x'' K_{-\alpha_i}$$

for some x', $x'' \in U(\epsilon)^+_{\beta-\alpha_i}$. Write $\tau(x') = y'\sigma'$ and $\tau(x'') = y''\sigma''$ for some y', $y'' \in \mathcal{U}(\epsilon)^+_{\beta-\alpha_i}$ and σ', $\sigma'' \in \Sigma$. Applying $\pi \circ \tau$ to the above identity, we get

$$0 = \pi(y')\mathrm{K}_{\alpha_i}\varsigma' + \pi(y'')\mathrm{K}_{-\alpha_i}\varsigma''$$

for some ς', $\varsigma'' \in \Sigma \setminus \{0\}$ since $\pi\tau(x) = \pi(y)\varsigma^{-1} = 0$. By the triangular decomposition of $\mathrm{U}(\epsilon)[\sigma]$, we obtain $\pi(y')\mathrm{K}_{\alpha_i}\varsigma' = \pi(y'')\mathrm{K}_{-\alpha_i}\varsigma'' = 0$, or equivalently $\pi(y') = \pi(y'') = 0$. Then the minimality of $\mathrm{ht}(\beta)$ implies $y' = y'' = 0$ and hence $\tau(xF_i - (-1)^{p(i)p(\beta)} F_i x) = 0$. Since τ is an isomorphism, this completes the proof. □

As a consequence, when $M \neq N$ we obtain algebra isomorphisms

$$^{\pm} : \mathbf{f}(\epsilon) := {}'\mathbf{f}(\epsilon)/\mathcal{I} \longrightarrow \mathcal{U}(\epsilon)^{\pm}$$

defined by $\theta_i^+ = e_i$, $\theta_i^- = f_i$ ($i \in I$). For each $\beta \in Q_+$, take a basis B_β of $\mathbf{f}(\epsilon)_\beta$ and its dual basis $B^*_\beta = \{b^* \,|\, b \in B_\beta\}$ with respect to the nondegenerate bilinear form on $\mathbf{f}(\epsilon)$. Put

$$\Theta_\beta = (q - q^{-1})^{\mathrm{ht}(\beta)} \sum_{b \in B_\beta} b^+ \otimes (b^*)^- \in \mathcal{U}(\epsilon)^+_\beta \otimes \mathcal{U}(\epsilon)^-_{-\beta}$$

with $\Theta_0 := 1 \otimes 1$.

Let us also take a completion

$$\mathcal{U}(\epsilon)^+ \widehat{\otimes} \mathcal{U}(\epsilon)^- = \bigoplus_{\gamma \in Q} \prod_{\gamma = \alpha + \beta} \mathcal{U}(\epsilon)^+_\alpha \otimes \mathcal{U}(\epsilon)^-_\beta .$$

Now the following theorem can be proved in a similar way with [78, Theorem 4.1.2].

Theorem 3.1.11. *The element*

$$\Theta = \sum_{\beta \in Q_+} \Theta_\beta \in \mathcal{U}(\epsilon)^+ \widehat{\otimes} \mathcal{U}(\epsilon)^-$$

satisfies the following properties.

(1) *For any $u \in \mathcal{U}(\epsilon)$, we have*

$$\Theta\Delta(u) = \overline{\Delta}(u)\Theta$$

where $\overline{\Delta}(u) \coloneqq \overline{\Delta(\overline{u})}$. Moreover, this property uniquely determines Θ_β.

(2) *Let $\overline{\Theta} = \sum \overline{\Theta}_\beta$ where $\overline{\Theta}_\beta \coloneqq (^- \otimes {}^-)(\Theta_\beta)$. Then $\overline{\Theta}\Theta = \Theta\overline{\Theta} = 1$ holds.*

We call the element Θ a *universal R-matrix.* In Section 4.2.2, we will use this element to construct a $\mathcal{U}(\epsilon)$-linear map from a tensor product of two $\mathcal{U}(\epsilon)$-modules to the opposite tensor product.

3.2 Finite type subalgebra and its polynomial representations

Definition 3.2.1. The generalized quantum group $\mathring{\mathcal{U}}(\epsilon)$ of *finite type A* is defined as the subalgebra of $\mathcal{U}(\epsilon)$ generated by e_i, f_i $(i \in I \setminus \{0\})$, k_μ $(\mu \in P_{\text{fin}})$.

Similarly defined subalgebra $\mathring{U}(\epsilon)$ of $U(\epsilon)$ is isomorphic to the quantum group $U_q(\mathfrak{gl}_{M|N})$ associated to the Lie superalgebra $\mathfrak{gl}_{M|N}$ with a Borel subalgebra labeled by ϵ. Since the isomorphism τ in Theorem 3.1.6 restricts to the one between $\mathring{\mathcal{U}}(\epsilon)$ and $\mathring{U}(\epsilon)$ (up to the extension by Σ), there exists a concrete connection between representations of $\mathring{\mathcal{U}}(\epsilon)$ and those of $U_q(\mathfrak{gl}_{M|N})$. In this context, let us give a quick review on certain finite-dimensional representations which appear in later chapters, called *polynomial representations*, of $\mathring{\mathcal{U}}(\epsilon)$.

For a $\mathring{\mathcal{U}}(\epsilon)$-module V and $\lambda \in P_{\text{fin}}$, define the λ-weight space of V by

$$V_\lambda = \{ v \in V \mid k_\mu v = \mathbf{q}(\lambda, \mu) v \text{ for } \mu \in P_{\text{fin}} \} .$$

Clearly, $e_i V_\lambda \subset V_{\lambda+\alpha_i}$ and $f_i V_\lambda \subset V_{\lambda-\alpha_i}$. We denote by $\mathrm{wt}(V)$ the set of weights of V. Since the Cartan parts of $\mathcal{U}(\epsilon)$ and $\mathring{\mathcal{U}}(\epsilon)$ are the same, we define the notion of weights of $\mathcal{U}(\epsilon)$-modules in exactly the same way.

Remark 3.2.2. For a $U_q(\mathfrak{gl}_{M|N})$-module W, the λ-weight space of W is defined by

$$W_\lambda = \left\{ w \in W \mid K_\mu w = q^{(\mu|\lambda)} w \text{ for } \mu \in P_{\mathrm{fin}} \right\}.$$

It agrees with our notion for $\mathring{\mathcal{U}}(\epsilon)$-modules via the isomorphism τ. Namely, let V be a $\mathring{\mathcal{U}}(\epsilon)$-module with the weight space decomposition $V = \bigoplus_{\lambda \in P_{\mathrm{fin}}} V_\lambda$. We extend the $\mathring{\mathcal{U}}(\epsilon)$-action on V to a $\mathring{\mathcal{U}}(\epsilon)[\sigma]$-action by assigning

$$\sigma_j v = (-1)^{\epsilon_j (\delta_j|\lambda)} v$$

for $j \in \mathbb{I}$ and $v \in V_\lambda$. Pulling it back through τ, we obtain a $\mathring{U}(\epsilon)[\sigma]$-module structure on V. Then the subspace V_λ is the λ-weight space under the $\mathring{U}(\epsilon)$-action. Indeed, for $v \in V_\lambda$ with $\lambda = \sum c_j \delta_j$ and $\mu = \sum d_j \delta_j \in P_{\mathrm{fin}}$, we have

$$K_\mu v = \tau(K_\mu) v = k_\mu \prod_{j \in \mathbb{I}} \sigma_j^{d_j} v = (-1)^{\sum \epsilon_j c_j d_j} k_\mu v = (-1)^{\sum \epsilon_j c_j d_j} \mathbf{q}(\mu, \lambda) v = q^{(\mu|\lambda)} v.$$

Let $\mathring{\mathcal{C}}(\epsilon)$ be the category of finite-dimensional polynomial representations of $\mathring{\mathcal{U}}(\epsilon)$. By definition, $\mathring{\mathcal{C}}(\epsilon)$ consists of finite-dimensional $\mathring{\mathcal{U}}(\epsilon)$-modules V with a weight space decomposition

$$V = \bigoplus_{\lambda \in P_{\geq 0}} V_\lambda.$$

We also denote by $\mathring{\mathcal{C}}^\ell(\epsilon)$ the full subcategory of $\mathring{\mathcal{C}}(\epsilon)$ of V such that every weight of V is of degree ℓ. Then we have

$$\mathring{\mathcal{C}}(\epsilon) = \bigoplus_{\ell \geq 0} \mathring{\mathcal{C}}^\ell(\epsilon).$$

The irreducible modules in $\mathring{\mathcal{C}}(\epsilon)$ are classified by their highest weights, which are parametrized by hook partitions. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is called an $(M|N)$-hook partition if $\lambda_{M+1} \leq N$, and let $\mathscr{P}_{M|N}$ denote the set of $(M|N)$-hook partitions.

To each $\lambda \in \mathscr{P}_{M|N}$, we assign a finite-dimensional irreducible highest weight $\mathring{\mathcal{U}}(\epsilon)$-module $V_\epsilon(\lambda)$ as follows (*cf.* [24, Section 2.4.1]). Define a tableau $H_{\lambda,\epsilon}$ of shape λ by the following recursive rule:

(1) Fill the first row (resp. column) with 1 if $\epsilon_1 = 0$ (resp. $\epsilon_1 = 1$).

(2) After filling a subdiagram μ of λ with $1, 2, \ldots, k$, fill the first row (resp. column) of λ/μ with $k+1$ if $\epsilon_{k+1} = 0$ (resp. $\epsilon_{k+1} = 1$).

Then $V_\epsilon(\lambda)$ is defined to have the highest weight $\sum_{i \in \mathbb{I}} m_i \delta_i$, where m_i is the number of i's in $H_{\lambda,\epsilon}$.

Example 3.2.3. The $\mathring{\mathcal{U}}(\epsilon)$-module $V_\epsilon((1))$ can be seen as a quantum analogue of the natural representation $\mathbb{C}^{M|N}$ of $\mathfrak{gl}_\epsilon$. Indeed, consider the n-dimensional $\Bbbk$-vector space

$$\mathcal{V} = \bigoplus_{i \in \mathbb{I}} \Bbbk \left|\mathbf{e}_i\right\rangle$$

on which $\mathring{\mathcal{U}}(\epsilon)$ acts by

$$k_\mu \left|\mathbf{e}_i\right\rangle = \mathbf{q}(\mu, \delta_i), \quad e_k \left|\mathbf{e}_i\right\rangle = \delta_{i,k+1} \left|\mathbf{e}_k\right\rangle, \quad f_k \left|\mathbf{e}_i\right\rangle = \delta_{i,k} \left|\mathbf{e}_{k+1}\right\rangle.$$

Then it is easy to check that $\mathcal{V}$ is an irreducible highest weight $\mathring{\mathcal{U}}(\epsilon)$-module with highest weight δ_1, hence isomorphic to $V_\epsilon((1))$.

Furthermore, it is known [7,75,77] that for each $\ell \geq 1$, $\mathcal{V}^{\otimes \ell}$ is semisimple and its simple components are exactly those $V_\epsilon(\lambda)$ for the $(M|N)$-hook partitions λ of ℓ. In particular, any tensor product of $V_\epsilon(\lambda)$'s is again semisimple, and the composition multiplicities are given by the usual Littlewood-Richardson coefficients. For example, we have

$$V_\epsilon((l)) \otimes V_\epsilon((m)) \cong \bigoplus_{t \in H_\epsilon(l,m)} V_\epsilon((l+m-t, t)) \tag{3.2.1}$$

where $H_\epsilon(l,m) = \{t \,|\, 0 \leq t \leq \min\{l,m\},\ (l+m-t,t) \in \mathscr{P}_{M|N}\}$. Note that this index set is nothing but

$$H_\epsilon(l,m) = \begin{cases} \{\max\{l+m-n,0\}, \max\{l+m-n,0\}+1, \ldots, \min\{l,m\}\} & \text{if } \epsilon = \epsilon_{0|n} \\ \{0,1,\ldots,\min\{l,m,n-1\}\} & \text{if } \epsilon = \epsilon_{1|n-1} \\ \{0,1,\ldots,\min\{l,m\}\} & \text{otherwise} \end{cases} \tag{3.2.2}$$

(recall that we are assuming $n \geq 4$).

Remark 3.2.4. The character of $V_\epsilon(\lambda)$ is given by a hook Schur polynomial [8]. When $\epsilon = \epsilon_{M|N}$, the existence of crystal base is also known, which is again explained in terms of semistandard Young tableaux [7, 77].

At the heart of the above results lies the following Schur–Weyl-type duality.

Theorem 3.2.5 ([77, Theorem 3.1]). *Let $\mathcal{V} = V_\epsilon((1))$ and $\mathcal{R} : \mathcal{V}^{\otimes 2} \longrightarrow \mathcal{V}^{\otimes 2}$ be a $\mathring{\mathcal{U}}(\epsilon)$-linear map defined by*

$$\mathcal{R}\left(|\mathbf{e}_i\rangle \otimes |\mathbf{e}_j\rangle\right) = \begin{cases} qq_i\,|\mathbf{e}_i\rangle \otimes |\mathbf{e}_i\rangle & \text{if } i = j \\ q\,|\mathbf{e}_j\rangle \otimes |\mathbf{e}_i\rangle & \text{if } i > j \\ (q^2-1)\,|\mathbf{e}_i\rangle \otimes |\mathbf{e}_j\rangle + q\,|\mathbf{e}_j\rangle \otimes |\mathbf{e}_i\rangle & \text{if } i < j. \end{cases}$$

Let $H_\ell(q^2)$ be the finite Hecke algebra (see Definition 4.3.10). Then $\mathcal{V}^{\otimes \ell}$ is a $(\mathring{\mathcal{U}}(\epsilon), H_\ell(q^2))$-bimodule whose right $H_\ell(q^2)$-action is given by $h_m = \mathcal{R}_m$, where $\mathcal{R}_m$ denotes the map given by applying $\mathcal{R}$ on the m-th and $(m+1)$-st factors of $\mathcal{V}^{\otimes \ell}$.

Furthermore, the functor

$$\begin{aligned} \mathcal{J}_\ell : H_\ell(q^2)\text{-mod} &\longrightarrow \mathring{\mathcal{C}}^\ell(\epsilon) \\ M &\longmapsto \mathcal{V}^{\otimes \ell} \otimes_{H_\ell(q^2)} M \end{aligned}$$

is an equivalence of categories if $\ell \leq n$.

Chapter 4

Super duality for polynomial representations

We begin our study of representations of quantum affine superalgebras from polynomial representations of $\mathcal{U}(\epsilon)$. The main result is an equivalence between monoidal categories $\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty})$ obtained as an inverse limit of categories $\mathcal{C}_{\mathbb{Z}}(\epsilon^{(k)})$ of polynomial representations of $\mathcal{U}(\epsilon^{(k)})$, for any (01)-sequence ϵ^{∞} of infinite length. In particular, from the equivalence with $\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})$ (or $\mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty})$) where $\overline{\epsilon}^{\infty} = (1^{\infty})$, we obtain a concrete connection between polynomial representations of quantum affine algebras and those of superalgebras. Such a super-duality-type equivalence (Theorem 4.3.28) is depicted by the following diagram:

$$
\begin{array}{ccccc}
 & & \mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}) & & \\
 & \overset{\mathcal{S}_{\infty|0}}{\swarrow}\ \simeq & & \simeq\ \overset{\mathcal{S}_{0|\infty}}{\searrow} & \\
\mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty}) & & & & \mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty}).
\end{array}
$$

For finite-dimensional representations of quantum affine algebras, the pioneering work [58] teaches us that fundamental representations and their normalized R-matrices are building blocks, and information on tensor product structure can be extracted from singularities (or spectral decomposition) of normalized R-matrices.

In the spirit of [58], we introduce fundamental representations $\mathcal{W}_{l,\epsilon}(x)$ in the category $\mathcal{C}(\epsilon)$ of polynomial representations of $\mathcal{U}(\epsilon)$, and construct normalized R-matrices $\mathcal{R}^{\mathrm{norm}}_{l,m}$. The spectral decomposition of $\mathcal{R}^{\mathrm{norm}}_{l,m}$ is computed, which is observed to be the same as the one in non-super cases. This allows us to generalize to arbitrary ϵ in a uniform manner, several important constructions from the representation theory of quantum affine algebras, such as fusion construction of irreducibles [53] and generalized quantum affine Schur–Weyl duality functor $\mathcal{F}_\epsilon$ [51].

To turn this analogy into a mathematical entity, we introduce truncation functors that relate $\mathcal{C}(\epsilon)$ for various ϵ. They are motivated from the super duality formalism for Lie algebras, and accordingly expected to be equivalences of categories, after taking an infinite rank limit. We prove that the degree ℓ component of the Schur–Weyl-type duality functor $\mathcal{F}_{\epsilon,\ell}$ is an equivalence whenever $n > \ell$, namely at high ranks. Using the notion of inverse limit categories, we lift those partial equivalences to monoidal equivalences between inverse limit categories $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$, which we interpret as a quantum affine analogue of a super duality.

Since the existence of the universal R-matrix is only known when $M \neq N$ (Theorem 3.1.11), we put

$$\mathcal{E}_r = \{\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \{0,1\}^r \mid |\{i|\epsilon_i = 0\}| \neq |\{i|\epsilon_i = 1\}|\}, \quad \mathcal{E} = \bigcup_{r \geq 4} \mathcal{E}_r,$$

and tacitly assume that ϵ is in $\mathcal{E}$. We also endow $\mathcal{E}$ with a partial order

$$\epsilon' < \epsilon \iff \epsilon' \text{ is a proper subsequence of } \epsilon.$$

The results of this chapter are based on [72].

4.1 Super duality for polynomial representations of $\mathfrak{gl}_n$

The category $\mathcal{F}_n$ of polynomial representations of $\mathfrak{gl}_n$ is semisimple, whose simple objects are parametrized by partitions λ of length not larger than n. Namely, we identify such a partition λ with a dominant weight

$$\lambda = \sum_{i=1}^{n} \lambda_i \delta_i \in P^+_{\mathrm{fin}}$$

where $\{\delta_i\}_{i=1}^n \subset \mathfrak{h}^*$ is the dual basis of $\{E_{ii}\}_{i=1}^n \subset \mathfrak{h}$, and then $L_n(\lambda)$ is the irreducible highest weight $\mathfrak{gl}_n$-module with highest weight λ.

A large part of this category can be understood through an algebra isomorphism

$$\begin{aligned} K(\mathcal{F}_n) &\xrightarrow{\cong} \Lambda(x_1, \ldots, x_n) \\ [L_n(\lambda)] &\longmapsto s_\lambda \end{aligned}$$

obtained by taking the character. Here $\Lambda_n = \Lambda(x_1, \ldots, x_n)$ denotes the ring of symmetric polynomials in $x_1, \ldots, x_n$. For example, that $\{L_n(\lambda)\}_{\ell(\lambda)\leq n}$ is a complete set of simples corresponds to the fact that $\{s_\lambda\}_{\ell(\lambda)\leq n}$ is a $\mathbb{Z}$-basis of Λ_n, the tensor product decomposition of $L_n(\lambda) \otimes L_n(\mu)$ is given by the Littlewood-Richardson rule $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$, and so on.

On the combinatorics side Λ_n, it is more natural to consider its inverse limit, the ring of symmetric functions Λ in infinitely many indeterminates $x_1, x_2, \ldots$. For instance, the basis of Λ_n is parametrized by partitions with length not larger than n, rather than all the partitions. It leads to a degeneration: in Λ, the product $s_{(1^l)} s_{(1^m)}$ is computed by the Pieri's formula, but in Λ_n the summands corresponding to partitions of length larger than n are missing in the same product. In contrast, such a degeneration does not occur in the product of single-row partitions $s_{(l)} s_{(m)}$. Certainly there exists an asymmetry in Λ_n that does not appear in the limit Λ. Indeed, Λ enjoys an algebra involution $s_\lambda \mapsto s_{\lambda^t}$.

On the representation theory side, $s_{(1^l)}$ (resp. $s_{(l)}$) is the character of the exterior power $\Lambda^l(\mathbb{C}^n)$ (resp. symmetric power $S^l(\mathbb{C}^n)$), where $\mathbb{C}^n$ is the natural representation of $\mathfrak{gl}_n$. Hence, the asymmetry arises from the difference of symmetric and exterior power (or, bosonic and fermionic). This can be remedied by considering representations of infinite rank Lie algebra $\mathfrak{gl}_\infty$, which corresponds to the inverse limit Λ on the combinatorics side.

Now the symmetry $s_\lambda \leftrightarrow s_{\lambda^t}$ in Λ should be understood as an exchange of bosons and fermions, at least at the level of heuristics. Remarkably, this can be made into a mathematical theorem, as an equivalence of categories (after taking inverse limits). The idea is to introduce an intermediating Lie superalgebra $\mathfrak{gl}_{n|n}$ and two truncation functors which pick out only bosons or fermions. Then the infinite rank limit assures that nothing is lost in the course of truncations, hence equivalences (see (1.2.1)). This method, developed in [20, 21], is called *super duality*, and yields an interesting and useful perspective on representation theory of Lie (super)algebras. In the remaining of this section, we explain how super duality is constructed in this easiest example.

Introduce the following index set

$$\widetilde{\mathbb{I}} = \left\{ \frac{1}{2} < 1 < \frac{3}{2} < 2 < \cdots \right\}$$

with $\mathbb{Z}/2\mathbb{Z}$-grading given by $|i| = 2i$. Let $\widetilde{V}$ be the infinite-dimensional vector superspace over $\mathbb{C}$ with a basis $\{v_i\}_{i\in\widetilde{\mathbb{I}}}$ with the induced $\mathbb{Z}/2\mathbb{Z}$-grading. We also set

$$\mathbb{I} = \widetilde{\mathbb{I}} \cap \mathbb{Z}, \quad \overline{\mathbb{I}} = \widetilde{\mathbb{I}} \cap \left(\frac{1}{2} + \mathbb{Z}\right)$$

and V (resp. $\overline{V}$) is defined to be the subspace of $\widetilde{V}$ spanned by v_i for $i \in \mathbb{I}$ (resp. $i \in \overline{\mathbb{I}}$).

Let $\widetilde{\mathfrak{g}} = \mathrm{End}(\widetilde{V})$ be the Lie superalgebra of linear endomorphisms on $\widetilde{V}$, with the standard basis $\{E_{rs}\}_{r,s\in\widetilde{\mathbb{I}}}$. Namely, E_{rs} is defined to be the linear map $v_i \mapsto \delta_{i,s} v_r$, whose parity is given by $|r| + |s|$. The Lie superalgebra $\widetilde{\mathfrak{g}}$ is the infinite rank general linear Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$, accompanied with the following data:

- Cartan subalgebra $\widetilde{\mathfrak{h}} = \bigoplus_{i\in\widetilde{\mathbb{I}}} \mathbb{C}E_{ii}$,
- Borel subalgebra $\widetilde{\mathfrak{b}} = \bigoplus_{r\leq s} \mathbb{C}E_{rs}$.

Observe that this Borel subalgebra corresponds to a (infinite) (01)-sequence (01010 . . .), and hence not a standard one.

We take $\mathfrak{g} = \mathrm{End}(V)$, $\overline{\mathfrak{g}} = \mathrm{End}(\overline{V})$ which are regarded as subalgebras of $\widetilde{\mathfrak{g}}$ naturally, and corresponding Cartan $\mathfrak{h}, \overline{\mathfrak{h}}$ and Borel $\mathfrak{b}, \overline{\mathfrak{b}}$. Both $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ are isomorphic to the Lie algebra $\mathfrak{gl}_\infty$, not super.

Next, we define module categories. Given $\lambda \in \mathscr{P}$, we define weights

$$\lambda = \sum_{i\in\mathbb{I}} \lambda_i\delta_i \in \mathfrak{h}^*, \quad \lambda^\natural = \sum_{s\in\overline{\mathbb{I}}} \lambda^t_{s+\frac{1}{2}}\delta_s \in \overline{\mathfrak{h}}^*,$$
$$\lambda^\theta = \sum_{r\in\widetilde{\mathbb{I}}} \theta(\lambda)_r\delta_r \in \widetilde{\mathfrak{h}}^*$$

where $\theta(\lambda) = \left(\theta(\lambda)_{\frac{1}{2}}, \theta(\lambda)_1, \theta(\lambda)_{\frac{3}{2}}, \dots\right)$ is defined by[1]

$$\theta(\lambda)_j = \max\{\lambda_j - j, 0\}, \quad \theta(\lambda)_{j-\frac{1}{2}} = \max\{\lambda^t_j - j + 1, 0\} \quad (j \in \mathbb{Z}).$$

Let $\widetilde{L}(\lambda)$ be the irreducible highest weight $\widetilde{\mathfrak{g}}$-module with highest weight λ^θ with respect to the Borel subalgebra $\widetilde{\mathfrak{b}}$, and similary $L(\lambda)$ over $\mathfrak{g}$ and $\overline{L}(\lambda)$ over $\overline{\mathfrak{g}}$. Hence, a partition λ simultaneously parametrizes irreducible representations of three different Lie

[1] Equivalently, λ^θ, λ, $\lambda^\natural$ are the weights of the tableaux $H_{\lambda,\epsilon}$ defined in Section 4.2 with respect to $\epsilon = (10101...)$, $(000...)$, $(111...)$ respectively.

(super)algebras. Also note that $\mathrm{ch}L(\lambda) = s_\lambda$ and $\mathrm{ch}\overline{L}(\lambda) = s_{\lambda^t}$, while $\widetilde{L}(\lambda)$ interpolates the correspondence $L(\lambda) \leftrightarrow \overline{L}(\lambda)$ (which amounts to $s_\lambda \leftrightarrow s_{\lambda^t}$).

The category $\widetilde{\mathcal{F}}$ is defined to be the category of $\widetilde{\mathfrak{g}}$-modules V such that

(1) $V = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} V_\lambda$ with $\dim V_\lambda < \infty$ and $\mathrm{wt}(V)$ is finitely dominated,

(2) V is a direct sum of $\widetilde{L}(\lambda)$'s for $\lambda \in \mathscr{P}$.

Similarly we define $\mathcal{F}$, $\overline{\mathcal{F}}$ for $\mathfrak{g}, \overline{\mathfrak{g}}$, respectively. In this case, $\mathcal{F} = \overline{\mathcal{F}}$ and is the category of polynomial representations of $\mathfrak{gl}_\infty$.

Finally, we define the truncation functors that relate the module categories introduced. Given a $\widetilde{\mathfrak{g}}$-module $\widetilde{V}$ with a weight space decomposition $\widetilde{V} = \bigoplus_{\gamma \in \widetilde{\mathfrak{h}}^*} \widetilde{V}_\gamma$, we form the subspaces

$$\mathfrak{tr}(\widetilde{V}) = \bigoplus_{\gamma \in \mathfrak{h}^*} \widetilde{V}_\gamma, \quad \overline{\mathfrak{tr}}(\widetilde{V}) = \bigoplus_{\gamma \in \overline{\mathfrak{h}}^*} \widetilde{V}_\gamma.$$

Then $\mathfrak{tr}(\widetilde{V})$ (resp. $\overline{\mathfrak{tr}}(\widetilde{V})$) is closed under the action of the subalgebra $\mathfrak{g}$ (resp. $\overline{\mathfrak{g}}$). Moreover, for a $\widetilde{\mathfrak{g}}$-module homomorphism $f : \widetilde{V} \to \widetilde{W}$, we obtain by restriction a $\mathfrak{g}$-linear map $\mathfrak{tr}(f) : M \to N$ and a $\overline{\mathfrak{g}}$-linear map $\overline{\mathfrak{tr}}(f) : \overline{M} \to \overline{N}$. Then using odd reflections and comparing characters, one can prove that

$$\mathfrak{tr}\left(\widetilde{L}(\lambda)\right) = L(\lambda), \quad \overline{\mathfrak{tr}}\left(\widetilde{L}(\lambda)\right) = \overline{L}(\lambda)$$

for any $\lambda \in \mathscr{P}$. Therefore, we obtain functors

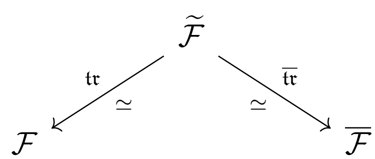
$$\mathfrak{tr} : \widetilde{\mathcal{F}} \longrightarrow \mathcal{F}, \quad \overline{\mathfrak{tr}} : \widetilde{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}$$

called truncations.

Now super duality asserts that

Theorem 4.1.1 ([20])**.** *The truncations* $\mathfrak{tr}$, $\overline{\mathfrak{tr}}$ *are equivalences of highest weight categories.*

Hence we obtain the following diagram:



and a nontrivial autoequivalence $\overline{\mathfrak{tr}} \circ (\mathfrak{tr})^{-1}$ on $\mathcal{F} = \overline{\mathcal{F}}$. At the level of Grothendieck ring $K(\mathcal{F})$, which is isomorphic to the ring of symmetric functions Λ via the character map, it induces the involution $s_\lambda \mapsto s_{\lambda^t}$. Thus, super duality provides a *categorification* of this important symmetry on Λ.

The goal of this chapter is to establish a quantum affine analogue of this diagram.

4.2 Finite-dimensional representations of $\mathcal{U}(\epsilon)$

4.2.1 Fundamental representations

As explained in Section 3.2, the notion of polynomial representations of $\mathring{\mathcal{U}}(\epsilon)$ directly generalizes to $\mathcal{U}(\epsilon)$. In accordance with Chapter 5, let us put it in the following way.

Definition 4.2.1. Let $\mathcal{C}(\epsilon)$ (resp. $\mathcal{C}^\ell(\epsilon)$) be the category of $\mathcal{U}(\epsilon)$-modules that belong to $\mathring{\mathcal{C}}(\epsilon)$ (resp. $\mathring{\mathcal{C}}^\ell(\epsilon)$) as $\mathring{\mathcal{U}}(\epsilon)$-modules.

Again we have

$$\mathcal{C}(\epsilon) = \bigoplus_{\ell \geq 0} \mathcal{C}^\ell(\epsilon).$$

Let us introduce a family of $\mathcal{U}(\epsilon)$-modules that play fundamental roles in the study of polynomial representations of $\mathcal{U}(\epsilon)$. Consider a supersymmetric Fock space

$$\mathcal{W}_\epsilon = \bigoplus_{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon)} \Bbbk \left| \mathbf{m} \right\rangle,$$

where

$$\mathbb{Z}_+^n(\epsilon) = \{\mathbf{m} = (m_1, \ldots, m_n) \,|\, m_i \in \mathbb{Z}_{\geq 0} \text{ if } \epsilon_i = 0,\, m_i \in \{0,1\} \text{ if } \epsilon_i = 1\}.$$

This space carries a natural $\mathcal{U}(\epsilon)$-action, with an arbitrary choice of $x \in \Bbbk^\times$, given by[2]

$$\begin{aligned} k_\mu \left| \mathbf{m} \right\rangle &= \mathbf{q}(\mu, \sum_{j \in \mathbb{I}} m_j \delta_j) \left| \mathbf{m} \right\rangle, \\ e_i \left| \mathbf{m} \right\rangle &= x^{\delta_{i0}} [m_{i+1}] \left| \mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} \right\rangle, \\ f_i \left| \mathbf{m} \right\rangle &= x^{-\delta_{i0}} [m_i] \left| \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} \right\rangle, \end{aligned}$$

[2]As a rule, we always assume $\left| \mathbf{m} \right\rangle = 0$ unless $\mathbf{m} \in \mathbb{Z}_+^n(\epsilon)$.

for $i \in I$ and $\mu \in P_{\mathrm{fin}}$. We denote this $\mathcal{U}(\epsilon)$-module by $\mathcal{W}_\epsilon(x)$.

Proposition 4.2.2. *The $\mathcal{U}(\epsilon)$-module $\mathcal{W}_\epsilon(x)$ has the following direct sum decomposition*

$$\mathcal{W}_\epsilon(x) = \bigoplus_{l \geq 0} \mathcal{W}_{l,\epsilon}(x), \quad \mathcal{W}_{l,\epsilon}(x) = \bigoplus_{|\mathbf{m}|=l} \Bbbk\, |\mathbf{m}\rangle$$

where $|\mathbf{m}| = \sum m_i$. Moreover, each $\mathcal{W}_{l,\epsilon}(x)$ is irreducible over $\mathcal{U}(\epsilon)$.

Definition 4.2.3. For $l \geq 0$, the $\mathcal{U}(\epsilon)$-module $\mathcal{W}_{l,\epsilon}(x)$ is called the *l-th fundamental representation with spectral parameter x.*

When $\epsilon = \epsilon_{0|n}$ and $0 \leq l \leq n$, $\mathcal{W}_{l,\epsilon}(x)$ is isomorphic to the l-th fundamental representation[3] $V(\varpi_l)$ over $U'_{-q^{-1}}(\widehat{\mathfrak{gl}}_n)$. At the other extreme $\epsilon = \epsilon_{n|0}$, $\mathcal{W}_{l,\epsilon}(x)$ becomes the Kirillov-Reshetikhin module (see Remark 4.3.16) over $U'_q(\widehat{\mathfrak{gl}}_n)$ corresponding to a single-row partition (l). Thus, our $\mathcal{W}_{l,\epsilon}(x)$ interpolates two most important finite-dimensional representations of quantum affine algebras of type A, while one is fermionic and the other is bosonic.

The following proposition records basic properties of fundamental representations.

Proposition 4.2.4. *The following properties hold.*

(1) *As a $\mathring{\mathcal{U}}(\epsilon)$-module, $\mathcal{W}_{l,\epsilon}(x) \cong V_\epsilon((l))$ and hence $\mathcal{W}_{l,\epsilon}(x) \in \mathcal{C}(\epsilon)$.*

(2) *For any $x, y \in \Bbbk^\times$, a tensor product $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$, as a $\mathring{\mathcal{U}}(\epsilon)$-module, is semisimple and decomposes into*

$$\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \cong \bigoplus_{t \in H_\epsilon(l,m)} V_\epsilon((l+m-t, t)). \tag{4.2.1}$$

Definition 4.2.5. The category $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ is defined to be the monoidal Serre subcategory of $\mathcal{C}(\epsilon)$ generated by $\mathcal{W}_{l,\epsilon}(q^{2n+l+1})$ for all $l \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$.

We also put $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon) \coloneqq \mathcal{C}_{\mathbb{Z}}(\epsilon) \cap \mathcal{C}^\ell(\epsilon)$, so that $\mathcal{C}_{\mathbb{Z}}(\epsilon) = \bigoplus \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon)$.

In other words, $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ is the smallest full subcategory containing all $\mathcal{W}_{l,\epsilon}(q^{2n+l+1})$ such that it is closed under taking subobjects, quotients, extensions and tensor products. Note

[3] Recall that we have set $V(\varpi_0) = V(\varpi_n) = \Bbbk$ in Example 2.2.3.

that each $\mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon)$ is not closed under tensor products. Rather, the degree is additive in taking tensor product, that is

$$V \otimes W \in \mathcal{C}_{\mathbb{Z}}^{\ell+\ell'}(\epsilon) \quad \text{if } V \in \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon),\, W \in \mathcal{C}_{\mathbb{Z}}^{\ell'}(\epsilon).$$

The following lemma tells us that $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ is already generated by $\mathcal{W}_{1,\epsilon}(q^{2n})$ for $n \in \mathbb{Z}$. Note that when $\epsilon = (1^n)$, the exact sequence below recovers the one in [1, Lemma B.1].

Lemma 4.2.6. *For $\ell \geq 2$, we have the following exact sequence:*

$$0 \to \mathcal{W}_{\ell,\epsilon}(1) \to \mathcal{W}_{1,\epsilon}(q^{1-\ell}) \otimes \mathcal{W}_{\ell-1,\epsilon}(q) \xrightarrow{R} \mathcal{W}_{\ell-1,\epsilon}(q) \otimes \mathcal{W}_{1,\epsilon}(q^{1-\ell}) \to \mathcal{W}_{\ell,\epsilon}(1) \to 0.$$

Proof. The middle map is given by the normalized R-matrix that will be introduced in the next subsection, and the other two are given explicitly. Then the exactness follows from the spectral decomposition of R-matrix. See Section 6.1.1 for detailed proof. □

From the discussion on R-matrices in the next subsection, it will become clear that to understand the structure of tensor products of fundamental representations, it is sufficient to consider the subcategory $\mathcal{C}_{\mathbb{Z}}(\epsilon)$. Moreover, it is well-known that every irreducible polynomial representation of $\mathfrak{gl}_{M|N}$ appears as a composition factor of tensor powers of the natural representation $\mathbb{C}^{M|N}$. Since the first fundamental representation $\mathcal{W}_{1,\epsilon}(x)$ is its quantum affine analogue, the study of polynomial representations of $\mathcal{U}(\epsilon)$ essentially reduces to the one of $\mathcal{C}_{\mathbb{Z}}(\epsilon)$.

Remark 4.2.7. Suppose $\epsilon = \epsilon_{0|n}$ and let us restrict ourselves to modules over $U'_{\widetilde{q}}(\widehat{\mathfrak{sl}}_n)$ ($\widetilde{q} = -q^{-1}$). The corresponding category $\mathcal{C}_{\mathbb{Z}}$ is called Hernandez-Leclerc category or skeleton subcategory [41]. By the result of [58] (see Theorem 2.2.4), any finite-dimensional irreducible representation of $U'_q(\widehat{\mathfrak{sl}}_n)$ is a tensor product of spectral parameter shifts of irreducibles in the category $\mathcal{C}_{\mathbb{Z}}$. Since a finite-dimensional irreducible representation of $U'_q(\widehat{\mathfrak{gl}}_n)$ can be obtained as a tensor product of an irreducible polynomial representation and a one-dimensional representation, it is indeed enough to study $\mathcal{C}_{\mathbb{Z}}(\epsilon_{0|n})$ to understand finite-dimensional representations of $U'_q(\widehat{\mathfrak{gl}}_n)$.

On the other hand, if ϵ is not homogeneous, then there are far more finite-dimensional $\mathfrak{gl}_\epsilon$-modules than polynomial representations. Consequently, $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ does not cover all the finite-dimensional representations of $\mathcal{U}(\epsilon)$ (*cf.* [96, Proposition 4.15]).

4.2.2 R-matrix

Next, we introduce a $\mathcal{U}(\epsilon)$-linear map $\mathcal{R}^{\mathrm{norm}}_{l,m}$ on $\mathcal{W}_l(z_1) \otimes \mathcal{W}_m(z_2)$, called the normalized R-matrix. Then information on tensor product structure is contained in the spectral decomposition of $\mathcal{R}^{\mathrm{norm}}_{l,m}$, as is well-known in the non-super cases. This map is constructed by applying to the tensor product the universal R-matrix Θ constructed in Chapter 3. To obtain a well-defined map, we use the affinization technique to obtain a well-defined intertwiner (see Section 2.2.2), following [58] together with the standard construction [78].

For $V \in \mathcal{C}(\epsilon)$, we define the *affinization* of V as

$$V_{\mathrm{aff}} = \Bbbk[z^{\pm 1}] \otimes V$$

for an indeterminate z, which is also a $\mathcal{U}(\epsilon)$-module by

$$e_i = z^{\delta_{i0}} \otimes e_i, \quad f_i = z^{-\delta_{i0}} \otimes f_i, \quad k_\mu = 1 \otimes k_\mu.$$

As in non-super case, V_{aff} is P^0-graded by

$$(V_{\mathrm{aff}})_\lambda = z^k \otimes V_{\mathrm{cl}(\lambda)} \quad \text{for } \lambda = \iota \circ \mathrm{cl}(\lambda) + k\boldsymbol{\delta}$$

where $\iota : P_{\mathrm{fin}} \longrightarrow P^0$ is the section of $\mathrm{cl} : P^0 \longrightarrow P_{\mathrm{fin}}$ defined by $\iota(\delta_i) = \boldsymbol{\delta}_i$ for $i \in \mathbb{I}$. Then the multiplication by z can be understood as a degree $\boldsymbol{\delta}$ automorphism of V_{aff}, and we define for $x \in \Bbbk^\times$

$$V_x = V_{\mathrm{aff}}/(z - x)V_{\mathrm{aff}}.$$

For example, we have $\mathcal{W}_{l,\epsilon}(y)_x \cong \mathcal{W}_{l,\epsilon}(xy)$.

For V, $W \in \mathcal{C}(\epsilon)$, let us take a completion

$$V_{\mathrm{aff}} \widehat{\otimes} W_{\mathrm{aff}} = \sum_{\lambda,\mu \in P^0} \prod_{\beta \in Q_+} (V_{\mathrm{aff}})_{\lambda+\beta} \otimes (W_{\mathrm{aff}})_{\mu-\beta}$$

of the tensor product $V_{\mathrm{aff}} \otimes W_{\mathrm{aff}}$, and also the opposite completion

$$V_{\mathrm{aff}} \widetilde{\otimes} W_{\mathrm{aff}} = \sum_{\lambda,\mu \in P^0} \prod_{\beta \in Q_+} (V_{\mathrm{aff}})_{\lambda-\beta} \otimes (W_{\mathrm{aff}})_{\mu+\beta}$$

so that $\mathcal{U}(\epsilon)^+ \widehat{\otimes} \mathcal{U}(\epsilon)^-$ and $\mathcal{U}(\epsilon)^- \widetilde{\otimes} \mathcal{U}(\epsilon)^+$ act on $V_{\mathrm{aff}} \widehat{\otimes} W_{\mathrm{aff}}$ and $V_{\mathrm{aff}} \widetilde{\otimes} W_{\mathrm{aff}}$ respectively.

Observe that since the sets of weights of V and W are bounded above, we have

$$V_{\mathrm{aff}}\widehat{\otimes}W_{\mathrm{aff}} = \Bbbk\,[\![z_1/z_2]\!] \otimes_{\Bbbk[z_1/z_2]} (V_{\mathrm{aff}} \otimes W_{\mathrm{aff}}),$$
$$W_{\mathrm{aff}}\widetilde{\otimes}V_{\mathrm{aff}} = \Bbbk\,[\![z_1/z_2]\!] \otimes_{\Bbbk[z_1/z_2]} (W_{\mathrm{aff}} \otimes V_{\mathrm{aff}}).$$

where we write $V_{\mathrm{aff}} = V_{z_1}$, $W_{\mathrm{aff}} = W_{z_2}$.

Let $\Pi_{\mathbf{q}} : V_{\mathrm{aff}}\widehat{\otimes}W_{\mathrm{aff}} \longrightarrow V_{\mathrm{aff}}\widehat{\otimes}W_{\mathrm{aff}}$ be defined by

$$\Pi_{\mathbf{q}}(v \otimes w) = \mathbf{q}(\mathrm{cl}(\mu), \mathrm{cl}(\nu))v \otimes w$$

for $v \in (V_{\mathrm{aff}})_\mu$ and $w \in (W_{\mathrm{aff}})_\nu$, and $s : V_{\mathrm{aff}}\widehat{\otimes}W_{\mathrm{aff}} \longrightarrow W_{\mathrm{aff}}\widetilde{\otimes}V_{\mathrm{aff}}$ the flip $v \otimes w \mapsto w \otimes v$. Repeating the proof of [78, Theorem 32.1.5] replacing ${}_f\Pi$ there with $\Pi_{\mathbf{q}}$, we obtain an intertwiner between two completions.

Theorem 4.2.8. *We have an isomorphism of $\mathcal{U}(\epsilon)$-modules*

$$\mathcal{R}^{\mathrm{univ}}_{V,W} \coloneqq \Theta \circ \Pi_{\mathbf{q}} \circ s : V_{\mathrm{aff}}\widehat{\otimes}W_{\mathrm{aff}} \longrightarrow W_{\mathrm{aff}}\widetilde{\otimes}V_{\mathrm{aff}}.$$

Restricting the domain, we also obtain a $\mathcal{U}(\epsilon)$-linear map

$$\mathcal{R}^{\mathrm{univ}}_{V,W} : V_{\mathrm{aff}} \otimes W_{\mathrm{aff}} \longrightarrow \Bbbk\,[\![z_1/z_2]\!] \otimes_{\Bbbk[z_1/z_2]} (W_{\mathrm{aff}} \otimes V_{\mathrm{aff}}),$$

and either is called a *universal R-matrix* as well. They satisfy the following important property: for $M, N, L \in \mathcal{C}(\epsilon)$, the following diagrams

$$M \otimes N \otimes L \xrightarrow[\mathrm{id}_M \otimes \mathcal{R}^{\mathrm{univ}}_{N,L}]{} M \otimes L \otimes N \xrightarrow[\mathcal{R}^{\mathrm{univ}}_{M,L} \otimes \mathrm{id}_N]{} L \otimes M \otimes N, \quad M \otimes N \otimes L \xrightarrow{\mathcal{R}^{\mathrm{univ}}_{M \otimes N, L}} L \otimes M \otimes N \tag{4.2.2}$$

$$M \otimes N \otimes L \xrightarrow[\mathcal{R}^{\mathrm{univ}}_{M,N} \otimes \mathrm{id}_L]{} N \otimes M \otimes L \xrightarrow[\mathrm{id}_N \otimes \mathcal{R}^{\mathrm{univ}}_{M,L}]{} N \otimes L \otimes M, \quad M \otimes N \otimes L \xrightarrow{\mathcal{R}^{\mathrm{univ}}_{M, N \otimes L}} N \otimes L \otimes M \tag{4.2.3}$$

commute, where we omit affinizations and scalar extensions. It can be proved just as in the non-super case, see [78, Section 32.2]. Note also that the Yang-Baxter equation follows from these two diagrams.

Let us explain more explicitly on intertwiners between tensor products of fundamental representations. We change the base field to $\mathbf{k}$ temporarily, and let

$$\mathcal{R}^{\mathrm{univ}}_{l,m}(z_1,z_2) : \mathcal{W}_{l,\epsilon}(z_1)\otimes\mathcal{W}_{m,\epsilon}(z_2) \longrightarrow \mathbf{k}\,[\![z_1/z_2]\!] \otimes_{\mathbf{k}[z_1/z_2]} (\mathcal{W}_{m,\epsilon}(z_2)\otimes\mathcal{W}_{l,\epsilon}(z_1))$$

be the universal R-matrix for $\mathcal{W}_{l,\epsilon}$ and $\mathcal{W}_{m,\epsilon}$, where $z_1,\ z_2$ are indeterminates. Recall from (4.2.1) that as a $\mathring{\mathcal{U}}(\epsilon)$-module,

$$\mathcal{W}_{l,\epsilon}(x)\otimes\mathcal{W}_{m,\epsilon}(y) \cong \bigoplus_{t\in H_\epsilon(l,m)} V_\epsilon((l+m-t,t)) \quad (x,y\in\mathbf{k}^\times).$$

For $s=\max H_\epsilon(l,m)$, since $\mathcal{R}^{\mathrm{univ}}_{l,m}$ is invertible, we have

$$\mathcal{R}^{\mathrm{univ}}_{l,m}(z_1,z_2)|_{V_\epsilon((l+m-s,s))} = \varphi_{l,m}(z_1/z_2)\mathrm{id}_{V_\epsilon((l+m-s,s))}$$

for some nonzero $\varphi_{l,m}(z_1/z_2)\in\mathbf{k}\,[\![z_1/z_2]\!]$, by Schur's lemma. Put

$$c_t(z) = \prod_{i=t+1}^{\min\{l,m\}} \frac{1-q^{l+m-2i+2}z}{z-q^{l+m-2i+2}}$$

for $t\in H_\epsilon(l,m)$. We define the *normalized R-matrix* by

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(z) = \varphi_{l,m}(z)^{-1}c_s(z)\mathcal{R}^{\mathrm{univ}}_{l,m}(z)$$

where $z=z_1/z_2$. It is a unique $\mathbf{k}[z_1^{\pm1},z_2^{\pm1}]\otimes\mathcal{U}(\epsilon)$-linear map normalized by

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(z)|_{V_\epsilon((l+m-s,s))} = c_s(z)\mathrm{id}_{V_s},$$

because of the following irreducibility of tensor products of fundamental representations.

Theorem 4.2.9. *For $l,m\in\mathbb{Z}_{\geq0}$, the tensor product $\mathcal{W}_{l,\epsilon}(x)\otimes\mathcal{W}_{m,\epsilon}(y)$ is irreducible for generic $x,y\in\mathbf{k}^\times$.*

Proof. It follows from the irreducibility of $\mathcal{W}_{l,\epsilon}(1)\otimes\mathcal{W}_{m,\epsilon}(1)$ [77, Theorem 4.7] and a general commutative algebra argument [55, Lemma 3.4.2]. □

Take a projection onto the t-th classical component

$$\mathcal{P}_t^{l,m} : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \longrightarrow V_\epsilon((l+m-t,t)) \subset \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x).$$

This is of course defined up to a scalar multiple, and we normalize it by choosing a $\mathring{\mathcal{U}}(\epsilon)$-highest weight vector

$$v(l,m,t) \in V_\epsilon((l+m-t,t)) \subset \mathcal{W}_{l,\epsilon} \otimes \mathcal{W}_{m,\epsilon}.$$

which will be explained in Section 4.3.1. Then again by Schur's lemma, we can write

$$\mathcal{R}_{l,m}^{\mathrm{norm}}(z) = \sum_{t \in H_\epsilon(l,m)} \rho_t(z) \mathcal{P}_t^{l,m}$$

for some $\rho_t(z) \in \mathbf{k}(z)$. This expression of $\mathcal{R}_{l,m}^{\mathrm{norm}}(z)$ is called the spectral decomposition of the normalized R-matrix, and known to contain much information on the structure of $\mathcal{W}_l(x) \otimes \mathcal{W}_m(y)$. We will compute the spectral decomposition in Section 4.3.1, by connecting it to the known one (2.2.1) in the non-super cases.

Theorem 4.2.10. *For $l, m \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{R}_{l,m}^{\mathrm{norm}}(z) = \sum_{t \in H_\epsilon(l,m)} \prod_{i=t+1}^{\min\{l,m\}} \frac{1 - q^{l+m-2i+2} z}{z - q^{l+m-2i+2}} \mathcal{P}_t^{l,m} \tag{4.2.4}$$

where $z = z_1/z_2$ and we understand the coefficient of $\mathcal{P}_{\min\{l,m\}}^{l,m}$ to be 1.

Remark 4.2.11. In [70], an intertwiner $R_{l,m}(z)$ on a tensor product $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$ is obtained as a result of 2D reduction of a solution of a tetrahedron equation. By the uniqueness of the normalized R-matrix, one can directly check that the map $R_{l,m}(z)$ coincides with our $\mathcal{R}_{l,m}^{\mathrm{norm}}(z)$. When $\epsilon = (1^N, 0^M)$, it is possible to compute the spectral decomposition of $R_{l,m}(z)$ in an explicit way [70, Section 6], and the formula for general ϵ follows as in the proof of Theorem 4.2.10 (*cf.* [74, Section 7]).

In our study of $\mathcal{C}(\epsilon)$ for general ϵ, a crucial observation is that the coefficients of the spectral decomposition (4.2.4) are the same for any ϵ. The only difference lies in classical decompositions, but they also coincide if the length of ϵ is large enough, as seen in (3.2.2).

Therefore, in the virtue of Theorem 2.2.4 we expect that the tensor product structure of given two modules over $\mathcal{U}(\epsilon)$ should be the same with the non-super one, at sufficiently high ranks. We will turn this idea into a mathematical statement in the following sections.

4.2.3 Fusion construction of irreducible polynomial representations

As a first step, we construct simple modules in $\mathcal{C}(\epsilon)$ by means of a fusion construction [56]. We prove the validity of the fusion construction for general ϵ, adapting the argument of [53]. Since this is done uniformly in ϵ, this gives a natural correspondence between irreducible polynomial representations over $\mathcal{U}(\epsilon)$ for any ϵ.

Let $V, W \in \mathcal{C}(\epsilon)$ and $\mathcal{R}^{\mathrm{univ}}_{V,W}$ the universal R-matrix on $V_{\mathrm{aff}} \otimes W_{\mathrm{aff}}$. We say that $\mathcal{R}^{\mathrm{univ}}_{V,W}$ is *rationally renormalizable* [53] if there exists $a \in \mathbf{k}((z_1/z_2))^\times$ such that $a\mathcal{R}^{\mathrm{univ}}_{V,W}$ takes values in $W_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$. If it is the case, then one can choose such a so that $a\mathcal{R}^{\mathrm{univ}}_{V,W}|_{z_1=c_1,z_2=c_2}$ does not vanish for any $c_1, c_2 \in \mathbf{k}^\times$. We put $\mathbf{r}_{V,W} = a\mathcal{R}^{\mathrm{univ}}|_{z_1=z_2=1}$.

For example, $\mathcal{R}^{\mathrm{univ}}_{l,m}$ is rationally renormalizable, which is obvious from the formula (4.2.4). Then for any simple $V, W \in \mathcal{C}_{\mathbb{Z}}(\epsilon)$, $\mathcal{R}^{\mathrm{univ}}_{V,W}$ is rationally renormalizable thanks to the following lemma.

Lemma 4.2.12 (*cf.* [60, Propositions 2.11 and 2.12]). *For $\mathcal{U}(\epsilon)$-modules V and W, $\mathcal{R}^{\mathrm{univ}}_{V,W}$ is rationally renormalizable in any one of the following cases:*

(1) *V (resp. W) is a subquotient of V_0 (resp. W_0) and $\mathcal{R}^{\mathrm{univ}}_{V_0,W}$ (resp. $\mathcal{R}^{\mathrm{univ}}_{V,W_0}$) is rationally renormalizable,*

(2) *$V = V_1 \otimes V_2$ (resp. $W = W_1 \otimes W_2$) and both $\mathcal{R}^{\mathrm{univ}}_{V_1,W}$ and $\mathcal{R}^{\mathrm{univ}}_{V_2,W}$ (resp. $\mathcal{R}^{\mathrm{univ}}_{V,W_1}$ and $\mathcal{R}^{\mathrm{univ}}_{V,W_2}$) are rationally renormalizable.*

As in Section 2.2.2, we want to prove that the image of the composition of $\mathbf{r}_{V,W}$ is simple unless it vanishes. We first consider the case of two modules.

Theorem 4.2.13. *Suppose that irreducible $V, W \in \mathcal{C}(\epsilon)$ are such that $\mathcal{R}^{\mathrm{univ}}_{V,V}$, $\mathcal{R}^{\mathrm{univ}}_{W,W}$ and $\mathcal{R}^{\mathrm{univ}}_{V,W}$ are rationally renormalizable with*

$$\mathbf{r}_{V,V} \in \mathbf{k}^\times \mathrm{id}_{V^{\otimes 2}} \quad or \quad \mathbf{r}_{W,W} \in \mathbf{k}^\times \mathrm{id}_{W^{\otimes 2}}.$$

Then the image of $\mathbf{r}_{V,W}$ is irreducible, and isomorphic to the head of $V \otimes W$ and the socle of $W \otimes V$.

Proof. For reader's convenience, we present the proof following [53]. We assume $\mathbf{r}_{W,W} \in \mathbf{k}^\times \mathrm{id}_{W^{\otimes 2}}$, for the proof for the other case being symmetric.

Take a nonzero submodule S of $W \otimes V$. Since $\mathcal{R}^{\mathrm{univ}}_{V,W}$ and $\mathcal{R}^{\mathrm{univ}}_{W,W}$ are rationally renormalizable, there exist $a, b \in \mathbf{k}((z_1/z_2))^\times$ such that

$$\mathbf{r}_{V,W} = a\mathcal{R}^{\mathrm{univ}}_{V,W}|_{z_1=z_2=1}, \quad \mathbf{r}_{W,W} = b\mathcal{R}^{\mathrm{univ}}_{W,W}|_{z_1=z_2=1}.$$

Then we also obtain

$$ab\mathcal{R}^{\mathrm{univ}}_{S,W} : S_{\mathrm{aff}} \otimes W_{\mathrm{aff}} \longrightarrow W_{\mathrm{aff}} \otimes S_{\mathrm{aff}},$$

and the following diagram

$$\begin{array}{ccccc} S_{\mathrm{aff}} \otimes W_{\mathrm{aff}} & & \xrightarrow{\quad ab\mathcal{R}^{\mathrm{univ}}_{S,W} \quad} & & W_{\mathrm{aff}} \otimes S_{\mathrm{aff}} \\ \big\downarrow & & & & \big\downarrow \\ W_{\mathrm{aff}} \otimes V_{\mathrm{aff}} \otimes W_{\mathrm{aff}} & \xrightarrow{\mathrm{id}_W \otimes a\mathcal{R}^{\mathrm{univ}}_{V,W}} & W_{\mathrm{aff}} \otimes W_{\mathrm{aff}} \otimes V_{\mathrm{aff}} & \xrightarrow{b\mathcal{R}^{\mathrm{univ}}_{W,W} \otimes \mathrm{id}_V} & W_{\mathrm{aff}} \otimes W_{\mathrm{aff}} \otimes V_{\mathrm{aff}}. \end{array}$$

which is commutative by (4.2.2), (4.2.3). As we specialize at 1, we obtain

$$\begin{array}{ccccc} S \otimes W & & \xrightarrow{\quad ab\mathcal{R}^{\mathrm{univ}}_{S,W}|_{z_1=z_2=1} \quad} & & W \otimes S \\ \big\downarrow & & & & \big\downarrow \\ W \otimes V \otimes W & \xrightarrow{\mathrm{id}_W \otimes \mathbf{r}_{V,W}} & W \otimes W \otimes V & \xrightarrow{\mathrm{id}_{W \otimes W \otimes V}} & W \otimes W \otimes V \end{array}$$

where we use the assumption $\mathbf{r}_{W,W} \in \mathbf{k}^\times \mathrm{id}_{W^{\otimes 2}}$. Consequently we derive $S \otimes W \subset W \otimes \mathbf{r}^{-1}_{V,W}(S)$.

Now we can find a submodule K of V such that $S \subset W \otimes K$ and $K \otimes W \subset \mathbf{r}^{-1}_{V,W}(S)$ [53, Lemma 3.10]. Since S is nonzero, so is K, which also implies $S = V$. But then $V \otimes W \subset \mathbf{r}^{-1}_{V,W}(S)$ and so the image of $\mathbf{r}_{V,W}$ is contained in S. Since we have taken S arbitrarily, this means that $\mathrm{im}\mathbf{r}_{V,W}$ is the unique simple submodule of $W \otimes V$, in other words its simple socle. □

By an induction on the number of tensor factors, we obtain the following corollary.

Corollary 4.2.14. *Suppose that irreducible $V_1, \ldots, V_t \in \mathcal{C}(\epsilon)$ are given such that such* *that $\mathcal{R}^{\mathrm{univ}}_{V_i,V_j}$ is rationally renormalizable with $\mathbf{r}_{V_i,V_i} \in \mathbf{k}\mathrm{id}_{V_i^{\otimes 2}}$ for any $1 \leq i, j \leq t$. Let*

$$\mathbf{r} : V_1 \otimes \cdots \otimes V_t \longrightarrow V_t \otimes \cdots \otimes V_1$$

be the composition of $\mathbf{r}_{V_i,V_j}$ associated with a reduced expression of the longest element of $\mathfrak{S}_t$. Then the image of $\mathbf{r}$ is irreducible unless it is zero.

Now let us specialize to fundamental representations. As explained, $\mathcal{R}^{\mathrm{univ}}_{l,m}$ is rationally renormalizable by taking a renormalization

$$\mathbf{r}_{l,m}(z_1/z_2) = d_{l,m}(z_1/z_2)\mathcal{R}^{\mathrm{norm}}_{l,m},$$

where

$$d_{l,m}(z) = \prod_{k=1}^{\min\{l,m\}} (z - q^{l+m-2k+2}) \tag{4.2.5}$$

is called the *denominator* of the normalized R-matrix. Note that if $c_1/c_2 \in \mathbf{k}^\times$ is not a zero of $d_{l,m}(z)$, then we can specialize the normalized R-matrix itself to obtain

$$R_{(l,m),\epsilon}(c_1, c_2) \coloneqq \mathcal{R}^{\mathrm{norm}}_{l,m}(c_1/c_2) : \mathcal{W}_{l,\epsilon}(c_1) \otimes \mathcal{W}_{m,\epsilon}(c_2) \longrightarrow \mathcal{W}_{m,\epsilon}(c_2) \otimes \mathcal{W}_{l,\epsilon}(c_1),$$

which is just a scalar multiple of $\mathbf{r}_{l,m}(c_1/c_2)$. In particular, they have the same image. Therefore, from the above corollary, we obtain the following fusion construction for $\mathcal{U}(\epsilon)$-modules.

Corollary 4.2.15. *Suppose $\boldsymbol{l} = (l_1, \dots, l_t) \in (\mathbb{Z}_{\geq 0})^t$ and $\mathbf{c} = (c_1, \dots, c_t) \in (\Bbbk^\times)^t$ are given such that c_i/c_j is not a zero of $d_{l_i,l_j}(z_i/z_j)$ for any $i < j$. Let*

$$R_{\boldsymbol{l},\epsilon}(\boldsymbol{c}) : \mathcal{W}_{l_1,\epsilon}(c_1) \otimes \cdots \otimes \mathcal{W}_{l_t,\epsilon}(c_t) \longrightarrow \mathcal{W}_{l_t,\epsilon}(c_t) \otimes \cdots \otimes \mathcal{W}_{l_1,\epsilon}(c_1) \tag{4.2.6}$$

be the composition of specializations $\mathcal{R}^{\mathrm{norm}}_{l_i,l_j}(c_i/c_j)$ associated with a reduced expression of the longest element of $\mathfrak{S}_t$. Then the image of $R_{\boldsymbol{l},\epsilon}(\boldsymbol{c})$ is irreducible unless it is zero.

Let $\mathcal{P}^+$ (resp. $\mathcal{P}^+_{\mathbb{Z}}$) be the set of pairs $(\boldsymbol{l}, \boldsymbol{c})$ such that

(1) $\boldsymbol{l} = (l_1, \dots, l_t) \in (\mathbb{Z}_{\geq 0})^t$ and $\boldsymbol{c} = (c_1, \dots, c_t) \in (\Bbbk^\times)^t$ (resp. $c_i \in q^{l_i+1+2\mathbb{Z}}$ for all i) for some $t \geq 1$,

(2) for any $i < j$, c_i/c_j is not a zero of $d_{l_i,l_j}(z_i/z_j)$.

For $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^+$, we can define $R_{\boldsymbol{l},\epsilon}(\boldsymbol{c})$ as above, and we set

- $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$: the image of $R_{\boldsymbol{l},\epsilon}(\boldsymbol{c})$,
- $\mathcal{P}^+(\epsilon) = \{(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^+ \,|\, \mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c}) \neq 0\}$, and similarly $\mathcal{P}^+_{\mathbb{Z}}(\epsilon)$.

Here we understand $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c}) = \mathcal{W}_{l_1,\epsilon}(c_1)$ when $t = 1$. Note that $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{C}^{\sum l_i}(\epsilon)$, for $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ being a quotient of

$$\mathcal{W}_{l_1,\epsilon}(c_1) \otimes \cdots \otimes \mathcal{W}_{l_t,\epsilon}(c_t) \in \mathcal{C}^{\sum l_i}(\epsilon).$$

In the next subsection, we will see that every irreducible module in the category $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ is indeed obtained by the fusion construction.

Finally, we record here an important property that follows by a similar argument with the proof of Theorem 4.2.13.

Proposition 4.2.16. *For $l, m \geq 1$ and $x, y \in \Bbbk^\times$, we have*

$$\mathrm{Hom}_{\mathcal{U}(\epsilon)}\left(\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y), \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x)\right) = \Bbbk \cdot \mathbf{r}_{l,m}(x/y).$$

Proof. The argument of [54, Proposition 3.2.9], which is for modules over quiver Hecke algebras, applies to our case as well. □

4.2.4 Generalized quantum affine Schur-Weyl duality

Next, we analyze the structure of the monoidal category $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ using the generalized quantum affine Schur–Weyl duality functor [51]. Since such a functor is defined on the poles of normalized R-matrices of a given family of representations, the construction is uniform for any ϵ, including the non-super cases $\epsilon_{M|0}$, $\epsilon_{0|N}$.

Let $R(\beta)$ be the quiver Hecke algebra of type A_∞, introduced in Section 2.3.2. Recall that $R(\beta)$ is defined by the data

$$P_{ij}(u, v) = (u - v)^{\delta_{i+1,j}}, \quad Q_{ij}(u, v) = \delta(i \neq j) P_{ij}(u, v) P_{ji}(v, u)$$

for $i, j \in J = \mathbb{Z}$.

Define $X : J \longrightarrow \Bbbk^\times$ by $X(i) = q^{-2i}$, so that $X(i)/X(j) = q^{-2(i-j)}$ is a zero of the denominator $d_{1,1}(z) = z - q^2$ if and only if $j = i + 1$.

Fix $\ell > 0$ and let $X_1, \ldots, X_\ell$ be indeterminates. For $\nu = (\nu_1, \ldots, \nu_\ell) \in J^\ell$, put

$$\mathbb{O}_\nu = \Bbbk \llbracket X_1 - X(\nu_1), \ldots, X_\ell - X(\nu_\ell) \rrbracket ,$$

the completion of the localization of $\Bbbk[X_1, \ldots, X_\ell]$ at $X_i = X(\nu_i)$ $(i = 1, \ldots, \ell)$. We also take its field of fractions $\mathbb{K}_\nu$ of $\mathbb{O}_\nu$. Note that for $f \in \Bbbk(X_1, \ldots, X_\ell)$ that is regular at $X_i = X(\nu_i)$ for all i, we may expand f formally to regard $f \in \mathbb{O}_\nu$.

For $\beta \in \mathbb{N}[J]$ with $\mathrm{ht}(\beta) = \ell$, we define associative $\Bbbk$-algebras

$$\mathbb{O}_\beta = \bigoplus_{\nu \in J^\beta} \mathbb{O}_\nu e(\nu), \quad \mathbb{K}_\beta = \bigoplus_{\nu \in J^\beta} \mathbb{K}_\nu e(\nu)$$

whose multiplications are given so that $e(\nu)$'s are orthogonal central idempotent elements.

For $V \coloneqq (\mathcal{W}_{1,\epsilon})_{\mathrm{aff}}$, $V^{\otimes \ell}$ can be regarded as a $\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}] \otimes \mathcal{U}(\epsilon)$-module where X_i corresponds to the action of z on the i-th component. We define

$$V_{\mathbb{O}}^{\otimes \beta} = \mathbb{O}_\beta \otimes_{\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]} V^{\otimes \ell}, \quad V_{\mathbb{K}}^{\otimes \beta} = \mathbb{K}_\beta \otimes_{\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]} V^{\otimes \ell},$$

regarding $V_{\mathbb{O}}^{\otimes \beta}$ as a subspace of $V_{\mathbb{K}}^{\otimes \beta}$.

Let r_m be the endomorphism on $V_{\mathbb{K}}^{\otimes \beta}$ induced from $\mathcal{R}_{1,1}^{\mathrm{norm}}$ on the m-th and $(m+1)$-st component on $V^{\otimes \ell}$, that is,

$$\begin{array}{ccc} \mathbb{K}_\nu e(\nu) \otimes_{\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]} V^{\otimes \ell} & \longrightarrow & \mathbb{K}_{s_m(\nu)} e(s_m(\nu)) \otimes_{\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]} V^{\otimes \ell} \\ f e(\nu) \otimes (v_1 \otimes \cdots \otimes v_\ell) & \longmapsto & s_m(f) e(s_m(\nu)) \otimes (\cdots \otimes \mathcal{R}_{1,1}^{\mathrm{norm}}(v_m \otimes v_{m+1}) \otimes \cdots) \end{array}$$

for $\nu \in J^\beta$, $f \in \mathbb{K}_\nu$ and $v_1 \otimes \cdots \otimes v_\ell \in V^{\otimes \ell}$. Since $\mathcal{R}_{1,1}^{\mathrm{norm}}$ is $\mathcal{U}(\epsilon)$-linear, so is r_m. Then it can be proved that there exists a right $R(\beta)$-module structure on $V_{\mathbb{K}}^{\otimes \beta}$ given by

$$\begin{aligned} e(\nu) &= \text{projection onto } \mathbb{K}_\nu e(\nu) \otimes_{\Bbbk[X_1^{\pm 1}, \ldots, X_\ell^{\pm 1}]} V^{\otimes \ell} \subset V_{\mathbb{K}}^{\otimes \beta}, \\ e(\nu) x_k &= e(\nu) X(\nu_k)^{-1} (X_k - X(\nu_k)), \\ e(\nu) \tau_m &= \begin{cases} e(\nu)(r_m - 1) \left(\frac{1}{x_m - x_{m+1}} \right) & \text{if } \nu_m = \nu_{m+1} \\ e(\nu) r_m P_{\nu_m, \nu_{m+1}}(x_{m+1}, x_m) & \text{if } \nu_m \neq \nu_{m+1}, \end{cases} \end{aligned}$$

which is compatible with the left $\mathcal{U}(\epsilon)$-action.

Proposition 4.2.17. *For $\beta \in \mathbb{N}[J]$, $V_{\mathbb{O}}^{\otimes \beta}$ is invariant under the $R(\beta)$-action.*

Proof. Since the spectral decomposition (4.2.4) coincides with the one for the case $\epsilon = \epsilon_{0|n}$, the argument in [51, Theorem 3.3] works equally well. □

Consequently, we obtain a $(\mathcal{U}(\epsilon), R(\beta))$-bimodule $V_{\mathbb{O}}^{\otimes \beta}$ and a functor

$$\mathcal{F}_{\epsilon,\beta} : M \longmapsto V_{\mathbb{O}}^{\otimes \beta} \otimes_{R(\beta)} M$$

taking a left $R(\beta)$-module M to produce a $\mathcal{U}(\epsilon)$-module $\mathcal{F}_{\epsilon,\beta}(M)$. We also put

$$\mathcal{F}_{\epsilon,\ell} = \bigoplus_{\mathrm{ht}(\beta)=\ell} \mathcal{F}_{\epsilon,\beta}, \quad \mathcal{F}_{\epsilon} = \bigoplus_{\ell \geq 0} \mathcal{F}_{\epsilon,\ell}.$$

Theorem 4.2.18 (*cf.* [51, Theorems 3.4, 3.8])**.** *The functor $\mathcal{F}_{\epsilon,\beta}$ is exact and induces*

$$\mathcal{F}_{\epsilon,\ell} : R(\ell)\text{-gmod} \longrightarrow \mathcal{C}^{\ell}(\epsilon), \quad \mathcal{F}_{\epsilon} : R\text{-gmod} \longrightarrow \mathcal{C}(\epsilon).$$

Moreover, the functor $\mathcal{F}_{\epsilon}$ is monoidal.

We shall describe the image of simple R-modules under $\mathcal{F}_{\epsilon}$. Considering the results in Section 2.3.2, we first do for the one-dimensional R-modules $L(a, b)$.

Proposition 4.2.19 (*cf.* [51, Proposition 4.9])**.** *For a segment (a, b) of length ℓ, we have*

$$\mathcal{F}_{\epsilon}(L(a, b)) \cong \mathcal{W}_{\ell,\epsilon}(q^{-a-b}).$$

Proof. By [51, Proposition 3.5], it holds when $a = b$. Then we use induction on ℓ as in the proof of [51, Proposition 4.9]. Namely, we apply $\mathcal{F}_{\epsilon}$ to the exact sequence in Proposition 2.3.5(3) with $a' = a = b$, $b' = b - 1$. Then the middle map is a nonzero multiple of $\mathcal{R}^{\mathrm{norm}}_{1,\ell-1}(q^{-\ell})$ by Proposition 4.2.16. Comparing it with the exact sequence in Lemma 4.2.6, we obtain the conclusion. □

Next, we prove that $\mathcal{F}_{\epsilon}$ maps renormalized R-matrices to normalized R-matrices.

Lemma 4.2.20. *Suppose that (a, b), (a', b') are segments of lengths ℓ, ℓ' respectively, such that $(a, b) \geq (a', b')$. Then for $c = q^{-a-b}$ and $c' = q^{-a'-b'}$, c/c' is not a zero of $d_{\ell,\ell'}(z)$. Moreover, the map*

$$\mathcal{F}_{\epsilon}(\mathbf{r}_{L(a,b),L(a',b')}) : \mathcal{W}_{\ell,\epsilon}(c) \otimes \mathcal{W}_{\ell',\epsilon}(c') \longrightarrow \mathcal{W}_{\ell',\epsilon}(c') \otimes \mathcal{W}_{\ell,\epsilon}(c)$$

is equal to a nonzero constant multiple of $\mathcal{R}^{\mathrm{norm}}_{\ell,\ell'}(c/c')$ *unless*

$$a' < a \le b' < b,\ M = 1 \ \text{and}\ N \le b' - a + 1.$$

Proof. As $(a,b) \ge (a',b')$, we have $a' + b' - a - b \le \ell' - \ell$, and then the first assertion follows from the denominator formula

$$d_{\ell,\ell'}(z) = (z - q^{\ell+\ell'})(z - q^{\ell+\ell'-2}) \cdots (z - q^{|\ell-\ell'|+2}).$$

For the second one, it suffices to prove that $\mathcal{F}_\epsilon(\mathbf{r}_{L(a,b),L(a',b')})$ is nonzero except in the prescribed case since

$$\mathrm{Hom}_{\mathcal{U}(\epsilon)}\left(\mathcal{W}_{\ell,\epsilon}(c) \otimes \mathcal{W}_{\ell',\epsilon}(c'), \mathcal{W}_{\ell',\epsilon}(c') \otimes \mathcal{W}_{\ell,\epsilon}(c)\right) = \mathbf{k} \cdot \mathcal{R}^{\mathrm{norm}}_{\ell,\ell'}(c/c')$$

from Proposition 4.2.16.

According to Proposition 2.3.5, $\mathbf{r} = \mathbf{r}_{L(a,b),L(a',b')}$ is not an isomorphism if and only if either $a' < a \le b' < b$ or $a = b' + 1$. In those cases, we have the following exact sequence

$$0 \longrightarrow \mathcal{W}_{\ell_1,\epsilon}(q^{-a'-b}) \otimes \mathcal{W}_{\ell_2,\epsilon}(q^{-a-b'}) \longrightarrow \mathcal{W}_{\ell,\epsilon}(c) \otimes \mathcal{W}_{\ell',\epsilon}(c') \longrightarrow \mathcal{W}_{\ell',\epsilon}(c') \otimes \mathcal{W}_{\ell,\epsilon}(c) \xrightarrow{\mathcal{F}_\epsilon(\mathbf{r})} \mathcal{W}_{\ell_1,\epsilon}(q^{-a'-b}) \otimes \mathcal{W}_{\ell_2,\epsilon}(q^{-a-b'}) \longrightarrow 0$$

by applying $\mathcal{F}_\epsilon$ to the exact sequences in Proposition 2.3.5(2),(3), where $\ell_1 = b - a' + 1$ and $\ell_2 = b' - a + 1$. Then $\mathcal{F}_\epsilon(\mathbf{r}) = 0$ if and only if

$$\dim\left(\mathcal{W}_{\ell_1,\epsilon}(q^{-a'-b}) \otimes \mathcal{W}_{\ell_2,\epsilon}(q^{-a-b'})\right) = \dim\left(\mathcal{W}_{\ell,\epsilon}(c) \otimes \mathcal{W}_{\ell',\epsilon}(c')\right),$$

or equivalently, the classical decompositions of $\mathcal{W}_{\ell_1,\epsilon}(q^{-a'-b}) \otimes \mathcal{W}_{\ell_2,\epsilon}(q^{-a-b'})$ and $\mathcal{W}_{\ell,\epsilon}(c) \otimes \mathcal{W}_{\ell',\epsilon}(c')$ coincide. From (3.2.2), this happens exactly when $a' < a \le b' < b$, $M = 1$ and $N \le b' - a + 1$. □

Hence, we obtain a super analogue of [51, Theorem 4.11].

Theorem 4.2.21. *Let* $((a_1, b_1), \ldots, (a_t, b_t))$ *be an ordered multisegment with* $\ell_k = b_k - a_k + 1$ *and* $\ell = \sum \ell_k$, *and* L *the corresponding irreducible* $R^J(\ell)$*-module. If* $N = |\{i | \epsilon_i = 1\}| \ge$

ℓ_k *for all* $k = 1, \dots, t$, *then* $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ *is nonzero and*

$$\mathcal{F}_\epsilon(L) \cong \mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$$

for $\boldsymbol{l} = (\ell_1, \dots, \ell_t)$ *and* $\boldsymbol{c} = (q^{-a_1-b_1}, \dots, q^{-a_t-b_t})$.

Proof. Since $\mathcal{F}_\epsilon$ is exact, everything follows once we check that $\mathcal{F}_\epsilon(\mathbf{r}_{L(a_i,b_i),L(a_j,b_j)})$ is nonzero for every $i < j$. That condition is ensured by the assumption $N \geq \ell_k$ for all k. Indeed, as $a_j < a_i \leq b_j < b_i$ implies $b_j - a_i + 1 < b_j - a_j + 1 = \ell_j \leq N$, the exception in Lemma 4.2.20 cannot happen. □

In particular, $\mathcal{F}_\epsilon$ sends simple R-modules to simple $\mathcal{U}(\epsilon)$-modules or zero. Together with the classification of simple R-modules (Proposition 2.3.4), this allows us to find all the irreducible objects in $\mathcal{C}_\mathbb{Z}(\epsilon)$.

Theorem 4.2.22. *For any irreducible* $V \in \mathcal{C}_\mathbb{Z}(\epsilon)$, *there exists* $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^+_\mathbb{Z}(\epsilon)$ *such that* $V \cong \mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$.

Proof. By definition of $\mathcal{C}_\mathbb{Z}(\epsilon)$ and Lemma 4.2.6, V is a composition factor of $\mathcal{W}_{1,\epsilon}(q^{-2j_1}) \otimes \cdots \otimes \mathcal{W}_{1,\epsilon}(q^{-2j_\ell})$ for some $j_1, \dots, j_\ell \in \mathbb{Z}$. Then Theorem 4.2.21 implies that V is isomorphic to $\mathcal{F}_\epsilon(M)$ for some composition factor M of $L(j_1) \circ \cdots \circ L(j_\ell)$. Such M can be obtained as the image of

$$\mathbf{r}^{w_0} : L(a_1, b_1) \circ \cdots \circ L(a_t, b_t) \longrightarrow L(a_t, b_t) \circ \cdots \circ L(a_1, b_1)$$

for some ordered multisegment $((a_1, b_1), \dots, (a_t, b_t))$ by Proposition 2.3.4. Thus, $\mathcal{F}_\epsilon(M) \cong \mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ for

$$\boldsymbol{l} = (b_1 - a_1 + 1, \dots, b_t - a_t + 1), \quad \boldsymbol{c} = (q^{-a_1-b_1}, \dots, q^{-a_t-b_t}),$$

and clearly $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}_\mathbb{Z}(\epsilon)$. □

4.3 Super duality

So far, we have reproduced several important constructions in the representation theory of quantum affine algebras, for $\mathcal{U}(\epsilon)$ in a uniform manner. In this section, we establish a concrete connection between $\mathcal{C}(\epsilon)$ and $\mathcal{C}(\epsilon')$ for $\epsilon' < \epsilon$. As we can choose $\epsilon' = \epsilon_{0|N}$ as well, this includes a connection to the module category of quantum affine algebras, and eventually explains the uniformity observed.

4.3.1 Truncation

Given $\epsilon \in \mathcal{E}_n$, let $\epsilon' = (\epsilon'_1, \dots, \epsilon'_{n-1})$ be the sequence obtained by removing the i-th entry ϵ_i from ϵ. Let $\mathbb{I}' = \{1, \dots, n-1\}$ which is $\mathbb{Z}/2\mathbb{Z}$-graded by ϵ', $I' = \{0, 1, \dots n-2\}$ and

$$P' = \bigoplus_{j \in \mathbb{I}'} \mathbb{Z}\delta'_j,$$

the weight lattice for $\mathcal{U}(\epsilon')$.

Theorem 4.3.1 ([77, Theorem 4.3])**.** *There exists a $\Bbbk$-algebra map $\phi^{\epsilon}_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon)$ defined on the generators e'_j, f'_j $(j \in I')$ and $k_{\delta'_l}$ $(l \in \mathbb{I}')$ of $\mathcal{U}(\epsilon')$ by*

$$k_{\delta'_l} \longmapsto \begin{cases} k_{\delta_l} & \text{if } 1 \le l \le i-1 \\ k_{\delta_{l+1}} & \text{if } i \le l \le n-1, \end{cases}$$

Case 1. If $2 \le i \le n-1$, then

$$(e'_j,\ f'_j) \longmapsto \begin{cases} (e_j,\ f_j) & \text{if } j = 0, 1, \dots, i-2 \\ \big([e_{i-1}, e_i]_{\mathbf{q}(\alpha_{i-1},\alpha_i)},\ [f_i, f_{i-1}]_{\mathbf{q}(\alpha_{i-1},\alpha_i)^{-1}}\big) & \text{if } j = i-1 \\ (e_{j+1},\ f_{j+1}) & \text{if } j = i, \dots, n-2, \end{cases}$$

Case 2. If $i = n$, then

$$(e'_j,\ f'_j) \longmapsto \begin{cases} (e_j,\ f_j) & \text{if } j \neq 0 \\ \big([e_{n-1}, e_0]_{\mathbf{q}(\alpha_{n-1},\alpha_0)},\ [f_0, f_{n-1}]_{\mathbf{q}(\alpha_{n-1},\alpha_0)^{-1}}\big) & \text{if } j = 0, \end{cases}$$

Case 3. If $i = 1$, then

$$(e'_j,\ f'_j) \longmapsto \begin{cases} \big([e_{n-1}, e_0]_{\mathbf{q}(\alpha_{n-1},\alpha_0)},\ [f_0, f_{n-1}]_{\mathbf{q}(\alpha_{n-1},\alpha_0)^{-1}}\big) & \text{if } j = 0 \\ (e_j,\ f_j) & \text{if } j \neq 0. \end{cases}$$

More generally, if $\epsilon' = (\epsilon'_1, \dots, \epsilon'_{n-r})$ is obtained from ϵ by removing $\epsilon_{i_1}, \dots, \epsilon_{i_r}$ for some $i_1 < \cdots < i_r$, we define a $\Bbbk$-algebra homomorphism $\phi^{\epsilon}_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon)$ as a successive composition of the above algebra homomorphism[4].

Now for $V \in \mathcal{C}(\epsilon)$, let

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(V) := \bigoplus_{(\mu|\delta_{i_1})=\cdots=(\mu|\delta_{i_r})=0} V_\mu.$$

Proposition 4.3.2 ([77, Proposition 4.4]). *The following properties hold.*

(1) *The subspace $\mathfrak{tr}^{\epsilon}_{\epsilon'}(V)$ of V is closed under the $\mathcal{U}(\epsilon')$-action induced by $\phi^{\epsilon}_{\epsilon'}$, and hence a $\mathcal{U}(\epsilon')$-module contained in the category $\mathcal{C}(\epsilon')$.*

(2) *As a $\mathcal{U}(\epsilon')$-module, we have $\mathfrak{tr}^{\epsilon}_{\epsilon'}(V \otimes W) \cong \mathfrak{tr}^{\epsilon}_{\epsilon'}(V) \otimes \mathfrak{tr}^{\epsilon}_{\epsilon'}(W)$.*

For any $\mathcal{U}(\epsilon)$-linear map $f : V \longrightarrow W$ for V, $W \in \mathcal{C}(\epsilon)$, the restriction of f on the subspaces

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(f) : \mathfrak{tr}^{\epsilon}_{\epsilon'}(V) \longrightarrow \mathfrak{tr}^{\epsilon}_{\epsilon'}(W)$$

is a well-defined $\mathcal{U}(\epsilon')$-module homomorphism. Therefore, we obtain a monoidal functor $\mathfrak{tr}^{\epsilon}_{\epsilon'}$, which can be easily seen to be exact as well.

Definition 4.3.3. The exact monoidal functor

$$\mathfrak{tr}^{\epsilon}_{\epsilon'} : \mathcal{C}(\epsilon) \longrightarrow \mathcal{C}(\epsilon')$$

is called a *truncation*.

We obtain a functor $\mathfrak{tr}^{\epsilon}_{\epsilon'} : \mathring{\mathcal{C}}(\epsilon) \longrightarrow \mathring{\mathcal{C}}(\epsilon')$ in the same manner, also called a truncation. Not only being exact and monoidal, truncations also preserve the ingredients we have used for the study of $\mathcal{C}(\epsilon)$.

Proposition 4.3.4 ([77, Propositions 4.5, 4.6]). *For $\epsilon' < \epsilon$, let M' and N' be the numbers of 0's and 1's in ϵ' respectively.*

(1) *For $\lambda \in \mathscr{P}_{M|N}$,*

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(V_\epsilon(\lambda)) \cong \begin{cases} V_{\epsilon'}(\lambda) & \textit{if } \lambda \in \mathscr{P}_{M'|N'} \\ 0 & \textit{otherwise.} \end{cases}$$

[4] A different ordering of $\epsilon_{i_1}, \ldots, \epsilon_{i_r}$ affects $\phi^{\epsilon}_{\epsilon'}$ only by conjugations by T_i in Proposition 3.1.3.

(2) *For* $l \in \mathbb{Z}_{\geq 0}$ *and* $x \in \Bbbk^{\times}$,

$$\mathbf{tr}^{\epsilon}_{\epsilon'}(\mathcal{W}_{l,\epsilon}(x)) \cong \begin{cases} \mathcal{W}_{l,\epsilon'}(x) & \text{if } (l) \in \mathscr{P}_{M'|N'} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, truncations are well-defined between subcategories $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ and $\mathcal{C}_{\mathbb{Z}}(\epsilon')$.

Lemma 4.3.5. *For* $\epsilon' < \epsilon$, *we have*

$$\mathbf{tr}^{\epsilon}_{\epsilon'}\left(\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}(z)\right) = \mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon'}(z).$$

Proof. Follows from the uniqueness of normalized R-matrices. □

In particular, truncations preserve the specializations $R_{(l,m),\epsilon}(x,y) = \mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}(x/y)$. Putting it all together, we conclude that truncations respect the fusion construction.

Theorem 4.3.6. *For* $\epsilon' < \epsilon$ *and* $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^{+}(\epsilon)$, *we have*

$$\mathbf{tr}^{\epsilon}_{\epsilon'}(\mathcal{W}_{\epsilon}(\boldsymbol{l}, \boldsymbol{c})) \cong \mathcal{W}_{\epsilon'}(\boldsymbol{l}, \boldsymbol{c}).$$

Moreover, since the bimodule $V_{\mathbb{O}}^{\otimes \beta}$ in Section 4.2.4 is a scalar extension of a tensor product of $(\mathcal{W}_{1,\epsilon})_{\mathrm{aff}}$, it is compatible with the truncation, and so is the duality functor $\mathcal{F}_{\epsilon}$.

Lemma 4.3.7. *For* $\epsilon' < \epsilon$, *there exists a natural isomorphism* $\mathbf{tr}^{\epsilon}_{\epsilon'} \circ \mathcal{F}_{\epsilon} \cong \mathcal{F}_{\epsilon'}$.

Those two compatibilities indicate that truncations correctly relates $\mathcal{C}(\epsilon)$ and $\mathcal{C}(\epsilon')$, which explain the uniformity of constructions in ϵ. In the next subsection, we will prove that $\mathbf{tr}^{\epsilon}_{\epsilon'}$ is indeed a (partial) equivalence, by establishing a similar claim for the duality functor $\mathcal{F}_{\epsilon}$.

To illustrate how truncation functors work, let us consider two applications. First, we shall prove the stability in ϵ of the classical decomposition of an irreducible module $\mathcal{W}_{\epsilon}(\boldsymbol{l}, \boldsymbol{c})$. Let $m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon)$ denote the composition multiplicity of $V_{\epsilon}(\lambda)$ in $\mathcal{W}_{\epsilon}(\boldsymbol{l}, \boldsymbol{c})$ as a $\mathring{\mathcal{U}}(\epsilon)$-module, for $\lambda \in \mathscr{P}_{M|N}$.

Theorem 4.3.8. *For* $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^{+}$ *and a partition* λ, *there exists* $m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})} \in \mathbb{Z}_{\geq 0}$ *such that*

(1) $m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})} = 0 \implies m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) = 0$ *for all* ϵ,

(2) $m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})} \neq 0 \implies m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) = m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})}$ *whenever* $m_{\lambda}^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) \neq 0$.

Proof. By Proposition 4.3.4(1), if there exists an $\epsilon \in \mathcal{E}$ such that $m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) \neq 0$, then $m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) = m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon')$ for any $\epsilon' > \epsilon$. If there are ϵ, ϵ' with $m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon), m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon') \neq 0$, then by taking $\epsilon'' > \epsilon, \epsilon'$, again we get

$$m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon) = m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon'') = m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon').$$

Hence the composition multiplicity $m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}(\epsilon)$ stabilizes to a nonnegative integer $m_\lambda^{(\boldsymbol{l},\boldsymbol{c})}$, as $\epsilon \gg 0$. □

In particular, to compute the character of $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ is equivalent to do the corresponding irreducible representation of $U_q'(\widehat{\mathfrak{sl}}_n)$, for large enough n.

Next, a similar argument proves the spectral decomposition (4.2.4) of $\mathcal{R}_{l,m}^{\mathrm{norm}}(z)$, as we now explain. We first clarify the normalization of the projection $\mathcal{P}_t^{l,m}$. Fix $\epsilon \in \mathcal{E}$ and $l, m \in \mathbb{Z}_{\geq 0}$. Take $\epsilon'' = (\epsilon_1'', \ldots, \epsilon_{n''}'') > \epsilon$ such that for $\epsilon' = \epsilon_{0|N''}$, we have

$$\begin{aligned} \mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y) &\cong \bigoplus_{0 \leq t \leq \min\{l,m\}} V_{\epsilon''}((l+m-t,t)), \\ \mathcal{W}_{l,\epsilon'}(x) \otimes \mathcal{W}_{m,\epsilon'}(y) &\cong \bigoplus_{0 \leq t \leq \min\{l,m\}} V_{\epsilon'}((l+m-t,t)), \end{aligned}$$

over $\mathring{\mathcal{U}}(\epsilon'')$ and $\mathring{\mathcal{U}}(\epsilon')$, respectively. That is, we may choose ϵ'' with many 1's, so that $H_{\epsilon''}(l,m) = \{0, 1, \ldots, \min\{l,m\}\}$.

Take a highest weight vector $v'(l,m,t)$ of $V_{\epsilon'}((l+m-t,t))$ in $\mathcal{W}_{l,\epsilon'}(x) \otimes \mathcal{W}_{m,\epsilon'}(y)$ uniquely determined by the condition

$$\begin{aligned} & v'(l,m,t) \in \mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}, \\ & v'(l,m,t) \equiv |\mathbf{e}_1 + \cdots + \mathbf{e}_t + \mathbf{e}_{m+1} + \cdots + \mathbf{e}_{l+m-t}\rangle \otimes |\mathbf{e}_1 + \cdots + \mathbf{e}_m\rangle \pmod{q^{-1}\mathcal{L}_{l,\epsilon'} \otimes \mathcal{L}_{m,\epsilon'}} \end{aligned}$$

where $\mathcal{L}_{l,\epsilon'}$ is a lower crystal lattice of $\mathcal{W}_{l,\epsilon'}$ spanned by $|\mathbf{m}\rangle$ over A_∞[5]. Define similarly $v'(m,l,t)$.

Recall that by definition of truncation, we may regard

$$V_{\epsilon'}((l+m-t,t)) = \mathfrak{tr}_{\epsilon'}^{\epsilon''} V_{\epsilon''}((l+m-t,t)) \subset V_{\epsilon''}((l+m-t,t)),$$

[5]Here, A_∞ denotes the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = \infty$.

and then we define a $\mathring{\mathcal{U}}(\epsilon'')$-linear map

$$\mathcal{P}_t^{l,m} : \mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y) \longrightarrow \mathcal{W}_{m,\epsilon''}(y) \otimes \mathcal{W}_{l,\epsilon''}(x)$$

to be the unique map that maps $v'(l,m,t')$ to $\delta_{tt'}v'(m,l,t)$. Again regarding

$$V_\epsilon((l+m-t,t)) \subset V_{\epsilon''}((l+m-t,t)) \quad (t \in H_\epsilon(l,m)),$$
$$\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \subset \mathcal{W}_{l,\epsilon''}(x) \otimes \mathcal{W}_{m,\epsilon''}(y),$$

we obtain the desired projection

$$\mathcal{P}_t^{l,m} : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \longrightarrow \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x)$$

by restriction. Note that $\mathcal{P}_t^{l,m}$ does not depend on the choice of ϵ'' and $x, y \in \Bbbk^\times$.

Proof of Theorem 4.2.10. Given $l, m \in \mathbb{Z}_{\geq 0}$ and ϵ, we fix ϵ' and ϵ'' as above. Write

$$\mathcal{R}^{\mathrm{norm}}_{l,m,\boldsymbol{\epsilon}}(z) = \sum_{t \in H_{\boldsymbol{\epsilon}}(l,m)} \rho_{t,\boldsymbol{\epsilon}}(z)\mathcal{P}_t^{l,m}$$

for $\boldsymbol{\epsilon} = \epsilon, \epsilon'$ or ϵ'', where $\mathcal{P}_t^{l,m}$ denotes the projection accordingly. Recall from (2.2.1) that the spectral decomposition for $\epsilon' = \epsilon_{0|N''}$ is known to be

$$\rho_{t,\epsilon'}(z) = \prod_{i=t+1}^{\min\{l,m\}} \frac{z - q^{l+m-2i+2}}{1 - q^{l+m-2i+2}z}.$$

Since $\mathfrak{tr}^{\epsilon''}_{\epsilon'}(\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon''}) = \mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon'}$ and $\mathfrak{tr}^{\epsilon''}_{\epsilon'}(\mathcal{P}_t^{l,m}) = \mathcal{P}_t^{l,m}$ for all $t = 0, \ldots, \min\{l,m\}$ by our choice of ϵ'', we obtain

$$\rho_{t,\epsilon''}(z) = \rho_{t,\epsilon'}(z).$$

Now that

$$\mathfrak{tr}^{\epsilon''}_{\epsilon}(\mathcal{P}_t^{l,m}) = \begin{cases} \mathcal{P}_t^{l,m} & \text{if } t \in H_\epsilon(l,m) \\ 0 & \text{otherwise,} \end{cases}$$

we truncate to ϵ the spectral decomposition of $\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon''}$ to conclude the proof. □

4.3.2 Equivalence of duality functors

The main result of this subsection is the following equivalence.

Theorem 4.3.9. *For $\ell < n$, we have an equivalence of categories*

$$\mathcal{F}_{\epsilon,\ell} : R^J(\ell)\text{-mod}_0 \longrightarrow \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon).$$

Together with Lemma 4.3.7, we understand that for given ℓ, the truncation

$$\mathfrak{tr}^{\epsilon}_{\epsilon'} : \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon) \longrightarrow \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon')$$

is an equivalence whenever the length of ϵ' is larger than ℓ. Therefore, *if ϵ' is infinite*, the whole functor $\mathfrak{tr} : \mathcal{C}_{\mathbb{Z}}(\epsilon) \to \mathcal{C}_{\mathbb{Z}}(\epsilon')$ should be an equivalence of monoidal categories. We will deal with the infinite rank issue in the next subsection.

To prove the theorem, we first identify the quiver Hecke algebra $R(\ell)$ with the affine Hecke algebra after completion. There is a well-known Schur–Weyl-type duality between affine Hecke algebras and quantum affine algebras of type A, and we establish its super analogues by adapting the approach of [17]. Then the desired equivalence is obtained by lifting this duality to quiver Hecke side.

Let us first recall the definition of the affine Hecke algebra.

Definition 4.3.10. For $\ell \geq 2$, the *affine Hecke algebra* $H^{\mathrm{aff}}_{\ell}(q^2)$ is the $\Bbbk$-algebra generated by $X_k^{\pm 1}$ $(1 \leq k \leq \ell)$ and h_m $(1 \leq m \leq \ell - 1)$ subject to the following relations:

$$\begin{gathered} h_m h_{m+1} h_m = h_{m+1} h_m h_{m+1}, \quad h_m h_{m'} = h_{m'} h_m \quad (|m - m'| > 1), \\ (h_m - q^2)(h_m + 1) = 0, \\ X_k X_{k'} = X_{k'} X_k, \quad X_k X_k^{-1} = X_k^{-1} X_k = 1, \\ h_m X_m h_m = q^2 X_{m+1}, \quad h_m X_k = X_k h_m \quad (k \neq m, m+1). \end{gathered}$$

We also put $H^{\mathrm{aff}}_0(q^2) = \Bbbk$ and $H^{\mathrm{aff}}_1(q^2) = \Bbbk[X^{\pm 1}]$.

The *finite Hecke algebra* $H_{\ell}(q^2)$ is the subalgebra generated by h_m $(1 \leq m \leq \ell - 1)$.

Since the braid relation holds between h_m, it is well-known that one can define h_w for $w \in \mathfrak{S}_{\ell}$ without ambiguity by choosing any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ and setting $h_w \coloneqq h_{i_1} \cdots h_{i_n}$. Moreover, $\{X_1^{a_1} \cdots X_{\ell}^{a_{\ell}} h_w\}_{a_i \in \mathbb{Z}_{\geq 0}, w \in \mathfrak{S}_{\ell}}$ (resp. $\{h_w\}_{w \in \mathfrak{S}_{\ell}}$) is a basis of $H^{\mathrm{aff}}_{\ell}(q^2)$ (resp. $H_{\ell}(q^2)$).

Next, we consider completions of $H_\ell^{\mathrm{aff}}(q^2)$. Set (see Section 4.2.4 for notations)

$$\mathbb{O}H_\ell^{\mathrm{aff}}(q^2) = \mathbb{O}_\ell \otimes_{\Bbbk[X_1^{\pm 1},\dots,X_\ell^{\pm 1}]} H_\ell^{\mathrm{aff}}(q^2) \cong \mathbb{O}_\ell \otimes_{\Bbbk} H_\ell(q^2),$$
$$\mathbb{K}H_\ell^{\mathrm{aff}}(q^2) = \mathbb{K}_\ell \otimes_{\mathbb{O}_\ell} \mathbb{O}H_\ell^{\mathrm{aff}}(q^2).$$

They are associative $\Bbbk$-algebras by

$$h_m f e(\nu) = s_m(f)e(s_m(\nu))h_m + \frac{(q^2-1)X_{m+1}(fe(\nu) - s_m(f)e(s_m(\nu)))}{X_{m+1}-X_m}$$

for $fe(\nu) \in \mathbb{K}_\ell$.

Introduce intertwining elements $\Phi_m \in \mathbb{K}H_\ell^{\mathrm{aff}}(q^2)$ $(m = 1, \dots, \ell-1)$ defined by

$$\Phi_m = h_m - \frac{(q^2-1)X_{m+1}}{X_{m+1}-X_m}.$$

As the name suggests, Φ_m satisfies the following properties which can be checked by straightforward but cumbersome computations:

$$\Phi_m f(X_m, X_{m+1})e(\nu) = f(X_{m+1}, X_m)e(s_m(\nu))\Phi_m,$$
$$\Phi_m\Phi_{m'} = \Phi_{m'}\Phi_m \quad (|m-m'|>1),$$
$$\Phi_m\Phi_{m+1}\Phi_m = \Phi_{m+1}\Phi_m\Phi_{m+1},$$
$$\Phi_m^2 e(\nu) = \frac{X_{m+1}-q^2X_m}{X_m - X_{m+1}} \cdot \frac{X_m - q^2X_{m+1}}{X_{m+1}-X_m} e(\nu),$$

for $f(X_m, X_{m+1}) \in \Bbbk(X_m, X_{m+1})$. Let us normalize Φ_m to

$$\widetilde{\Phi}_m = \frac{X_m - X_{m+1}}{X_{m+1} - q^2 X_m}\Phi_m$$

so that $\widetilde{\Phi}_m^2 = 1$.

Now let

$$\widehat{R}^J(\ell) = \Bbbk\llbracket x_1, \dots x_\ell \rrbracket \otimes_{\Bbbk[x_1,\dots,x_\ell]} R^J(\ell)$$

be a completion of $R^J(\ell)$, with the naturally extended multiplication.

Theorem 4.3.11 ([11,88]). *There is a* $\Bbbk$*-algebra isomorphism*

$$\Psi : \widehat{R}^J(\ell) \longrightarrow \mathbb{O}H_\ell^{\mathrm{aff}}(q^2)$$

defined by

$$\begin{aligned}
\Psi(e(\nu)) &= e(\nu),\\
\Psi(e(\nu)x_k) &= e(\nu)X(\nu_k)^{-1}(X_k - X(\nu_k)),\\
\Psi(e(\nu)\tau_m) &= \begin{cases} e(\nu)(\widetilde{\Phi}_m - 1)\left(X_m X(\nu_m)^{-1} - X_{m+1}X(\nu_{m+1})^{-1}\right)^{-1} & \text{if } \nu_{m+1} = \nu_m \\ e(\nu)\widetilde{\Phi}_m\left(X_m X(\nu_m)^{-1} - X_{m+1}X(\nu_{m+1})^{-1}\right) & \text{if } \nu_{m+1} = \nu_m + 1 \\ e(\nu)\widetilde{\Phi}_m & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $H_\ell^{\mathrm{aff}}(q^2)$-mod$_J$ denote the category of finite-dimensional $H_\ell^{\mathrm{aff}}(q^2)$-modules such that the eigenvalues of X_k $(1 \leq k \leq \ell)$ lies in the set $\{X(j) = q^{-2j}\}_{j\in J} = q^{2\mathbb{Z}}$. Our choice of a completion implies an equivalence

$$H_\ell^{\mathrm{aff}}(q^2)\text{-mod}_J \simeq \mathbb{O}H_\ell^{\mathrm{aff}}(q^2)\text{-mod}$$

where the right hand side denotes the category of finite-dimensional $\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)$-modules on which $e(\nu)$ acts as the projection to the simultaneous generalized eigenspace corresponding to ν.

Similarly, the category $R^J(\ell)$-mod$_0$ of (not necessarily graded) finite-dimensional $R^J(\ell)$-modules on which x_k acts nilpotently is equivalent to the category $\widehat{R}^J(\ell)$-mod of (not necessarily graded) finite-dimensional $\widehat{R}^J(\ell)$-modules.

To sum up, the algebra isomorphism Ψ induces an equivalence

$$\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)\text{-mod} \simeq \widehat{R}^J(\ell)\text{-mod}$$

and hence we obtain

$$\Psi^* : H_\ell^{\mathrm{aff}}(q^2)\text{-mod}_J \xrightarrow{\simeq} R^J(\ell)\text{-mod}_0.$$

We shall use this equivalence to identify $\mathcal{F}_\epsilon$ with another duality functor arising from affine Hecke algebras, which we now explain.

Recall that the functor $\mathcal{F}_{\epsilon,\ell}$ is given by the tensor product with the $(\mathcal{U}(\epsilon), R^J(\ell))$-bimodule $V_{\mathbb{O}}^{\otimes\ell} = \mathbb{O}_\ell \otimes_{\Bbbk[X_1^{\pm1},\dots,X_\ell^{\pm1}]} V^{\otimes\ell}$, where $V^{\otimes\ell} = \mathcal{W}_{1,\epsilon}(X_1) \otimes \cdots \mathcal{W}_{1,\epsilon}(X_\ell)$ for indeterminates $X_1, \dots, X_\ell$. The application of the normalized R-matrix on m-th and $(m+1)$-st factors gives rise to the linear map

$$R_m : V^{\otimes \ell} \longrightarrow \Bbbk(X_m, X_{m+1}) \otimes_{\Bbbk[X_m^{\pm 1}, X_{m+1}^{\pm 1}]} V^{\otimes \ell}.$$

Then one can prove the following lemma by verifying the defining relations of $H_\ell^{\mathrm{aff}}(q^2)$.

Lemma 4.3.12. *The $\mathcal{U}(\epsilon)$-module $V^{\otimes \ell}$ carries a right $H_\ell^{\mathrm{aff}}(q^2)$-module structure given by*

$$\begin{aligned}(f \otimes v)X_k &= (X_k f) \otimes v,\\ (f \otimes v)h_m &= (f \otimes v)\left(\frac{X_{m+1} - q^2 X_m}{X_m - X_{m+1}} R_m + \frac{(q^2-1)X_{m+1}}{X_{m+1} - X_m}\right)\end{aligned}$$

for $f \otimes v \in V^{\otimes \ell}$, $1 \le k \le \ell$ and $1 \le m \le \ell - 1$.

Consequently we obtain a $(\mathcal{U}(\epsilon), \mathbb{O}H_\ell^{\mathrm{aff}}(q^2))$-bimodule $V_{\mathbb{O}}^{\otimes \ell}$ and the functor

$$\begin{aligned}\mathcal{F}_{\epsilon,\ell}^* : H_\ell^{\mathrm{aff}}(q^2)\text{-mod}_J &\longrightarrow \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon)\\ M &\longmapsto V_{\mathbb{O}}^{\otimes \ell} \otimes_{\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)} M.\end{aligned}$$

Comparing the above formulas, this $\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)$-action is compatible under Ψ with the $\widehat{R}^J(\ell)$-module structure given in Section 4.2.4. Therefore, we arrive at a natural isomorphism

$$\mathcal{F}_{\epsilon,\ell}^* \cong \mathcal{F}_{\epsilon,\ell} \circ \Psi^*,$$

and then Theorem 4.3.9 follows immediately from the following super analogue of [17]. Since its proof is rather technical and irrelevant to our discussion, we put it in a separate Section 6.1.2.

Theorem 4.3.13. *For $\ell < n$, the functor*

$$\begin{aligned}\mathcal{F}_{\epsilon,\ell}^* : H_\ell^{\mathrm{aff}}(q^2)\text{-mod}_J &\longrightarrow \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon)\\ M &\longmapsto V_{\mathbb{O}}^{\otimes \ell} \otimes_{\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)} M\end{aligned}$$

is an equivalence of categories.

Corollary 4.3.14. *Let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and its subsequence $\epsilon' = (\epsilon'_1, \dots, \epsilon'_{n'})$ be given. If*

$n' > \ell$, then the truncation induces an equivalence of categories

$$\mathfrak{tr}^{\epsilon}_{\epsilon'} : \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon) \longrightarrow \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon').$$

As an application, we obtain short exact sequences called *T-system*, for generalized quantum groups. For $r, s \in \mathbb{Z}_{\geq 0}$ and $c \in \Bbbk^{\times}$, define the *Kirillov–Reshetikhin type module*

$$\mathcal{W}^{r,s}_{\epsilon}(c) \coloneqq \mathcal{W}_{\epsilon}(\boldsymbol{l}, \boldsymbol{c}) \quad \text{where } \boldsymbol{l} = (r, \ldots, r) \in (\mathbb{Z}_{\geq 0})^{s},\ \boldsymbol{c} = (cq^{-2(s-1)}, \ldots, cq^{-2}, c).$$

When ϵ is homogeneous, $\mathcal{W}^{r,s}_{\epsilon}(c)$ recovers the usual Kirillov–Reshetikhin module for quantum affine algebras of type $A^{(1)}_{n-1}$ (see Remark 4.3.16 below). In particular, $\mathcal{W}^{r,s}_{\epsilon}(c) \cong V_{\epsilon}((s^{r}))$ as a $\mathring{\mathcal{U}}(\epsilon)$-module unless $(s^{r}) \notin \mathscr{P}_{M|N}$, in which case $\mathcal{W}^{r,s}_{\epsilon}(c) = 0$.

Proposition 4.3.15. *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{W}^{r,s+1}_{\epsilon}(c) \otimes \mathcal{W}^{r,s-1}_{\epsilon}(cq^{-2}) \longrightarrow \mathcal{W}^{r,s}_{\epsilon}(c) \otimes \mathcal{W}^{r,s}_{\epsilon}(cq^{-2}) \longrightarrow \bigotimes_{r'=r\pm 1} \mathcal{W}^{r',s}_{\epsilon}(cq^{-1}) \longrightarrow 0.$$

for any $r, s \in \mathbb{Z}_{\geq 0}$ and $c \in \Bbbk^{\times}$.

Proof. Since the result for arbitrary c can be obtained by the shift of spectral parameter, we may take $c = q^{r-1}$ so that the whole sequence belongs to the subcategory $\mathcal{C}^{2rs}_{\mathbb{Z}}(\epsilon)$.

In case of $\epsilon = \epsilon_{0|n}$, the existence of such a short sequence is well-known under the name of T-system [37, 68, 82]. For general ϵ, take ϵ' such that $\epsilon' > \epsilon$ and the number N' of 1's in ϵ' is larger than $2rs$.

$$\mathcal{C}^{2rs}_{\mathbb{Z}}(\epsilon) \xleftarrow{\ \mathfrak{tr}^{\epsilon'}_{\epsilon}\ } \mathcal{C}^{2rs}_{\mathbb{Z}}(\epsilon') \xrightarrow[\ \simeq\ ]{\ \mathfrak{tr}^{\epsilon'}_{\epsilon_{0|N'}}\ } \mathcal{C}^{2rs}_{\mathbb{Z}}(\epsilon_{0|N'})$$

The choice of ϵ' together with Corollary 4.3.14 implies that $\mathfrak{tr}^{\epsilon'}_{\epsilon_{0|N'}}$ is an equivalence, so that we can lift the T-system in $\mathcal{C}^{rs}_{\mathbb{Z}}(\epsilon_{0|N'})$ to $\mathcal{C}^{rs}_{\mathbb{Z}}(\epsilon')$. Since $\mathfrak{tr}^{\epsilon'}_{\epsilon}$ is exact, applying it to the lifted sequence in $\mathcal{C}^{rs}_{\mathbb{Z}}(\epsilon')$ gives us the desired exact sequence by Theorem 4.3.6. □

Remark 4.3.16. The Kirillov–Reshetikhin module $W^{(r)}_{s,a}$ associated with $r \in \{1, \ldots, n-1\}$ and $s \in \mathbb{Z}_{\geq 1}$ over the quantum affine algebra $U'_{q}(\widehat{\mathfrak{sl}}_{n})$ is defined to be the finite-dimensional irreducible module whose Drinfeld polynomial $(P_i(u))_{i=1,\dots,n-1}$ is given by

$$P_i(u) = \begin{cases} \prod_{k=1}^{s}(1 - aq^{2k-2}u) & \text{if } i = r \\ q & \text{if } i \neq r. \end{cases}$$

Then for $r < n$, $\mathcal{W}_{r,\epsilon_{0|n}}(c)$ corresponds to $W_{1,a}^{(r)}$ [84, Remark 3.3] for $a = -o(r)(-q)^{-n}\widetilde{c}$, where $\widetilde{\ }: \Bbbk \to \Bbbk$ is an automorphism induced from $\widetilde{q} = -q^{-1}$ and $o : I \setminus \{0\} \to \{\pm 1\}$ is chosen such that $o(i) = -o(j)$ whenever $a_{ij} \neq 0$. Hence, $\mathcal{W}_{\epsilon}^{r,s}(c)$ corresponds to $W_{s,aq^{2-2s}}^{(r)}$.

It is known that KR modules of quantum affine algebras possess a number of nice properties, most notably the existence of crystal bases [86] and the T-system. The T-system, a family of short exact sequences, is now understood as an exchange relation in the theory of cluster algebras, and hence KR modules are a starting point towards a *monoidal categorification* of cluster algebras [41, 60, 62]. We expect that KR modules for quantum affine superalgebra play a similarly important role (*e.g.* see [75] for crystals when $\epsilon = \epsilon_{M|N}$).

4.3.3 Inverse limit category

We interpret the infinite rank limit involved in super duality as the inverse limit of categories. In this section, we construct inverse limits of $\mathcal{C}(\epsilon)$ and record general properties, following the exposition of [31].

For the remaining of this chapter, we fix $\epsilon^\infty = (\epsilon_i)_{i\geq 1} = (\epsilon_1, \epsilon_2, \dots)$, with infinitely many 0's and 1's for convenience. Take an ascending chain $(\epsilon^{(k)})_{k\geq 1}$ of subsequences of ϵ^∞ such that $\epsilon^\infty = \lim_k \epsilon^{(k)}$ and $\epsilon^{(k)} \in \mathcal{E}$ for all k.

Definition 4.3.17. Define the inverse limit category

$$\mathcal{C}(\epsilon^\infty) = \varprojlim \mathcal{C}(\epsilon^{(k)})$$

to be the category whose

(1) objects are pairs $\mathbb{V} = ((V_k)_{k\geq 1}, (f_k)_{k\geq 1})$ such that

$$V_k \in \mathcal{C}(\epsilon^{(k)}), \quad f_k : \mathfrak{tr}_k^{k+1}(V_{k+1}) \xrightarrow{\cong} V_k \quad (k \geq 1)$$

where $\mathfrak{tr}_k^{k+1} := \mathfrak{tr}_{\epsilon^{(k)}}^{\epsilon^{(k+1)}}$ and f_k is a $\mathcal{U}(\epsilon^{(k)})$-module isomorphism;

(2) morphisms from $\mathbb{V} = ((V_k), (f_k))$ to $\mathbb{W} = ((W_k), (g_k))$ are sequences $\phi = (\phi_k)_{k\geq 1}$ of $\phi_k \in \mathrm{Hom}_{\mathcal{U}(\epsilon^{(k)})}(V_k, W_k)$ which make the following diagram commute for all k

$$\begin{array}{ccc} \mathfrak{tr}_k^{k+1}(V_{k+1}) & \xrightarrow{\ f_k\ } & V_k \\ {\scriptstyle \mathfrak{tr}_k^{k+1}(\phi_{k+1})}\downarrow & & \downarrow{\scriptstyle \phi_k} \\ \mathfrak{tr}_k^{k+1}(W_{k+1}) & \xrightarrow{\ g_k\ } & W_k. \end{array}$$

For each $k \geq 1$, we associate a functor

$$\mathfrak{tr}_k : \mathcal{C}(\epsilon^\infty) \longrightarrow \mathcal{C}(\epsilon^{(k)})$$

given by $\mathfrak{tr}_k(\mathbb{V}) = V_k$ and $\mathfrak{tr}_k(\phi) = \phi_k$, for an object $\mathbb{V} = ((V_k), (f_k))$ and a morphism $\phi = (\phi_k)$.

The inverse limit category is an abelian category with a monoidal structure given by

$$\mathbb{V} \otimes \mathbb{W} = ((V_k \otimes W_k)_{k\geq 1}, (f_k \otimes g_k)_{k\geq 1})$$

where $\mathbb{V} = ((V_k), (f_k))$ and $\mathbb{W} = ((W_k), (g_k))$, and $\mathfrak{tr}_k$ is exact. The following property also follows from standard arguments.

Lemma 4.3.18. *Given a category $\mathcal{C}$ with a family of functors $(F_k : \mathcal{C} \longrightarrow \mathcal{C}(\epsilon^{(k)}))_{k\geq 1}$ such that $\mathfrak{tr}_k^{k+1} \circ F_{k+1} \cong F_k$ for all $k \geq 1$, there exists a functor*

$$F = \varprojlim F_k : \mathcal{C} \longrightarrow \mathcal{C}(\epsilon^\infty)$$

such that $\mathfrak{tr}_k \circ F \cong F_k$ for all $k \geq 1$. Moreover, F is exact if every F_k is exact. If $\mathcal{C}$ is a monoidal category and every F_k is monoidal, then so is F.

Remark 4.3.19. If we take another ascending chain $(\widetilde{\epsilon}^{(k)})_{k\geq 1}$ for ϵ^∞ and construct the inverse limit category $\widetilde{\mathcal{C}}(\epsilon^\infty) = \varprojlim \mathcal{C}(\widetilde{\epsilon}^{(k)})$, then we have an equivalence of categories $\mathcal{C}(\epsilon^\infty) \simeq \widetilde{\mathcal{C}}(\epsilon^\infty)$. Indeed, given any finite subsequence ϵ' of ϵ^∞, we can find $\epsilon^{(k)} > \epsilon'$ so that we can define a truncation $\mathfrak{tr}_{\epsilon'} := \mathfrak{tr}_{\epsilon}^{\epsilon^{(k)}} \circ \mathfrak{tr}_k$, which does not depend on the choice of k. Then these truncations are assembled to a functor $\mathcal{C}(\epsilon^\infty) \to \widetilde{\mathcal{C}}(\epsilon^\infty)$ by Lemma 4.3.18, and vice versa. Therefore, the category $\mathcal{C}(\epsilon^\infty)$ is independent of the choice of an ascending chain $(\epsilon^{(k)})$ and the requirement $\epsilon^{(k)} \in \mathcal{E}$ does not affect $\mathcal{C}(\epsilon^\infty)$ at all.

As observed in Section 4.3.1, fundamental representations, normalized R-matrices and their specializations are compatible with truncations. Hence we may take an object

$$\mathcal{W}_{l,\epsilon^\infty}(x) = \big((\mathcal{W}_{l,\epsilon^{(k)}}(x))_{k\geq 1}, (f_k)_{k\geq 1}\big) \in \mathcal{C}(\epsilon^\infty)$$

for $l \in \mathbb{Z}_{\geq 0}$ and $x \in \Bbbk^\times$, and a morphism

$$R_{(l_1,l_2),\epsilon^\infty}(c_1,c_2) = \big(R_{(l_1,l_2),\epsilon^{(k)}}(c_1,c_2)\big)_{k\geq 1} : \mathcal{W}_{l_1,\epsilon^\infty}(c_1)\otimes\mathcal{W}_{l_2,\epsilon^\infty}(c_2) \longrightarrow \mathcal{W}_{l_2,\epsilon^\infty}(c_2)\otimes\mathcal{W}_{l_1,\epsilon^\infty}(c_1)$$

for $c_1, c_2 \in \Bbbk^\times$ such that $d_{l_1,l_2}(c_1/c_2) \neq 0$.

For $(\boldsymbol{l},\boldsymbol{c}) \in \mathcal{P}^+$, we also put

$$\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l},\boldsymbol{c}) = \mathrm{im} R_{\boldsymbol{l},\epsilon^\infty}(\boldsymbol{c}) = \big(\mathcal{W}_{\boldsymbol{l},\epsilon^{(k)}}(\boldsymbol{c})\big)_{k\geq 1} \in \mathcal{C}(\epsilon^\infty),$$

where $R_{\boldsymbol{l},\epsilon^\infty}(\boldsymbol{c})$ is the composition of $R_{(l_i,l_j),\epsilon^\infty}(c_i,c_j)$ as in the finite rank cases. Observe that $\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l},\boldsymbol{c})$ is nonzero for any $(\boldsymbol{l},\boldsymbol{c}) \in \mathcal{P}^+$. Indeed, if we take a large enough k so that $l_i < N_k$ for all i, then $\mathfrak{tr}^{\epsilon^{(k)}}_{\epsilon_{0|N_k}} \mathcal{W}_{\epsilon^{(k)}}(\boldsymbol{l},\boldsymbol{c}) \cong \mathcal{W}_{\epsilon_{0|N_k}}(\boldsymbol{l},\boldsymbol{c}) \neq 0$, and so $\mathcal{W}_{\epsilon^{(k')}}(\boldsymbol{l},\boldsymbol{c}) \neq 0$ for all $k' > k$. Then the simplicity of $\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l},\boldsymbol{c})$ is immediate from the following easy lemma.

Lemma 4.3.20. *An object $\mathbb{V} \in \mathcal{C}(\epsilon^\infty)$ is simple if it is nonzero and for all k, $\mathfrak{tr}_k\mathbb{V}$ is simple or zero.*

Proof. Let $\mathbb{U}$ be a nonzero subobject of $\mathbb{V}$, with $\mathfrak{tr}_k\mathbb{U}$ regarded as a submodule of $\mathfrak{tr}_k\mathbb{V}$ for all k. For minimal k_0 such that $\mathfrak{tr}_{k_0}\mathbb{U} \neq 0$, we have $\mathfrak{tr}_k\mathbb{U} = \mathfrak{tr}_k\mathbb{V}$ for all $k \geq k_0$ due to the irreducibility of $\mathfrak{tr}_k\mathbb{V}$. But then

$$\mathfrak{tr}_k\mathbb{U} = \mathfrak{tr}_k^{k_0}\mathfrak{tr}_{k_0}\mathbb{U} = \mathfrak{tr}_k^{k_0}\mathfrak{tr}_{k_0}\mathbb{V} = \mathfrak{tr}_k\mathbb{V}$$

for all $k \leq k_0$ as well. □

Proposition 4.3.21. *For $(\boldsymbol{l},\boldsymbol{c}) \in \mathcal{P}^+$, $\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l},\boldsymbol{c})$ is a simple object in $\mathcal{C}(\epsilon^\infty)$.*

We can also define the inverse limit $\mathring{\mathcal{C}}(\epsilon^\infty)$ of $\mathring{\mathcal{C}}(\epsilon^{(k)})$, in which category the analogue of Lemma 4.3.20 holds with the same proof. Namely, we have simple objects

$$V_{\epsilon^\infty}(\lambda) = \Big((V_{\epsilon^{(k)}}(\lambda)), (\mathring{f}_k)\Big)$$

for $\lambda \in \mathscr{P}$, where $V_{\epsilon^{(k)}}(\lambda)$ is understood to be zero if $\lambda \notin \mathscr{P}_{M_k|N_k}$. Applying Lemma 4.3.18 to forgetful functors, we may regard $\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l}, \boldsymbol{c})$ as an object in $\mathring{\mathcal{C}}(\epsilon^\infty)$. Then we obtain the following classical decomposition of simple objects in $\mathcal{C}(\epsilon^\infty)$.

Proposition 4.3.22. *For $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^+$, $\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l}, \boldsymbol{c})$ is a semisimple object in $\mathring{\mathcal{C}}(\epsilon^\infty)$ and*

$$\mathcal{W}_{\epsilon^\infty}(\boldsymbol{l}, \boldsymbol{c}) = \bigoplus_{\lambda \in \mathscr{P}} V_{\epsilon^\infty}(\lambda)^{\oplus m_\lambda^{(\boldsymbol{l}, \boldsymbol{c})}}$$

where $m_\lambda^{(\boldsymbol{l}, \boldsymbol{c})}$ is the multiplicity in Theorem 4.3.8.

Remark 4.3.23. In the formalism of super duality for classical Lie superalgebras, the inverse limit of module categories can be understood as a module category of infinite rank Lie superalgebras. We expect that our limit $\mathcal{C}(\epsilon^\infty)$ is also closely related to certain module category of infinite rank quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_\infty)$ (*e.g.* [38]).

Definition 4.3.24. (1) $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$ is the full subcategory of $\mathcal{C}(\epsilon^\infty)$ consisting of $\mathbb{V} = (V_k)$ such that $V_k \in \mathcal{C}_{\mathbb{Z}}(\epsilon^{(k)})$ for all k and the composition length of V_k stabilizes for sufficiently large k.

(2) For $\ell \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$ is defined to be the full subcategory of $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$ consisting of $\mathbb{V} = (V_k)$ with $V_k \in \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^{(k)})$ for all k.

Note that a finite length condition is imposed in the definition of $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$, and so it is not exactly the inverse limit of the categories $\mathcal{C}_{\mathbb{Z}}(\epsilon^{(k)})$. Still, we have the following result.

Proposition 4.3.25. (1) *We have $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty) = \bigoplus_{\ell \geq 0} \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$.*

(2) *For $\ell \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty) = \varprojlim \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^{(k)})$, the inverse limit category associated with $\{\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^{(k)})\}_{k \geq 1}$ defined as in Definition 4.3.17.*

Proof. (1) We prove that any $\mathbb{V} = (V_k) \in \mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$ is a direct sum of objects from $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$ for finitely many ℓ. Write $V_k = \bigoplus_\ell V_k^\ell$ for $V_k^\ell \in \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon)$. If we set $\ell_k = \max\{\ell \,|\, V_k^\ell \neq 0\}$, then the sequence $(\ell_k)_{k \geq 1}$ is bounded above. Otherwise, $\ell(V_k)$ grows indefinitely as $\mathfrak{tr}_k^{k'} V_{k'}^{\ell_k} = V_k^{\ell_k} \neq 0$, which is impossible by definition of $\mathcal{C}_{\mathbb{Z}}(\epsilon^\infty)$. Hence we have $\mathbb{V} = \bigoplus_{\ell \leq \ell_n} \mathbb{V}^\ell$ for sufficiently large n, where $\mathbb{V}^\ell = (V_k^\ell)_{k \geq 1} \in \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$.

(2) The inverse limit $\mathcal{A} \coloneqq \varprojlim \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^{(k)})$, clearly contains $\mathcal{C}^\ell_{\mathbb{Z}}(\epsilon^\infty)$ as a subcategory. Take $\mathbb{V} = (V_k) \in \mathcal{A}$, and let us check that the composition length $\ell(V_k)$ of V_k is stabilized for

large k. Recall from Theorem 4.2.22 that the simple objects in $\mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k)})$ are of the form $\mathcal{W}_{\epsilon^{(k)}}(\boldsymbol{l}, \boldsymbol{c})$ for $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}_{\mathbb{Z}}^{+}(\epsilon^{(k)})$ with $\ell = \sum l_i$. Moreover, as a $\mathring{\mathcal{U}}(\epsilon^{(k)})$-module, $\mathcal{W}_{\epsilon^{(k)}}(\boldsymbol{l}, \boldsymbol{c})$ is a direct sum of $V_{\epsilon^{(k)}}(\lambda)$ for partitions λ of ℓ. But if the number of 0's in $\epsilon^{(k)}$ exceed ℓ, then every partition λ of ℓ are in $\mathscr{P}_{\epsilon^{(k)}}$ and $V_{\epsilon^{(k)}}(\lambda)$ is nonzero. Thus the composition length of V_k as a $\mathring{\mathcal{U}}(\epsilon^{(k)})$-module is stabilized once $\epsilon^{(k)}$ has at least ℓ 0's, and then so is $\ell(V_k)$. □

Remark 4.3.26. Recall that we know all the simple objects in the category $\mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k)})$ (Theorem 4.2.22), from which it also follows that $\mathfrak{tr}_k^{k'}$ sends simples to simples or zero. Using a criterion [31, Lemma 4.1.5] for simplicity of objects in inverse limit categories and the parametrization [16, 58] (see Section 2.2.2) of finite-dimensional irreducible representations of $U_q'(\widehat{\mathfrak{sl}}_n)$ (see also [33] for $U_q'(\widehat{\mathfrak{gl}}_n)$), we obtain a complete classification of simple objects in $\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty})$. Namely, they are of the form $\mathcal{W}_{\epsilon^{\infty}}(\boldsymbol{l}, \boldsymbol{c})$ for $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}_{\mathbb{Z}}^{+}$, and the pair $(\boldsymbol{l}, \boldsymbol{c})$ is uniquely determined up to permutation.

4.3.4 Super duality

Finally, we are ready to establish the super duality. By Lemma 4.3.18, replacing $\mathcal{C}(\epsilon^{\infty})$ with $\mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{\infty})$, the duality functors $(\mathcal{F}_{\epsilon^{(k)},\ell})_{k\geq 1}$ now induce exact functors

$$\mathcal{F}_{\epsilon^{\infty},\ell} : R(\ell)\text{-mod}_0 \longrightarrow \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{\infty}),$$
$$\mathcal{F}_{\epsilon^{\infty}} = \bigoplus_{\ell\geq 0} \mathcal{F}_{\epsilon^{\infty},\ell} : R\text{-mod}_0 \longrightarrow \mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}),$$

where $\mathcal{F}_{\epsilon^{\infty}}$ is monoidal in addition. The following result, which follows from a general property of inverse limit categories, is the final step.

Proposition 4.3.27. *The functor $\mathcal{F}_{\epsilon^{\infty},\ell}$ is an equivalence of categories, and so $\mathcal{F}_{\epsilon^{\infty}}$ is a monoidal equivalence.*

Proof. Recall that for sufficiently large k, $\mathcal{F}_{\epsilon^{(k)},\ell}$ and $\mathfrak{tr}_k^{k+1} : \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k+1)}) \longrightarrow \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k)})$ are equivalences by Theorem 4.3.9 and Corollary 4.3.14, respectively. The latter one also implies that $\mathfrak{tr}_k : \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k+1)}) \longrightarrow \mathcal{C}_{\mathbb{Z}}^{\ell}(\epsilon^{(k)})$ is an equivalence for large k, by [31, Lemma 3.1.3]. Since $\mathcal{F}_{\epsilon^{(k)},\ell} \cong \mathfrak{tr}_k \circ \mathcal{F}_{\epsilon^{\infty},\ell}$ for any $k \geq 1$, the result follows. □

Put

$$\underline{\epsilon}^{\infty} = (\underline{\epsilon}_i)_{i\geq 1} = (0, 0, 0, \dots), \quad \overline{\epsilon}^{\infty} = (\overline{\epsilon}_i)_{i\geq 1} = (1, 1, 1, \dots).$$

Let $\mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty})$ and $\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})$ be defined as above, associated with ascending chains $(\underline{\epsilon}^{(k)} = \epsilon_{k|0})$ and $(\overline{\epsilon}^{(k)} = \epsilon_{0|k})$ respectively. Then we can find an increasing sequence $\{r_k\}$ and $\{s_k\}$ such that $\underline{\epsilon}^{(k)} < \epsilon^{(r_k)}$ and $\overline{\epsilon}^{(k)} < \epsilon^{(s_k)}$ and we set

$$\mathcal{S}_{k|0} = \mathfrak{tr}^{\epsilon^{(r_k)}}_{\epsilon_{k|0}} \circ \mathfrak{tr}_{r_k}, \quad \mathcal{S}_{0|k} = \mathfrak{tr}^{\epsilon^{(s_k)}}_{\epsilon_{0|k}} \circ \mathfrak{tr}_{s_k},$$

which does not depend on r_k, s_k.

$$\begin{array}{ccc} & \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon^{\infty}) & \\ {}^{\mathcal{S}_{k|0}}\swarrow & & \searrow^{\mathcal{S}_{0|k}} \\ \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon_{k|0}) & & \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon_{0|k}) \end{array}$$

Then they induce exact functors $\mathcal{S}_{\infty|0}$ and $\mathcal{S}_{0|\infty}$

$$\begin{array}{ccc} & \mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon^{\infty}) & \\ {}^{\mathcal{S}_{\infty|0}}\swarrow & & \searrow^{\mathcal{S}_{0|\infty}} \\ \mathcal{C}^{\ell}_{\mathbb{Z}}(\underline{\epsilon}^{\infty}) & & \mathcal{C}^{\ell}_{\mathbb{Z}}(\overline{\epsilon}^{\infty}), \end{array}$$

which sum up to exact monoidal functors

$$\begin{array}{ccc} & \mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}) & \\ {}^{\mathcal{S}_{\infty|0}}\swarrow & & \searrow^{\mathcal{S}_{0|\infty}} \\ \mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty}) & & \mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty}), \end{array}$$

Now we come to a climax of this chapter, which can be viewed as a quantum affine analogue of the super duality introduced in Section 4.1.

Theorem 4.3.28. *The functors $\mathcal{S}_{\infty|0}$ and $\mathcal{S}_{0|\infty}$ are equivalences of monoidal categories.*

As an application, let us give a description of Grothendieck ring of the category $\mathcal{C}_{\mathbb{Z}}(\epsilon)$ for any ϵ. Recall that the Grothendieck ring of $\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty})$ is $\mathbb{Z}_{\geq 0}$-graded:

$$K(\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty})) = \bigoplus_{\ell \geq 0} K(\mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon^{\infty})).$$

Let $S = \{(l,a) \,|\, l \in \mathbb{Z}_{\geq 0},\, a \in l+1+2\mathbb{Z}\}$ and

$$R = \mathbb{Z}[t_{l,a}]_{(l,a)\in S}$$

the polynomial ring generated by $t_{l,a}$, graded by $\deg t_{l,a} = l$.

Proposition 4.3.29. *There is an isomorphism of graded rings*

$$\begin{aligned} K(\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty})) &\longrightarrow R \\ [\mathcal{W}_{l,\epsilon^{\infty}}(q^a)] &\longmapsto t_{l,a}. \end{aligned}$$

Proof. For $k \geq 1$, it is well-known [33] (see also [34, Corollary 2]) that $K(\mathcal{C}_{\mathbb{Z}}(\epsilon_{0|k}))$ is isomorphic to

$$R_k \coloneqq \mathbb{Z}[t_{l,a}]_{1\leq l\leq k,(l,a)\in S} \subset R$$

by matching $\left[\mathcal{W}_{l,\epsilon_{0|k}}(q^a)\right]$ with $t_{l,a}$, which respects the grading. Since $\mathfrak{tr}^{\epsilon_{0|k+1}}_{\epsilon_{0|k}}$ induces a map $R_{k+1} \longrightarrow R_k$ given by $t_{k+1,a} = 0$ for all a, we derive

$$K(\mathcal{C}^{\ell}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})) = \varprojlim K(\mathcal{C}^{\ell}_{\mathbb{Z}}(\epsilon_{0|k})) \cong \varprojlim R^{\ell}_k = R^{\ell}$$

and so as a graded ring, $K(\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})) \cong R$. Since $K(\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})) \cong K(\mathcal{C}_{\mathbb{Z}}(\epsilon^{\infty}))$ by Theorem 4.3.28, we obtain the desired isomorphism. ☐

In particular, it induces a surjective ring homomorphism

$$\begin{aligned} R &\longrightarrow K(\mathcal{C}_{\mathbb{Z}}(\epsilon^{(k)})) \\ t_{l,a} &\longmapsto \left[\mathcal{W}_{l,\epsilon^{(k)}}(q^a)\right] \end{aligned}$$

by composing with truncation $\mathfrak{tr}_k$. Hence, if there is a relation that holds in sufficiently large ranks in the Grothedieck ring of the category of finite-dimensional modules over $U'_q(\widehat{\mathfrak{gl}}_n)$, then we can lift it to $K(\mathcal{C}_{\mathbb{Z}}(\epsilon))$ for any ϵ. The T-system in the previous subsection is one such example.

Corollary 4.3.30. *Let $\epsilon \in \mathcal{E}$ and $(\boldsymbol{l},\boldsymbol{c}) \in \mathcal{P}^+_{\mathbb{Z}}(\epsilon)$ be given. If we have*

$$\left[\mathcal{W}_{\epsilon_{0|k}}(\boldsymbol{l},\boldsymbol{c})\right] = \chi\left(\left\{\left[\mathcal{W}_{l,\epsilon_{0|k}}(q^a)\right]\right\}\right)$$

for some $\chi \in R$ and a sufficiently large k, then we obtain the same identity in $K(\mathcal{C}_{\mathbb{Z}}(\epsilon))$:

$$[\mathcal{W}_{\epsilon}(\boldsymbol{l}, \boldsymbol{c})] = \chi\left(\{[\mathcal{W}_{l,\epsilon}(q^a)]\}\right).$$

Remark 4.3.31. Recall that in the super duality in Section 4.1, we have $\mathfrak{g} = \overline{\mathfrak{g}} = \mathfrak{gl}_{\infty}$ and so super duality induces an autoequivalence on the category $\mathcal{O} = \overline{\mathcal{O}}$ which categorifies the involution $s_{\lambda} \mapsto s_{\lambda^t}$ on the ring of symmetric functions.

In our case, there exists a $\mathbb{Q}$-algebra isomorphism $\widetilde{\cdot} : \mathcal{U}(\epsilon_{0|n}) \longrightarrow \mathcal{U}(\epsilon_{n|0})$ given by

$$\widetilde{q} = -q^{-1}, \quad \widetilde{e}_i = e_i, \quad \widetilde{f}_i = f_i, \quad \widetilde{k}_{\mu} = k_{\mu}.$$

This induces an equivalence between $\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty})$ and $\mathcal{C}_{\mathbb{Z}}(\underline{\epsilon}^{\infty})$. If we identify their Grothendieck rings under this equivalence, then our equivalence $\mathcal{S}_{\infty|0} \circ \mathcal{S}_{0|\infty}^{-1}$ from super duality induces an involution on $K(\mathcal{C}_{\mathbb{Z}}(\overline{\epsilon}^{\infty}))$, which can be viewed as a quantum affine analogue of the above involution on the ring of symmetric functions.

Chapter 5

Oscillator representations of $U_q(\widehat{\mathfrak{gl}}_n)$

In this chapter, we initiate the study of q-oscillator representations of $U_q(\widehat{\mathfrak{gl}}_n)$. They are (level 0) infinite-dimensional representations, but still share similar tensor product structures with finite-dimensional representations. On one hand, such a similarity stems from the fact that the spectral decompositions of R-matrices for q-oscillators are very close to the ones for finite-dimensional representations. On the other hand, they can be viewed as another bosonic counterpart of finite-dimensional representations, and hence should be intimately related to them under the super duality philosophy.

We define the category $\widehat{\mathcal{O}}_{\mathrm{osc}}$ of q-oscillator representations of $U_q(\widehat{\mathfrak{gl}}_n)$ as affinizations of the one $\mathcal{O}_{\mathrm{osc}}$ of q-oscillator representations of $U_q(\mathfrak{gl}_n)$ (*cf.* Definition 4.2.1). Namely, the role of polynomial $\mathring{\mathcal{U}}(\epsilon)$-modules in Chapter 4 is now played by q-oscillator representations of $U_q(\mathfrak{gl}_n)$, which are q-analogues of oscillator representations of $\mathfrak{gl}_n$.

Hence, to study $\widehat{\mathcal{O}}_{\mathrm{osc}}$, we first need to reproduce the results in Section 3.2 for q-oscillators of $U_q(\mathfrak{gl}_n)$. This is done in Section 5.2 whose main result is the decomposition (5.2.2), from which we deduce the semisimplicity and the decomposition of tensor products of two irreducible q-oscillator representations corresponding to single rows.

Then we may adopt the same approach as in Chapter 4. Fundamental q-oscillator representations $\mathcal{W}_l^{\mathrm{osc}}(x)$ and normalized R-matrices are defined. We compute the spectral decomposition of normalized R-matrices, which allows us to construct irreducible objects in $\widehat{\mathcal{O}}_{\mathrm{osc}}$ by fusion, as in Section 4.2.3. Consequently, we obtain a natural correspondence between irreducible q-oscillator representations and irreducible finite-dimensional representations obtained from the same data $(\boldsymbol{l}, \boldsymbol{c})$ by fusion constructions, which is more striking than the one in Chapter 4 in that q-oscillator representations are infinite-dimensional.

As a first step toward a *quantum affine super duality* for this correspondence, we introduce a analogous category $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$ for generalized quantum group $\mathcal{U}(\boldsymbol{\epsilon})$, where $\boldsymbol{\epsilon} = (010\ldots 10)$ is an alternating (01)-sequence originated from classical super duality. We can repeat the above constructions in the super case, and define truncation functors from $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$ to $\widehat{\mathcal{O}}_{\mathrm{osc}}$, and to a category of finite-dimensional representations of $U_q(\widehat{\mathfrak{gl}}_n)$. However, to establish a super-duality-type equivalence seems to be much harder, as is the same in classical theory. Instead we shall give some evidences, including T-systems and a relation to finite-dimensional representations of $\mathcal{U}(\epsilon_{M|N})$.

The results of this chapter is based on [73], with a more uniform account as in [74].

5.1 Howe duality and oscillator representations of $\mathfrak{gl}_n$

To motivate q-oscillator representations, let us briefly review a pair of Howe dualities, from which a nice correspondence between oscillators and finite-dimensional representations of $\mathfrak{gl}_n$ is obtained.

The celebrated skew Howe duality of type A refers to the following $(\mathfrak{gl}_n, GL_\ell)$-bimodule decomposed as a direct sum of simples:

$$W^{\otimes \ell} := \Lambda(\mathbb{C}^n)^{\otimes \ell} = \bigoplus_{\substack{\ell(\lambda)\leq \ell \\ \ell(\lambda^t)\leq n}} V_{\mathfrak{gl}_n}(\lambda^t) \otimes V_{GL_\ell}(\lambda) \tag{5.1.1}$$

where $V_{GL_\ell}(\lambda)$ (resp. $V_{\mathfrak{gl}_n}(\lambda^t)$) is the finite-dimensional irreducible representation of GL_ℓ (resp. $\mathfrak{gl}_n$) associated with $\lambda \in \mathscr{P}$ (resp. λ^t). The joint action of $\mathfrak{gl}_n$ and GL_ℓ on the tensor power of *fermionic Fock space* W has various nice properties, such as double centralizer property, semisimplicity and multiplicity-freeness. This gives rise to a method to understand representations of $\mathfrak{gl}_n$ occurring in the bimodule in terms of the representation theory of GL_ℓ, and vice versa.

More precisely, let $\overline{\mathcal{F}}_n$ be the (semisimple) category generated by $V_{\mathfrak{gl}_n}(\lambda^t)$ for all $\lambda \in \mathscr{P}$ with $\ell(\lambda^t) \leq n$, which is closed under tensor products. We denote by $K(GL_\ell)$ the Grothendieck group of the category of polynomial representations of GL_ℓ, and then $K(GL) = \bigoplus_{\ell\geq 0} K(GL_\ell)$ has a coalgebra structure given by the branching rule. Then the Grothendieck ring of $\overline{\mathcal{F}}_n$ is a homomorphic image of the dual of the coalgebra $K(GL)$, hence the name *duality*.

On the other hand, if we replace $\Lambda(\mathbb{C}^n)$ with the symmetric algebra $S(\mathbb{C}^n)$, we obtain another duality called the Howe duality of type A,

$$S(\mathbb{C}^n)^{\otimes \ell} = \bigoplus_{\ell(\lambda) \leq \ell, n} V_{\mathfrak{gl}_n}(\lambda) \otimes V_{GL_\ell}(\lambda).$$

Exactly the same argument applies to the category $\mathcal{F}_n$ generated by $V_{\mathfrak{gl}_n}(\lambda)$ with $\ell(\lambda) \leq n$. In fact, $\mathcal{F}_n = \overline{\mathcal{F}}_n$ and is the category of polynomial representations of $\mathfrak{gl}_\mathfrak{n}$. Thus we obtain two ring surjections

$$\begin{array}{ccc} & K(GL)^* & \\ \swarrow & & \searrow \\ K(\mathcal{F}_n) & & K(\mathcal{F}_n), \end{array}$$

where $[V_{\mathfrak{gl}_n}(\lambda^t)]$ and $[V_{\mathfrak{gl}_n}(\lambda)]$ correspond to the same element in $K(GL)^*$ (*cf.* Section 4.1).

In general, suppose we have two dualities $(\mathfrak{g}, G_\ell)$ on $W^{\otimes \ell}$ and $(\overline{\mathfrak{g}}, G_\ell)$ on $\overline{W}^{\otimes \ell}$ for Lie (super)algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ and a Lie group (or algebra) G_ℓ for all $\ell \geq 1$. If the irreducible G_ℓ-modules occurring in both dualities are the same, then we can expect that the semisimple monoidal categories generated by irreducible $\mathfrak{g}$- and $\overline{\mathfrak{g}}$-modules in W and $\overline{W}$ have parallel structures.

We shall recall another Howe duality [44] which is paired with the skew Howe duality (5.1.1) in the above sense. Let us fix $r \in \{2, 3, \ldots, n-2\}$. A new *bosonic Fock space* is the same as $S(\mathbb{C}^n)$ as a vector space, but with a twisted $\mathfrak{gl}_n$-action: Put

$$W^{\text{osc}} := S(\mathbb{C}^{r*} \oplus \mathbb{C}^{n-r}) = \mathbb{C}[x_1^*, \ldots, x_r^*, x_{r+1}, \ldots, x_n].$$

It has a $\mathfrak{gl}_r \oplus \mathfrak{gl}_{n-r}$-action induced from the natural $\mathfrak{gl}_{n-r}$-module $\mathbb{C}^{n-r}$ and the dual $\mathbb{C}^{r*}$ of the one of $\mathfrak{gl}_r$. To extend it to a $\mathfrak{gl}_n$-action, we only need to define the actions of e_r and f_r, and they are given by

$$e_r = -\frac{\partial}{\partial x_r^*}\frac{\partial}{\partial x_{r+1}}, \quad f_r = x_r^* x_{r+1}.$$

To describe the irreducible $\mathfrak{gl}_n$-modules appearing in $(W^{\text{osc}})^{\otimes \ell}$, let

$$\mathcal{P}(GL_\ell) = \{\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell \mid \lambda_1 \geq \cdots \geq \lambda_\ell\}$$

be the set of generalized partitions of length ℓ. It is well-known that $\mathcal{P}(GL_\ell)$ parametrizes the finite-dimensional irreducible representations of GL_ℓ (with integral weights), and we denote by $V_{GL_\ell}(\lambda)$ the one associated with $\lambda \in \mathcal{P}(GL_\ell)$. Let us also set $\mathcal{P}(GL_0) = \{\emptyset\}$.

When $\lambda \in \mathcal{P}(GL_\ell)$ can be written as

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_s > \lambda_{s+1} = \cdots = \lambda_t = 0 > \lambda_{t+1} \geq \cdots \geq \lambda_\ell)$$

for some $s < t$, we put

$$\lambda^+ = (\lambda_1 \geq \cdots \geq \lambda_s), \quad \lambda^- = (-\lambda_\ell \geq \cdots \geq -\lambda_{t+1}).$$

If instead $\lambda_\ell > 0$ (resp. $\lambda_1 < 0$), then we put $\lambda^+ = \lambda$, $\lambda^- = \emptyset$ (resp. $\lambda^+ = \emptyset$, $\lambda^- = (-\lambda_\ell, \ldots, -\lambda_1)$). Clearly, $\lambda^\pm$ are partitions[1]. Set

$$\mathcal{P}(GL_\ell)_{(r,n-r)} = \{\lambda \in \mathcal{P}(GL_\ell) \,|\, \ell(\lambda^-) \leq r,\, \ell(\lambda^+) \leq n-r\}.$$

For $\lambda \in \mathcal{P}(GL_\ell)_{(r,n-r)}$, we define a weight

$$\varpi_{\lambda,r} = -\ell\varpi_r + \sum_{i=1}^{s} \lambda_i \delta_{r+i} + \sum_{j=t+1}^{\ell} \lambda_j \delta_{r-(j-\ell)} \in P_{\text{fin}} \tag{5.1.2}$$

where $\varpi_r = \delta_1 + \cdots + \delta_r$ is the r-th fundamental weight of $\mathfrak{gl}_\mathfrak{n}$.

We denote by V^λ the irreducible highest weight $\mathfrak{gl}_n$-module of highest weight $\varpi_{\lambda,r}$. By convention, we put $V^\emptyset = \mathbb{C}$, the trivial representation. Since $\varpi_{\lambda,r}$ is never dominant, every V^λ is infinite-dimensional. More precisely, it is contained in the parabolic BGG category associated with the Levi subalgebra of $\mathfrak{gl}_n$ generated by e_i, f_i ($i \neq r$) and $\mathfrak{h}$. Now we can state the desired duality.

Theorem 5.1.1 ([44])**.** *There exists a GL_ℓ-action on $(W^{\text{osc}})^{\otimes \ell}$ which commutes with the $\mathfrak{gl}_n$-action, and as a $(\mathfrak{gl}_n, GL_\ell)$-module $(W^{\text{osc}})^{\otimes \ell}$ has the following multiplicity-free decomposition*

$$(W^{\text{osc}})^{\otimes \ell} = \bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(r,n-r)}} V^\lambda \otimes V_{GL_\ell}(\lambda).$$

We call V^λ an irreducible *oscillator representation* of $\mathfrak{gl}_n$. They were first studied in

[1]Partitions are distinguished from generalized partitions, in that they consist only of positive integers.

connection with unitarizable representations of real Lie groups, in different names such as Segal-Shale-Weil representations or metaplectic representations [63].

Let $O_{\rm osc}$ be the category of $\mathfrak{gl}_n$-modules V such that

(1) $V = \bigoplus_{\mu \in P_{\rm fin}} V_\mu$ with $\dim V_\mu < \infty$ and $\mathrm{wt}(V)$ is finitely dominated,

(2) $V = \bigoplus_{\ell \geq 0} V_\ell$ where V_ℓ is a direct sum of V^λ's for $\lambda \in \mathcal{P}(GL_\ell)_{(r,n-r)}$, and $V_\ell = 0$ for all sufficiently large ℓ.

Thanks to the duality, $O_{\rm osc}$ is a semisimple monoidal category whose monoidal structure is given by the branching rule of GL_ℓ. Namely, for given $\lambda \in \mathcal{P}(GL_\ell)_{(r,n-r)}$ and $\mu \in \mathcal{P}(GL_{\ell'})_{(r,n-r)}$, we have

$$V^\lambda \otimes V^\mu = \bigoplus_{\nu \in \mathcal{P}(GL_{\ell+\ell'})_{(r,n-r)}} (V^\nu)^{\oplus c^\nu_{\lambda\mu}}$$

where $c^\nu_{\lambda\mu}$ is the multiplicity of $V_{GL_\ell}(\lambda) \otimes V_{GL_{\ell'}}(\mu)$ in the restriction of the $GL_{\ell+\ell'}$-module $V_{GL_{\ell+\ell'}}(\nu)$ to $GL_\ell \times GL_{\ell'}$.

On the other hand, we know from (5.1.1) that the branching multiplicity $c^\nu_{\lambda\mu}$ is also equal to the multiplicity of $V_{\mathfrak{gl}_n}(\nu)$ in $V_{\mathfrak{gl}_n}(\lambda) \otimes V_{\mathfrak{gl}_n}(\mu)$, if λ, μ, ν are partitions. Hence to compute $c^\nu_{\lambda\mu}$ in general, we have to take a tensor product with enough power of the determinant representation and then compute the tensor product multiplicity. In turn, tensor products of two irreducible oscillator representations are in general of infinite length, as shown in the following example.

Example 5.1.2. For $l, m \in \mathbb{Z}$, we have

$$V^{(l)} \otimes V^{(m)} = V^{(l_1,l_2)} \oplus V^{(l_1+1,l_2-1)} \oplus \cdots = \bigoplus_{t \in \mathbb{Z}_{\geq 0}} V^{(l_1+t,l_2-t)} \tag{5.1.3}$$

where $l_1 = \max\{l,m\}$ and $l_2 = \min\{l,m\}$.

Now recall that this pair of Howe dualities consists of an exterior power and a symmetric power, or a fermionic and a bosonic Fock space. This alludes to the possibility of supersymmetric construction, with accompanied super duality explaining the correspondence of V^λ and $V_{\mathfrak{gl}_n}(\lambda)$. Let us briefly explain how to construct the super duality, which is actually more intricate than the one in Section 4.1.

Introduce the following (01)-sequences:

$$\boldsymbol{\epsilon} = \epsilon^{(a,b)} := (\epsilon^{(a)}, \epsilon^{(b)}, 0), \quad \epsilon^{(a)} = (\underbrace{0, 1, 0, 1, \ldots, 0, 1}_{2a}) \tag{5.1.4}$$

for $a, b \geq 1$. We put $r_{\boldsymbol{\epsilon}} = 2a$, which plays the role of r in the above construction for $\mathfrak{gl}_n$. Let us also take the following subsequences of $\boldsymbol{\epsilon}$:

$$\underline{\boldsymbol{\epsilon}} = (0^a, 0^{b+1}), \quad \overline{\boldsymbol{\epsilon}} = (1^a, 1^b), \quad \varepsilon = (0^a, 1^b) \tag{5.1.5}$$

with $r_{\underline{\boldsymbol{\epsilon}}} = r_{\overline{\boldsymbol{\epsilon}}} = r_\varepsilon = a$.

For $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}, \varepsilon$, let $\mathfrak{gl}_\epsilon^e$ be the central extension of $\mathfrak{gl}_\epsilon$ at α_{r_ϵ} (see Section 5.2.2 for a precise definition). Let $\mathcal{O}_\epsilon$ be a (version of) parabolic BGG category of $\mathfrak{gl}_\epsilon^e$, associated with the Levi subalgebra $\mathfrak{l}_\epsilon$ generated by e_i, f_i ($i \in I \setminus \{r_\epsilon\}$) and $\mathfrak{h}$, defined as in Section 4.1. The extra central extension is to define truncation functors, which constitute the following diagram:

$$\begin{array}{ccccccc} & & & \mathcal{O}_{\boldsymbol{\epsilon}} & & & \\ & & \swarrow & & \searrow & & \\ & \mathcal{O}_{(\epsilon^{(a)}, 1^b)} & & & & \mathcal{O}_{(0^a, \epsilon^{(b)}, 0)} & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{O}_{\overline{\boldsymbol{\epsilon}}} & & & \mathcal{O}_{\epsilon_{a|b}} & & & \mathcal{O}_{\underline{\boldsymbol{\epsilon}}}. \end{array} \tag{5.1.6}$$

Here the categories in the middle row are defined in a similar manner.

The category O_{osc} is a full subcategory of $\mathcal{O}_{\underline{\boldsymbol{\epsilon}}}$. Moreover, for each $\lambda \in \mathcal{P}(GL_\ell)_{(r,n-r)}$, one can construct a super analogue $V_{\boldsymbol{\epsilon}}^\lambda \in \mathcal{O}_{\boldsymbol{\epsilon}}$ such that $\mathfrak{tr}_{\underline{\boldsymbol{\epsilon}}}^{\boldsymbol{\epsilon}}\left(V_{\boldsymbol{\epsilon}}^\lambda\right) = V_{\underline{\boldsymbol{\epsilon}}}^\lambda$, while $\mathfrak{tr}_{\overline{\boldsymbol{\epsilon}}}^{\boldsymbol{\epsilon}}\left(V_{\boldsymbol{\epsilon}}^\lambda\right)$ is the finite-dimensional irreducible $\mathfrak{gl}_{a+b}$-module that corresponds to the same GL_ℓ-module as $V_{\underline{\boldsymbol{\epsilon}}}^\lambda$ in the above pair of Howe dualities.

Taking limits $a, b \to \infty$ properly, we obtain equivalences of categories

$$\begin{array}{ccccc} & & \mathcal{O}_{\boldsymbol{\epsilon}} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{O}_{\overline{\boldsymbol{\epsilon}}} & & \mathcal{O}_{\epsilon_{\infty|\infty}} & & \mathcal{O}_{\underline{\boldsymbol{\epsilon}}}, \end{array}$$

under which the (inverse limit of) category O_{osc} is equivalent to the (inverse limit of) category of finite-dimensional representations.

The goal of this chapter is to establish a quantum affine analogue of the above correspondence of irreducible oscillators and irreducible finite-dimensional representations, and give some ideas towards the corresponding quantum affine super duality.

Remark 5.1.3. The name *oscillator* representation is coined by Howe, as a partner of spin representations (see the end of [44, Section 2]). Indeed, oscillator representations of $\mathfrak{sp}_{2n}$ are constructed by replacing the Clifford algebra in a realization of spin representations of $\mathfrak{so}_{2n}$ with the Weyl algebra. Moreover, the corresponding Howe duality for oscillator representations is obtained by directly switching the exterior power in the skew Howe duality of type D (for spin representations) to the symmetric power, without a twist.

Recently, a quantum affine analogue of this spin-oscillator correspondence is obtained in [74], namely between finite-dimensional representations of $U'_q(X_n^{(1)})$ and q-oscillator representations of $U'_q(Y_n^{(1)})$ for $(X,Y) = (C,D)$ or (D,C). This enables a computation of spectral decompositions of normalized R-matrices for fundamental representations of $U'_q(\widehat{\mathfrak{sp}}_{2n})$, which was unknown before.

5.2 Oscillator representations of $U_q(\mathfrak{gl}_n)$

We first introduce q-oscillator representations of $U_q(\mathfrak{gl}_n)$. Since the constructions are uniform, we may consider super cases at the same time.

For the remaining of this chapter, we assume the following notations, some of which override the ones in Chapter 3:

- We will freely use the notations in Section 5.1.
- Each $\epsilon \in \mathcal{E}_n$ is implicitly accompanied with a choice of $r \in I \setminus \{0, 1, n-1\}$, and we put

$$\mathbb{I}^- = \{1, \dots, r\}, \quad \mathbb{I}^+ = \{r+1, \dots, n\},$$
$$\epsilon_- = (\epsilon_1, \dots, \epsilon_r), \quad \epsilon_+ = (\epsilon_{r+1}, \dots, \epsilon_n).$$

- $P_{\mathrm{fin},\epsilon} = \mathbb{Z}\Lambda_{r,\epsilon} \oplus \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_n$ with a symmetric bilinear form $(\,\cdot\,|\,\cdot\,)$ given by

$$(\delta_i, \delta_j) = (-1)^{\epsilon_i}\delta_{ij}, \quad (\Lambda_{r,\epsilon}|\Lambda_{r,\epsilon}) = 0, \quad (\delta_i, \Lambda_{r,\epsilon}) = \begin{cases} 0 & \text{if } i \in \mathbb{I}^- \\ 1 & \text{if } i \in \mathbb{I}^+. \end{cases}$$

- $P_{\geq 0,\epsilon} = \mathbb{Z}\Lambda_{r,\epsilon} - \sum_{i\in\mathbb{I}^-}\mathbb{Z}_{\geq 0}\delta_i + \sum_{j\in\mathbb{I}^+}\mathbb{Z}_{\geq 0}\delta_j$.
- $\mathbf{q}(\,\cdot\,,\,\cdot\,)$, $\widehat{\mathbf{q}}(\,\cdot\,,\,\cdot\,)$: $\Bbbk^\times$-valued symmetric biadditive forms on $P_{\mathrm{fin},\epsilon}$ defined by

$$\mathbf{q}(\lambda,\mu) = \prod q_i^{c_i d_i}, \quad \widehat{\mathbf{q}}(\lambda,\mu) = q^{\sum_{i\in\mathbb{I}^+}(\ell' c_i + \ell d_i)}\mathbf{q}(\lambda,\mu)$$

  for $\lambda = \ell\Lambda_{r,\epsilon} + \sum c_i\delta_i$, $\mu = \ell'\Lambda_{r,\epsilon} + \sum d_i\delta_i \in P_{\mathrm{fin},\epsilon}$.
- For $\epsilon' = \epsilon \setminus \{\epsilon_i\}$, we revise the algebra homomorphism $\phi^{\epsilon}_{\epsilon'} : \mathcal{U}(\epsilon') \to \mathcal{U}(\epsilon)$ to $\widetilde{\phi}^{\epsilon}_{\epsilon'}$ and the truncation $\mathfrak{tr}^{\epsilon}_{\epsilon'}(V)$ of a $\mathcal{U}(\epsilon)$-module V, according to the change of the weight lattice from P_{fin} to $P_{\mathrm{fin},\epsilon}$. See Theorem 5.2.4 and (5.2.1) below.
- We redefine $\mathcal{U}(\epsilon)$ to be the algebra generated by e_i, f_i $(i \in I)$ and k_μ $(\mu \in P_{\mathrm{fin},\epsilon})$ with the same defining relations. The finite type subalgebra $\mathring{\mathcal{U}}(\epsilon)$ is defined accordingly.
- $\mathring{\mathcal{U}}(\epsilon_-,\epsilon_+)$: the subalgebra of $\mathcal{U}(\epsilon)$ generated by e_i, f_i $(i \neq 0, r)$ and k_μ $(\mu \in P_{\mathrm{fin},\epsilon})$.
- $V_{(\epsilon_-,\epsilon_+)}(\lambda,\mu) = V_{\epsilon_-}(-\lambda) \otimes V_{\epsilon_+}(\mu)$: irreducible $\mathring{\mathcal{U}}(\epsilon_-,\epsilon_+)$-module, where $V_{\epsilon_-}(-\lambda)$ is the dual of a $\mathring{\mathcal{U}}(\epsilon_-)$-module $V_{\epsilon_-}(\lambda)$.

The algebra $\mathcal{U}(\epsilon)$ defined in Chapter 3 is a subalgebra of the redefined one, generated by e_i, f_i $(i \in I)$ and k_{δ_j} $(j \in \mathbb{I})$. See also Remark 5.2.10.

5.2.1 Fock space and fundamental q-oscillator representations

Consider again the supersymmetric Fock space

$$\mathcal{W}_\epsilon = \bigoplus_{\mathbf{m}\in\mathbb{Z}^n_+(\epsilon)} \Bbbk\,|\mathbf{m}\rangle$$

but now with different decomposition

$$\mathcal{W}_\epsilon = \bigoplus_{l\in\mathbb{Z}} \mathcal{W}^{\mathrm{osc}}_{l,\epsilon}, \quad \mathcal{W}^{\mathrm{osc}}_{l,\epsilon} = \bigoplus_{l(\mathbf{m})=l} \Bbbk\,|\mathbf{m}\rangle$$

where we put

$$|\mathbf{m}|_- = m_1 + \cdots + m_r, \quad |\mathbf{m}|_+ = m_{r+1} + \cdots + m_n,$$
$$|\mathbf{m}| = |\mathbf{m}|_+ + |\mathbf{m}|_-\,, \quad l(\mathbf{m}) = |\mathbf{m}|_+ - |\mathbf{m}|_-\,.$$

Given $x \in \Bbbk^\times$, let us assign the following action of generators of $\mathcal{U}(\epsilon)$ on $\mathcal{W}_\epsilon$:

$$
\begin{aligned}
& k_{\Lambda_{r,\epsilon}} |\mathbf{m}\rangle = q^{\sum_{j\in\mathbb{I}^+} m_j} |\mathbf{m}\rangle \,, \\
& k_{\delta_i} |\mathbf{m}\rangle = q_i^{-m_i} |\mathbf{m}\rangle \quad (i \in \mathbb{I}^-), \\
& k_{\delta_j} |\mathbf{m}\rangle = q_j^{m_j} q\, |\mathbf{m}\rangle \quad (j \in \mathbb{I}^+), \\
& e_0 |\mathbf{m}\rangle = x\, |\mathbf{m} + \mathbf{e}_1 + \mathbf{e}_n\rangle \,, \\
& f_0 |\mathbf{m}\rangle = -x^{-1}[m_1][m_n]\, |\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_n\rangle \,, \\
& \left.\begin{aligned} & e_i |\mathbf{m}\rangle = [m_i]\, |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \\ & f_i |\mathbf{m}\rangle = [m_{i+1}]\, |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \end{aligned}\right. \qquad (i \in \mathbb{I}^-), \\
& e_r |\mathbf{m}\rangle = -[m_r][m_{r+1}]\, |\mathbf{m} - \mathbf{e}_r - \mathbf{e}_{r+1}\rangle \,, \\
& f_r |\mathbf{m}\rangle = |\mathbf{m} + \mathbf{e}_r + \mathbf{e}_{r+1}\rangle \,, \\
& \left.\begin{aligned} & e_j |\mathbf{m}\rangle = [m_{j+1}]\, |\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \\ & f_j |\mathbf{m}\rangle = [m_j] \end{aligned}\right. \qquad (j \in \mathbb{I}^+).
\end{aligned}
$$

Proposition 5.2.1. *For $x \in \Bbbk^\times$, the above formula defines a $\mathcal{U}(\epsilon)$-action on $\mathcal{W}_\epsilon$.*

Moreover, for each $l \in \mathbb{Z}$, the subspace $\mathcal{W}^{\rm osc}_{l,\epsilon}$ is closed under the $\mathcal{U}(\epsilon)$-action and irreducible over $\mathcal{U}(\epsilon)$.

We denote the resulting $\mathcal{U}(\epsilon)$-module by $\mathcal{W}^{\rm osc}_{\epsilon}(x)$ and $\mathcal{W}^{\rm osc}_{l,\epsilon}(x)$ respectively, and call the latter the *l-th fundamental q-oscillator representation.* When we regard them as $\mathring{\mathcal{U}}(\epsilon)$-modules, we simply omit x for being irrelevant.

Note that the weight of $|\mathbf{m}\rangle \in \mathcal{W}^{\rm osc}_\epsilon$ is

$$
\mathrm{wt}\, |\mathbf{m}\rangle = \Lambda_{r,\epsilon} - \sum_{i\le r} m_i\delta_i + \sum_{j>r} m_j\delta_j \in P_{\mathrm{fin},\epsilon}.
$$

As a $\mathring{\mathcal{U}}(\epsilon)$-module, $\mathcal{W}^{\rm osc}_{l,\epsilon}$ is an irreducible highest weight representation generated by a highest weight vector

$$
v_l = \begin{cases} |l\mathbf{e}_{r+1}\rangle & \text{if } l \ge 0 \\ |-l\mathbf{e}_r\rangle & \text{if } l < 0. \end{cases}
$$

Remark 5.2.2. Suppose $\epsilon_i = 1$ for all i, in which case $\mathcal{W}^{\rm osc}_{l,\epsilon}(x)$ is finite-dimensional, and even zero unless $-r \le l \le n-r$. If $-r \le l \le n-r$, then as a $U'_{-q^{-1}}(\widehat{\mathfrak{gl}}_n)$-module, $\mathcal{W}^{\rm osc}_{l,\epsilon}(x)$ is the $(l+r)$-th fundamental representation $V(\varpi_{l+r})_x$. Here $U'_{-q^{-1}}(\widehat{\mathfrak{gl}}_n)$ is identified with the subalgebra of $\mathcal{U}(\epsilon)$ generated by e_i, f_i, k_{δ_j} (see Remark 5.2.10).

On the other hand, when $\epsilon_i = 0$ for all i, again we find $U'_q(\widehat{\mathfrak{gl}}_n)$ as a subalgebra of $\mathcal{U}(\epsilon)$, and then the $U_q(\mathfrak{gl}_n)$-module $\mathcal{W}^{\rm osc}_\epsilon$ can be seen as a q-analogue of the bosonic Fock space $W^{\rm osc}$ in the previous section.

Remark 5.2.3. In [67], more general q-oscillator representations of $U'_q(\widehat{\mathfrak{sl}}_n)$, which are level one in the context of level-rank duality, are introduced. By general we mean that the distribution of particles and holes can be arbitrary, while our definition is the case with first r holes, and then $n-r$ particles.

Suppose ϵ' is obtained from ϵ by removing ϵ_i. We redefine truncation functors to make it compatible with q-oscillator representations. First, let us consider a slight twist of the algebra homomorphism $\phi^\epsilon_{\epsilon'}$ from Theorem 4.3.1.

Theorem 5.2.4. *There exists a $\Bbbk$-algebra homomorphism $\widetilde{\phi}^\epsilon_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon)$ defined on the generators e'_j, f'_j ($j \in I'$) and k_μ ($\mu \in P_{{\rm fin},\epsilon'}$) by*

$$k_{\Lambda_{r',\epsilon'}} \longmapsto k_{\Lambda_{r,\epsilon}}, \quad k_{\delta'_l} \longmapsto \begin{cases} k_{\delta_l} & \text{if } 1 \leq l \leq i-1 \\ k_{\delta_{l+1}} & \text{if } i \leq l \leq n-1, \end{cases}$$

Case 1. If $2 \leq i \leq r$, then

$$(e_j,\, f_j) \longmapsto \begin{cases} (e_j,\, f_j) & \text{if } 0 \leq j \leq i-2 \\ \left([e_i, e_{i-1}]_{\mathbf{q}(\alpha_{i-1},\alpha_i)^{-1}},\, [f_{i-1}, f_i]_{\mathbf{q}(\alpha_{i-1},\alpha_i)}\right) & \text{if } j = i-1 \\ (e_{j+1},\, f_{j+1}) & \text{if } i \leq j \leq n-2, \end{cases}$$

Case 2. If $r+1 \leq i \leq n-1$,

$$(e_j,\, f_j) \longmapsto \begin{cases} (e_j,\, f_j) & \text{if } 0 \leq j \leq i-2 \\ \left([e_{i-1}, e_i]_{\mathbf{q}(\alpha_{i-1},\alpha_i)},\, [f_i, f_{i-1}]_{\mathbf{q}(\alpha_{i-1},\alpha_i)^{-1}}\right) & \text{if } j = i-1 \\ (e_{j+1},\, f_{j+1}) & \text{if } i \leq j \leq n-2, \end{cases}$$

Case 3. If $i = n$, then

$$(e_j,\, f_j) \longmapsto \begin{cases} (e_j,\, f_j) & \text{if } j \neq 0 \\ \left([e_{n-1}, e_0]_{\mathbf{q}(\alpha_{n-1},\alpha_0)},\, [f_0, f_{n-1}]_{\mathbf{q}(\alpha_{n-1},\alpha_0)^{-1}}\right) & \text{if } j = 0, \end{cases}$$

Case 4. If $i = 1$, then

$$(e_j,\, f_j) \longmapsto \begin{cases} \left([e_1, e_0]_{\mathbf{q}(\alpha_0,\alpha_1)^{-1}},\, [f_0, f_1]_{\mathbf{q}(\alpha_0,\alpha_1)}\right) & \text{if } j = 0 \\ (e_{j+1},\, f_{j+1}) & \text{if } j \neq 0. \end{cases}$$

More generally, if $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{n-r})$ is obtained from ϵ by removing $\epsilon_{i_1}, \ldots, \epsilon_{i_r}$ for some $i_1 < \cdots < i_r$, we define a $\Bbbk$-algebra homomorphism $\widetilde{\phi}^{\epsilon}_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon)$ as a successive composition of the above algebra homomorphism.

For a $\mathcal{U}(\epsilon)$-module V with $\mathrm{wt}(V) \subset P_{\geq 0,\epsilon}$, define

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(V) = \bigoplus_{\substack{\mu \in \mathrm{wt}(V) \\ (\mathrm{pr}(\mu)|\delta_{i_1}) = \cdots = (\mathrm{pr}(\mu)|\delta_{i_t}) = 0}} V_\mu \tag{5.2.1}$$

where $\mathrm{pr} : P_{\geq 0} \subset \mathbb{Z}\Lambda_{r,\epsilon} \oplus \bigoplus \mathbb{Z}\delta_i \longrightarrow \bigoplus \mathbb{Z}\delta_i$ is the projection $\Lambda_{r,\epsilon} \mapsto 0$. In the same manner as in Section 4.3.1, we obtain an exact monoidal functor $\mathfrak{tr}^{\epsilon}_{\epsilon'}$ defined on $\mathcal{U}(\epsilon)$-modules with weights in $P_{\geq 0,\epsilon}$. Again, it is easy check the following lemmas.

Lemma 5.2.5. *For $l \in \mathbb{Z}$ and $x \in \Bbbk^\times$, we have as a $\mathcal{U}(\epsilon')$-module,*

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(\mathcal{W}^{\mathrm{osc}}_{l,\epsilon}(x)) \cong \begin{cases} \mathcal{W}^{\mathrm{osc}}_{l,\epsilon'}(x) & \text{if } (l) \in \mathcal{P}(GL_1)_{(\epsilon'_-,\epsilon'_+)} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2.6. *For $\mu \in \mathscr{P}_{M_-|N_-}$ and $\nu \in \mathscr{P}_{M_+|N_+}$, we have as a $\mathring{\mathcal{U}}(\epsilon'_-, \epsilon'_+)$-module*

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}\left(V_{(\epsilon_-,\epsilon_+)}(\mu,\nu)\right) \cong \begin{cases} V_{(\epsilon'_-,\epsilon'_+)}(\mu,\nu) & \text{if } \mu \in \mathscr{P}_{M'_-|N'_-} \text{ and } \nu \in \mathscr{P}_{M'_+|N'_+} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 4.3.4, for $\lambda_\pm \in \mathscr{P}_{M_\pm|N_\pm}$, we have

$$\mathfrak{tr}^{\epsilon_\pm}_{\epsilon'_\pm}\left(V_{\epsilon_\pm}(\lambda_\pm)\right) \cong \begin{cases} V_{\epsilon'_\pm}(\lambda_\pm) & \text{if } \lambda \in \mathscr{P}_{M'_\pm|N'_\pm} \\ 0 & \text{otherwise.} \end{cases}$$

□

5.2.2 Oscillator representations of $U_q(\mathfrak{gl}_n)$

The main result of this section is Theorem 5.2.14 and (5.2.2), which are reproductions of Proposition 4.3.4 and (3.2.1) for $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$ or $\overline{\boldsymbol{\epsilon}}$ (see (5.1.4), (5.1.5) for notation). Since q-oscillator representations in the case $\epsilon = \overline{\boldsymbol{\epsilon}}$ will turn out to be finite-dimensional, we only need to prove them for $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$.

Unlike polynomial representations, where quantum Schur–Weyl duality (Theorem 3.2.5) was available, we do not have a quantum version of the Howe duality. Instead, we make use of the classical limit to prove the case $\epsilon = \underline{\boldsymbol{\epsilon}}$, and then truncations to lift it to the remaining one $\epsilon = \boldsymbol{\epsilon}$.

Proposition 5.2.7. *For $\ell \geq 1$, $\mathcal{W}_\epsilon^{\rm osc}(x)^{\otimes \ell}$ is semisimple as a $\mathring{\mathcal{U}}(\epsilon)$-module.*

Proof. Introduce a nondegenerate symmetric bilinear form on $\mathcal{W}_\epsilon^{\rm osc}(x)$

$$(|\mathbf{m}\rangle, |\mathbf{m}'\rangle) = \delta_{\mathbf{m},\mathbf{m}'} q^{-\sum_{i=1}^n \frac{m_i(m_i-1)}{2}} \prod_{i=1}^n [m_i]!$$

for $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}_+^n(\epsilon)$, and an anti-involution η on $\mathring{\mathcal{U}}(\epsilon)$

$$\eta(k_\mu) = k_\mu,$$

$$\eta(e_i) = \begin{cases} (-q^2)^{\epsilon_i - \epsilon_{i+1}} q_i f_i k_{\alpha_i}^{-1} & \text{if } i < r \\ (-q^2)^{\epsilon_r - 1} q_r f_r k_{\alpha_r}^{-1} & \text{if } i = r \\ q_i f_i k_{\alpha_i}^{-1} & \text{if } i > r, \end{cases}$$

$$\eta(f_i) = \begin{cases} (-q^2)^{\epsilon_{i+1} - \epsilon_i} q_i^{-1} k_{\alpha_i} e_i & \text{if } i < r \\ (-q^2)^{1-\epsilon_r} q_r^{-1} k_{\alpha_r} e_r & \text{if } i = r \\ q_i^{-1} k_{\alpha_i} e_i & \text{if } i > r \end{cases}$$

for $\mu \in P_{\text{fin},\epsilon}$ and $i \in I \setminus \{0\}$. Then one can check that

(1) $(\eta \otimes \eta) \circ \Delta = \Delta \circ \eta$,

(2) $(xv, w) = (v, \eta(x)w)$ for $x \in \mathring{\mathcal{U}}(\epsilon)$ and $v, w \in \mathcal{W}_\epsilon^{\rm osc}(x)$,

(3) For the A_∞-lattice

$$\mathcal{L}_\infty = \bigoplus_{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon)} A_\infty |\mathbf{m}\rangle \subset \mathcal{W}_\epsilon^{\rm osc},$$

we have $(\mathcal{L}_\infty, \mathcal{L}_\infty) \subset A_\infty$, and the induced form on $\mathcal{L}_\infty / q^{-1}\mathcal{L}_\infty$ is positive-definite.

Now the semisimplicity can be shown along the argument of [7, Theorem 2.12]. □

Let us introduce q-analogues of irreducible oscillator representations of $\mathfrak{gl}_n$. For $\lambda \in \mathcal{P}(GL_\ell)$, define $\Lambda_{\lambda,\epsilon} \in P_{\geq 0,\epsilon}$ as follows:

(1) If $\epsilon_{r+1}=0$ (resp. $\epsilon_{r+1}=1$), then fill the first row (resp. column) of λ^+ with $r+1$. After filling a subdiagram $\mu\subset\lambda^+$ with $r+1,\ldots,r+k$, fill the first row (resp. column) of λ^+/μ with $r+k+1$ if $\epsilon_{r+k+1}=0$ (resp. $\epsilon_{r+k+1}=1$).

(2) Fill λ^- in the same way with $r, r-1, r-2, \ldots$.

(3) Let m_i be the number of occurrences of i's in $\lambda^\pm$, and define

$$\Lambda_{\lambda,\epsilon}=\ell\Lambda_{r,\epsilon}-\sum_{i\in\mathbb{I}^-}m_i\delta_i+\sum_{j\in\mathbb{I}^+}m_j\delta_j.$$

In other words, $(m_{r+1},m_{r+2},\ldots)$ (resp. $(m_r,m_{r-1},\ldots)$) is the content of the tableau H_{λ^+,ϵ^+} (resp. H_{λ^-,ϵ^-}). Hence the weight $\Lambda_{\lambda,\epsilon}$ is well-defined if and only if $\lambda^\pm\in\mathscr{P}_{M^\pm|N^\pm}$, and we put

$$\mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}=\{\lambda\in\mathcal{P}(GL_\ell)\,|\,\lambda^\pm\in\mathscr{P}_{M^\pm|N^\pm}\}.$$

For example, we have

$$\mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}=\begin{cases}\mathcal{P}(GL_\ell)_{(r,n-r)} & \text{if } \epsilon=\epsilon_{n|0}\\ \{\lambda\in\mathcal{P}(GL_\ell)\,|\,n-r\geq\lambda_1,\ \lambda_\ell\geq -r\} & \text{if } \epsilon=\epsilon_{0|n}.\end{cases}$$

Definition 5.2.8. For $\lambda\in\mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$, we denote by $\mathcal{V}_\epsilon^\lambda$ the irreducible highest weight $\mathring{\mathcal{U}}(\epsilon)$-module with highest weight $\Lambda_{\lambda,\epsilon}$.

Example 5.2.9. The l-th fundamental q-oscillator representation $\mathcal{W}_{l,\epsilon}^{\rm osc}(x)$ is isomorphic to $\mathcal{V}_\epsilon^{(l)}$ as a $\mathring{\mathcal{U}}(\epsilon)$-module.

Next, we consider classical limits. Let β be a 2-cocycle on $\mathfrak{gl}_\epsilon$ defined by $\beta(X,Y)=\mathrm{str}([J,X]Y)$ for $J=\sum_{j\in\mathbb{I}^+}E_{jj}$. Define a central extension $\mathfrak{gl}_\epsilon^e=\mathfrak{gl}_\epsilon\bigoplus\mathbb{C}c$ with respect to β, that is c is central in $\mathfrak{gl}_\epsilon^e$ and

$$[X,Y]=XY-(-1)^{p(X)p(Y)}YX+\beta(X,Y)c$$

for homogeneous $X,Y\in\mathfrak{gl}_\epsilon$. In particular, we have

$$[E_{i,i+1},E_{i+1,i}]=E_{i,i}-(-1)^{\epsilon_i+\epsilon_{i+1}}E_{i+1,i+1}-(-1)^{\epsilon_r}\delta_{ir}c=:h_i\quad(i\in I\setminus\{0\}).$$

The dual weight lattice $P^\vee_{\mathrm{fin},\epsilon} = \mathbb{Z}c \oplus \bigoplus \mathbb{Z}E_{ii}$ of $\mathfrak{gl}^e_\epsilon$ is indeed in a perfect pairing with $P_{\mathrm{fin},\epsilon}$ given by

$$\langle \delta_i, E_{jj}\rangle = \delta_{ij}, \quad \langle \delta_i, c\rangle = \langle \Lambda_{r,\epsilon}, E_{ii}\rangle = 0, \quad \langle \Lambda_{r,\epsilon}, c\rangle = 1,$$

from which we obtain an isomorphism of abelian groups $\phi : P_{\mathrm{fin},\epsilon} \longrightarrow P^\vee_{\mathrm{fin},\epsilon}$ given by

$$\phi(\delta_i) = (-1)^{\epsilon_i} E_{ii} + \delta(i \in \mathbb{I}^+)c, \quad \phi(\Lambda_{r,\epsilon}) = \sum_{j \in \mathbb{I}^+} E_{jj}$$

such that $(\lambda|\mu) = \langle \lambda, \phi(\mu)\rangle$ for $\lambda, \mu \in P_{\mathrm{fin},\epsilon}$. Note that $\phi(\alpha_i) = (-1)^{\epsilon_i} h_i$.

We identify $\mathfrak{gl}_\epsilon$ with the subalgebra of $\mathfrak{gl}^e_\epsilon$ generated by $E_{i,i+1}, E_{i+1,i}$ $(i = 1, \ldots, n-1)$ and $\phi(\delta_j)$ $(j = 1, \ldots, n)$. When we restrict to this subalgebra, the weight lattice $P_{\mathrm{fin},\epsilon}$ for $\mathfrak{gl}^e_\epsilon$ degenerates to P_{fin} through $\Lambda_{r,\epsilon} \equiv (-1)^{\epsilon_{r+1}}\delta_{r+1} + \cdots + (-1)^{\epsilon_n}\delta_n$..

Remark 5.2.10. When $\epsilon = \underline{\boldsymbol{\epsilon}}$, $\mathcal{U}(\underline{\boldsymbol{\epsilon}}) = U(\underline{\boldsymbol{\epsilon}})$ can be identified under the isomorphism ϕ with $U_q'(\widehat{\mathfrak{gl}}^e_{a+b+1})$, the quantum group associated with a Cartan datum (see Section 2.2.1)

$$(A^{(1)}_{a+b},\ P_{\mathrm{fin},\epsilon},\ \{\alpha_i\},\ P^\vee_{\mathrm{fin},\epsilon},\ \{h_i = \phi(\alpha_i)\}).$$

Since the classical limit is not well-defined for $\mathring{\mathcal{U}}(\epsilon)$, we first have to regard $\mathcal{V}^\lambda_\epsilon$ as a module over the quantum superalgebra $\mathring{U}(\epsilon)$ using the algebra isomorphism τ. Here $\mathring{U}(\epsilon)$ is also redefined as the algebra generated by E_i, F_i $(i \in I \setminus \{0\})$ and K_μ $(\mu \in P_{\mathrm{fin},\epsilon})$ with the same relations as in Definition 3.1.4, and the algebra isomorphism τ in Theorem 3.1.6 restricts to

$$\tau : \mathring{U}(\epsilon)[\sigma] \longrightarrow \mathring{\mathcal{U}}(\epsilon)[\sigma].$$

Suppose V is a highest weight $\mathring{\mathcal{U}}(\epsilon)$-module with highest weight $\Lambda_{\lambda,\epsilon}$ for some $\lambda \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$. Pulling V back through τ, we obtain a $\mathring{U}(\epsilon)$-module V^τ and let $V^\tau_{\mathcal{A}}$ be the $\mathcal{A}$-span of $\{F_{i_1} \cdots F_{i_s} v \,|\, i_1, \ldots, i_s \in I \setminus \{0\},\ s \geq 0\}$ for a highest weight vector v. Define the classical limit $\overline{V^\tau} = V^\tau_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$, where $\mathbb{C}$ is an $\mathcal{A}$-algebra by $q = 1$. Then we obtain an algebra homomorphism $U(\mathfrak{gl}^e_\epsilon) \longrightarrow \mathrm{End}(\overline{V^\tau})$ given by the induced action of $\mathring{U}(\epsilon)$, namely

$$\begin{aligned} &E_{i,i+1} \longmapsto E_i, \quad E_{i+1,i} \longmapsto F_i, \\ &\phi(\delta_i) \longmapsto \frac{K_{\delta_i} - K^{-1}_{\delta_i}}{q - q^{-1}}, \quad \phi(\Lambda_{r,\epsilon}) \longmapsto \frac{K_{\Lambda_{r,\epsilon}} - K^{-1}_{\Lambda_{r,\epsilon}}}{q - q^{-1}}. \end{aligned}$$

Now one can easily verify the following lemma.

Lemma 5.2.11. *As a $\mathfrak{gl}_\epsilon^e$-module, $\overline{V^\tau}$ is a highest weight module of highest weight $\Lambda_{\lambda,\epsilon}$.*

In the proofs below, we shall use another truncations, associated with a sequence $\epsilon^{(a+k,b+k)}$ for $k \geq 1$ that contains $\epsilon^{(a,b)}$ as a subsequence obtained by removing the first and the last $2k$ entries in $\epsilon^{(a+k)}$ and $(\epsilon^{(b+k)}, 0)$ respectively. We will use the notation $\widetilde{\boldsymbol{\epsilon}} := \epsilon^{(a+k,b+k)}$ to emphasize this truncation. Note that if $\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$, then $\lambda \in \mathcal{P}(GL_\ell)_{(\widetilde{\boldsymbol{\epsilon}}_-,\widetilde{\boldsymbol{\epsilon}}_+)}$ as well and $\Lambda_{\lambda,\boldsymbol{\epsilon}} = \Lambda_{\lambda,\widetilde{\boldsymbol{\epsilon}}}$.

Proposition 5.2.12. *For $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$, any highest weight $\mathring{\mathcal{U}}(\epsilon)$-submodule of $(\mathcal{W}_\epsilon^{\rm osc})^{\otimes \ell}$ is isomorphic to $\mathcal{V}_\epsilon^\lambda$ for some $\lambda \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$.*

Proof. Let V be a highest weight $\mathring{\mathcal{U}}(\epsilon)$-submodule of $(\mathcal{W}_\epsilon^{\rm osc})^{\otimes \ell}$, which is irreducible by the semisimplicity of $(\mathcal{W}_\epsilon^{\rm osc})^{\otimes \ell}$ (Proposition 5.2.7).

When $\epsilon = \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$, it can be shown using Lemma 5.2.11 that the classical limit of $(\mathcal{W}_\epsilon^{\rm osc})^{\otimes \ell}$ is isomorphic to $(\mathcal{W}^{\rm osc})^{\otimes \ell}$, $W^{\otimes \ell}$, respectively. Then the classical limit of V is a highest weight submodule, and hence isomorphic to V^λ for some $\lambda \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$ by dualities Theorem 5.1.1, (5.1.1) respectively.

Finally assume $\epsilon = \boldsymbol{\epsilon}$, and let v be a highest weight vector of V. Note that if v is in the component $V_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}(\mu,\nu)$, then $|\mu|$ and $|\nu|$ are minimal among other $\mathring{\mathcal{U}}(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)$-components of V. We may further assume that a and b are large enough, so that $V_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}(\mu,\nu) \neq 0$. Indeed, take $\tilde{\boldsymbol{\epsilon}} = \epsilon^{(a+k,b+k)}$ for $k > 0$, and identify $v \in (\mathcal{W}_{\boldsymbol{\epsilon}}^{\rm osc})^{\otimes \ell} \subset (\mathcal{W}_{\widetilde{\boldsymbol{\epsilon}}}^{\rm osc})^{\otimes \ell}$. Then it is easy to check that v is also a $\mathring{\mathcal{U}}(\widetilde{\boldsymbol{\epsilon}})$-highest weight vector, and we may choose V to be the $\mathring{\mathcal{U}}(\widetilde{\boldsymbol{\epsilon}})$-submodule generated by v from the beginning.

Now we have $\mathfrak{tr}_{\overline{\boldsymbol{\epsilon}}}^{\boldsymbol{\epsilon}}(V_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}(\mu,\nu)) = V_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}(\mu,\nu) \neq 0$. If we let w be a $\mathring{\mathcal{U}}(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)$-highest weight vector of $V_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}(\mu,\nu)$, then by the minimality of $|\mu|$ and $|\nu|$, w is also a $\mathring{\mathcal{U}}(\overline{\boldsymbol{\epsilon}})$-highest weight vector of $\mathfrak{tr}_{\overline{\boldsymbol{\epsilon}}}^{\boldsymbol{\epsilon}}(V)$. Since $(\mathcal{W}_{\overline{\boldsymbol{\epsilon}}}^{\rm osc})^{\otimes \ell}$ is finite-dimensional, the weight of w is of the form $\Lambda_{\lambda,\overline{\boldsymbol{\epsilon}}}$ for some $\lambda \in \mathcal{P}(GL_\ell)_{\overline{\boldsymbol{\epsilon}}}$. This implies that the weight of v is $\Lambda_{\lambda,\boldsymbol{\epsilon}}$, as desired. □

In particular, we understand that the classical limit of $\mathcal{V}_{\underline{\boldsymbol{\epsilon}}}^\lambda$ is, as a $\mathfrak{gl}_{a+b+1}$-module, isomorphic to the tensor product of the irreducible oscillator representation V^λ and the trivial representation $\mathbb{C}$ with $I = 1$.

Lemma 5.2.13. *Let $\widetilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}^{(a+k,b+k)}$ for $k \geq 1$.*

(1) *For $\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$ such that $\mathcal{V}_{\boldsymbol{\epsilon}}^\lambda \subset (\mathcal{W}_{\boldsymbol{\epsilon}}^{\rm osc})^{\otimes \ell}$, we have $\mathcal{V}_{\widetilde{\boldsymbol{\epsilon}}}^\lambda \subset (\mathcal{W}_{\widetilde{\boldsymbol{\epsilon}}}^{\rm osc})^{\otimes \ell}$.*

(2) *For* $\lambda \in \mathcal{P}(GL_\ell)_{(\widetilde{\boldsymbol{\epsilon}}_-,\widetilde{\boldsymbol{\epsilon}}_+)}$ *such that* $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}} \subset (\mathcal{W}^{\rm osc}_{\widetilde{\boldsymbol{\epsilon}}})^{\otimes \ell}$*, we have*

$$\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) \cong \begin{cases} \mathcal{V}^\lambda_{\boldsymbol{\epsilon}} & \text{if } \lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) Let v be a $\mathring{\mathcal{U}}(\boldsymbol{\epsilon})$-highest weight vector of $\mathcal{V}^\lambda_{\boldsymbol{\epsilon}} \subset (\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell} \subset (\mathcal{W}^{\rm osc}_{\widetilde{\boldsymbol{\epsilon}}})^{\otimes \ell}$. As in the proof of Proposition 5.2.12, v is also a $\mathring{\mathcal{U}}(\widetilde{\boldsymbol{\epsilon}})$-highest weight vector. Since v has weight $\Lambda_{\lambda,\boldsymbol{\epsilon}} = \Lambda_{\lambda,\widetilde{\boldsymbol{\epsilon}}}$, we have $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}} \subset (\mathcal{W}^{\rm osc}_{\widetilde{\boldsymbol{\epsilon}}})^{\otimes \ell}$.

(2) First assume $\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$. Then a highest weight vector v of $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}$ belongs to $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}})$ and is a $\mathring{\mathcal{U}}(\boldsymbol{\epsilon})$-highest weight vector as well, which implies $\mathcal{V}^\lambda_{\boldsymbol{\epsilon}} \subset \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}})$. If this containment is proper, we can find another $\mathring{\mathcal{U}}(\boldsymbol{\epsilon})$-highest weight vector $w \in \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) \setminus \Bbbk v$ by semisimplicity of $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) \subset (\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell}$. But again as above, w is also a $\mathring{\mathcal{U}}(\widetilde{\boldsymbol{\epsilon}})$-highest weight vector in $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}$ of weight less than $\Lambda_{\lambda,\boldsymbol{\epsilon}}$, which is absurd.

Now suppose $\lambda \notin \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$. Then there exists $i \in \mathbb{I}_{\widetilde{\boldsymbol{\epsilon}}} \setminus \mathbb{I}_{\boldsymbol{\epsilon}}$ such that $({\rm pr}_{\mathbb{I}_{\widetilde{\boldsymbol{\epsilon}}}}(\Lambda_{\lambda,\widetilde{\boldsymbol{\epsilon}}})|\delta_i) \neq 0$, where $\mathbb{I}_{\widetilde{\boldsymbol{\epsilon}}} = \{1,2,\ldots,2a+2b+4k+1\}$ is the index set for $\widetilde{\boldsymbol{\epsilon}}$. Since $\Lambda_{\lambda,\widetilde{\boldsymbol{\epsilon}}}$ is the highest weight of $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}$, for each $\mu \in {\rm wt}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}})$ one can easily find $j \in \mathbb{I}_{\widetilde{\boldsymbol{\epsilon}}} \setminus \mathbb{I}_{\boldsymbol{\epsilon}}$ such that $({\rm pr}_{\mathbb{I}_{\widetilde{\boldsymbol{\epsilon}}}}(\mu)|\delta_j) \neq 0$. Hence $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) = 0$. □

Now we arrive at the main result of this subsection.

Theorem 5.2.14. *For* $\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$*, we have*

$$\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\boldsymbol{\epsilon}'}(\mathcal{V}^\lambda_{\boldsymbol{\epsilon}}) \cong \begin{cases} \mathcal{V}^\lambda_{\boldsymbol{\epsilon}'} & \text{if } \lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}'_-,\boldsymbol{\epsilon}'_+)} \\ 0 & \text{otherwise,} \end{cases}$$

for $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}$ *or* $\overline{\boldsymbol{\epsilon}}$*.*

Proof. First note that for $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$,

$$(\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}'})^{\otimes \ell} \cong \bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}'_-,\boldsymbol{\epsilon}'_+)}} (\mathcal{V}^\lambda_{\boldsymbol{\epsilon}'})^{\oplus d^\lambda}$$

by the semisimplicity, classical limits and classical dualities. We also have from Proposition 5.2.12,

$$(\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell} \cong \bigoplus_{\lambda \in S} (\mathcal{V}^\lambda_{\boldsymbol{\epsilon}})^{\oplus d^\lambda_{\boldsymbol{\epsilon}}}$$

for some $S \subset \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$ and $d^\lambda_{\boldsymbol{\epsilon}} \in \mathbb{Z}_{\geq 0}$.

Step 1. We first prove the assertion for $\lambda \in S$, when $\boldsymbol{\epsilon}' = \overline{\boldsymbol{\epsilon}}$. Let $V = \mathcal{V}^\lambda_{\boldsymbol{\epsilon}}$ with a highest weight vector v.

Suppose $\lambda \in \mathcal{P}(GL_\ell)_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}$. Then the argument in the proof of Proposition 5.2.12 tells us that $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(V)$ contains a $\mathring{\mathcal{U}}(\overline{\boldsymbol{\epsilon}})$-highest weight vector of highest weight $\Lambda_{\lambda,\overline{\boldsymbol{\epsilon}}}$, and so $\mathcal{V}^\lambda_{\overline{\boldsymbol{\epsilon}}} \subset \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(V)$. To prove the equality, we consider the classical limit $\overline{V^\tau}$, which is a highest weight $U(\mathfrak{gl}^e_{\boldsymbol{\epsilon}})$-module with highest weight $\Lambda_{\lambda,\boldsymbol{\epsilon}}$. The truncation $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(\overline{V^\tau})$ is defined in the same way as in (5.2.1), and using the argument of [20, Lemma 3.5], it can be proved that $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(\overline{V^\tau})$ is also a highest weight module with highest weight $\Lambda_{\lambda,\overline{\boldsymbol{\epsilon}}}$. Since $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(\overline{V^\tau})$ is a finite-dimensional $\mathfrak{gl}^e_{a+b}$-module, this implies that it is irreducible and so the character of $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(\overline{V^\tau})$ coincides with that of $\mathcal{V}^\lambda_{\overline{\boldsymbol{\epsilon}}}$. This gives the desired equality.

Next, suppose that $\lambda \notin \mathcal{P}(GL_\ell)_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}$. Taking $\widetilde{\boldsymbol{\epsilon}} = \epsilon^{(a+k,b+k)}$ with $\lambda \in \mathcal{P}(GL_\ell)_{(\overline{\widetilde{\boldsymbol{\epsilon}}}_-,\overline{\widetilde{\boldsymbol{\epsilon}}}_+)}$, we have $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}} \subset (\mathcal{W}^{\rm osc}_{\widetilde{\boldsymbol{\epsilon}}})^{\otimes \ell}$ and $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) = V$ by Lemma 5.2.13. On the other hand, we just have proved that $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\overline{\widetilde{\boldsymbol{\epsilon}}}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) \cong \mathcal{V}^\lambda_{\overline{\widetilde{\boldsymbol{\epsilon}}}}$, and it can be easily seen that $\mathfrak{tr}^{\overline{\widetilde{\boldsymbol{\epsilon}}}}_{\overline{\boldsymbol{\epsilon}}}(\mathcal{V}^\lambda_{\overline{\widetilde{\boldsymbol{\epsilon}}}}) = 0$. Now

$$\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}(V) = \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}} \circ \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) = \mathfrak{tr}^{\overline{\widetilde{\boldsymbol{\epsilon}}}}_{\overline{\boldsymbol{\epsilon}}} \circ \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\overline{\widetilde{\boldsymbol{\epsilon}}}}(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}) = \mathfrak{tr}^{\overline{\widetilde{\boldsymbol{\epsilon}}}}_{\overline{\boldsymbol{\epsilon}}}(\mathcal{V}^\lambda_{\overline{\widetilde{\boldsymbol{\epsilon}}}}) = 0$$

as expected.

Step 2. We claim $S = \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$ and $d^\lambda_{\boldsymbol{\epsilon}} = d^\lambda$. Indeed, given $\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$, we again take $\widetilde{\boldsymbol{\epsilon}}$ as above and then $\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}} \subset (\mathcal{W}^{\rm osc}_{\widetilde{\boldsymbol{\epsilon}}})^{\otimes \ell}$ by *Step 1.* Applying $\mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}$, we get $\mathcal{V}^\lambda_{\boldsymbol{\epsilon}} = \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}\left(\mathcal{V}^\lambda_{\widetilde{\boldsymbol{\epsilon}}}\right) \subset (\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell}$ and hence $\lambda \in S$.

Step 3. Finally, suppose $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}$ and let $\lambda \in \mathcal{P}(GL_\ell)_{(\underline{\boldsymbol{\epsilon}}_-,\underline{\boldsymbol{\epsilon}}_+)}$ be given. Since $\mathcal{P}(GL_\ell)_{(\underline{\boldsymbol{\epsilon}}_-,\underline{\boldsymbol{\epsilon}}_+)}$ is contained in $\mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$, $\mathcal{V}^\lambda_{\boldsymbol{\epsilon}} \subset (\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell}$ and its highest weight vector v is in the component $V_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}(\lambda^-,\lambda^+)$. Once again, a highest weight vector $w \in V_{(\underline{\boldsymbol{\epsilon}}_-,\underline{\boldsymbol{\epsilon}}_+)}(\lambda^-,\lambda^+) \subset \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}(\mathcal{V}^\lambda_{\boldsymbol{\epsilon}})$ is also a $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vector by the minimality of $|\lambda^-| + |\lambda^+|$. Hence w generates $\mathcal{V}^\lambda_{\underline{\boldsymbol{\epsilon}}} \subset \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}(\mathcal{V}^\lambda_{\boldsymbol{\epsilon}})$. Now it remains to compare both ends of the following identity

$$\bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(\underline{\boldsymbol{\epsilon}}_-,\underline{\boldsymbol{\epsilon}}_+)}} (\mathcal{V}^\lambda_{\underline{\boldsymbol{\epsilon}}})^{\oplus d^\lambda} = (\mathcal{W}^{\rm osc}_{\underline{\boldsymbol{\epsilon}}})^{\otimes \ell} = \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}\left((\mathcal{W}^{\rm osc}_{\boldsymbol{\epsilon}})^{\otimes \ell}\right) = \bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}} \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}\left(\mathcal{V}^\lambda_{\boldsymbol{\epsilon}}\right)^{\oplus d^\lambda},$$

keeping in mind that $\mathcal{V}^\lambda_{\underline{\boldsymbol{\epsilon}}} \subset \mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}(\mathcal{V}^\lambda_{\boldsymbol{\epsilon}})$ and $\mathcal{P}(GL_\ell)_{(\underline{\boldsymbol{\epsilon}}_-,\underline{\boldsymbol{\epsilon}}_+)} \subset \mathcal{P}(GL_\ell)_{(\boldsymbol{\epsilon}_-,\boldsymbol{\epsilon}_+)}$. □

Corollary 5.2.15. *For $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$ or $\overline{\boldsymbol{\epsilon}}$, we have the following $\mathring{\mathcal{U}}(\epsilon)$-module decomposition*

$$(\mathcal{W}^{\rm osc}_\epsilon)^{\otimes \ell} \cong \bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}} (\mathcal{V}^\lambda_\epsilon)^{\oplus d^\lambda} \tag{5.2.2}$$

where $d^\lambda = \dim V_{GL_\ell}(\lambda)$.

In particular, we obtain from (5.1.3)

$$\mathcal{W}^{\rm osc}_{l,\epsilon} \otimes \mathcal{W}^{\rm osc}_{m,\epsilon} = \bigoplus_{t\geq 0} \mathcal{V}_\epsilon^{(l_1+t,l_2-t)} \tag{5.2.3}$$

as a $\mathring{\mathcal{U}}(\epsilon)$-module, for $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$.

Definition 5.2.16. For $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$, let $\mathcal{O}_{{\rm osc},\epsilon}$ be the category of $\mathring{\mathcal{U}}(\epsilon)$-modules V such that

(1) $V = \bigoplus_{\mu\in P} V_\mu$ with $\dim V_\mu < \infty$ and ${\rm wt}(V)$ is finitely dominated,

(2) $V = \bigoplus_{\ell\geq 0} V_\ell$ where V_ℓ is a direct sum of $\mathcal{V}_\epsilon^\lambda$'s for $\lambda \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$, and $V_\ell = 0$ for all sufficiently large ℓ.

Again we put $\mathcal{V}_\epsilon^\emptyset = \mathbb{k}$. Here we do not require that V itself is of finite length. Indeed, already $\mathcal{V}_\epsilon^{(l)} \otimes \mathcal{V}_\epsilon^{(m)}$ has infinitely many irreducible components. On the other hand, the multiplicity of each $\mathcal{V}_\epsilon^\lambda$ in $V \in \mathcal{O}_{{\rm osc},\epsilon}$ is finite, due to the finite-dimensionality of weight spaces of V.

Proposition 5.2.17. *The category $\mathcal{O}_{{\rm osc},\epsilon}$ is a semisimple monoidal category.*

5.3 Oscillator representations of $U_q(\widehat{\mathfrak{gl}}_n)$

We proceed to affine types. In the remaining of this chapter, ϵ stands for either $\boldsymbol{\epsilon}$, $\underline{\boldsymbol{\epsilon}}$, or $\overline{\boldsymbol{\epsilon}}$, unless otherwise stated.

5.3.1 Category $\widehat{\mathcal{O}}_{{\rm osc},\epsilon}$

Definition 5.3.1. The category $\widehat{\mathcal{O}}_{{\rm osc},\epsilon}$ of *q-oscillator representations* of $\mathcal{U}(\epsilon)$ is defined to be the category of $\mathcal{U}(\epsilon)$-modules V such that V belongs to $\mathcal{O}_{{\rm osc},\epsilon}$ as a $\mathring{\mathcal{U}}(\epsilon)$-module.

The category $\widehat{\mathcal{O}}_{{\rm osc},\epsilon}$ is closed under taking submodules, quotients and tensor products (Proposition 5.2.17). Moreover, since ${\rm wt}(V) \subset P_{\geq 0,\epsilon}$ for $V \in \widehat{\mathcal{O}}_{{\rm osc},\epsilon}$, the truncation (5.2.1) is well-defined on $\widehat{\mathcal{O}}_{\mathrm{osc},\epsilon}$, and then we obtain exact monoidal functors

$$\begin{array}{ccc} & \widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}} & \\ {\scriptstyle \mathrm{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}} \swarrow & & \searrow {\scriptstyle \mathrm{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}} \\ \widehat{\mathcal{O}}_{\mathrm{osc},\underline{\boldsymbol{\epsilon}}} & & \widehat{\mathcal{O}}_{\mathrm{osc},\overline{\boldsymbol{\epsilon}}} \end{array}$$

by Theorem 5.2.14.

In the context of super duality, the category $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$ serves as a module category of intermediating superalgebra. In the following subsections, we will see that irreducible representations of $U_q'(\widehat{\mathfrak{gl}}_n)$ in category $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}'}$ for $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$ are indeed interpolated by irreducible objects in $\widehat{\mathcal{O}}_{\mathrm{osc},\boldsymbol{\epsilon}}$.

Note that since $\mathcal{P}(GL_\ell)_{(\overline{\boldsymbol{\epsilon}}_-,\overline{\boldsymbol{\epsilon}}_+)}$ is a finite set for all $\ell \geq 0$, the $\mathcal{U}(\overline{\boldsymbol{\epsilon}})$-modules in $\widehat{\mathcal{O}}_{\mathrm{osc},\overline{\boldsymbol{\epsilon}}}$ are finite-dimensional. Moreover, $\widehat{\mathcal{O}}_{\mathrm{osc},\overline{\boldsymbol{\epsilon}}}$ contains all finite-dimensional fundamental representations $\mathcal{W}^{\mathrm{osc}}_{l,\overline{\boldsymbol{\epsilon}}}(x) \cong V(\varpi_{l+r})_x$. Hence it is a category of finite-dimensional representations of $U'_{\widetilde{q}}(\widehat{\mathfrak{gl}}^{e}_{a+b})$, whose image under the forgetful functor (with respect to $U'_{\widetilde{q}}(\widehat{\mathfrak{gl}}_{a+b}) \subset U'_{\widetilde{q}}(\widehat{\mathfrak{gl}}^{e}_{a+b})$) is exactly the category of finite-dimensional representations of $U_{\widetilde{q}}(\widehat{\mathfrak{gl}}_{a+b})$.

On the other hand, $\widehat{\mathcal{O}}_{\mathrm{osc},\underline{\boldsymbol{\epsilon}}}$ (or its image under the forgetful functor) is by definition the category of q-oscillator representations of $U_q'(\widehat{\mathfrak{gl}}_{a+b+1})$. A conjectural super duality asserts that this category has a parallel structure with a category of finite-dimensional representations, although almost every object in $\widehat{\mathcal{O}}_{\mathrm{osc},\underline{\boldsymbol{\epsilon}}}$ is infinite-dimensional.

Restricted to the subalgebra $U_q'(\widehat{\mathfrak{sl}}_{a+b+1})$, the category $\widehat{\mathcal{O}}_{\mathrm{osc},\underline{\boldsymbol{\epsilon}}}$ is a full subcategory of the affinization $\widehat{\mathcal{O}}$ [36] of the BGG category $\mathcal{O}$ for the quantum group $U_q(\mathfrak{sl}_{a+b+1})$. Mukhin and Young [80] generalized to $\widehat{\mathcal{O}}$ several basic results on finite-dimensional representations of $U_q'(\widehat{\mathfrak{g}})$: they introduced the notion of ℓ-highest weight modules and q-characters in $\widehat{\mathcal{O}}$, classified the irreducibles by their ℓ-highest weights, and so on.

It should be interesting to understand our q-oscillator representations in this context. Here we present the ℓ-highest weight of the fundamental q-oscillator representations of $U_q'(\widehat{\mathfrak{sl}}_n)$, whose proof can be found in Section 6.2.1.

Proposition 5.3.2. *As a $U_q'(\widehat{\mathfrak{sl}}_n)$-module, the l-th fundamental q-oscillator representation*

$\mathcal{W}_l^{\rm osc}(1)$ is an ℓ-highest weight module with ℓ-highest weight $\Psi = (\Psi_i(z))_{i\in I\setminus\{0\}}$ given by

$$\Psi_i(z) = \sum_{k\geq 0} \Psi_{i,k} z^k = \begin{cases} (q^{-l}+u)(1+q^{-l}u)^{-1} & \text{if } l<0 \text{ and } i=r-1 \\ (u+q^{-|l|-1})(1+q^{-|l|-1}u)^{-1} & \text{if } i=r \\ (q^{l}+u)(1+q^{l}u)^{-1} & \text{if } l\geq 0 \text{ and } i=r+1 \\ 1 & \text{otherwise,} \end{cases}$$

where $u = (-q^{-1})^n z$.

Remark 5.3.3. Observe that the component $\Psi_i(z)$ of the ℓ-highest weight of $\mathcal{W}_0^{\rm osc}(-(-q)^n)$ is trivial for all i, except

$$\Psi_r(z) = q^{-1}\frac{1-qz}{1-q^{-1}z}.$$

This is the reciprocal of the ℓ-highest weight of the r-th finite-dimensional fundamental representation $V(\varpi_r)$ of $U_q'(\widehat{\mathfrak{sl}}_n)$ (up to a spectral parameter shift).

Recall that the ℓ-highest weight of the KR module $W^{(r)}_{s,q^{1-s}}$ (Remark 4.3.16) is

$$\Psi_r(z) = q^s \frac{\prod_{k=1}^s (1-q^{2k-2}q^{-1}q^{1-s}z)}{\prod_{k=1}^s (1-q^{2k-2}qq^{1-s}z)} = q^s\frac{1-q^{-s}z}{1-q^s z}.$$

Hence, the ℓ-highest weight of $\mathcal{W}_0^{\rm osc}(x)$ can be seen as the one of the KR module with $s=-1$, which is of course not defined.

Recently, Zhang constructed a generic asymptotic limit of KR modules, say $W^{(r)}_{x;c}$ for $c\neq 0$ [100]. It is an infinite-dimensional $U_q'(\widehat{\mathfrak{g}})$-module[2] contained in the category $\widehat{\mathcal{O}}$, and our $\mathcal{W}_0^{\rm osc}(x)$ is indeed one example of such a limit, $W^{(r)}_{x;q^{-1}}$. On one hand, it is a module-theoretic realization of the analytic continuation considered in [80]. On the other hand, it is a generalization of the asymptotic limit construction of prefundamental representations over the Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\widehat{\mathfrak{g}})$ [40], which recovers them by the non-generic limit $c=0$.

5.3.2 R-matrix and spectral decomposition

The construction of universal R-matrices in Section 4.2.2 also works for q-oscillator representations. Namely, one can define the affinization $V_{\rm aff} = V\otimes\mathbb{C}[z^{\pm 1}]$ of $V\in\widehat{\mathcal{O}}_{{\rm osc},\epsilon}$, and

[2] In [100], the construction is originally for $\mathfrak{g}=\mathfrak{gl}_{M|N}$, and the one for simple Lie algebras $\mathfrak{g}$ is treated in its appendix.

then the universal R-matrix

$$\mathcal{R}^{\mathrm{univ}}_{V,W} : V_{\mathrm{aff}} \otimes W_{\mathrm{aff}} \longrightarrow W_{\mathrm{aff}} \widetilde{\otimes} V_{\mathrm{aff}}$$

for $V, W \in \widehat{\mathcal{O}}_{\mathrm{osc},\epsilon}$.

Let us focus on fundamental q-oscillator representations: let $\mathcal{R}^{\mathrm{univ}}_{l,m,\epsilon} := \mathcal{R}^{\mathrm{univ}}_{\mathcal{W}^{\mathrm{osc}}_{l,\epsilon},\mathcal{W}^{\mathrm{osc}}_{m,\epsilon}}$. We have

$$\mathcal{R}^{\mathrm{univ}}_{l,m,\epsilon}(v_l \otimes v_m) = a(z_1/z_2)(v_m \otimes v_l)$$

for some $a(z) \in \Bbbk[\![z]\!]^{\times}$. Put

$$c(z) = \begin{cases} \prod_{i=1}^{\min\{|l|,|m|\}} \frac{1-q^{|l-m|+2i}z}{z-q^{|l-m|+2i}} & \text{if } lm > 0 \\ 1 & \text{if } lm \le 0, \end{cases}$$

and define the normalized R-matrix

$$\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon} = c(z)a(z)^{-1}\mathcal{R}^{\mathrm{univ}}_{l,m,\epsilon}.$$

We remark that $\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}(v_l \otimes v_m) = c(z)(v_m \otimes v_l)$, that is $\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}$ is not the identity on the tensor product $v_l \otimes v_m$ of ℓ-highest weight vectors if $lm > 0$.

For $t \in \mathbb{Z}_{\ge 0}$, let us define a $\mathring{\mathcal{U}}(\epsilon)$-linear map $\mathcal{P}^{l,m}_t : \mathcal{W}^{\mathrm{osc}}_{l,\epsilon} \otimes \mathcal{W}^{\mathrm{osc}}_{m,\epsilon} \longrightarrow \mathcal{W}^{\mathrm{osc}}_{m,\epsilon} \otimes \mathcal{W}^{\mathrm{osc}}_{l,\epsilon}$ as in Section 4.3.1. Namely, put $\widetilde{\boldsymbol{\epsilon}} = \epsilon^{(a+k,b+k)}$ for $k > 0$ such that $\mathcal{V}^{(l_1+t,l_2-t)}_{\widetilde{\overline{\boldsymbol{\epsilon}}}} \neq 0$. Take a $\mathring{\mathcal{U}}(\widetilde{\overline{\boldsymbol{\epsilon}}})$-highest weight vector $v_0(l,m,t)$ of $\mathcal{V}^{(l_1+t,l_2-t)}_{\widetilde{\overline{\boldsymbol{\epsilon}}}}$ in $\mathcal{W}^{\mathrm{osc}}_{l,\widetilde{\overline{\boldsymbol{\epsilon}}}} \otimes \mathcal{W}^{\mathrm{osc}}_{m,\widetilde{\overline{\boldsymbol{\epsilon}}}}$ (and so in $\mathcal{W}^{\mathrm{osc}}_{l,\widetilde{\boldsymbol{\epsilon}}} \otimes \mathcal{W}^{\mathrm{osc}}_{m,\widetilde{\boldsymbol{\epsilon}}}$, identifying $\mathcal{W}^{\mathrm{osc}}_{l,\widetilde{\overline{\boldsymbol{\epsilon}}}} = \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\widetilde{\overline{\boldsymbol{\epsilon}}}}(\mathcal{W}^{\mathrm{osc}}_{l,\widetilde{\boldsymbol{\epsilon}}}) \subset \mathcal{W}^{\mathrm{osc}}_{l,\widetilde{\boldsymbol{\epsilon}}}$) that corresponds to the one in its crystal base at $q = \infty$. Then we choose a projection $\mathcal{P}^{l,m}_t$ for $\epsilon = \widetilde{\boldsymbol{\epsilon}}$ normalized by

$$\mathcal{P}^{l,m}_t(v_0(l,m,t')) = \delta_{t,t'} v_0(m,l,t),$$

and the one for $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$ is obtained by truncation from $\widetilde{\boldsymbol{\epsilon}}$.

Now we ask for the spectral decomposition

$$\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}(z) = \sum_{t \ge 0} \rho_t(z)\mathcal{P}^{l,m}_t$$

for $\rho_t(z) \in \Bbbk(z)$. The strategy is similar with the case of polynomial representations: we prove the irreducibility of generic tensor products of two fundamental q-oscillator representations, which implies that the truncation preserves the normalized R-matrix. Then we lift the known spectral decomposition of the normalized R-matrix $\mathcal{R}^{\rm norm}_{l,m,\overline{\boldsymbol{\epsilon}}}$, *i.e.* for finite-dimensional fundamental representations.

Let us first prove the irreducibility of $\mathcal{W}^{\rm osc}_{l,\epsilon}(x)\otimes\mathcal{W}^{\rm osc}_{m,\epsilon}(y)$ for generic $x,y\in\Bbbk^\times$ in the case of $\epsilon=\underline{\boldsymbol{\epsilon}}$. From here to the proof of Theorem 5.3.7, we put $n=a+b+1$ and $r=a$ as in Section 5.1.

Let $l,m\in\mathbb{Z}$ be given. We shall find all $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vectors u_i of $\mathcal{W}^{\rm osc}_l(x)\otimes\mathcal{W}^{\rm osc}_m(y)$, and then establish a connection between them under the action of $\mathcal{U}(\underline{\boldsymbol{\epsilon}})$.

Set $l_1=\max\{l,m\}$, $l_2=\min\{l,m\}$ and

$$L=\max\{-l_1,l_2,0\}=\begin{cases} l_2 & \text{if } l_2\geq 0\\ -l_1 & \text{if } l_1\leq 0\\ 0 & \text{if } l_1\geq 0\geq l_2.\end{cases}$$

Equivalently, L is the smallest nonnegative integer such that $l_1+L\geq 0\geq l_2-L$. Then the tensor product decomposition (5.2.3) becomes

$$\mathcal{W}^{\rm osc}_l(x)\otimes\mathcal{W}^{\rm osc}_m(y)=\mathcal{V}^{(l)}\otimes\mathcal{V}^{(m)}\cong\bigoplus_{i=-L}^{\infty}\mathcal{V}^{(l_1+L+i,l_2-L-i)}. \tag{5.3.1}$$

Let us write $v_l=|\mathbf{m}\rangle$, $v_m=|\mathbf{m}'\rangle$ for $\mathbf{m},\mathbf{m}'\in(\mathbb{Z}_{\geq 0})^n$. Put

$$v^+_{a,b}=|\mathbf{m}+a(\mathbf{e}_r+\mathbf{e}_{r+1})\rangle\otimes|\mathbf{m}'+b(\mathbf{e}_r+\mathbf{e}_{r+1})\rangle$$

for $a,b\geq 0$ and

$$v^-_{a,b}=\begin{cases}|\mathbf{m}+a(-\mathbf{e}_{r+1}+\mathbf{e}_{r+2})\rangle\otimes|\mathbf{m}'+b(-\mathbf{e}_{r+1}+\mathbf{e}_{r+2})\rangle & \text{if } l_2\geq 0\\ |\mathbf{m}+a(\mathbf{e}_{r-1}-\mathbf{e}_r)\rangle\otimes|\mathbf{m}'+b(\mathbf{e}_{r-1}-\mathbf{e}_r)\rangle & \text{if } l_1\leq 0\end{cases}$$

for $a,b\geq 0$ with $a+b\leq L$. Note that $v^\pm_{0,0}=v_l\otimes v_m$.

Lemma 5.3.4. *The $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vector u_i of $\mathcal{V}^{(l_1+L+i,l_2-L-i)}$ in $\mathcal{W}^{\rm osc}_l(x)\otimes\mathcal{W}^{\rm osc}_m(y)$ is given by*

(1) *for* $i \geq 0$,

$$u_i = \sum_{j=0}^{i} \left[(-1)^j \prod_{k=1}^{j} \left(q^{-(|m|+2i-2k+1)} \frac{[|m|+i+1-k][i+1-k]}{[|l|+k][k]} \right) \right] v^+_{j,i-j},$$

(2) *for* $-L \leq i \leq -1$,

$$u_i = \sum_{j=0}^{-i} \left[(-1)^j \prod_{k=1}^{j} \left(q^{|m|+2i+2k} \frac{[-i+1-k]}{[k]} \right) \right] v^-_{j,-i-j}.$$

Proof. It is straightforward to verify $e_r u_i = e_{r+1} u_i = 0$, while $e_k u_i = 0$ for $k \neq 0, r, r+1$ is clear. □

Next, we describe how the $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vectors are related under the $\mathcal{U}(\underline{\boldsymbol{\epsilon}})$-action. Put

$$\begin{aligned}
\boldsymbol{F}^+ &= (e_{r+1} \cdots e_{n-2} e_{n-1})(e_{r-1} \cdots e_2 e_1) e_0, \\
\boldsymbol{E}^+ &= f_0 (f_1 \cdots f_{r-2} f_{r-1})(f_{n-1} \cdots f_{r+2} f_{r+1}), \\
\boldsymbol{F}^- &= \begin{cases} e_r (e_{r+2} \cdots e_{n-2} e_{n-1})(e_{r-1} \cdots e_2 e_1) e_0 & \text{if } l_2 \geq 0 \\ e_r (e_{r+1} \cdots e_{n-2} e_{n-1})(e_{r-2} \cdots e_2 e_1) e_0 & \text{if } l_1 \leq 0, \end{cases} \\
\boldsymbol{E}^- &= \begin{cases} f_0 (f_1 \cdots f_{r-2} f_{r-1})(f_{n-1} \cdots f_{r+3} f_{r+2}) f_r & \text{if } l_2 \geq 0 \\ f_0 (f_1 \cdots f_{r-3} f_{r-2})(f_{n-1} \cdots f_{r+2} f_{r+1}) f_r & \text{if } l_1 \leq 0. \end{cases}
\end{aligned}$$

The following lemma can be proved by a direct computation as well.

Lemma 5.3.5. *In* $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$, *we have the following identities.*

(1) *For* $a, b \geq 0$,

$$\begin{aligned}
\boldsymbol{F}^+ v^+_{a,b} &= y v^+_{a,b+1} + x q^{-|m|-2b-1} v^+_{a+1,b}, \\
\boldsymbol{E}^+ v^+_{a,b} &= -x^{-1} [a+|l|][a] v^+_{a-1,b} - y^{-1} q^{2a+|l|+1} [b+|m|][b] v^+_{a,b-1}.
\end{aligned}$$

(2) *For* $a, b \geq 0$ *with* $a + b \leq L$,

$$\boldsymbol{F}^- v^-_{a,b} = -y[|m|-b] v^-_{a,b+1} - x q^{|m|-2b} [|l|-a] v^-_{a+1,b},$$

$$\boldsymbol{E}^- v^-_{a,b} = -x^{-1}[a]v^-_{a-1,b} - y^{-1}q^{-|l|+2a}[b]v^-_{a,b-1}.$$

Using the identities, we show that all the $\mathring{\mathcal{U}}(\underline{\epsilon})$-highest weight vectors are generated from u_0, for generic x, y.

Lemma 5.3.6. *For generic $x, y \in \Bbbk^\times$, we have*

(1) *for $i \geq 0$, $\boldsymbol{F}^+ u_i \in \Bbbk^\times u_{i+1} + \Bbbk f_r u_i + \delta(i \neq 0)\Bbbk f_r^{(2)} u_{i-1}$,*

(2) *for $-L \leq i \leq 0$, when $l_2 \geq 0$, $\boldsymbol{F}^- u_i \in \Bbbk^\times u_{i-1} + \Bbbk f_{r+1} u_i + \delta(i \neq 0)\Bbbk f_{r+1}^{(2)} u_{i+1}$,*

(3) *for $-L \leq i \leq 0$, when $l_1 \leq 0$, $\boldsymbol{F}^- u_i \in \Bbbk^\times u_{i-1} + \Bbbk f_{r-1} u_i + \delta(i \neq 0)\Bbbk f_{r-1}^{(2)} u_{i+1}$.*

More precisely, if we regard x and y as indeterminates, then the coefficient of $u_{i\pm 1}$ in $\boldsymbol{F}^\pm u_i$ is a nonzero polynomial in x and y.

Proof. We only prove (1), leaving the other two to the reader. Considering the classical decomposition (5.3.1) and weights, we can write

$$\boldsymbol{F}^+ u_i = \sum_{k \geq 0} C_i^k f_r^{(k)} u_{i+1-k}$$

for some $C_i^k \in \Bbbk$. First, one can verify by a direct computation that

$$e_r^2 \boldsymbol{F}^+ u_i = [2][|m| + i][i](yq^{|m|-2i+1} - xq^{|l|+1})u_{i-1},$$

which implies $C_i^k = 0$ whenever $k > 2$, and also the identity

$$C_i^2 = \frac{[2][|m| + i][i]}{[|l| + |m| + 2i + 1][|l| + |m| + 2i]}(yq^{|m|-2i+1} - xq^{|l|+1}).$$

Next, by comparing the coefficients of $v^+_{i,0}$ of both sides of

$$e_r \boldsymbol{F}^+ u_i = -C_i^1[|l| + |m| + 2i]u_i - C_i^2[|l| + |m| + 2i + 1]f_r u_{i-1},$$

we obtain a formula for C_i^1. Then we substitute C_i^1, C_i^2 in

$$\boldsymbol{F}^+ u_i = C_i^0 u_{i+1} + C_i^1 f_r u_i + C_i^2 f_r^{(2)} u_{i-1}$$

with the obtained formulas, and then compare the coefficients of $v^+_{i+1,0}$, to conclude that C^0_i is a nonzero polynomial in x, y (regarded as indeterminates). $\square$

Theorem 5.3.7. *For generic $x, y \in \Bbbk^\times$, the tensor product $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$ is an irreducible $\mathcal{U}(\underline{\boldsymbol{\epsilon}})$-module.*

Proof. Taking a nonzero $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-submodule K of $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$, we claim

(1) K contains $u_0 = v_l \otimes v_m$,

(2) u_0 generates $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$ for generic x, y.

We first prove (1). Since $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$ is semisimple over $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$, K contains a $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vector, say u_i.

Suppose $-L \le i \le -1$. We first observe that $\boldsymbol{E}^- u_i$ is a nonzero scalar multiple of u_{i+1}. Indeed, that $\boldsymbol{E}^- u_i \neq 0$ can be verified by computing the coefficient of $v^-_{0,-i-1}$ in $\boldsymbol{E}^- u_i$, using Lemma 5.3.5. Since $\boldsymbol{E}^- u_i$ has the same weight as u_{i+1} and $\mathcal{W}^{\rm osc}_l(x) \otimes \mathcal{W}^{\rm osc}_m(y)$ is multiplicity-free over $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$, to conclude it remains to check that $e_j \boldsymbol{E}^- u_i = 0$ for all $j \in I \setminus \{0\}$. Actually, it is enough to do it for $j = r, r+1$ if $l_2 \ge 0$ or $j = r, r-1$ if $l_1 \le 0$ by a weight comparison. If $l_2 \ge 0$, then

$$\begin{aligned} e_r \boldsymbol{E}^- u_i &= f_0 (f_1 \cdots f_{r-2} f_{r-1})(f_{n-1} \cdots f_{r+3} f_{r+2}) \frac{k_r - k_r^{-1}}{q - q^{-1}} u_i \\ &= -[l+m+i+2] f_0 (f_1 \cdots f_{r-2} f_{r-1})(f_{n-1} \cdots f_{r+3} f_{r+2}) u_i = 0, \\ e_{r+1} \boldsymbol{E}^- u_i &= e_{r+1} f_0 (f_1 \cdots f_{r-2} f_{r-1})(f_{n-1} \cdots f_{r+3} f_{r+2}) f_r u_i = \boldsymbol{E}^- e_{r+1} u_i = 0. \end{aligned}$$

The case $l_1 \le 0$ is similar. Consequently, $u_{i+1} \in \Bbbk^\times \boldsymbol{E}^- u_i \subset K$, and so $u_0 \in K$. One can do analogously in the case $i > 0$, arguing with $\boldsymbol{E}^+ u_i$.

Provided $u_0 \in K$, an easy induction proves that $u_i \in K$ for $i > 0$ (resp. $-L \le i \le -1$), thanks to Lemma 5.3.6 (1) (resp. (2)). This completes the proof of the irreducibility. $\square$

Theorem 5.3.8. *For $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$, the tensor product $\mathcal{W}^{\rm osc}_{l,\epsilon}(x) \otimes \mathcal{W}^{\rm osc}_{m,\epsilon}(y)$ is an irreducible $\mathcal{U}(\epsilon)$-module for generic $x, y \in \Bbbk^\times$.*

Proof. Since $\mathcal{W}^{\rm osc}_{l,\overline{\boldsymbol{\epsilon}}}(x) \cong V(\varpi_{r+l})_x$ if $0 \le r+l \le N$ and zero otherwise, the irreducibility in the case $\epsilon = \overline{\boldsymbol{\epsilon}}$ is well-known [58].

For the remaining case $\epsilon = \boldsymbol{\epsilon}$, first identify

$$\mathcal{W}^{\rm osc}_{l,\underline{\boldsymbol{\epsilon}}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\underline{\boldsymbol{\epsilon}}}(y) \subset \mathcal{W}^{\rm osc}_{l,\boldsymbol{\epsilon}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\boldsymbol{\epsilon}}(y)$$

as a vector space. Given a nonzero $v \in \mathcal{W}^{\rm osc}_{l,\boldsymbol{\epsilon}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\boldsymbol{\epsilon}}(y)$, we can find a nonzero $v' \in \mathcal{W}^{\rm osc}_{l,\underline{\boldsymbol{\epsilon}}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\underline{\boldsymbol{\epsilon}}}(y)$ by applying finitely many E_i, F_{i-1} (for $i \leq r$) or E_{i-1}, F_i (for $i > r$) for some i's with $\epsilon_i = 1$. Since $\mathcal{W}^{\rm osc}_{l,\underline{\boldsymbol{\epsilon}}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\underline{\boldsymbol{\epsilon}}}(y)$ is irreducible, v' generates $\mathcal{W}^{\rm osc}_{l,\underline{\boldsymbol{\epsilon}}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\underline{\boldsymbol{\epsilon}}}(y)$ and in particular, all the classical components $\mathcal{V}^{(l_1+t,l_2-t)}_{\underline{\boldsymbol{\epsilon}}}$ $(t \geq 0)$. Again by the irreducibility of $\mathcal{V}^{(l_1+t,l_2-t)}_{\boldsymbol{\epsilon}}$, the submodule generated by v contains all $\mathcal{V}^{(l_1+t,l_2-t)}_{\boldsymbol{\epsilon}}$, namely $\mathcal{W}^{\rm osc}_{l,\boldsymbol{\epsilon}}(x) \otimes \mathcal{W}^{\rm osc}_{m,\boldsymbol{\epsilon}}(y)$. $\square$

Therefore, the normalized R-matrix

$$\mathcal{R}^{\rm norm}_{l,m,\epsilon} : \mathcal{W}^{\rm osc}_{l,\epsilon}(z_1) \otimes \mathcal{W}^{\rm osc}_{m,\epsilon}(z_2) \longrightarrow \Bbbk(z_1, z_2) \otimes_{\Bbbk[z_1^{\pm 1}, z_2^{\pm 1}]} \mathcal{W}^{\rm osc}_{m,\epsilon}(z_2) \otimes \mathcal{W}^{\rm osc}_{l,\epsilon}(z_1)$$

is uniquely characterized as the $\Bbbk[z_1^{\pm 1}, z_2^{\pm 1}] \otimes \mathcal{U}(\epsilon)$-linear map satisfying $\mathcal{R}^{\rm norm}_{l,m,\epsilon}(v_l \otimes v_m) = c(z)(v_m \otimes v_l)$.

Lemma 5.3.9. *For $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}$ or $\overline{\boldsymbol{\epsilon}}$, we have*

$$\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\boldsymbol{\epsilon}'} \left(\mathcal{R}^{\rm norm}_{l,m,\boldsymbol{\epsilon}} \right) = \mathcal{R}^{\rm norm}_{l,m,\boldsymbol{\epsilon}'}.$$

Now the spectral decomposition follows from the argument in Section 4.2.2.

Theorem 5.3.10. *For $\epsilon = \boldsymbol{\epsilon}$ or $\underline{\boldsymbol{\epsilon}}$, we have*

$$\mathcal{R}^{\rm norm}_{l,m,\epsilon} = \sum_{t \geq 0} \prod_{i=1}^{t} \frac{1 - q^{|l-m|+2i} z}{z - q^{|l-m|+2i}} \mathcal{P}^{l,m}_t \tag{5.3.2}$$

up to a scalar multiple. Here the coefficient of $\mathcal{P}^{l,m}_0$ is understood to be 1.

Remark 5.3.11. In the proof of Theorem 5.3.7, we have seen that $\boldsymbol{E}^{\pm} u_i$ is again an highest weight vector $u_{i \mp 1}$, up to a nonzero scalar multiple. Hence it is possible to directly compute the spectral decomposition of $\mathcal{R}^{\rm norm}_{l,m,\underline{\boldsymbol{\epsilon}}}$, see [74, Theorem 7.12] for example.

5.3.3 Fusion construction of irreducible q-oscillator representations

We apply the fusion construction to fundamental q-oscillator representations to obtain a family of irreducible representations in the category $\widehat{\mathcal{O}}_{\mathrm{osc},\epsilon}$.

Given $l_1, l_2 \in \mathbb{Z}$, suppose $c_1, c_2 \in \Bbbk^\times$ are such that $c_1/c_2 \notin q^{|l_1-l_2|+2\mathbb{Z}_{>0}}$. According to the spectral decomposition (5.3.2), we can specialize the normalized R-matrix to obtain a $\mathcal{U}(\epsilon)$-linear map

$$R_{(l_1,l_2),\epsilon}(c_1,c_2) = \mathcal{R}^{\mathrm{norm}}_{l_1,l_2,\epsilon}(c_1/c_2) : \mathcal{W}^{\mathrm{osc}}_{l_1,\epsilon}(c_1) \otimes \mathcal{W}^{\mathrm{osc}}_{l_2,\epsilon}(c_2) \longrightarrow \mathcal{W}^{\mathrm{osc}}_{l_2,\epsilon}(c_2) \otimes \mathcal{W}^{\mathrm{osc}}_{l_1,\epsilon}(c_1).$$

Let $\mathcal{P}^{+,\mathrm{osc}}$ be the set of pairs $(\boldsymbol{l}, \boldsymbol{c})$ such that

(1) $\boldsymbol{l} = (l_1, \ldots, l_\ell) \in \mathbb{Z}^\ell$ and $\boldsymbol{c} = (c_1, \ldots, c_\ell) \in (\Bbbk^\times)^\ell$ for some $\ell \geq 1$,

(2) $c_i/c_j \notin q^{|l_i-l_j|+2\mathbb{Z}_{>0}}$ for all $i < j$.

For $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^{+,\mathrm{osc}}$ with $\ell \geq 2$, we define a $\mathcal{U}(\epsilon)$-linear map

$$R_{\boldsymbol{l},\epsilon}(\boldsymbol{c}) : \mathcal{W}^{\mathrm{osc}}_{l_1,\epsilon}(c_1) \otimes \cdots \otimes \mathcal{W}^{\mathrm{osc}}_{l_\ell,\epsilon}(c_\ell) \longrightarrow \mathcal{W}^{\mathrm{osc}}_{l_\ell,\epsilon}(c_\ell) \otimes \cdots \otimes \mathcal{W}^{\mathrm{osc}}_{l_1,\epsilon}(c_1)$$

by taking the composition of $R_{(l_i,l_j),\epsilon}(c_i,c_j)$ associated with a reduced expression of the longest element of $\mathfrak{S}_\ell$.

Note from (5.3.2) that $R_{(l,l),\epsilon}(c,c)$ is a nonzero scalar multiple of the identity map on $(\mathcal{W}^{\mathrm{osc}}_{l,\epsilon}(c))^{\otimes 2}$. Although $\mathcal{R}^{\mathrm{norm}}_{l,m,\epsilon}$ is not rationally renormalizable, we may argue as in the proof of Theorem 4.2.13 to prove an analogue for q-oscillator representations.

Theorem 5.3.12. *For $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^{+,\mathrm{osc}}$, the image of $R_{\boldsymbol{l},\epsilon}(\boldsymbol{c})$ is an irreducible representation in $\widehat{\mathcal{O}}_{\mathrm{osc},\epsilon}$ unless it is zero.*

We put

$$\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c}) = \mathrm{im} R_{\boldsymbol{l},\epsilon}(\boldsymbol{c})$$

and when $\ell = 1$, $\mathcal{W}_\epsilon((l_1),(c_1)) := \mathcal{W}^{\mathrm{osc}}_{l_1,\epsilon}(c_1)$. Again, it is not easy to determine exactly when $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ is nonzero in general, but we have the following criterion as in the case of polynomial representations.

Proposition 5.3.13. *Let $\epsilon = \boldsymbol{\epsilon}, \underline{\boldsymbol{\epsilon}}$ or $\overline{\boldsymbol{\epsilon}}$. For $(\boldsymbol{l}, \boldsymbol{c}) \in \mathcal{P}^{+,\mathrm{osc}}$, $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ is nonzero if $\boldsymbol{l}^+ \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$, where $\boldsymbol{l}^+$ denotes the rearrangement of $\boldsymbol{l}$ into a weakly decreasing sequence. In particular, $\mathcal{W}_\epsilon(\boldsymbol{l}, \boldsymbol{c})$ is nonzero for all sufficiently large a, b.*

Proof. First suppose $\epsilon = \overline{\boldsymbol{\epsilon}}$. The assumption $\boldsymbol{l}^+ \in \mathcal{P}(GL_\ell)_{(\epsilon_-,\epsilon_+)}$ ensures that $0 \leq l_i + a \leq a + b$ for all i, so that $\mathcal{W}^{\mathrm{osc}}_{l_i,\overline{\boldsymbol{\epsilon}}}(c_i) \cong V(\varpi_{l_i+a})_{c_i}$. If we let w_i be a dominant extremal weight vector of $V(\varpi_{l_i+a})_{c_i}$, then $\mathcal{R}^{\rm norm}_{l_i,l_j,\overline{\boldsymbol{\epsilon}}}$ is normalized so that it maps $w_i \otimes w_j$ to $w_j \otimes w_i$. In particular, $R_{\boldsymbol{l},\overline{\boldsymbol{\epsilon}}}(\boldsymbol{c})$ maps $w_1 \otimes \cdots \otimes w_\ell$ to $w_\ell \otimes \cdots \otimes w_1$ and so $\mathcal{W}_{\overline{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c}) \neq 0$.

For the cases $\epsilon = \boldsymbol{\epsilon}$ or $\underline{\boldsymbol{\epsilon}}$, we take $\widetilde{\boldsymbol{\epsilon}} = \epsilon^{(a+k,b+k)}$ for $k > 0$ such that $\boldsymbol{l}^+ \in \mathcal{P}(GL_\ell)_{\overline{\widetilde{\boldsymbol{\epsilon}}}}$. The above argument tells us that $\mathcal{W}_{\overline{\widetilde{\boldsymbol{\epsilon}}}}(\boldsymbol{l},\boldsymbol{c})$ contains a classical component $\mathcal{V}^{\boldsymbol{l}^+}_{\overline{\widetilde{\boldsymbol{\epsilon}}}}$ generated by $w_\ell \otimes \cdots \otimes w_1$. By Theorem 5.2.14, $\mathcal{V}^{\boldsymbol{l}^+}_{\widetilde{\boldsymbol{\epsilon}}} \subset \mathcal{W}_{\widetilde{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c})$ as well. One can directly check that $\mathcal{V}^{\boldsymbol{l}^+}_{\boldsymbol{\epsilon}} \subset \mathfrak{tr}^{\widetilde{\boldsymbol{\epsilon}}}_{\boldsymbol{\epsilon}}(\mathcal{V}^{\boldsymbol{l}^+}_{\widetilde{\boldsymbol{\epsilon}}})$ so that $\mathcal{W}_{\boldsymbol{\epsilon}}(\boldsymbol{l},\boldsymbol{c}) \neq 0$, and again by Theorem 5.2.14 we have $\mathcal{W}_{\underline{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c}) \neq 0$. □

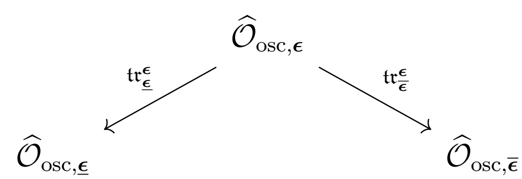
5.3.4 Correspondence of irreducibles and super duality

Since truncations to $\underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$ preserve fundamental q-oscillator representations and normalized R-matrices, we finally obtain the following correspondence.

Theorem 5.3.14. *For $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}$ or $\overline{\boldsymbol{\epsilon}}$, we have*

$$\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\boldsymbol{\epsilon}'}(\mathcal{W}_{\boldsymbol{\epsilon}}(\boldsymbol{l},\boldsymbol{c})) = \mathcal{W}_{\boldsymbol{\epsilon}'}(\boldsymbol{l},\boldsymbol{c}).$$

Therefore, $\mathcal{W}_{\boldsymbol{\epsilon}}(\boldsymbol{l},\boldsymbol{c})$ interpolates the finite-dimensional irreducible $U'_{-q^{-1}}(\widehat{\mathfrak{gl}}_{a+b})$-module $\mathcal{W}_{\overline{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c})$[3] and the irreducible q-oscillator $U_q(\widehat{\mathfrak{gl}}_{a+b+1})$-modules $\mathcal{W}_{\underline{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c})$ if they are nonzero. To sum up, we have the following diagram



in which irreducible q-oscillator representations and irreducible finite-dimensional representations correspond naturally. In the spirit of super duality, we expect that the truncations $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\underline{\boldsymbol{\epsilon}}}$, $\mathfrak{tr}^{\boldsymbol{\epsilon}}_{\overline{\boldsymbol{\epsilon}}}$ between the categories $\widehat{\mathcal{O}}_{\rm osc,\boldsymbol{\epsilon}}$ and $\widehat{\mathcal{O}}_{\rm osc,\boldsymbol{\epsilon}'}$ for $\boldsymbol{\epsilon}' = \underline{\boldsymbol{\epsilon}}, \overline{\boldsymbol{\epsilon}}$ become equivalences of categories, after taking a suitable limit of $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$.

As an evidence, we propose an exact sequence which should be viewed as a T-system for q-oscillator representations. For $l \in \mathbb{Z}$, $s \in \mathbb{Z}_{\geq 0}$ and $c \in \Bbbk^\times$, we introduce KR-type

[3]Although the condition of $\mathcal{P}^{+,\rm osc}$ on spectral parameters is stronger than the one of $\mathcal{P}^+$, every finite-dimensional irreducible module can still be obtained as $\mathcal{W}_{\overline{\boldsymbol{\epsilon}}}(\boldsymbol{l},\boldsymbol{c})$ for some $(\boldsymbol{l},\boldsymbol{c}) \in \mathcal{P}^{+,\rm osc}$.

modules in $\widehat{\mathcal{O}}_{\mathrm{osc},\underline{\boldsymbol{\epsilon}}}$,

$$\mathcal{W}^{l,s}_{\mathrm{osc}}(c) = \mathcal{W}_{\underline{\boldsymbol{\epsilon}}}(\boldsymbol{l}, \boldsymbol{c}), \quad \boldsymbol{l} = (l, \ldots, l),\ \boldsymbol{c} = (cq^{2-2s}, \ldots, cq^{-2}, c).$$

Note that the corresponding irreducible representation in $\widehat{\mathcal{O}}_{\mathrm{osc},\overline{\boldsymbol{\epsilon}}}$ is indeed the KR module $\mathcal{W}^{(l+r)}_{s,c}$, which implies the following classical irreducibility via truncation.

Proposition 5.3.15. *As a $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-module,*

$$\mathcal{W}^{l,s}_{\mathrm{osc}}(c) \cong \begin{cases} \mathcal{V}^{(l^s)} & \text{if } (l^s) \in \mathcal{P}(GL_s)_{(r,n-r)} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{W}^{l,s}_{\mathrm{osc}}(c)$ is nonzero if and only if $(l^s) \in \mathcal{P}(GL_s)_{(r,n-r)}$, as can be seen from Proposition 5.3.13 as well. In particular, $\mathcal{W}^{0,s}_{\mathrm{osc}}(c) \neq 0$ for any $s \geq 1$.

Conjecture 5.3.16. *There exists a short exact sequence in $\widehat{\mathcal{O}}_{\mathrm{osc}}$:*

$$0 \longrightarrow \mathcal{W}^{l,s+1}_{\mathrm{osc}}(1) \otimes \mathcal{W}^{l,s-1}_{\mathrm{osc}}(q^{-2}) \longrightarrow \mathcal{W}^{l,s}_{\mathrm{osc}}(1) \otimes \mathcal{W}^{l,s}_{\mathrm{osc}}(q^{-2}) \longrightarrow \bigotimes_{l'=l\pm 1} \mathcal{W}^{l',s}_{\mathrm{osc}}(q^{-1}) \longrightarrow 0.$$

The corresponding exact sequence in $\widehat{\mathcal{O}}_{\mathrm{osc},\overline{\boldsymbol{\epsilon}}}$ is the usual T-system of quantum affine algebras (Proposition 4.3.15). We will give the proof for the base case $s = 1$ in Section 6.2.2, and we expect that the general case can be proved by an induction in s.

Indeed, the known T-system for quantum affine algebras tells us via truncation that the classical decompositions of the tensor products in the sequence match. Moreover, from the known denominator formula for tensor products of KR modules (see *e.g.* [85]), one can prove using Theorem 5.3.12 that the tensor products $\mathcal{W}^{l,s+1}_{\mathrm{osc}}(1) \otimes \mathcal{W}^{l,s-1}_{\mathrm{osc}}(q^{-2})$ and $\bigotimes_{l'=l\pm 1} \mathcal{W}^{l',s}_{\mathrm{osc}}(q^{-1})$ are irreducible. Hence, it remains to construct nonzero $\mathcal{U}(\epsilon)$-linear maps whose composition vanishes, for which induction is expected to work.

Finally, we conclude this chapter with asremark on the corresponding category of $\mathcal{U}(\varepsilon)$-modules, for $\varepsilon = \epsilon_{a|b}$. Recall from Section 5.1 that in the classical picture, the equivalence between the category of oscillator representations and the one of finite-dimensional representations is obtained by two super dualities, and a module category of $\mathfrak{gl}_\varepsilon = \mathfrak{gl}_{a|b}$ is at the middle of them (see (5.1.6)).

Observe from definition that besides $\epsilon = \overline{\boldsymbol{\epsilon}}$, $\mathcal{W}^{\mathrm{osc}}_{l,\varepsilon}$ is finite-dimensional as well, and zero if $l > b$. Moreover, the proofs in Section 5.2.2 can be repeated with $\epsilon = \varepsilon$ ($\boldsymbol{\epsilon}' = \varepsilon$ in

Theorem 5.2.14), thanks to the following classical duality.

Theorem 5.3.17 ([23, Theorem 3.3])**.** *There exists a $(\mathfrak{gl}_{a|b}, GL_\ell)$-bimodule structure on $(W_\varepsilon^{\rm osc})^{\otimes \ell}$ with the following multiplicity-free decomposition into simple bimodules decomposition*

$$(W_\varepsilon^{\rm osc})^{\otimes \ell} = \bigoplus_{\lambda \in \mathcal{P}(GL_\ell)_{(\varepsilon_-, \varepsilon_+)}} V_\varepsilon^\lambda \otimes V_{GL_\ell}(\lambda).$$

Consequently, we obtain the category $\widehat{\mathcal{O}}_{{\rm osc},\varepsilon}$ of q-oscillator representations of $\mathcal{U}(\varepsilon)$, which are in fact finite-dimensional due to the finite dominance condition on weights. More specifically, if one restricts to the subalgebra corresponding to $\mathfrak{gl}_\varepsilon$ (rather than $\mathfrak{gl}_\varepsilon^e$), then the irreducible q-oscillator representations $\mathcal{V}_\varepsilon^\lambda$ are exactly the duals of the irreducible polynomial representations, namely those of nonpositive integral weights.

Therefore, our study of q-oscillator representations is somehow dual or complementary to the one of polynomial representations in Chapter 4. At this stage, it is hard to investigate the structure of tensor products of a polynomial representation and a dual of one, for instance as it is not semisimple even over the finite type subalgebra (*cf.* [99]). Nevertheless, it should be an interesting problem to understand this unique structure of tensor products of representations of quantum affine superalgebras, under the philosophy of super duality.

Chapter 6

Proofs

6.1 Chapter 4

6.1.1 Proof of Lemma 4.2.6

In this section we prove Lemma 4.2.6. The proof consists of direct calculations as indicated in [1, Lemma B.1], but we give details for the reader's convenience as it is little more involved.

We claim that there exists an exact sequence of the following form for each $\ell \geq 2$:

$$0 \longrightarrow \mathcal{W}_{\ell,\epsilon}(1) \xrightarrow{\psi_1} \mathcal{W}_{1,\epsilon}(q^{1-\ell}) \otimes \mathcal{W}_{\ell-1,\epsilon}(q) \xrightarrow{\mathcal{R}} \mathcal{W}_{\ell-1,\epsilon}(q) \otimes \mathcal{W}_{1,\epsilon}(q^{1-\ell}) \xrightarrow{\psi_2} \mathcal{W}_{\ell,\epsilon}(1) \longrightarrow 0, \tag{6.1.1}$$

for some $\mathcal{U}(\epsilon)$-linear maps ψ_1 and ψ_2 and $\mathcal{R} = \mathcal{R}^{\mathrm{norm}}_{1,\ell-1}(q^{-\ell})$. Recall from Theorem 4.2.10 that

$$\mathcal{R}^{\mathrm{norm}}_{1,\ell-1}(z) = \mathcal{P}_1 + \frac{1-zq^\ell}{z-q^\ell}\mathcal{P}_0, \tag{6.1.2}$$

which is equal to $\mathcal{P}_1$ when $z = q^{-\ell}$.

We may assume that $\epsilon_1 = 0$. Indeed, the result for arbitrary ϵ follows by choosing $\epsilon' > \epsilon$ with $\epsilon'_1 = 0$ and truncating the exact sequence (6.1.1) for ϵ' to ϵ, keeping Proposition 4.3.2, 4.3.4 and Lemma 4.3.5 in mind.

Recall that when $\epsilon_1 = 0$, the $\mathring{\mathcal{U}}(\epsilon)$-highest weight vectors of $V_\epsilon((\ell))$ and $V_\epsilon((\ell-1,1))$ in the decomposition $\mathcal{W}_{1,\epsilon}(x) \otimes \mathcal{W}_{\ell-1,\epsilon}(y) = V_\epsilon((\ell)) \oplus V_\epsilon((\ell-1,1))$ are given by

$$|\mathbf{e}_1\rangle \otimes |(\ell-1)\mathbf{e}_1\rangle\,, \quad |\mathbf{e}_1\rangle \otimes |(\ell-2)\mathbf{e}_1 + \mathbf{e}_2\rangle - q^{\ell-1}\,|\mathbf{e}_2\rangle \otimes |(\ell-1)\mathbf{e}_1\rangle \tag{6.1.3}$$

respectively. On the other hand, when $\epsilon_1 = 1$ the highest weight vectors become more complicated, which is the reason why we assume $\epsilon_1 = 0$.

Let us define ψ_1 and ψ_2 by

$$\psi_1(|\mathbf{m}\rangle) = \sum_{1\le k\le n} |\mathbf{e}_k\rangle \otimes |\mathbf{m}-\mathbf{e}_k\rangle \left([m_k] \prod_{k<j\le n} q^{m_j} \right)$$
$$\psi_2\left(|\mathbf{m}\rangle \otimes |\mathbf{e}_k\rangle\right) = |\mathbf{m}+\mathbf{e}_k\rangle \prod_{k<j\le n} q^{-m_j}$$

for $|\mathbf{m}\rangle$ and $1 \le k \le n$. Here we also understand $|\mathbf{m}\rangle = 0$ whenever $\mathbf{m} \notin \mathbb{Z}^n_+(\epsilon)$. Note that when $\epsilon = (1^N)$, ψ_1 and ψ_2 coincide with the maps in [1, Lemma B.1] up to a constant multiple.

Lemma 6.1.1. *The maps ψ_1 and ψ_2 are $\mathcal{U}(\epsilon)$-linear.*

Proof. Since the proof is rather straightforward, let us show that ψ_1 commutes with e_i $(i \in I)$, and leave the other details to the reader.

Case 1. Suppose that $i \in I \setminus \{0\}$. First we have

$$\begin{aligned} e_i\psi_1 |\mathbf{m}\rangle =& \sum [m_k] \prod_{j>k} q^{m_j} e_i\left(|\mathbf{e}_k\rangle \otimes |\mathbf{m}-\mathbf{e}_k\rangle\right) \\ =& \sum_{k\ne i,i+1} [m_k] \prod_{j>k} q^{m_j} [m_{i+1}] |\mathbf{e}_k\rangle \otimes |\mathbf{m}-\mathbf{e}_k+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle \\ &+ [m_{i+1}] \prod_{j>i+1} q^{m_j} [m_{i+1}-1] |\mathbf{e}_{i+1}\rangle \otimes |\mathbf{m}+\mathbf{e}_i-2\mathbf{e}_{i+1}\rangle \\ &+ [m_{i+1}] \prod_{j>i+1} q^{m_j} \cdot q_i^{-m_i} q_{i+1}^{m_{i+1}-1} |\mathbf{e}_i\rangle \otimes |\mathbf{m}-\mathbf{e}_{i+1}\rangle \\ &+ [m_i] \prod_{j>i} q^{m_j} [m_{i+1}] |\mathbf{e}_i\rangle \otimes |\mathbf{m}-\mathbf{e}_{i+1}\rangle . \end{aligned} \tag{6.1.4}$$

Let $(\star)$ denote the sum of last two terms, that is,

$$(\star) = [m_{i+1}] \prod_{j>i+1} q^{m_j} \cdot q_i^{-m_i} q_{i+1}^{m_{i+1}-1} |\mathbf{e}_i\rangle \otimes |\mathbf{m}-\mathbf{e}_{i+1}\rangle + [m_i] \prod_{j>i} q^{m_j} [m_{i+1}] |\mathbf{e}_i\rangle \otimes |\mathbf{m}-\mathbf{e}_{i+1}\rangle .$$

Suppose first that $e_i |\mathbf{m}\rangle = 0$. Note that $e_i |\mathbf{m}\rangle = 0$ if and only if $m_{i+1} = 0$ or $m_{i+1} \ne 0$, $m_i = 1 = \epsilon_i$. If $m_{i+1} = 0$, then $e_i\psi_1 |\mathbf{m}\rangle = 0$. In the other case, $|\mathbf{m}-\mathbf{e}_k+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle$ is nonzero if and only if $k = i$. So (6.1.4) is equal to

$$(\star) = [m_{i+1}] \prod_{j>i+1} q^{m_j} \left([m_i] \, q^{m_{i+1}} + q_i^{-m_i} q_{i+1}^{m_{i+1}-1} \right) |\mathbf{e}_i\rangle \otimes |\mathbf{m} - \mathbf{e}_{i+1}\rangle \, .$$

Since

$$\begin{aligned} [m_i] \, q^{m_{i+1}} + q_i^{-m_i} q_{i+1}^{m_{i+1}-1} &= [1] \, q^{m_{i+1}} + (-q) q_{i+1}^{m_{i+1}-1} \\ &= \begin{cases} q^{m_{i+1}} + (-q) q^{m_{i+1}-1} & \text{if } \epsilon_{i+1} = 0, \\ q + (-q)(-q^{-1})^0 & \text{if } \epsilon_{i+1} = 1 = m_{i+1}, \end{cases} \\ &= 0, \end{aligned}$$

we have $e_i \psi_1 |\mathbf{m}\rangle = 0 = \psi_1 e_i |\mathbf{m}\rangle$ whenever $e_i |\mathbf{m}\rangle = 0$.

Next suppose that $e_i |\mathbf{m}\rangle \neq 0$ (necessarily $m_{i+1} \neq 0$). We have

$$\begin{aligned} \psi_1 e_i |\mathbf{m}\rangle &= [m_{i+1}] \, \psi_1 |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \\ &= [m_{i+1}] \sum_{k \neq i, i+1} [m_k] \prod_{j>k} q^{m_j} |\mathbf{e}_k\rangle \otimes |\mathbf{m} - \mathbf{e}_k + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \\ &\quad + [m_{i+1}] \, [m_{i+1} - 1] \prod_{j>i+1} q^{m_j} |\mathbf{e}_{i+1}\rangle \otimes |\mathbf{m} - 2\mathbf{e}_{i+1} + \mathbf{e}_i\rangle \\ &\quad + [m_{i+1}] \, [m_i + 1] \, q^{-1} \prod_{j>i} q^{m_j} |\mathbf{e}_i\rangle \otimes |\mathbf{m} - \mathbf{e}_{i+1}\rangle \, . \end{aligned}$$

It is equal to (6.1.4) if

$$(\star) = [m_{i+1}] \, [m_i + 1] \, q^{-1} \prod_{j>i} q^{m_j} |\mathbf{e}_i\rangle \otimes |\mathbf{m} - \mathbf{e}_{i+1}\rangle \, . \tag{6.1.5}$$

Indeed, we have two possibilities: either $m_i = 0$ or $m_i \neq 0$ with $\epsilon_i = 0$. In the first case, we have

$$(\star) = [m_{i+1}] \prod_{j>i+1} q^{m_j} \cdot q_{i+1}^{m_{i+1}-1} |\mathbf{e}_i\rangle \otimes |\mathbf{m} - \mathbf{e}_{i+1}\rangle \, ,$$

and the product can be written as

$$\prod_{j>i+1} q^{m_j} \cdot q_{i+1}^{m_{i+1}-1} = \begin{cases} \prod_{j>i} q^{m_j} \cdot q^{-1} & \text{if } \epsilon_{i+1} = 0 \\ \prod_{j>i+1} q^{m_j} & \text{if } \epsilon_{i+1} = 1 = m_{i+1}. \end{cases}$$
$$= \prod_{j>i} q^{m_j} \cdot q^{-1},$$

which implies (6.1.5). In the other case, as $\prod_{j>i+1} q^{m_j} \cdot q_{i+1}^{m_{i+1}-1} = \prod_{j>i} q^{m_j} \cdot q^{-1}$ by the same reason, we have

$$\begin{aligned}(\star) &= \prod_{j>i} q^{m_j} \cdot q^{-1} \left|\mathbf{e}_i\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_{i+1}\right\rangle [m_{i+1}] \left([m_i]\, q + q^{-m_i}\right) \\ &= \prod_{j>i} q^{m_j} \cdot q^{-1} \left|\mathbf{e}_i\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_{i+1}\right\rangle [m_{i+1}] [m_i + 1] .\end{aligned}$$

Hence (6.1.5) holds.

Case 2. Suppose that $i = 0$. The proof is similar except that we should consider spectral parameters. First, we have

$$\begin{aligned} e_0 \psi_1 \left|\mathbf{m}\right\rangle &= \sum [m_k] \prod_{j>k} q^{m_j} e_0 \left(\left|\mathbf{e}_k\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_k\right\rangle\right) \\ &= \sum_{k \neq 1,n} [m_k] \prod_{j>k} q^{m_j} [m_1] \left|\mathbf{e}_k\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_k + \mathbf{e}_n - \mathbf{e}_1\right\rangle \cdot q \\ &\quad + [m_1] \prod_{j>1} q^{m_j} [m_1 - 1] \left|\mathbf{e}_1\right\rangle \otimes \left|\mathbf{m} - 2\mathbf{e}_1 + \mathbf{e}_n\right\rangle \cdot q \\ &\quad + [m_1] \prod_{j>1} q^{m_j} \cdot q_n^{-m_n} q_1^{m_1-1} \left|\mathbf{e}_n\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_1\right\rangle \cdot q^{1-\ell} \\ &\quad + [m_n] [m_1] \left|\mathbf{e}_n\right\rangle \otimes \left|\mathbf{m} - \mathbf{e}_1\right\rangle \cdot q. \end{aligned}$$

Note that since $\ell = \sum m_j$, we have in the third term above

$$\prod_{j>1} q^{m_j} \cdot q_n^{-m_n} q_1^{m_1-1} q^{1-\ell} = q^{-m_1} q_n^{-m_n} q_1^{m_1-1} q.$$

Similarly we have

$$\psi_1 e_0 \left|\mathbf{m}\right\rangle = [m_1]\, \psi_1 \left|\mathbf{m} + \mathbf{e}_n - \mathbf{e}_1\right\rangle \cdot 1$$

$$\begin{aligned}
&= [m_1] \sum_{k \neq 1,n} [m_k] \prod_{j>k} q^{m_j} \cdot q \, |\mathbf{e}_k\rangle \otimes |\mathbf{m} - \mathbf{e}_k + \mathbf{e}_n - \mathbf{e}_1\rangle \\
&\quad + [m_1] \, [m_1 - 1] \prod_{j>1} q^{m_j} \cdot q \, |\mathbf{e}_1\rangle \otimes |\mathbf{m} - 2\mathbf{e}_1 + \mathbf{e}_n\rangle \\
&\quad + [m_1] \, [m_n + 1] \, |\mathbf{e}_n\rangle \otimes |\mathbf{m} - \mathbf{e}_1\rangle \, .
\end{aligned}$$

Now the same argument applies as in *Case 1.* If $e_0 \, |\mathbf{m}\rangle = 0$, then either $m_1 = 0$ or $m_1 \neq 0$ with $m_n = 1 = \epsilon_n$. In the first case, we clearly have $\psi_1 e_0 \, |\mathbf{m}\rangle = e_0 \psi_1 \, |\mathbf{m}\rangle = 0$. In the latter case, we have

$$e_0 \psi_1 \, |\mathbf{m}\rangle = [m_1] \left([1] \, q + q^{-m_1} (-q^{-1})^{-1} q_1^{m_1 - 1} q \right) |\mathbf{e}_n\rangle \otimes |\mathbf{m} - \mathbf{e}_1\rangle = 0$$

as $q - q q^{1-m_1} q_1^{m_1-1}$ vanishes regardless of ϵ_1.

Next, if $e_0 \, |\mathbf{m}\rangle \neq 0$ and $m_1 \neq 0$, then again we have $\psi_1 e_0 \, |\mathbf{m}\rangle = e_0 \psi_1 \, |\mathbf{m}\rangle$ since

$$\begin{aligned}
[m_1] \left([m_n] \, q + q^{-m_1} q_n^{-m_n} q_1^{m_1-1} q \right) &= \begin{cases} [m_1] \left(0 + q^{-m_1} q_1^{m_1-1} q \right) & \text{if } m_n = 0 \\ [m_1] \left([m_n] \, q + q^{-m_n} q^{-m_1} q_1^{m_1-1} q \right) & \text{if } m_n \neq 0, \, \epsilon_n = 0 \end{cases} \\
&= [m_1] \, [m_n + 1] \, .
\end{aligned}$$

This completes the proof. □

Lemma 6.1.2. *(1) ψ_1 is injective and $\mathcal{R} \circ \psi_1 = 0$.*

(2) ψ_2 is surjective and $\psi_2 \circ \mathcal{R} = 0$.

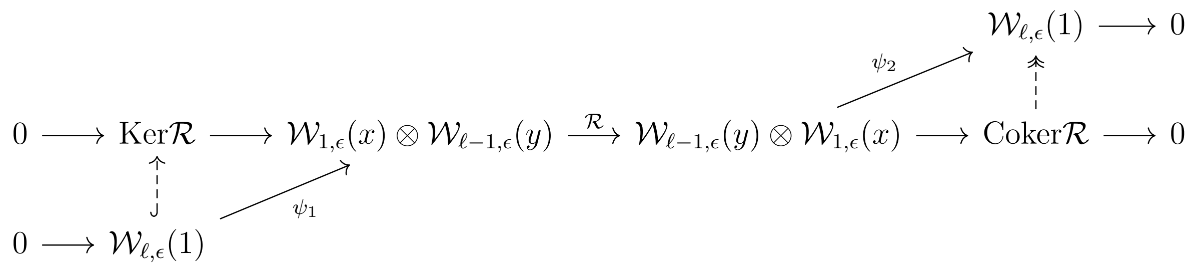
Proof. (1) It is clear that ψ_1 is injective since ψ_1 is nonzero and $\mathcal{W}_{\ell,\epsilon}(1)$ is irreducible.

By definition, we have $\psi_1(|\ell \mathbf{e}_1\rangle) = C \, |\mathbf{e}_1\rangle \otimes |(\ell - 1)\mathbf{e}_1\rangle = C v_1$ for a nonzero constant C (6.1.3). The $\mathring{\mathcal{U}}(\epsilon)$-highest weight vector v_1 is sent to zero by $\mathcal{R}$ since $\mathcal{R} = \mathcal{P}_1$ by (6.1.2). This implies that $\mathcal{R} \circ \psi_1 = 0$.

(2) Since ψ_2 is nonzero and $\mathcal{W}_{\ell,\epsilon}(1)$ is irreducible, it is surjective. Note that $v_2 = |(\ell - 1)\mathbf{e}_1\rangle \otimes |\mathbf{e}_2\rangle - q \, |(\ell - 2)\mathbf{e}_1 + \mathbf{e}_2\rangle \otimes |\mathbf{e}_1\rangle$ generates $\mathrm{Im}\mathcal{R}$, which is isomorphic to $V((\ell - 1, 1))$. Since $\psi_2(v_2) = 0$, we have $\psi_2 \circ \mathcal{R} = 0$. □

Lemma 6.1.3. *The sequence* (6.1.1) *is exact.*

Proof. By the previous lemmas and the universal mapping properties of Ker and Coker, we have the following commutative diagram of $\mathcal{U}(\epsilon)$-modules:



Hence two vertical arrows are isomorphisms. This implies that (6.1.1) is exact. □

6.1.2 Proof of Theorem 4.3.13

We assume that $\ell < n$. Put $\mathcal{F} = \mathcal{F}^*_{\epsilon,\ell}$. We first show that

$$\begin{aligned} \mathcal{F}^*_{\epsilon,\ell} : H^{\mathrm{aff}}_\ell(q^2)\text{-mod} &\longrightarrow \mathcal{C}^\ell(\epsilon) \\ M &\longmapsto \mathcal{V}^{\otimes \ell} \otimes_{H_\ell(q^2)} M \end{aligned} \tag{6.1.6}$$

is an equivalence of categories, almost following the arguments in [16, Section 4.3–4.6]. The exception is a part of Lemma 6.1.5 that uses the even Serre relation, which is replaced here with a more direct computation not involving Serre relations.

The following easy lemma is essential for the later argument.

Lemma 6.1.4 (*cf.* [16, Lemma 4.3])**.**

(1) *Let M be a finite-dimensional $H_\ell(q^2)$-module. If $v \in \mathcal{V}^{\otimes \ell}$ has nonzero components in each isotypical component of $\mathcal{J}_\ell(M)$, then the* $\mathbf{k}$*-linear map*

$$\begin{aligned} M &\longrightarrow \mathcal{V}^{\otimes \ell} \otimes_{H_\ell(q^2)} M = \mathcal{J}_\ell(M)\ , \\ m &\longmapsto v \otimes m \end{aligned}$$

is injective.

(2) *Let $\{\, v_i := |\mathbf{e}_i\rangle \mid i = 1, \dots, n \,\}$ be the standard basis of $\mathcal{V}$. If $i_1, \dots, i_\ell \in \{1, \dots, n\}$ are distinct, then the $\mathring{\mathcal{U}}(\epsilon)$-module $\mathcal{V}^{\otimes \ell}$ is generated by a single vector $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$. In particular, the vector satisfies the condition in (1).*

We first prove that $\mathcal{F}$ is essentially surjective. Suppose that $W \in \mathcal{C}^{\ell}(\epsilon)$ is given. By Theorem 3.2.5, there exists a $H_\ell(q^2)$-module M for which $W \cong \mathcal{J}_\ell(M) = \mathcal{V}^{\otimes \ell} \otimes M$ as a $\mathring{\mathcal{U}}(\epsilon)$-module. We shall extend the $H_\ell(q^2)$-action on M to $H_\ell^{\mathrm{aff}}(q^2)$ so that $W \cong \mathcal{V}^{\otimes \ell} \otimes_{H_\ell(q^2)} M \cong V_{\mathbb{O}}^{\otimes \ell} \otimes_{\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)} M$ as a $\mathcal{U}(\epsilon)$-module.

For $1 \leq j \leq \ell$, set $v^{(j)} = v_2 \otimes \cdots \otimes v_j \otimes v_n \otimes v_{j+1} \otimes \cdots \otimes v_\ell$. Regarding $\mathcal{V}^{\otimes \ell} \otimes_{H_\ell(q^2)} M$ as a $\mathcal{U}(\epsilon)$-module, the weight of $f_0\left(v^{(j)} \otimes m\right)$ is $\delta_1 + \cdots + \delta_\ell \in P$. As

$$\{v_{i_1} \otimes \cdots \otimes v_{i_\ell} \,|\, 1 \leq i_1, \ldots, i_\ell \leq \ell \text{ are distinct}\} \tag{6.1.7}$$

is a basis of $\left(\mathcal{V}^{\otimes \ell}\right)_{\delta_1 + \cdots + \delta_\ell}$, we can write as

$$f_0\left(v^{(j)} \otimes m\right) = \sum_{\mathbf{i}} \left(v_{i_1} \otimes \cdots \otimes v_{i_\ell}\right) \otimes m_{\mathbf{i}}, \tag{6.1.8}$$

where the sum is over $\mathbf{i} = (i_1, \ldots, i_\ell)$ such that $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$ belongs to (6.1.7), and $m_{\mathbf{i}} \in M$. In fact, considering the $H_\ell(q^2)$-action by $\mathcal{R}$ in Theorem 3.2.5, for each $\mathbf{i} = (i_1, \ldots, i_\ell)$ in (6.1.8), there exists $h_{\mathbf{i}} \in H_\ell(q^2)$ such that

$$v_{i_1} \otimes \cdots \otimes v_{i_\ell} = \left(v_2 \otimes \cdots \otimes v_j \otimes v_1 \otimes v_{j+1} \otimes \cdots \otimes v_\ell\right) h_{\mathbf{i}}.$$

Hence (6.1.8) is reduced to

$$f_0(v^{(j)} \otimes m) = \left(v_2 \otimes \cdots \otimes v_j \otimes v_1 \otimes v_{j+1} \otimes \cdots \otimes v_\ell\right) \otimes m', \tag{6.1.9}$$

for some $m' \in M$. By Lemma 6.1.4, such m' is unique. Therefore we obtain a $\mathbf{k}$-linear endomorphism $\alpha_j^- \in \mathrm{End}(H_\ell(q^2))$ sending m to m'. Considering e_0-action instead yields α_j^+. So we have

$$\begin{aligned} e_0\left(v^{(j)} \otimes m\right) &= \left(\Delta_j(e_0) v^{(j)}\right) \otimes \alpha_j^+(m) = \sum_{1 \leq i \leq \ell} \left(\Delta_i(e_0) v^{(j)}\right) \otimes \alpha_i^+(m), \\ f_0\left(v^{(j)} \otimes m\right) &= \left(\Delta_j(f_0) v^{(j)}\right) \otimes \alpha_j^-(m) = \sum_{1 \leq i \leq \ell} \left(\Delta_i(f_0) v^{(j)}\right) \otimes \alpha_i^-(m), \end{aligned} \tag{6.1.10}$$

where $\Delta_i(e_0)$ and $\Delta_i(f_0)$ are given by

$$\Delta_i(e_0) = 1^{\otimes i-1} \otimes e_0 \otimes \left(k_0^{-1}\right)^{\otimes \ell - i},$$

$$\Delta_i(f_0) = k_0^{\otimes i-1} \otimes f_0 \otimes 1^{\otimes \ell - i},$$

acting on $\mathcal{V}^{\otimes \ell}$. Note that $\Delta_i(e_0)v^{(j)} = 0$ unless $i = j$. Indeed, $v^{(j)}$ in (6.1.10) can be replaced by arbitrary $v \in \mathcal{V}^{\otimes \ell}$.

Lemma 6.1.5. *For $v \in \mathcal{V}^{\otimes \ell}$ and $m \in M$, we have*

$$e_0(v \otimes m) = \sum_{1 \le j \le \ell} (\Delta_j(e_0)v) \otimes \alpha_j^+(m),$$
$$f_0(v \otimes m) = \sum_{1 \le j \le \ell} (\Delta_j(f_0)v) \otimes \alpha_j^-(m).$$

Proof. We only prove the case for f_0 since the other case is similar. Take $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell}$. If none of i_j is equal to n, then $\Delta_j(f_0)v = 0$ for any j. On the other hand, we have $f_0(v \otimes m) = 0$ since $\delta_{i_1} + \cdots + \delta_{i_\ell} - \delta_n + \delta_1$ is not a weight of $\mathcal{V}^{\otimes \ell}$. Hence the identity holds.

For each pair of sequences

$$\mathbf{j} = (j_1 < j_2 < \cdots < j_r), \quad \mathbf{j}' = (j_1' < j_2' < \cdots < j_s')$$

in $\{1, \ldots, \ell\}$, which are disjoint, let $\mathcal{V}^{(\mathbf{j},\mathbf{j}')}$ be the subspace of $\mathcal{V}^{\otimes \ell}$ spanned by vectors of the form $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, where $i_{j_t} = 1$ $(1 \le t \le r)$, $i_{j_t'} = n$ $(1 \le t \le s)$ and $i_j \ne 1, n$ for others. Clearly $\mathcal{V}^{\otimes \ell} = \bigoplus \mathcal{V}^{(\mathbf{j},\mathbf{j}')}$, so that we may prove the identity for v in each $\mathcal{V}^{(\mathbf{j},\mathbf{j}')}$.

In addition, it is enough to check the identity for $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathcal{V}^{(\mathbf{j},\mathbf{j}')}$ with no $v_2, \ldots, v_{n-1}$ appearing more than once, due to Lemma 6.1.4(2) (with respect to the subalgebra of $\mathring{\mathcal{U}}(\epsilon)$ generated by e_i, f_i and $k_i^{\pm 1}$ for $i = 2, \ldots, n-1$). There is always such a vector since $\ell < n$.

We shall prove the identity by induction on s. We start with $s = 1$, and use induction on r. The case when $r = 0$ and $s = 1$ has already been done when we define $\alpha_j^{\pm}$ with $v = v^{(j)}$.

Suppose that it is true for $r - 1$. Choose $v = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathcal{V}^{(\mathbf{j},\mathbf{j}')}$ such that only $v_3, \ldots, v_{n-1}$ appear as a factor of v without repetition (which is possible as $s, r \ge 1$). Let v' be the vector obtained from v by replacing the last v_1 (that is, v_{j_r}) by v_2 so that v' has one less v_1 than v. By our choice of v, $e_1 v' = v$. Then we compute as

$$f_0(v \otimes m) = f_0 e_1(v' \otimes m) = e_1 f_0(v' \otimes m) = e_1 \sum_j (\Delta_j(f_0)v') \otimes \alpha_j^-(m)$$

$$
\begin{aligned}
&= e_1\left(q_1^{-\left|\left\{t\,|\,t<r,\,j_t<j_1'\right\}\right|}v''\right)\otimes\alpha_{j_1'}^-(m)\\
&= q_1^{-\left|\left\{t\,|\,t<r,\,j_t<j_1'\right\}\right|}(e_1v'')\otimes\alpha_{j_1'}^-(m)\\
&= q_1^{-\left|\left\{t\,|\,t<r,\,j_t<j_1'\right\}\right|}q_1^{-\delta(j_r<j_1')}\left[(1^{\otimes j_r-1}\otimes e_1\otimes 1^{\otimes \ell-j_r})v''\right]\otimes\alpha_{j_1'}^-(m)\\
&= q_1^{-\left|\left\{t\,|\,t\le r,\,j_t<j_1'\right\}\right|}\left[(1^{\otimes j_r-1}\otimes e_1\otimes 1^{\otimes \ell-j_r})v''\right]\otimes\alpha_{j_1'}^-(m)\\
&= \left(\Delta_{j_1'}(f_0)v\right)\otimes\alpha_{j_1'}^-(m)=\sum_j\left(\Delta_j(f_0)v\right)\otimes\alpha_j^-(m).
\end{aligned}
$$

Here the third equality follows from induction hypothesis on r, v'' is the resulting vector of replacing (the unique) v_n factor of v' by v_1, the last equality holds since v has exactly one v_n factor, and $\delta(P)$ is 1 if the statement P is true and 0 otherwise.

Now assume the result for $s-1$ and let us prove it for $s\ge 2$. Choose $v=v_{i_1}\otimes\cdots\otimes v_{i_\ell}\in\mathcal{V}^{(\mathbf{j},\mathbf{j}')}$ such that v_{n-1} does not appear as a factor of v and for each $i=2,\ldots,n-2$, v_i occurs at most once (which is possible as $s\ge 2$). We shall compute $[e_{n-1},f_{n-1}]\,f_0(v\otimes m)$ in two different ways.

We first have

$$[e_{n-1},f_{n-1}]\,f_0(v\otimes m)=\frac{q_n^{1-s}-q_n^{s-1}}{q-q^{-1}}f_0(v\otimes m), \tag{6.1.11}$$

since $[e_{n-1},f_{n-1}]=\frac{k_{n-1}-k_{n-1}^{-1}}{q-q^{-1}}$ and the weight of $f_0(v\otimes m)$ is

$$\sum_{i_k\ne n-1,n}\delta_{i_k}+s\delta_n+\delta_1-\delta_n.$$

Next, by similar arguments for (6.1.9), $f_0\left(v\otimes m\right)$ can be written as a sum of $v_{\mathbf{k}}\otimes m_{\mathbf{k}}$ for some $m_{\mathbf{k}}\in M$ and $v_{\mathbf{k}}=v_{k_1}\otimes\cdots\otimes v_{k_\ell}$ with none of v_{k_i} is equal to v_{n-1}. Hence $f_{n-1}f_0\left(v\otimes m\right)=0$ and so

$$[e_{n-1},f_{n-1}]\,f_0(v\otimes m)=-f_{n-1}e_{n-1}f_0(v\otimes m)=-f_{n-1}f_0e_{n-1}(v\otimes m). \tag{6.1.12}$$

We first compute

$$e_{n-1}(v\otimes m)=(e_{n-1}v)\otimes m=\left(q_n^{s-1}v'^{,1}+q_n^{s-2}v'^{,2}+\cdots+v'^{,s}\right)\otimes m,$$

where $v'^{,p}$ is obtained from v by replacing j_p'-th factor (which is v_n) by v_{n-1}. The vector

$v'^{,p}$ has one less v_n's than v, so that the induction hypothesis deduces

$$f_0 e_{n-1}(v \otimes m) = \sum_{p=1}^{s} q_n^{s-p} f_0(v'^{,p} \otimes m) = \sum_{p=1}^{s} q_n^{s-p} \sum_t (\Delta_t(f_0) v'^{,p}) \otimes \alpha_t^-(m).$$

By definition of $\Delta_t(f_0)$,

$$\Delta_t(f_0) v'^{,p} = \begin{cases} q_n^{u-1-\delta(u>p)} q_1^{-|\{k|j_k<t\}|} v''^{,p,u} & \text{if } t = j'_u \text{ for some } u \neq p, \\ 0 & \text{otherwise.} \end{cases}$$

where $v''^{,p,u}$ is obtained from $v'^{,p}$ by replacing j'_u-th factor (which is v_n) by v_1. Since any nonzero $v''^{,p,u}$ has exactly one v_{n-1},

$$\begin{aligned} f_{n-1} f_0 e_{n-1}(v \otimes m) &= f_{n-1} \sum_{p=1}^{s} q_n^{s-p} \sum_t (\Delta_t(f_0) v'^{,p}) \otimes \alpha_t^-(m) \\ &= \sum_{p=1}^{s} q_n^{s-p} \left[\sum_{u \neq p} q_n^{u-1-\delta(u>p)} q_1^{-|\{k|j_k<j'_u\}|} (f_{n-1} v''^{,p,u}) \otimes \alpha_{j'_u}^-(m) \right] \\ &= \sum_{p=1}^{s} q_n^{s-p} \left[\sum_{u \neq p} q_n^{u-1-\delta(u>p)} q_1^{-|\{k|j_k<j'_u\}|} q_n^{1+\delta(u<p)-p} v^u \otimes \alpha_{j'_u}^-(m) \right] \end{aligned}$$

where v^u is obtained from v by replacing j'_u-th factor (which is v_n) by v_1. Now for $1 \leq u \leq s$, the coefficient of $v^u \otimes \alpha_{j'_u}^-(m)$ is

$$\begin{aligned} &\sum_{p<u} q_n^{s-p} q_n^{u-2} q_1^{-|\{k|j_k<j'_u\}|} q_n^{1-p} + \sum_{p>u} q_n^{s-p} q_n^{u-1} q_1^{-|\{k|j_k<j'_u\}|} q_n^{2-p} \\ =& q_1^{-|\{k|j_k<j'_u\}|} \left[\sum_{p=1}^{u-1} q_n^{s-p} q_n^{u-2} q_n^{1-p} + \sum_{p=u}^{s-1} q_n^{s-p-1} q_n^{u-1} q_n^{1-p} \right] \\ =& q_1^{-|\{k|j_k<j'_u\}|} \sum_{p=1}^{s-1} q_n^{s+u-1-2p} = q_n^{u-1} q_1^{-|\{k|j_k<j'_u\}|} \frac{q_n^{s-1} - q_n^{1-s}}{q_n - q_n^{-1}}. \end{aligned}$$

Finally, combining the computation of (6.1.11) and (6.1.12) we obtain

$$f_0(v \otimes m) = \frac{q - q^{-1}}{q_n^{s-1} - q_n^{1-s}} [f_{n-1}, e_{n-1}] f_0(v \otimes m) = \frac{q - q^{-1}}{q_n^{s-1} - q_n^{1-s}} f_{n-1} f_0 e_{n-1}(v \otimes m)$$

$$= \sum_{u=1}^{s} \left(\Delta_{j'_u}(f_0)v\right) \otimes \alpha^-_{j'_u}(m) = \sum_{t=1}^{\ell} \left(\Delta_t(f_0)v\right) \otimes \alpha^-_t(m).$$

since $\Delta_{j'_u}(f_0)v = q_n^{u-1} q_1^{-|\{k|j_k<j'_u\}|} v^u$ and $\Delta_t(f_0)v = 0$ for $t \neq j'_u$. This completes the induction. □

Now, we define

$$X_j^{\pm 1} m = \alpha_j^{\pm}(m) \quad (m \in M,\ 1 \leq j \leq \ell). \tag{6.1.13}$$

Lemma 6.1.6. *M is an $H_\ell^{\mathrm{aff}}(q^2)$-module with respect to* (6.1.13)*, and W is isomorphic to $V^{\otimes \ell} \otimes_{H_\ell^{\mathrm{aff}}(q^2)} M$ as a $\mathcal{U}(\epsilon)$-module.*

Proof. The proof is almost identical to the one in [16], and we leave it to the reader. □

This completes the proof for essential surjectivity of $\mathcal{F}$.

Lemma 6.1.7. *The functor $\mathcal{F}$ is fully faithful.*

Proof. First, $\mathcal{F}$ is faithful since $\mathcal{J}_\ell$ is faithful. So it suffices to show that $\mathcal{F}_\ell$ is surjective on morphisms.

Suppose that $F : \mathcal{F}_\ell(M) \to \mathcal{F}_\ell(M')$ is a $\mathcal{U}(\epsilon)$-linear map for M, $M' \in H_\ell^{\mathrm{aff}}(q^2)$-mod. Since $\mathcal{J}_\ell$ is an equivalence, there is a $H_\ell(q^2)$-linear map $f : M \to M'$ such that $\mathcal{J}_\ell(f) = F$. Since F is $\mathcal{U}(\epsilon)$-linear, $e_0 F\left(v \otimes m\right) = F\left(e_0\left(v \otimes m\right)\right)$. The left hand side is equal to

$$e_0 F\left(v \otimes m\right) = e_0\left(v \otimes f(m)\right) = \sum_j \Delta_j(e_0)v \otimes X_j f(m),$$

while the right hand side is

$$F\left(e_0\left(v \otimes m\right)\right) = F\left(\sum_j \Delta_j(e_0)v \otimes X_j m\right) = \sum_j \Delta_j(e_0)v \otimes f(X_j m).$$

Now for each i, we can choose a vector $v(i)$ so that $\Delta_j(e_0)v = 0$ unless $j = i$, and at the same time $\Delta_i(e_0)v$ is of the form $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, whose factors are all distinct v_k's. For example, we may take $v(1) = v_1 \otimes v_2 \otimes \cdots \otimes v_\ell$. Putting $v = v(i)$ in the above identities, we obtain $X_i f(m) = f(X_i m)$ by Lemma 6.1.4. Hence f is $H_\ell^{\mathrm{aff}}(q^2)$-linear as well. □

Therefore, $\mathcal{F}$ in (6.1.6) is an equivalence of categories. Since every simple object in $H_\ell^{\mathrm{aff}}(q^2)$-mod is a quotient of $L(a_1) \circ \cdots \circ L(a_\ell)$ for some $a_1, \ldots, a_\ell \in \mathbf{k}$, where $\circ$ is a convolution product, and $\mathcal{F}(L(a_1) \circ \cdots \circ L(a_\ell)) = \mathcal{W}_1(a_1) \otimes \cdots \otimes \mathcal{W}_1(a_\ell)$, $\mathcal{F}$ induces the equivalence

$$\mathcal{F}^*_{\epsilon,\ell} : H^{\mathrm{aff}}_\ell(q^2)\text{-mod}_{\mathbb{Z}} \longrightarrow \mathcal{C}^\ell_{\mathbb{Z}}(\epsilon).$$

This completes the proof of Theorem 4.3.13.

6.2 Chapter 5

6.2.1 Proof of Proposition 5.3.2

Let us first recall the notion of ℓ-weights of representations of quantum affine algebras. The quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ has another set of generators $x^\pm_{i,t}, k_i^{\pm 1}, h^\pm_{i,s}$ ($i \in I \setminus \{0\}$, $t \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$) (see [5] for defining relations).

Let $\psi_{i,k}$ ($i \in I \setminus \{0\}, k \geq 0$) be the element determined by the following identity of formal power series in z:

$$\sum_{k=0}^{\infty} \psi_{i,k} z^k = k_i \exp\left((q - q^{-1}) \sum_{s=1}^{\infty} h_{i,s} z^s \right).$$

A $U'_q(\widehat{\mathfrak{sl}}_n)$-module V is called an ℓ-highest weight module if it is generated by an ℓ-highest weight vector v, that is, $x^+_{i,k} v = 0$ and $\psi_{i,k} v = \Psi_{i,k} v$ for all $i \in I \setminus \{0\}$, $k \geq 0$ and some scalars $\Psi_{i,k}$. Collecting those scalars in power series $\Psi_i(z) = \sum \Psi_{i,k} z^k$, the tuple $\Psi = (\Psi_i(z))_{i \in I \setminus \{0\}}$ is called the *ℓ-highest weight* of V.

Every (type 1) finite-dimensional irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$-module is an ℓ-highest weight module. Conversely, an ℓ-highest weight module is finite-dimensional if its ℓ-highest weight $\Psi = (\Psi_i(z))_{i \in I \setminus \{0\}}$ is of the form

$$\Psi_i(z) = q^{\deg P_i} \frac{P_i(q^{-2} z)}{P_i(z)},$$

for uniquely determined polynomials $P_i(z) \in \Bbbk[z]$ with constant term 1 [16, Theorem 3.3].

Now let us regard $\mathcal{W}^{\mathrm{osc}}_l(1)$ as a $U'_q(\widehat{\mathfrak{sl}}_n)$-module by restriction. We shall prove that $v_l \in \mathcal{W}^{\mathrm{osc}}_l(1)$ is an ℓ-highest weight vector with the ℓ-highest weight given in Proposition 5.3.2. Since $x^+_{i,t} v_l = 0$ for all $i \in I \setminus \{0\}$ and $t \in \mathbb{Z}$ by a weight consideration, the problem is to compute the action of $\psi_{i,k}$ on v_l. To do this, we recall the following lemmas expressing $\psi_{i,k}$ in terms of *root vectors* $E_{k\delta - \alpha_i}$. We refer the reader to [5] for unexplained notations and definitions below.

Fix a map $o : I \setminus \{0\} \to \{\pm 1\}$ such that $o(i+1) = -o(i)$ for all $i \in I \setminus \{0\}$.

Lemma 6.2.1 ([5, Lemma 1.5]). *For $i \in I \setminus \{0\}$ and $k > 0$, we have*

$$\psi_{i,k} = o(i)^k (q - q^{-1}) k_i \left(E_{k\delta - \alpha_i} e_i - q^{-2} e_i E_{k\delta - \alpha_i} \right).$$

Lemma 6.2.2 ([46, Lemma 4.3]). *For $i \in I \setminus \{0\}$ and $k > 0$, we have*

$$\begin{aligned} E_{(k+1)\delta - \alpha_i} = -\frac{1}{q + q^{-1}} \big(& E_{\delta - \alpha_i} e_i E_{k\delta - \alpha_i} - q^{-2} e_i E_{\delta - \alpha_i} E_{k\delta - \alpha_i} \\ & - E_{k\delta - \alpha_i} E_{\delta - \alpha_i} e_i + q^{-2} E_{k\delta - \alpha_i} e_i E_{\delta - \alpha_i} \big). \end{aligned}$$

Lemma 6.2.3 ([46, Lemma 4.7]). *For $i \in I \setminus \{0\}$, we have*

$$E_{\delta - \alpha_i} = (-q^{-1})^{n-2} (e_{i+1} \dots e_{n-1})(e_{i-1} \dots e_2 e_1) e_0 + \sum_{j_1, \dots, j_{n-1}} C_{j_1, \dots, j_{n-1}}(q) e_{j_1} \dots e_{j_{n-1}},$$

where the sum is over the sequences $(j_1, \dots, j_{n-1}) \in I^{n-1}$ such that $\sum_{k=1}^{n-1} \alpha_{j_k} = \sum_{j \in I \setminus \{i\}} \alpha_j$ with $j_{n-1} \neq 0$ and $C_{j_1, \dots, j_{n-1}}(q) \in \pm q^{\mathbb{Z}_{\leq 0}}$.

Proof of Proposition 5.3.2. We claim that v_l is an ℓ-highest weight vector with the given ℓ-highest weight. By weight consideration, $x^+_{i,t} v_l = 0$ for all $i \in I \setminus \{0\}$, $t \in \mathbb{Z}$. Let us show that that v_l is a simultaneous eigenvector of $\psi_{i,k}$ with the eigenvalues $\Psi_{i,k}$ above.

First assume $l \geq 0$. Since $e_j v_l = 0$ for all $j \in I \setminus \{0\}$, we have by Lemma 6.2.3,

$$\begin{aligned} E_{\delta - \alpha_r} v_l &= (-q^{-1})^{n-2} (e_{r+1} \cdots e_{n-1})(e_{r-1} \cdots e_1) e_0 v_l \\ &= (-q^{-1})^{n-2} \left| \mathbf{e}_r + (l+1) \mathbf{e}_{r+1} \right\rangle, \end{aligned}$$

and then

$$\begin{aligned} E_{2\delta - \alpha_r} v_l &= -\frac{1}{q + q^{-1}} \left(E_{\delta - \alpha_r} e_r E_{\delta - \alpha_r} - q^{-2} e_r E^2_{\delta - \alpha_r} + q^{-2} E_{\delta - \alpha_r} e_r E_{\delta - \alpha_r} \right) v_l \\ &= -\frac{1}{[2]} \left(-(-q^{-1})^{2n-4} [l+1](1 + q^{-2}) + q^{-2} (-q^{-1})^{2n-4} [2][l+2] \right) \left| \mathbf{e}_r + (l+1) \mathbf{e}_{r+1} \right\rangle \\ &= (-q^{-1})^{2n-4} (q^{-1}[l+1] - q^{-2}[l+2]) \left| \mathbf{e}_r + (l+1) \mathbf{e}_{r+1} \right\rangle \\ &= -(-q^{-1})^{2n-4} q^{-l-3} \left| \mathbf{e}_r + (l+1) \mathbf{e}_{r+1} \right\rangle. \end{aligned}$$

Repeating similar computation and using Lemma 6.2.2, we have

$$E_{k\delta-\alpha_r} v_l = (-1)^{k-1}(-q^{-1})^{(n-2)k}(q^{-l-3})^{k-1} \left|\mathbf{e}_r + (l+1)\mathbf{e}_{r+1}\right\rangle ,$$

and by Lemma 6.2.1

$$\begin{aligned}\psi_{r,k} v_l &= o(r)^k (q-q^{-1}) k_r \left(E_{k\delta-\alpha_r} e_r - q^{-2} e_r E_{k\delta-\alpha_r}\right) v_l \\ &= o(r)^k (q-q^{-1}) q^{-l-1} q^{-2} [l+1] (-1)^{k-1} (-q^{-1})^{(n-2)k} (q^{-l-3})^{k-1} v_l \\ &= o(r)^k (q^{-l-1} - q^{l+1})(-q^{-1})^{nk}(-q^{-l-1})^k v_l.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\Psi_r(z) v_l = \sum_{k\geq 0} \psi_{r,k} z^k v_l &= \left(q^{-l-1} - (q^{l+1} - q^{-l-1}) \sum_{k\geq 1} \left\{ o(r)(-q^{-1})^n(-q^{-l-1})z \right\}^k \right) v_l \\ &= \left(q^{-l-1} - (q^{l+1} - q^{-l-1}) \frac{-q^{-l-1}u}{1+q^{-l-1}u} \right) v_l = \frac{u + q^{-l-1}}{1+q^{-l-1}u} v_l,\end{aligned}$$

where $u = o(r)(-q^{-1})^n z$. The computation of $\Psi_{r+1}(z)$ is similar, where we begin from

$$\begin{aligned}E_{\delta-\alpha_{r+1}} v_l &= (-q^{-1})^{n-2}(e_{r+2}\ldots e_{n-1})(e_r \ldots e_2 e_1) e_0 v_l \\ &= -(-q^{-1})^{n-2}[l] \left|(l-1)\mathbf{e}_{r+1} + \mathbf{e}_{r+2}\right\rangle .\end{aligned}$$

For $i \neq r,\ r+1$, it is obvious since $E_{\delta-\alpha_i} v_l = 0$. The case when $l < 0$ can be dealt with by the same computation. □

6.2.2 Proof of Conjecture 5.3.16 for $s = 1$

Recall the sequence to be proved (5.3.16) (with $s = 1$):

$$0 \longrightarrow \mathcal{W}^{l,2}_{\mathrm{osc}}(1) \longrightarrow \mathcal{W}^{\mathrm{osc}}_l(1) \otimes \mathcal{W}^{\mathrm{osc}}_l(q^{-2}) \xrightarrow{\psi} \mathcal{W}^{\mathrm{osc}}_{l+1}(q^{-1}) \otimes \mathcal{W}^{\mathrm{osc}}_{l-1}(q^{-1}) \longrightarrow 0.$$

Let ψ be a linear map defined by

$$\psi : |\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle \longmapsto -\sum_{i\leq r} [m_i] \prod_{k\leq i} q^{m'_k - m_k} \left|\mathbf{m} - \mathbf{e}_i\right\rangle \otimes \left|\mathbf{m} + \mathbf{e}_i\right\rangle$$

$$+\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\left|\mathbf{m}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_j\right\rangle,$$

which will be shown below to be a $\mathcal{U}(\underline{\boldsymbol{\epsilon}})$-module homomorphism. Then ψ is automatically surjective as its codomain $\mathcal{W}_{l+1}^{\mathrm{osc}}(q^{-1})\otimes\mathcal{W}_{l-1}^{\mathrm{osc}}(q^{-1})$ is irreducible by Theorem 5.3.12.

The remaining map is a canonical inclusion, as $\mathcal{W}_{\mathrm{osc}}^{l,2}(1)$ is constructed as the image of

$$R_{l,l}(q^{-2},1):\mathcal{W}_l^{\mathrm{osc}}(q^{-2})\otimes\mathcal{W}_l^{\mathrm{osc}}(1)\longrightarrow\mathcal{W}_l^{\mathrm{osc}}(1)\otimes\mathcal{W}_l^{\mathrm{osc}}(q^{-2}).$$

Once we show that $\mathcal{W}_{\mathrm{osc}}^{l,2}(1)$ is in the kernel of ψ (as a submodule of $\mathcal{W}_l^{\mathrm{osc}}(1)\otimes\mathcal{W}_l^{\mathrm{osc}}(q^{-2})$), the conclusion follows by comparing the classical decompositions of the modules appearing in the sequence (see (5.3.1), Proposition 5.3.15). To sum up, the proof now reduces to showing the following two lemmas.

Lemma 6.2.4. *The map ψ is $\mathcal{U}(\underline{\boldsymbol{\epsilon}})$-linear.*

Proof. Since the proof consists of straightforward computations, let us check only the following three cases and leave the others to the reader.

Case 1. $[\psi,e_a]=0$ for $1\le a<r$: We compute

$$\begin{aligned}
&\psi(e_a(|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle))=\psi\begin{pmatrix}q^{m'_a-m'_{a+1}}[m_a]\left|\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1}\right\rangle\otimes|\mathbf{m}'\rangle\\ +[m'_a]\,|\mathbf{m}\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1}\right\rangle\end{pmatrix}\\
&=q^{m'_a-m'_{a+1}}[m_a]\\
&\quad\cdot\begin{pmatrix}-\sum_{i\le r}[(\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1})_i]\prod_{k\le i}q^{m'_k-(\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1})_k}\left|\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1}-\mathbf{e}_i\right\rangle\otimes\left|\mathbf{m}'+\mathbf{e}_i\right\rangle\\ +\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\left|\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_j\right\rangle\end{pmatrix}\\
&\quad+[m'_a]\begin{pmatrix}-\sum_{i\le r}[m_i]\prod_{k\le i}q^{(\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1})_k-m_k}\left|\mathbf{m}-\mathbf{e}_i\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1}+\mathbf{e}_i\right\rangle\\ +\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\left|\mathbf{m}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1}-\mathbf{e}_j\right\rangle\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&e_a(\psi(|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle))=e_a\begin{pmatrix}-\sum_{i\le r}[m_i]\prod_{k\le i}q^{m'_k-m_k}\left|\mathbf{m}-\mathbf{e}_i\right\rangle\otimes\left|\mathbf{m}+\mathbf{e}_i\right\rangle\\ +\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\left|\mathbf{m}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_j\right\rangle\end{pmatrix}\\
&=-\sum_{i\le r}[m_i]\prod_{k\le i}q^{m'_k-m_k}\begin{pmatrix}q^{-(\mathbf{m}'+\mathbf{e}_i)_a-(\mathbf{m}'+\mathbf{e}_i)_{a+1}}[(\mathbf{m}-\mathbf{e}_i)_a]\left|\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1}-\mathbf{e}_i\right\rangle\otimes\left|\mathbf{m}'+\mathbf{e}_i\right\rangle\\ +[(\mathbf{m}'+\mathbf{e}_i)_a]\left|\mathbf{m}-\mathbf{e}_i\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1}+\mathbf{e}_i\right\rangle\end{pmatrix}\\
&\quad+\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\begin{pmatrix}q^{m'_a-m'_{a+1}}[m_a]\left|\mathbf{m}-\mathbf{e}_a+\mathbf{e}_{a+1}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_j\right\rangle\\ +[m'_a]\left|\mathbf{m}+\mathbf{e}_j\right\rangle\otimes\left|\mathbf{m}'-\mathbf{e}_a+\mathbf{e}_{a+1}-\mathbf{e}_j\right\rangle\end{pmatrix}.
\end{aligned}$$

Let us compare the coefficients of $|\mathbf{m}-\mathbf{e}_a\rangle\otimes|\mathbf{m}'+\mathbf{e}_{a+1}\rangle$. In the first one, it is

$$\begin{aligned}&-q^{m'_a-m'_{a+1}}[m_a][m_{a+1}+1]\prod_{k\le a+1}q^{m'_k-m_k}-[m'_a][m_a]\prod_{k\le a}q^{m'_k-m_k}\\&\qquad=\frac{[m_a]}{q-q^{-1}}\prod_{k\le a}q^{m'_k-m_k}\left(-q^{m'_a+1}+q^{m'_a-2m_{a+1}-1}-q^{m'_a-1}+q^{-m'_a-1}\right),\end{aligned}$$

and in the other,

$$\begin{aligned}&-[m_{a+1}]\prod_{k\le a+1}q^{m'_k-m_k}q^{m'_a-m'_{a+1}-1}[m_a]-[m_a]\prod_{k\le a}q^{m'_k-m_k}[m'_a+1]\\&\qquad=\frac{[m_a]}{q-q^{-1}}\prod_{k\le a}q^{m'_k-m_k}\left(-q^{m'_a-1}+q^{m'_a-2m_{a+1}-1}-q^{m'_a+1}+q^{-m'_a-1}\right)\end{aligned}$$

and so they are the same. Other coefficients are easier to check, so we omit them.

Case 2. $[\psi,e_r]=0$: Let us compare

$$\begin{aligned}&\psi(e_r(|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle))=\psi\begin{pmatrix}-q^{m'_r+m'_{r+1}+1}[m_r][m_{r+1}]\,|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}\rangle\otimes|\mathbf{m}'\rangle\\-[m'_r][m'_{r+1}]\,|\mathbf{m}\rangle\otimes|\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1}\rangle\end{pmatrix}\\&=-q^{m'_r+m'_{r+1}+1}[m_r][m_{r+1}]\\&\quad\cdot\begin{pmatrix}-\sum_{i\le r}[(\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1})_i]\prod_{k\le i}q^{m'_k-(\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1})_k}\,|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_i\rangle\\+\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-(\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1})_k}\,|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle\end{pmatrix}\\&\quad-[m'_r][m'_{r+1}]\\&\quad\cdot\begin{pmatrix}-\sum_{i\le r}[m_i]\prod_{k\le i}q^{(\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1})_k-m_k}\,|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_i\rangle\\+\sum_{j>r}[(\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1})_j]\prod_{j<k}q^{(\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1})_k-m_k}\,|\mathbf{m}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1}-\mathbf{e}_j\rangle\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&e_r(\psi(|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle))=e_r\begin{pmatrix}-\sum_{i\le r}[m_i]\prod_{k\le i}q^{m'_k-m_k}\,|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}+\mathbf{e}_i\rangle\\+\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\,|\mathbf{m}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle\end{pmatrix}\\&=\sum_{i\le r}[m_i]\prod_{k\le i}q^{m'_k-m_k}\end{aligned}$$

$$
\begin{aligned}
&\cdot\begin{pmatrix} q^{(\mathbf{m}'+\mathbf{e}_i)_r+m'_{r+1}+1}[(\mathbf{m}-\mathbf{e}_i)_r][m_{r+1}]\,|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_i\rangle \\ +[(\mathbf{m}'+\mathbf{e}_i)_r][m'_{r+1}]\,|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_i\rangle \end{pmatrix}\\
&-\sum_{j>r}[m'_j]\prod_{j<k}q^{m'_k-m_k}\\
&\cdot\begin{pmatrix} q^{m'_r+(\mathbf{m}'-\mathbf{e}_j)_{r+1}+1}[m_r][(\mathbf{m}+\mathbf{e}_j)_{r+1}]\,|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle \\ +[m'_r][(\mathbf{m}'-\mathbf{e}_j)_{r+1}]\,|\mathbf{m}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_j\rangle \end{pmatrix}.
\end{aligned}
$$

First, the coefficient of $|\mathbf{m}-\mathbf{e}_r\rangle\otimes|\mathbf{m}'-\mathbf{e}_{r+1}\rangle$ in the former is

$$
-q^{m'_r+m'_{r+1}+1}[m_r][m_{r+1}][m'_{r+1}]\prod_{r+1<k}q^{m'_k-m_k}+[m'_r][m'_{r+1}][m_r]\prod_{k\le r}q^{m'_k-m_k}q^{-1}
$$

Since $|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle\in(\mathcal{W}_l^{\mathrm{osc}})^{\otimes 2}$, we have $\sum_{k\le r}(m'_k-m_k)=\sum_{k>r}(m'_k-m_k)$ so that

$$
\prod_{r+1<k}q^{m'_k-m_k}=q^{m_{r+1}-m'_{r+1}}\prod_{k\le r}q^{m'_k-m_k}.
$$

Hence the above coefficient can be rewritten as

$$
\frac{[m_r][m'_{r+1}]}{q-q^{-1}}\prod_{k\le r}q^{m'_k-m_k}\left(-q^{m'_r+2m_{r+1}+1}+q^{m'_r+1}+q^{m'_r-1}-q^{-m'_r-1}\right)
$$

and similary the one in the latter is

$$
\begin{aligned}
&[m_r]\prod_{k\le r}q^{m'_k-m_k}[m'_r+1][m'_{r+1}]-[m'_{r+1}]\prod_{r+1<k}q^{m'_k-m_k}q^{m'_r+m'_{r+1}}[m_r][m_{r+1}+1]\\
&\qquad=\frac{[m_r][m'_{r+1}]}{q-q^{-1}}\prod_{k\le r}q^{m'_k-m_k}\left(q^{m'_r+1}-q^{-m'_r-1}-q^{m'_r+2m_{r+1}+1}+q^{m'_r-1}\right)
\end{aligned}
$$

respectively, which are equal. The other coefficients are done as follows:

- $|\mathbf{m}-2\mathbf{e}_r-\mathbf{e}_{r+1}\rangle\otimes|\mathbf{m}'+\mathbf{e}_r\rangle$:

$$
\begin{aligned}
&q^{m'_r+m'_{r+1}+1}[m_r-1][m_r][m_{r+1}]\prod_{k\le r}q^{m'_k-m_k}q\\
&\qquad=[m_r]\prod_{k\le r}q^{m'_k-m_k}q^{m'_r+m'_{r+1}+2}[m_r-1][m_{r+1}],
\end{aligned}
$$

- $|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}-\mathbf{e}_i\rangle \otimes |\mathbf{m}'+\mathbf{e}_i\rangle$ $(i \neq r)$:

$$q^{m'_r+m'_{r+1}+1}[m_r][m_{r+1}][m_i]\prod_{k\leq i} q^{m'_k-m_k} = [m_i][m_r][m_{r+1}]q^{m'_r+m'_{r+1}+1}\prod_{k\leq i} q^{m'_k-m_k},$$

- $|\mathbf{m}+\mathbf{e}_{r+1}\rangle \otimes |\mathbf{m}'-\mathbf{e}_r-2\mathbf{e}_{r+1}\rangle$:

$$[m'_r][m'_{r+1}][m'_{r+1}-1]\prod_{r+1<k} q^{m'_k-m_k} = [m'_{r+1}][m'_r][m'_{r+1}-1]\prod_{r+1<k} q^{m'_k-m_k},$$

- $|\mathbf{m}+\mathbf{e}_j\rangle \otimes |\mathbf{m}'-\mathbf{e}_r-\mathbf{e}_{r+1}-\mathbf{e}_j\rangle$ $(j \neq r+1)$:

$$[m'_r][m'_{r+1}][m'_j]\prod_{j<k} q^{m'_k-m_k} = [m'_j][m'_r][m'_{r+1}]\prod_{j<k} q^{m'_k-m_k},$$

- $|\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_j\rangle \otimes |\mathbf{m}'-\mathbf{e}_j\rangle$ $(j \neq r+1)$:

$$q^{m'_r+m'_{r+1}+1}[m_r][m_{r+1}][m'_j]\prod_{j<k} q^{m'_k-m_k} = [m'_j]\prod_{j<k} q^{m'_k-m_k}[m_r][m_{r+1}],$$

- $|\mathbf{m}-\mathbf{e}_i\rangle \otimes |\mathbf{m}-\mathbf{e}_r-\mathbf{e}_{r+1}+\mathbf{e}_i\rangle$ $(i \neq r)$:

$$[m'_r][m'_{r+1}][m_i]\prod_{k\leq i} q^{m'_k-m_k} = [m_i]\prod_{k\leq i} q^{m'_k-m_k}[m'_r][m'_{r+1}].$$

Case 3. $[\psi, e_0] = 0$: Taking care of spectral parameters, we obtain

$$\psi(e_0(|\mathbf{m}\rangle \otimes |\mathbf{m}'\rangle)) = \psi\begin{pmatrix} q^{-m'_1-m'_n-1}\,|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n\rangle \otimes |\mathbf{m}'\rangle \\ +q^{-2}\,|\mathbf{m}\rangle \otimes |\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n\rangle \end{pmatrix}$$
$$= q^{-m'_1-m'_n-1}\begin{pmatrix} -\sum_{i\leq r}[(\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n)_i]\prod_{k\leq i} q^{m'_k-(\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n)_k}\,|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_i\rangle \otimes |\mathbf{m}'+\mathbf{e}_i\rangle \\ +\sum_{j>r}[m'_j]\prod_{j<k} q^{m'_k-(\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n)_k}\,|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n+\mathbf{e}_j\rangle \otimes |\mathbf{m}'-\mathbf{e}_j\rangle \end{pmatrix}$$
$$+\, q^{-2}\begin{pmatrix} -\sum_{i\leq r}[m_i]\prod_{k\leq i} q^{(\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n)_k-m_k}\,|\mathbf{m}-\mathbf{e}_i\rangle \otimes |\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n+\mathbf{e}_i\rangle \\ +\sum_{j>r}[(\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n)_j]\prod_{j<k} q^{(\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n)_k-m_k}\,|\mathbf{m}+\mathbf{e}_j\rangle \otimes |\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_j\rangle \end{pmatrix}$$

and

$$e_0(\psi(|\mathbf{m}\rangle\otimes|\mathbf{m}'\rangle)) = e_0\begin{pmatrix} -\sum_{i\le r}[m_i]\prod_{k\le i} q^{m'_k-m_k}\,|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}+\mathbf{e}_i\rangle \\ +\sum_{j>r}[m'_j]\prod_{j<k} q^{m'_k-m_k}\,|\mathbf{m}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle\end{pmatrix}$$

$$= -\sum_{i\le r}[m_i]\prod_{k\le i} q^{m'_k-m_k}\begin{pmatrix} q^{-(\mathbf{m}'+\mathbf{e}_i)_1-m'_n-1}q^{-1}\,|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_i\rangle \\ +q^{-1}\,|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n+\mathbf{e}_i\rangle\end{pmatrix}$$

$$+\sum_{j>r}[m'_j]\prod_{j<k} q^{m'_k-m_k}\begin{pmatrix} q^{-m'_1-(\mathbf{m}'-\mathbf{e}_j)_n-1}q^{-1}\,|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n+\mathbf{e}_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle \\ +q^{-1}\,|\mathbf{m}+\mathbf{e}_j\rangle\otimes|\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_j\rangle\end{pmatrix}.$$

The coefficient of $|\mathbf{m}+\mathbf{e}_n\rangle\otimes|\mathbf{m}'+\mathbf{e}_1\rangle$ in the former is

$$-q^{-m'_1-m'_n-1}[m_1+1]q^{m'_1-m_1-1}+q^{-2}[m'_n+1] = \frac{-q^{-m'_n-1}+q^{-2m_1-m'_n-3}+q^{m'_n-1}-q^{-m'_n-3}}{q-q^{-1}}$$

while the one in the latter is

$$-[m_1]q^{m'_1-m_1}q^{-m'_1-m'_n-2}q^{-1}+q^{-1}[m'_n] = \frac{-q^{-m'_n-3}+q^{-2m_1-m'_n-3}+q^{m'_n-1}-q^{-m'_n-1}}{q-q^{-1}},$$

which coincide. The other coefficients are easier:

- $|\mathbf{m}+\mathbf{e}_1+2\mathbf{e}_n\rangle\otimes|\mathbf{m}'-\mathbf{e}_n\rangle\,:\; q^{-m'_1-m'_n-1}[m'_n] = [m'_n]q^{-m'_1-m'_n+1}q^{-2}$,
- $|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n+e_j\rangle\otimes|\mathbf{m}'-\mathbf{e}_j\rangle$ $(j\neq n)$:

$$q^{-m'_1-m'_n-1}[m'_j]\prod_{j<k}q^{m'_k-m_k}q^{-1} = [m'_j]\prod_{j<k}q^{m'_k-m_k}q^{-m'_1-m'_n-2},$$

- $|\mathbf{m}-\mathbf{e}_1\rangle\otimes|\mathbf{m}'+2\mathbf{e}_1+\mathbf{e}_n\rangle\,:\; q^{-2}[m_1]q^{m'_1-m_1}q = [m_1]q^{m'_1-m_1}q^{-1}$,
- $|\mathbf{m}-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n+\mathbf{e}_i\rangle$ $(i\neq 1)$:

$$q^{-2}[m_i]\prod_{k\le i}q^{m'_k-m_k}q = [m_i]\prod_{k\le i}q^{m'_k-m_k}q^{-1},$$

- $|\mathbf{m}+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_i\rangle\otimes|\mathbf{m}'+\mathbf{e}_i\rangle$ $(i\neq 1)$:

$$q^{-m'_1-m'_n-1}[m_i]\prod_{k\le i}q^{m'_k-m_k}q^{-1} = [m_i]\prod_{k\le i}q^{m'_k-m_k}q^{-m'_1-m'_n-2},$$

- $|\mathbf{m}+\mathbf{e}_j\rangle \otimes |\mathbf{m}'+\mathbf{e}_1+\mathbf{e}_n-\mathbf{e}_j\rangle$ $(j \neq n)$:

$$q^{-2}[m'_j]\prod_{j<k} q^{m'_k-m_k} q = [m'_j]\prod_{j<k} q^{m'_k-m_k} q^{-1}.$$

□

Recall the notations from Section 5.3.2. Since $\mathcal{W}^{l,2}_{\mathrm{osc}}(1)$ is isomorphic to $\mathcal{V}^{(l,l)}$ as a $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-module, it is generated by the $\mathring{\mathcal{U}}(\underline{\boldsymbol{\epsilon}})$-highest weight vector $u_{-L} \in \mathcal{W}^{\mathrm{osc}}_l(1) \otimes \mathcal{W}^{\mathrm{osc}}_l(q^{-2})$ as a submodule. Thus it is enough to prove the following statement.

Lemma 6.2.5. $\psi(u_{-L}) = 0$.

Proof. Let us check it when $l > 0$ (so that $L = l$), leaving the other case $l < 0$ to the reader. Recall from Lemma 5.3.4 that $u_{-l} = \sum_{p=0}^{l} A_p v^-_{p,l-p}$ where

$$\begin{aligned} v^-_{p,l-p} &= |(l-p)\mathbf{e}_{r+1} + p\mathbf{e}_{r+2}\rangle \otimes |p\mathbf{e}_{r+1} + (l-p)\mathbf{e}_{r+2}\rangle\,, \\ A_p &= (-1)^p \prod_{k=1}^{p}\left(q^{-l+2k}\frac{[l+1-k]}{[k]}\right). \end{aligned}$$

By definition of ψ, we have

$$\begin{aligned} \psi(v^-_{p,l-p}) =&[p]q^{l-2p}\,|(l-p+1)\mathbf{e}_{r+1} + p\mathbf{e}_{r+2}\rangle \otimes |(p-1)\mathbf{e}_{r+1} + (l-p)\mathbf{e}_{r+2}\rangle \\ &+ [l-p]\,|(l-p)\mathbf{e}_{r+1} + (p+1)\mathbf{e}_{r+2}\rangle \otimes |p\mathbf{e}_{r+1} + (l-p-1)\mathbf{e}_{r+2}\rangle \end{aligned}$$

and then

$$\begin{aligned} \psi(u_{-l}) &= \sum_{p=0}^{l} A_p \psi(v^-_{p,l-p}) \\ &= \sum_{p=0}^{l} \begin{pmatrix} A_{p+1}[p+1]q^{l-2p-2} \\ +A_p[l-p] \end{pmatrix} |(l-p)\mathbf{e}_{r+1} + (p+1)\mathbf{e}_{r+2}\rangle \otimes |p\mathbf{e}_{r+1} + (l-p-1)\mathbf{e}_{r+2}\rangle \\ &= 0 \end{aligned}$$

as desired. □

Bibliography

[1] T. Akasaka, M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. Res. Inst. Math. Sci. **33** (1997) 839–867.

[2] V. Bazhanov, S. Lukyanov, A. Zamolodchikov, *Integrable structure of conformal field theory. II. Q-operator and DDV equation*, Comm. Math. Phys. **190** (1997) no. 2, 247–278.

[3] V. Bazhanov, Z. Tsuboi, *Baxter's Q-operators for supersymmetric spin chains*, Nuclear Phys. B **805** (2008) no. 3, 451—516.

[4] J. Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994) no. 3, 555–568.

[5] J. Beck, V. Chari, A. Pressley, *An algebraic characterization of the affine canonical basis*, Duke Math. J. **99** (1999) 455–487.

[6] N. Beisert, W. Galleas, T. Matsumoto, *A quantum affine algebra for the deformed Hubbard chain*, J. Phys. A **45** (2012) no. 36, 365206, 20 pp.

[7] G. Benkart, S.-J. Kang, M. Kashiwara, *Crystal bases for the quantum superalgebra* $U_q(\mathfrak{gl}(m,n))$, J. Amer. Math. Soc. **13** (2000) 295-331.

[8] A. Berele, A. Regev, *Hook Young diagrams with applications to combinatorics and representations of Lie superalgebras*, Adv. Math. **64** (1987) 118–175.

[9] I. Bernstein, A. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I.*, Ann. Sci. École Norm. Sup. (4) **10** (1977) no. 4, 441–472.

[10] J. Brundan, *Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. **16** (2003) 185–231.

[11] J. Brundan, A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (3) (2009) 451–484.

[12] J. Brundan, I. Losev, B. Webster, *Tensor product categorifications and the super Kazhdan-Lusztig conjecture*, Int. Math. Res. Not. (2017) 6329–6410.

[13] J. Brundan, C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup*, J. Eur. Math. Soc. **14** (2012) 373–419.

[14] V. Chari, D. Hernandez, *Beyond Kirillov-Reshetikhin modules*, in Quantum affine algebras, extended affine Lie algebras, and their applications, Contemp. Math. **506**, Amer. Math. Soc., Providence, RI, (2010) 49–81.

[15] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, (1994) xvi+651 pp.

[16] V. Chari, A. Pressley, *Quantum affine algebras and their representations*, in Representations of groups (Banff, AB, 1994), CMS Conference Proceedings, Vol. 16, American Mathematical Society, Providence, RI (1995) 59–78.

[17] V. Chari, A. Pressley, *Quantum affine algebras and affine Hecke algebras*, Pacific J. Math. **174** (2) (1996) 295–326.

[18] S.-J. Cheng, J.-H. Kwon, *Howe duality and Kostant's homology formula for infinite dimensional Lie superalgebras*, Int. Math. Res. Not. IMRN 2008, Art. ID rnn 085, 52pp.

[19] S.-J. Cheng, J.-H. Kwon, W. Wang, *Irreducible characters of Kac-Moody Lie superalgebras*, Proc. Lond. Math. Soc. (3) **110** (2015) no. 1, 108–132.

[20] S.-J. Cheng, N. Lam, *Irreducible characters of general linear superalgebra and super duality*, Comm. Math. Phys. **298** (2010) 645–672.

[21] S.-J. Cheng, N. Lam, W. Wang, *Super duality and irreducible characters of ortho-symplectic Lie superalgebras*, Invent. Math. **183** (2011) 189–224.

[22] S.-J. Cheng, N. Lam, W. Wang, *The Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras*, Duke Math. J. **164** (2015) 617–695.

[23] S.-J. Cheng, N. Lam, R. B. Zhang, *Character formula for infinite-dimensional unitarizable modules of the general linear superalgebra*, J. Algebra, **273** (2004) 780–805.

[24] S.-J. Cheng, W. Wang, *Dualities and Representations of Lie Superalgebras*, Graduate Studies in Mathematics 144, Amer. Math. Soc., 2013.

[25] S.-J. Cheng, W. Wang, R. B. Zhang, *Super duality and Kazhdan–Lusztig polynomials*, Trans. Amer. Math. Soc. **360** (2008) 5883–5924.

[26] I. V. Cherednik, *A new interpretation of Gel'fand-Tzetlin bases*, Duke Math. J. **54** (1987) no. 2, 563–577.

[27] E. Date, M. Okado, *Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type* $A_n^{(1)}$, Internat. J. Modern Phys. A **9** (1994) 399–417.

[28] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, (1987) 798–820.

[29] V. G. Drinfeld, *A new realization of Yangians and of quantum affine algebras*, (Russian) Dokl. Akad. Nauk SSSR **296** (1987) no. 1, 13–17.

[30] T. Enright, R. Howe, N. Wallach, *Classification of unitary highest weight modules*, in Representation Theory of Reductive Groups (P. Trombi, Ed.), Birkhäuser, Boston (1983) 97–144.

[31] I. Entova-Aizenbud, *Notes on restricted inverse limits of categories*, preprint (2015) arXiv:1504.01121.

[32] E. Frenkel, D. Hernandez, *Baxter's relations and spectra of quantum integrable models*, Duke Math. J. **164** (2015) 2407–2460.

[33] E. Frenkel, E. Mukhin, *The Hopf algebra* $\mathrm{Rep}U_q\widehat{\mathfrak{gl}}_\infty$, Selecta Math. (N.S.) **8** (2002) no. 4, 537–635.

[34] E. Frenkel, N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, Recent Developments in Quantum Affine Algebras and related topics, Contemp. Math. **248** (1999) 163–205.

[35] V. Ginzburg, N. Reshetikhin, E. Vasserot, *Quantum groups and flag varieties*, Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992), Contemp. Math., 175, Amer. Math. Soc., Providence, RI, 1994, 101–130.

[36] D. Hernandez, *Representations of quantum affinizations and fusion product*, Transform. Groups **10** (2005) 163–200.

[37] D. Hernandez, *The Kirillov-Reshetikhin conjecture and solutions of T-systems*, J. Reine Angew. Math. **596** (2006) 63–87.

[38] D. Hernandez, *The algebra* $U_q(\widehat{sl}_\infty)$ *and applications*, J. Algebra **329** (2011) 147–162.

[39] D. Hernandez, *Advances relating to R-matrices and their applications [following Maulik-Okounkov, Kang-Kashiwara-Kim-Oh,...]*, Séminaire Bourbaki. Vol. 2016/2017. Exposés 1120–1135. Astérisque 2019, no. 407, Exp. No. 1129, 297–331.

[40] D. Hernandez, M. Jimbo, *Asymptotic representations and Drinfeld rational fractions*, Compositio Math. **148** (2012) 1593–1623.

[41] D. Hernandez, B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010) 265–341.

[42] D. Hernandez, B. Leclerc, *Cluster algebras and category $\mathcal{O}$ for representations of Borel subalgebras of quantum affine algebras*, Algebra Number Theory **10** (2016) 2015–2052.

[43] J. Hong, S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, **42**, American Mathematical Society, Providence, RI, (2002) xviii+307 pp.

[44] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989) 539–570.

[45] R. Howe, E.-C. Tan, J. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2004) 1601–1626.

[46] I.-S. Jang, J.-H. Kwon, E. Park, *Unipotent quantum coordinate ring and prefundamental representations for types $A_n^{(1)}$ and $D_n^{(1)}$*, Int. Math. Res. Not. IMRN **2023**, no. 2, 1119–1172.

[47] M. Jimbo, *A q-difference analogue of $U(\widehat{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) no. 1, 63–69.

[48] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977) no. 1, 8–96.

[49] V. G. Kac, *Infinite-dimensional Lie algebras*, Third edition. Cambridge University Press, Cambridge (1990) xxii+400 pp.

[50] S.-J. Kang, M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, Invent. Math. **190** (2012) no. 3, 699–742.

[51] S.-J. Kang, M. Kashiwara, M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras*, Invent. math. **211** (2018) 591–685.

[52] S.-J. Kang, M. Kashiwara, M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras, II*, Duke Math. J. **164** (2015) 1549–1602.

[53] S.-J. Kang, M. Kashiwara, M. Kim, S.-j. Oh, *Simplicity of heads and socles of tensor products*, Compos. Math. **151** (2015) 377–396.

[54] S.-J. Kang, M. Kashiwara, M. Kim, S.-j. Oh, *Monoidal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018) 349–426.

[55] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, *Affine crystals and vertex models*, Internat. J. Modern Phys. A **7** (suppl. 1A) (1992) 449–484.

[56] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, *Perfect crystals of quantum affine Lie algebras*, Duke Math. J. **68** (1992) no. 3, 499–607.

[57] S-J. Kang, E. Park, *Irreducible modules over Khovanov-Lauda-Rouquier algebras of type A_n and semistandard tableaux*, J. Algebra **339** (2011) 223–251.

[58] M. Kashiwara, *On level zero representations of quantum affine algebras*, Duke Math. J. **112** (2002) 117–175.

[59] M. Kashiwara, M. Kim, *Laurent phenomenon and simple modules of quiver Hecke algebras*, Compos. Math. **155.12** (2019) 2263–2295.

[60] M. Kashiwara, M. Kim, S.-j. Oh, E. Park, *Monoidal categorification and quantum affine algebras*, Compos. Math. **156** (2020) 1039–1077.

[61] M. Kashiwara, M. Kim, S.-j. Oh, E. Park, *PBW theory for quantum affine algebras*, preprint (2020) arXiv:2011.14253, to appear in J. Eur. Math. Soc.

[62] M. Kashiwara, M. Kim, S.-j. Oh, E. Park, *Monoidal categorification and quantum affine algebras II*, preprint (2021) arXiv:2103.10067.

[63] M. Kashiwara, M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978) 1–47.

[64] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995. xii+531 pp.

[65] M. Khovanov, A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2011) 309–347.

[66] M. Kim, *Khovanov-Lauda-Rouquier algebras and R-matrices*, Ph.D. thesis, Seoul National University (2012).

[67] A. Kuniba, *Tetrahedron equation and quantum R matrices for q-oscillator representations mixing particles and holes*, SIGMA Symmetry Integrability Geom. Methods Appl. 14 (2018) Paper No. 067, 23pp.

[68] A. Kuniba, T. Nakanishi, J. Suzuki, *Functional relations in solvable lattice models: I. Functional relations and representation theory*, Internat. J. Modern Phys. A **9** (1994) 5215–5266.

[69] A. Kuniba, M. Okado, *Tetrahedron equation and quantum R matrices for infinite-dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$*, J. Phys. A **46** (2013) 485203.

[70] A. Kuniba, M. Okado, S. Sergeev, *Tetrahedron equation and generalized quantum groups*, J. Phys. A **48** (2015) 304001 (38p).

[71] J.-H. Kwon, *Rational semistandard tableaux and character formula for the Lie superalgebra $\widehat{\mathfrak{gl}}_{\infty|\infty}$*, Adv. Math. **217** (2008) 713–739.

[72] J.-H. Kwon, S.-M. Lee, *Super duality for quantum affine algebras of type A*, Int. Math. Res. Not. IMRN **2022**, no. 23, 18446–18525.

[73] J.-H. Kwon, S.-M. Lee, *Affinization of q-oscillator representations of $U_q(\mathfrak{gl}_n)$*, preprint (2022) arXiv:2203.12862, to appear in Lett. Math. Phys.

[74] J.-H. Kwon, S.-M. Lee, M. Okado, *Oscillator representations of quantum affine orthosymplectic superalgebras*, preprint (2023) arXiv:2304.06215.

[75] J.-H. Kwon, M. Okado, *Kirillov-Reshetikhin modules over generalized quantum groups of type A*, Publ. Res. Inst. Math. Sci. **57** (2021) no. 3-4, 993–1039.

[76] J.-H. Kwon, M. Okado, *Higher level q-oscillator representations for $U_q(C_n^{(1)})$, $U_q(C^{(2)}(n+1))$ and $U_q(B^{(1)}(0,n))$*, Comm. Math. Phys. **385** (2021) 1041–1082.

[77] J.-H. Kwon, J. Yu, *R-matrix for generalized quantum groups of type A*, J. Algebra, **566** (2021) 309–341.

[78] G. Lusztig, *Introduction to quantum groups*, Progr. Math. **110**, Birkhäuser, 1993.

[79] S. Machida, *Quantized superalgebras and generalized quantum groups*, M.S. Thesis (2017) Osaka City University.

[80] E. Mukhin, C. A. S. Young, *Affinization of category $\mathcal{O}$ for quantum groups*, Trans. Amer. Math. Soc. **366** (2014) 4815–4847.

[81] I. Musson, *Lie superalgebras and enveloping algebras*, Graduate Studies in Mathematics, 131. American Mathematical Society, Providence, RI, 2012. xx+488 pp.

[82] H. Nakajima, *t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003) 259–274.

[83] H. Nakajima, *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Ann. of Math. **160** (2004) 1057–1097.

[84] H. Nakajima, *Extremal weight modules of quantum affine algebras*, Adv. Stud. Pure Math. **40** (2004) 343–369.

[85] S.-j. Oh, T. Scrimshaw, *Simplicity of tensor products of Kirillov–Reshetikhin modules: nonexceptional affine and G types*, preprint (2019) arXiv:1910.10347.

[86] M. Okado, A. Schilling, *Existence of Kirillov-Reshetikhin crystals for nonexceptional types*, Represent. Theory **12** (2008) 186–207.

[87] J. Perk, C. Schultz, *New families of commuting transfer matrices in q-state vertex models*, Phys. Lett. A **84** (1981) no. 8, 407–410.

[88] R. Rouquier, *2-Kac-Moody algebras*, preprint (2008) arXiv:0812.5023.

[89] V. Serganova, *Characters of irreducible representations of simple Lie superalgebras*, Doc. Math. Extra Volume ICM II (1998) 583–593.

[90] B. S. Shastry, *Exact integrability of the one-dimensional Hubbard model*, Phys. Rev. Lett. **56** (1986) no. 23, 2453–2455.

[91] M. Varagnolo, E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. **659** (2011) 67–100.

[92] M. Vazirani, *Parameterizing Hecke algebra modules: Bernstein-Zelevinsky multisegments, Kleshchev multipartitions, and crystal graphs*, Transform. Groups **7** (2002) no. 3, 267–303.

[93] H. Yamane, *Quantized enveloping algebras associated to simple Lie superalgebras and universal R-matrices*, Publ. Res. Inst. Math. Sci. **30** (1994) 15–84.

[94] H. Yamane, *On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras*, Publ. Res. Inst. Math. Sci. **35** (1999) 321–390.

[95] A. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980) no. 2, 165–210.

[96] H. Zhang, *Representations of quantum affine superalgebras*, Math. Z. **278** (2014) 663–703.

[97] H. Zhang, *Universal R-matrix of quantum affine* $\mathfrak{gl}(1,1)$, Lett. Math. Phys. **105** (2015) no. 11, 1587–1603.

[98] H. Zhang, *RTT realization of quantum affine superalgebras and tensor products*, Int. Math. Res. Not. **2016** (2016) 1126–1157.

[99] H. Zhang, *Fundamental representations of quantum affine superalgebras and R-matrices*, Transform. Groups **22** (2017) 559–590.

[100] H. Zhang, *Asymptotic representations of quantum affine superalgebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **13** (2017) Paper No. 066, 25 pp.

[101] H. Zhang, *Length-two representations of quantum affine superalgebras and Baxter operators*, Comm. Math. Phys. **358** (2018) no. 2, 815–862.

국문초록

본 학위논문은 양자 아핀 (초)대수의 표현론에 관한 초 쌍대성에 기반을 둔 새로운 접근을 제시한다. 구체적으로는, 주어진 범주를 분석하기 위하여 우선 보손 혹은 페르미온 짝을 찾고, 그 초대칭 유사체를 구성하여 보손 측과 페르미온 측을 이어주는 함자를 찾는 방법이다. 이때 주요한 역할을 하는 것이 R-행렬과 그 스펙트럴 분해로, 이를 통해 각각의 경우를 기존의 잘 알려진 유한차원 표현론과 유사한 방식으로 분석할 수 있다.

본 학위논문에서는 A형 양자 아핀 (초)대수의 두 가지 모듈 범주를 고려하고자 한다. 첫째로 다루어지는 것은 다항식 표현들의 범주로, 이 경우에는 유용한 슈어–바일 류의 쌍대성을 이용하여 특히 통일적인 분석이 가능하였다. 양자 아핀 초대수에 대한 범주와 기존의 양자 아핀 대수에 대한 범주를 직접적으로 연결하는 함자를 건설하였고, 이로부터 범주들의 역극한 사이의 범주 동치를 얻게 된다.

둘째로, q-진동자 표현이라 불리는 무한차원 표현들의 범주를 도입하였고, 유한차원 기약 표현들과 자연스럽게 대응되는 기약 q-진동자 표현들을 찾았다. q-진동자 표현들은 유한차원 표현들의 보손 짝으로 볼 수 있기 때문에 양자 아핀 초대수에 대한 유사한 모듈 범주를 도입함으로써 이 대응을 설명할 수 있었으며, 이러한 초대칭 유사체를 통한 q-진동자와 유한차원 표현들 사이의 연결은 초 쌍대성의 철학에 의해 범주들의 동치로 이어질 것으로 예상된다.

주요어휘: 초 쌍대성, 양자 아핀 대수, 일반 선형 리 초대수, R-행렬, 슈어–바일 쌍대성, 진동자 표현
학번: 2017-22587