이학석사 학위논문

Ward's and Mass-one equation for Almost-Hermitian Random Matrix

(유사-에르미트 랜덤행렬의 와드와 매스-원 공식)

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Yongwoo Lee

Dissertation Director : Professor Insuk Seo

Department of Mathematical Sciences
Seoul National University

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Abstract

Ward's and Mass-one equation for Almost-Hermitian Random Matrix

Yongwoo Lee

Department of Mathematical Sciences
The Graduate School
Seoul National University

We consider the microscopic scaling limit of non-Hermitian random matrix, especially, almost-Hermitian random matrix with unitary invariance. The scaling limit for the edge regime has already been obtained in pioneering work of Bender in [8].

Ward's equation has been used in proving the edge universality conjecture for random normal matrix model, under additional assumption. Under the same assumption, universality has been verified for the bulk scaling limit of almost-Hermitian model using Ward's equation. However, not many are known for the edge scaling limit of almost-Hermitian matrix model. In this thesis, we prove that the limiting kernel for the edge regime satisfies Ward's and mass-one equations.

Key words: Almost-Hermitian random matrix, Ward's equation, Mass-one equation
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Contents

Chapter 1

Introduction

Almost-Hermitian random matrix model (AGUE) is a random matrix model that lies in between Hermitian and non-Hermitian matrix models. A study of local statistics for AGUE was pioneered by Fyodorov, Khoruzhenko and Sommers [12] for the bulk statistic, and pioneered by Bender [8] for the edge statistics. AGUE exhibits new types of universality classes that was not observed from Hermitian and non-Hermitian models.

Universality is one of the most important property studied in random matrix theory. It means that a correlation function, which determines distribution of eigenvalues, is independent of choice of exact probability measure for matrix elements, but only dependent on a few universal parameters, such as invariance properties or hermiticity, etc. Universality has been proved for wide classes of random matrices, for example, Gaussian unitary ensemble (GUE), and Ginibre unitary ensemble (GinUE), and many other more. However, it still remains open for more general type of random matrix models, and thus it is often referred as universality conjecture.

One successful way to prove universality is using Ward's equation. For instance, in the bulk regime of random normal matrix model, universality conjecture is recently proved in [4], especially using Ward's equation. Similarly, in [5], universality conjecture was partially proved in the soft edge regime of random normal matrix model using Ward's equation, but under additional assumption. Without the assumption, the soft edge universality is proved in [15] by developing advanced asymptotic theory for orthogonal polynomials.

Ward's and mass-one equations are fundamental equations which are expected to be true for limiting kernels, but it is non-trivial for limiting kernels to satisfy the equations. Especially, it was not verified in the literature that the edge limiting kernels of AGUE satisfy Ward's equation. These equations are fundamental in following senses. Ward's equation is derived from Ward identities which is also known as loop equations. These identities are consequences of reparametrization invariance property of normalization constants related to joint intensity functions. Mass-one equation is also fundamental in the sense that it follows from basic property of a correlation kernel.

Main results of this thesis is proofs for that the edge limiting kernels for AGUE satisfies Ward's and mass-one equations, and stated in Theorem 5.3.3 and Theorem 5.4.2. The organization of this thesis is as follows. In Chapter 2, we start with examples of random matrix model that are closely related to AGUE, and collect theorems that can be applied for general random matrix model. Chapter 3 is devoted to explain determinantal structure that underlies unitary ensembles, and then introduce Ward's and mass-one equation. In Chapter 4, we survey universality of local scaling limits for certain classes of random matrices, which covers GUE and GinUE but not AGUE. Finally, in Chapter 5, we define AGUE and state the main results. Specifically, we prove Ward's and mass-one equation for the limiting kernels of AGUE in the edge regime.

Chapter 2

Random matrix models

Almost-Hermitian random matrix is closely related to Gaussian unitary ensemble (GUE) and Ginibre unitary ensemble (GinUE), so we begin with introducing these ensembles. In Section 2.1, we introduce GUE and Elliptic GinUE which is a generalised model of GinUE. In Section 2.2, we introduce Frostman's type theorem which justify notion of limiting spectrum.

2.1 GUE and Elliptic GinUE

Consider a space of n by n non-Hermitian (resp. Hermitian) matrices with complex entries. We can endow a probability measure on the space by assigning probability measure for each element, independently.

Definition 2.1.1. *A random non-Hermitian (resp. Hermitian) complex matrix M is said to be the* Ginibre unitary ensemble (GinUE) *(resp.* Gaussian unitary ensemble (GUE)*) if*

$$d\mathbb{P}_{n,n}(M) = \frac{1}{\mathcal{Z}_n} \exp\left\{ -n \sum_{j,k=1}^{n} |m_{jk}|^2 \right\}, \qquad M = \left[m_{jk}\right]_{j,k=1}^{n},$$

where $\mathcal{Z}_n$ is a normalization constant

$$\mathcal{Z}_n = \int_{\mathbb{C}^{n^2}} \exp\left\{ -n \sum_{j,k=1}^{n} |m_{jk}|^2 \right\} \prod_{j,k=1}^{n} dm_{jk}.$$

with dm_{jk} denoting the Lebesgue measure on $\mathbb{C}$.

We remark that the given definitions are normalised version of GUE and GinUE. The word "unitary" in the names comes from a fact that the probability measures are invariant under the map $M \mapsto UMU^*$ for any complex unitary matrix U. In case of GinUE, the measure is invariant under the map $M \mapsto UMV^*$ for any complex unitary matrix U and V.

For each randomly picked M according to $\mathbb{P}_{n,n}$, there is a random sample of n-points, $\{\zeta_j\}_{j=1}^n$, where ζ_j's are the eigenvalues of M. If we consider $\mathbb{C}^n$ as a space of n eigenvalues of M, then $\mathbb{P}_{n,n}$ induces a probability measure $\mathbb{P}_n$ on $\mathbb{C}^n$. Then, $\mathbb{P}_n$ can be represented in forms of *Boltzmann-Gibbs law.*

Proposition 2.1.2. *Let $\mathbb{P}_n$ be the induced probability measure from GinUE (resp. GUE) as mentioned above. Then, $\mathbb{P}_n$ is given by*

$$d\mathbb{P}_n(\zeta_1, \dots, \zeta_n) = \frac{1}{Z_n} e^{-H_n(\zeta_1,\dots,\zeta_n)}, \tag{2.1}$$

where Z_n is a normalization constant

$$Z_n = \int_{\mathbb{C}^n} e^{-H_n(\zeta_1,\dots,\zeta_n)} d\zeta_1 \cdots d\zeta_n,$$

and H_n, the Hamiltonian, is defined by

$$H_n(\zeta_1, \dots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q_n(\zeta_i), \tag{2.2}$$

where $Q_n : \mathbb{C} \to \mathbb{R}$, which is called external field*, is given by $Q_n(\zeta) = |\zeta|^2$ (resp. $Q_n(\zeta) = |\zeta|^2$ if $\zeta \in \mathbb{R}$, and $Q_n(\zeta) = \infty$ otherwise).*

Proof. We refer [9, 11] for the proof. □

The eigenvalues are not independent, but repelling each other. This behavior can be observed in Figure 2.1, in comparison between random samplings of eigenvalues from GinUE and the uniform distribution on the unit disk.

The repulsion between eigenvalues of GinUE (and GUE) is due to logarithmic terms in (2.2). The logarithmic terms can be interpreted by 2-dimensional electrostatics. An electrically charged 2-dimensional particle generates a potential which is proportional to logarithmic distance from the particle. Thus, if we consider n-identically charged particle system where the

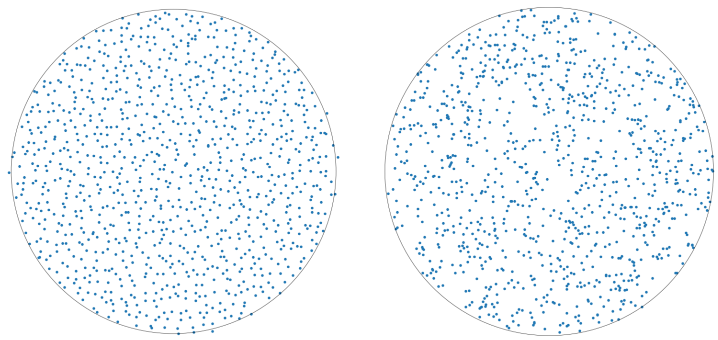

Figure 2.1: Random samplings from GinUE (left) and the uniform distribution on the unit disk(right).

particles are located at $\{\zeta_j\}_{j=1}^n$, then the electric potential between the particles are proportional to $\sum_{j\neq k}\log 1/|\zeta_j-\zeta_k|$. For this reason, the eigenvalues of GinUE (and GUE) can be interpreted as identically charged 2-dimensional electrical particles under external field Q_n.

A natural generalization of GinUE is taking in account more generalized external field Q_n. However, (2.1) may not be well-defined if no restriction on external field is given. A trivial example, *e.g.* $Q_n \equiv 0$, shows that (2.1) can be ill-defined for certain external field. We will further discuss about this problem in the next section. Setting a side of this problem, we now define *Elliptic Ginibre unitary ensemble.*

Definition 2.1.3. *For $\tau \in [0,1)$, a random complex n by n matrix A_τ is said to be* Elliptic Ginibre unitary ensemble *or* Elliptic GinUE *if*

$$A_\tau = \sqrt{1+\tau}H_1 + i\sqrt{1-\tau}H_2 \tag{2.3}$$

where H_1 and H_2 are independent GUEs.

Proposition 2.1.4. *The distribution of eigenvalue of Elliptic GinUE is represented by* (2.1) *with an external field $Q_n \equiv Q_\tau$ given by*

$$Q_\tau(\zeta) = \frac{1}{1+\tau}\xi^2 + \frac{1}{1-\tau}\eta^2, \qquad \zeta = \xi + i\eta \in \mathbb{C}. \tag{2.4}$$

Proof. We refer [2, 13] for the proof. □

Almost-Hermitian matrix is a Elliptic GinUE model with varying τ. That is, we consider a sequence of τ_n. By taking $\tau_n \to 1$, Elliptic GinUE becomes more alike to GUE rather then GinUE. See Chapter 5 for more detail.

2.2 Limiting spectrum

In this section, we collect theorems related to limiting spectrum for general matrix models. Limiting spectrum is not main of interest in this thesis, but we included it for completeness. Many of these results are due to logarithmic potential theory, and based on [14, 16, 18].

Let $\mathcal{B}(D)$ be a collection of positive unit Borel measure on a domain D. We typically consider cases with D being $\mathbb{R}$, $\mathbb{C}$, or a compact set.

Definition 2.2.1. *A* weighted logarithmic energy *of an external field* $Q : D \to \mathbb{R} \cup \{\infty\}$ *is a functional* $I_Q : \mathcal{B}(D) \to \mathbb{R}$ *given by*

$$I_Q(\sigma) := \iint_{D^2} \log \frac{1}{|\zeta_1 - \zeta_2|} \, d\sigma(\zeta_1) d\sigma(\zeta_2) + \int_D Q \, d\sigma, \qquad \sigma \in \mathcal{B}(D). \tag{2.5}$$

Finding explicit "minimizer" of weighted logarithmic energy of an external field is active research area in these days, and many are not known. The existence of the minimizer is not trivial, and it is not true for general external field. For example, if we set no restriction on Q, then infimum value of weighted logarithmic energy can vary from $-\infty$ to ∞ according to choice of Q. Therefore, we restrict our attention to a collection $\mathcal{Q}(D)$ of external fields where each external field $Q : D \to \mathbb{R} \cup \{\infty\}$ satisfies the following conditions:

1. Q is lower semi-continuous;
2. $\inf\{I_Q(\sigma) : \sigma \in \mathcal{B}(D)\} < \infty$;
3. $\liminf_{\zeta \to \infty} Q(\zeta) / \log |\zeta|^2 > 1$.

Even with these conditions, we can still cover wide range of external fields on $\mathbb{C}$. Especially, Condition 2 and 3 are necessary for weighted logarithmic energy to have a meaningful infimum value. Condition 3 can be weaken to $Q(\zeta) - \log |\zeta|^2 \to \infty$ as $|\zeta| \to \infty$, but we will satisfy with the condition given above.

It is known from logarithmic potential theory that there exists a unique *equilibrium measure* which minimize weighted logarithmic energy for an external field in $\mathcal{Q}(D)$.

Theorem 2.2.2. *For $Q \in \mathcal{Q}(D)$, there exists a unique measure $\hat{\sigma} \in \mathcal{B}(D)$ which is called* equilibrium measure *such that*

$$I_Q(\hat{\sigma}) = \inf_{\sigma \in \mathcal{B}(D)} I_Q(\sigma),$$

and $\hat{\sigma}$ is compactly supported and its support is called droplet*. Furthermore, if Q is a C^2-function and $D = \mathbb{C}$, then $\hat{\sigma}$ equals to*

$$d\hat{\sigma} = \Delta Q \cdot \mathbf{1}_S \cdot dA,$$

where Δ is a quarter of the Laplacian*, S is the droplet of $\hat{\sigma}$, and dA is a $1/\pi$-normalized Lebesgue measure on $\mathbb{C}$.*

Proof. We refer [18] for the proof. □

Example 2.2.3. *A droplet of elliptic GinUE is*

$$S_\tau = \left\{ \zeta \in \mathbb{C} : \frac{1}{(1+\tau)^2}\xi^2 + \frac{1}{(1-\tau)^2}\eta^2 \leq 1, \text{ where } \xi = \operatorname{Re}\zeta, \eta = \operatorname{Im}\zeta \right\}, \tag{2.6}$$

where we can see the origin of "elliptic" in the name of the ensemble. We refer [13, 19] for the proof.

As an analogue to (2.5), we define discrete version of weighted logarithmic energy for an averaged empirical measure by

$$\sigma_{n,\zeta} \coloneqq \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_j} \in \mathcal{B}(D),$$

where $\zeta = (\zeta_1, \ldots, \zeta_n) \in D^n$. Note that $I_Q(\sigma_{n,\zeta}) = \infty$, so instead (2.5), we define *discrete weighted logarithmic energy* by

$$J_Q(\sigma_n) \coloneqq \frac{2}{n(n-1)} \sum_{1 \leq j < k \leq n} \log \frac{1}{|\zeta_j - \zeta_k|} + \frac{2}{n(n-1)} \sum_{j=1}^{n} Q(\zeta_j). \tag{2.7}$$

We call $\hat{\sigma}_n$ a *weighted Fekete points* if it minimize (2.7). Note that (2.7) does not have a unique minimizer in general. For example, given a rotation symmetric external field, rotating a Fekete point configuration with respect to the origin gives another Fekete point configurations for any $n > 1$. Furthermore, calculation of Fekete points is quite difficult problem. However, Fekete points are related to equilibrium measure in the following sense.

Theorem 2.2.4. *For a Fekete points $\hat{\sigma}_n$ and the equilibrium measure $\hat{\sigma}$,*

$$\hat{\sigma}_n \to \hat{\sigma} \qquad \text{as } n \to \infty,$$

in the weak-star convergence.

Proof. We refer [18] for the proof. □

Recall (2.1) for the precise definition of $\mathbb{P}_n$. For any $k = 1, \ldots, n$, we define a marginal probability measure $\mathbb{P}_n^{(k)}$ by

$$\mathbb{P}_n^{(k)}(\zeta) = \mathbb{P}_n(\zeta \times D^{n-k}),$$

where $\zeta \in D^k$. In [16], the author proved a convergence of the marginal measure to a product of the equilibrium measures for $D \subset \mathbb{R}$, and extended to $D \subset \mathbb{C}$ in [14].

Theorem 2.2.5. *For a marginal probability measure $\mathbb{P}_n^{(k)}$ and the equilibrium measure $\hat{\sigma}$,*

$$\mathbb{P}_n^{(k)} \to \otimes_{j=1}^{k} \hat{\sigma} \qquad \text{as } n \to \infty,$$

in the weak-star convergence, where $\otimes$ denote product of measures.

Proof. We refer [14] and [16] for the proof. □

Chapter 3

Determinantal structure

It is extremely hard to analyze the Boltzmann-Gibbs measure given as (2.1) by its own. However, with a help from determinantal structure that lies behind the measure, it is possible to adapt theory of orthogonal polynomials.

In this chapter, we discuss about determinantal structure that underlies GinUE and GUE. In Section 3.1 and 3.2, we introduce correlation kernel which determines determinantal structure behind (2.1). In Section 3.3, we introduce n-level Ward's and the mass-one equation which are the main topic of this thesis.

3.1 Determinantal point process

Given a suitable function f on $\mathbb{C}^n$, we define a *canonical average* of f with respect to $\mathbb{P}_n$ by

$$\int_{\mathbb{C}^n} f(\lambda_1, \dots, \lambda_n)\mathbb{P}_n(d\lambda_1, \dots, d\lambda_n).$$

A function class in which we are interested is a class of generalized functions

$$f_\zeta(\lambda) = \sum_{j=1}^{n} \delta(\zeta - \lambda_j),$$

where $\zeta \in \mathbb{C}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and δ is the Dirac-delta function. Even though f_λ is a generalized function, a canonical average of f_λ is well-defined, because probability density function of $\mathbb{P}_n$ is Schwartz function.

A canonical average of f_λ is called a *one-point intensity function* $\mathbf{R}_{n,1}$:

$\mathbb{C} \to \mathbb{R}$. Explicitly, it is given by

$$\mathbf{R}_{n,1}(\zeta) \coloneqq \int_{\mathbb{C}^n} f_\zeta(\lambda_1, \ldots, \lambda_n)\mathbb{P}_n(d\lambda_1, \ldots, d\lambda_n) = n \int_{\mathbb{C}^{n-1}} \mathbb{P}_n(\zeta, d\lambda_2, \ldots, d\lambda_n),$$

where the last equality follows from $\mathbb{P}_n$ being invariant under permutation of coordinates. Generalizing the notion of one-point intensity function, we define the following.

Definition 3.1.1. *A* k-point intensity function $\mathbf{R}_{n,k} : \mathbb{C}^k \to \mathbb{R}$ *is defined by*

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) \coloneqq \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} \mathbb{P}_n(\zeta_1, \ldots, \zeta_k, d\lambda_{k+1}, \ldots, d\lambda_n). \tag{3.1}$$

In particular, if $k = 1$, then we write $\mathbf{R}_n$ in place of $\mathbf{R}_{n,1}$.

Note that

$$\frac{\mathbf{R}_{n,k+1}(\zeta_1, \ldots, \zeta_{k+1})}{\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k)} = (n-k)\,\mathbb{P}(\lambda_{k+1} = \zeta_{k+1} | \lambda_1 = \zeta_1, \ldots, \lambda_k = \zeta_k),$$

so $\mathbf{R}_{n,k+1}(\zeta_1, \ldots, \zeta_{k+1})/\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k)$ can be regarded as density at ζ_{k+1} provided that there are particles at $\zeta_1, \ldots, \zeta_k$.

Theorem 3.1.2. *There exist a* correlation kernel $\mathbf{K}_n : \mathbb{C}^2 \to \mathbb{C}$ *such that*

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) = \det\big[\mathbf{K}_n(\zeta_i, \zeta_j)\big]_{i,j=1}^{n},$$

for all $1 \le k \le n$.

In fact, the correlation kernel can be explicitly written as

$$\mathbf{K}_n(\zeta_1, \zeta_2) = \sum_{j=1}^{n} q_j(\zeta_1)\bar{q_j}(\zeta_2)e^{-nQ(\zeta_1)/2 - nQ(\zeta_2)/2},$$

where q_j is the j-th orthogonal polynomial with respect to a measure $e^{-nQ(\zeta)}d\zeta$. These are far from being trivial, and we refer [18] for details.

We remark that $\mathbf{K}_n$ is not uniquely determined by $\mathbf{R}_{n,k}$'s. For example, $\mathbf{K}_n(\zeta_1, \zeta_2)$ and $e^{i(\zeta_1 + \bar{\zeta_2})} \cdot \mathbf{K}_n(\zeta_1, \zeta_2)$ give the same $\mathbf{R}_{n,k}$ for all k. In general, a Hermitian function $c : \mathbb{C}^2 \to \mathbb{C}$ is called a *cocylce* if $c(\zeta_1, \zeta_2) = g(\zeta_1)g(\bar{\zeta_2})$ for some continuous unimodular function g. Then, $\mathbf{K}_n$ and $c \cdot \mathbf{K}_n$ yield the same $\mathbf{R}_{n,k}$'s by the determinantal structure.

3.2 Rescaled correlation kernel

Consider an eigenvalue configuration $\{\zeta_j\}_{j=1}^n$ with respect to $\mathbb{P}_n$. We are interested in a microscopic behavior of the eigenvalues configuration. Therefore, we rescale the eigenvalues near an n-dependent *zooming point* $p_n \in \mathbb{C}$ by

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q_n(p_n)} \cdot (\zeta_j - p_n), \tag{3.2}$$

where $e^{i\theta_n}$ is set to be the outer normal to the boundary of the droplet if p_n is in the boundary of the droplet, or $\theta_n = 0$ otherwise. Here and after, whenever we consider rescaled system, we will always assume that $\Delta Q_n(p_n) >$ const. > 0, so that (3.2) does make sense. In this chapter and Chapter 4, we only consider $p_n = p_*$ being independent of n. However, in Chapter 5, especially for the "edge regime", we have to consider p_n being n-dependent.

The zooming scale $\sqrt{n\Delta Q_n(p_n)}$ is natural scaling to obtain universality in the following sense. Recall that the limiting spectrum of $\mathbb{P}_n$ is a compactly supported measure with density ΔQ_n. Roughly speaking, on the support of limiting spectrum, expected number of particles in a unit circle is approximately $n\Delta Q_n$. Thus, the zooming scale, $\sqrt{n\Delta Q_n(p_n)}$, is chosen so that average intensity becomes independent of n and Q_n in the rescaled system $\{z_j\}_{j=1}^n$.

Rescaling in (3.2) results rescalings of intensity functions and a correlation kernel. By change of variable, we get *rescaled intensity functions* $R_{n,k}$ by

$$R_{n,k}(z_1, \dots, z_k) = \frac{1}{(n\Delta Q_n(p_n))^k}\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k),$$

and a *rescaled correlation kernel* K_n by

$$K_n(z_1, z_2) = \frac{1}{n\Delta Q_n(p_n)}\mathbf{K}_n(\zeta_1, \zeta_2), \tag{3.3}$$

where $z_i = \sqrt{n\Delta Q_n(p_n)}(\zeta_i - p_n)$ for $i = 1, \dots, k$. Note that *determinantal structure*

$$R_{n,k}(z_1, \dots, z_k) = \det\big[K_n(z_j, z_l)\big]_{j,l=1}^k,$$

remains unchanged via rescaling.

3.3 Ward's and mass-one equation

In this section, we state Ward's equation and mass-one equations for n-level correlation kernel. Materials in this section heavily relies on [5].

Since Ward's and mass-one equation is the main object of this thesis, we cover them in full detail. We first illustrate *mass-one equation.*

Proposition 3.3.1. *A correlation kernel K_n satisfies* mass-one equation*:*

$$\int_{\mathbb{C}} |K_n(z,w)|^2 dA(w) = K_n(z,z).$$

Proof. From the determinantal relation between R_n and K_n,

$$K_n(z,z) = R_n(z) > 0,$$

for all $z \in \mathbb{C}$. The positiveness is direct consequence of (2.1) and (3.1). Then,

$$\int_{\mathbb{C}} \frac{|K_n(z,w)|^2}{K_n(z,z)} dA(w) = \int_{\mathbb{C}} \frac{R_n(z)R_n(w) - R_{n,2}(z,w)}{R_n(z)} dA(w) = 1,$$

where the first equality follows from the determinantal structure, and the second follows from the remark below (3.1.1). Then, the conclusion follows. $\square$

We now introduce n-level Ward's identities. We denote a collection of compactly supported smooth functions by $C_c^\infty(\mathbb{C})$. The following proof is based on a proof given in [7].

Proposition 3.3.2. *Let $\psi \in C_c^\infty(\mathbb{C})$ be a test-function. Then, Ward's identity:*

$$\mathbb{E}_n[W_n^+[\psi]] := \mathbb{E}_n \mathrm{I_n}[\psi] - \mathbb{E}_\mathrm{n} \mathrm{II_n}[\psi] + \mathbb{E}_\mathrm{n} \mathrm{III_n}[\psi] = 0,$$

holds where

$$\mathrm{I_n}[\psi] = \frac{1}{2} \sum_{\mathrm{j \neq k}} \frac{\psi(\zeta_\mathrm{j}) - \psi(\zeta_\mathrm{k})}{\zeta_\mathrm{j} - \zeta_\mathrm{k}}, \qquad \mathrm{II_n}[\psi] = \mathrm{n} \sum_{\mathrm{j=1}}^{\mathrm{n}} \partial \mathrm{Q_n}(\zeta_\mathrm{j}) \cdot \psi(\zeta_\mathrm{j}),$$

$$\mathrm{III_n}[\psi] = \sum_{\mathrm{j=1}}^{\mathrm{n}} \partial \psi(\zeta_\mathrm{j}). \tag{3.4}$$

Proof. For any $j = 1, \dots, n$, by integration by parts

$$\begin{aligned}\mathbb{E}_n[\partial\psi(\zeta_j)] &= \frac{1}{Z_n}\int_{\mathbb{C}^n} \partial\psi(\zeta_j)\, e^{-H_n(\zeta_1,\dots,\zeta_n)}\, d\zeta_1\cdots d\zeta_n \\ &= \frac{1}{Z_n}\int_{\mathbb{C}^n} \partial_j H_n(\zeta_1,\dots,\zeta_n)\,\psi(\zeta_j)\, e^{-H_n(\zeta_1,\dots,\zeta_n)}\, d\zeta_1\cdots d\zeta_n \\ &= \mathbb{E}_n[\partial_j H_n \cdot \psi(\zeta_j)].\end{aligned}$$

From (2.1), we have

$$\partial_j H_n(\zeta_1,\dots,\zeta_n) = \sum_{k\neq j} \frac{1}{\zeta_j - \zeta_k} + \partial Q_n(\zeta_j).$$

Then, summation of $\mathbb{E}_n[\partial\psi(\zeta_j)]$ over j gives the desired result. □

Ward's identity is equivalent to a normalization constant remaining unchanged under coordinate perturbation. We refer [4, 5] for the further detail of this explanation. In the literature, Ward's identity is also referred as *loop equation*.

Before we state Ward's equation, we first need to introduce the *Berezin kernel* of K_n by

$$B_n(z,w) = \frac{|K_n(z,w)|^2}{K_n(z,z)}, \tag{3.5}$$

and Cauchy transform of the Berezin kernel by

$$C_n(z) = \int_{\mathbb{C}} \frac{B_n(z,w)}{z-w} dA(w). \tag{3.6}$$

As explained in the proof of Proposition 3.3.1, $K_n(z,z)$ is positive everywhere, so Berezin kernel is well-defined.

n-level Ward's equation is formulated as the following.

Proposition 3.3.3. *Ward's Identity in proposition 3.3.2 implies,*

$$\bar{\partial} C_n(z) = R_n(z) - 1 - \Delta \log R_n(z) + o(1)$$

in the sense of distribution, where $o(1) \to 0$ as $n \to \infty$.

Proof. For a test function ψ, consider a sequence of test functions $\{\psi_n\}$ given by $\psi_n(r_n z + p_n) = \psi(z)$. From (3.1) and (3.4), by the symmetry of $\mathbb{P}_n$ for each coordinate,

$$\begin{aligned}
\mathbb{E}_n \mathrm{I_n}[\psi_\mathrm{n}] &= \int_{\mathbb{C}^n} \sum_{j\neq k} \frac{\psi_n(\zeta_j)}{\zeta_j - \zeta_k} \cdot d\mathbb{P}_n(\zeta_1, \dots, \zeta_n) \\
&= \iint_{\mathbb{C}^2} \frac{\psi_n(\zeta)}{\zeta - \zeta'} \mathbf{R}_{n,2}(\zeta, \zeta') \cdot dA(\zeta) \cdot dA(\zeta') \\
&= \iint_{\mathbb{C}^2} \frac{\psi_n(r_n z + p_n)}{(r_n z + p_n) - (r_n w + p_n)} r_n^{-4} R_{n,2}(z, w) \cdot r_n^2 dA(z) \cdot r_n^2 dA(w) \\
&= \int_{\mathbb{C}} \psi(z) \cdot r_n^{-1} \int_{\mathbb{C}} \frac{R_{n,2}(z, w)}{z - w} \cdot dA(w) \cdot dA(z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}_n \mathrm{II_n}[\psi_\mathrm{n}] &= \int_{\mathbb{C}^n} n \sum_{j=1}^{n} \partial Q_n(\zeta_j) \cdot \psi_n(\zeta_j) \cdot d\mathbb{P}_n(\zeta_1, \dots, \zeta_n) \\
&= \int_{\mathbb{C}} n\partial Q_n(\zeta) \cdot \psi_n(\zeta) \mathbf{R}_n(\zeta) \cdot dA(\zeta) \\
&= \int_{\mathbb{C}} n\partial Q_n(r_n z + p_n) \cdot \psi_n(r_n z + p_n) r_n^{-2} R_n(z) \cdot r_n^2 dA(z) \\
&= \int_{\mathbb{C}} \psi(z) \cdot n\partial Q_n(r_n z + p_n) \cdot R_n(z) \cdot dA(z).
\end{aligned}$$

Finally, note that $\partial \psi_n(\zeta) \cdot \dfrac{\partial \zeta}{\partial z} = \partial \psi(z)$, so we have

$$\begin{aligned}
\mathbb{E}_n \mathrm{III_n}[\psi_\mathrm{n}] &= \int_{\mathbb{C}^n} \sum_{j=1}^{n} \partial \psi_n(\zeta_j) \cdot d\mathbb{P}_n(\zeta_1, \dots, \zeta_n) \\
&= \int_{\mathbb{C}} \partial \psi_n(\zeta) \mathbf{R}_n(\zeta) \cdot dA(\zeta) \\
&= \int_{\mathbb{C}} r_n^{-1} \partial \psi(z) r_n^{-2} R_n(z) \cdot r_n^2 dA(z) \\
&= \int_{\mathbb{C}} \partial \psi(z) \cdot r_n^{-1} R_n(z) \cdot dA(z) \\
&= -\int_{\mathbb{C}} \psi(z) \cdot r_n^{-1} \partial R_n(z) \cdot dA(z).
\end{aligned}$$

Putting together, Ward's identity is equivalent to

$$r_n^{-1}\int_{\mathbb{C}} \frac{R_{n,2}(z,w)}{z-w}\,dA(w) - n\partial Q_n(r_n + p_n)R_n(z) - r_n^{-1}\partial R_n(z) = 0,$$

in the sense of distribution. Furthermore, since $R_{n,2}(z,w) = R_n(z)(R_n(w) - B_n(z,w))$, we have

$$\begin{aligned} &r_n^{-1}R_n(z)\int_{\mathbb{C}} \frac{B_n(z,w)}{z-w}\,dA(w) \\ &\qquad = r_n^{-1}R_n(z)\int_{\mathbb{C}} \frac{R_n(w)}{z-w}\,dA(w) - n\partial Q_n(r_n + p_n)R_n(z) - r_n^{-1}\partial R_n(z), \end{aligned}$$

and dividing both side by $r_n^{-1}R_n(z)$, and then differentiating by $\bar{\partial}$ with respect to z, we have,

$$\bar{\partial}C_n(z) = R_n(z) - nr_n^2\Delta Q_n(r_n z + p_n) - \Delta \log R_n(z).$$

Finally, substituting $r_n = (n\Delta Q_n(p_n))^{-\frac{1}{2}}$ gives the conclusion. $\square$

This proof is based on a proof introduced in [7].

Chapter 4

Universality of scaling limit

In this chapter, we mainly discuss about correlation kernels when number of eigenvalues goes to infinity. Existence of limiting kernel is far from being trivial, and this may not be true in general. Universality of limiting kernels has been proved for large classes of random matrices, but it is still unknown for wide range of classes. The main purpose of this chapter is collecting known results about various universality classes that are related to almost-Hermitian random matrices.

Section 4.1 is devoted to cover universality classes that arise in non-Hermitian random matrix theory, especially random normal matrices. In Section 4.2, we will briefly introduce universality classes for Wigner ensembles.

4.1 Random Normal Matrix: non-Hermitian case

We recall that K_n is a rescaled correlation kernel as in (3.3) via microscale as in (3.2). Throughout this chapter, we will assume that the zooming point $p_n = p_*$ is independent of n. Otherwise stated, we refer notation given in Section 3.2. Then, the following is known.

Theorem 4.1.1. *For a sequence of correlation kernels K_n, every subsequence of (K_n) has a further subsequence that converges locally uniformly, up to cocycles.*

Proof. This is a consequence of Montel's theorem. See [5] for details. □

Let K be the subsequential limit in Theorem 4.1.1. Then, we can extend n-level Ward's equation for K. As an analogue to (3.5) and (3.6), we can define *Berezin kernel* for K by

$$B(z,w) = \frac{|K(z,w)|^2}{K(z,z)} \tag{4.1}$$

and its Cauchy transform

$$C(z) = \int_{\mathbb{C}} \frac{B(z,w)}{z-w} dA(w), \tag{4.2}$$

if provided that $R(z) \coloneqq K(z,z) > 0$.

Theorem 4.1.2. *Let K be a subsequential limit in Theorem 4.1.1. Then, either $R = 0$ or $R > 0$ everywhere. Furthermore, if $R > 0$, then Ward's equation:*

$$\bar{\partial} C = R - 1 - \Delta \log R,$$

holds in the sense of distribution.

Proof. For the proof of dichotomy theorem of R, see [5]. Proofs for Ward's equation is similar to it of Theorem 5.4.2. □

We can not directly extend mass-one equality for the subsequential limits, but we still have inequality :

Theorem 4.1.3. *Let K be a subsequential limit in Theorem 4.1.1. Then, mass-one inequality*

$$\int_{\mathbb{C}} |K(z,w)|^2 dA(w) \leq K(z,z),$$

holds.

Proof. It is a consequence of Proposition 3.3.1 using Fatou's lemma. □

Now we introduce universality related to the limiting correlation kernel. We first state universality theorems for random normal matrices, and gives reference for the theorems.

Theorem 4.1.4. *Let p_* be a interior point of a droplet. Then, K_n converges locally uniformly to*

$$G(z,w) = e^{z\bar{w} - \frac{|z|^2+|w|^2}{2}}, \tag{4.3}$$

up to cocycles.

Proof. We refer [3] for the proof. □

The limiting kernel G is called *Ginibre kernel* which is named after Jean Ginibre. The next is often referred as *(soft) edge universality.*

Theorem 4.1.5. *Let p_* be a regular boundary point of a droplet. Then, K_n converges locally uniformly to*

$$G(z,w)\,\mathrm{erfc}(\frac{z+\bar{w}}{\sqrt{2}}), \tag{4.4}$$

up to cocycles, where erfc *is a complementary error function*

$$\mathrm{erfc}(z) \coloneqq \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{t^2}{2}} dt.$$

Proof. We refer [5] for the proof using Ward's equation under additional assumption. See also [15]. □

Theorem 4.1.4 has been proved using Ward's equation in [4], and Theorem 4.1.5 has been proved under additional assumption, called *translation invariance*, using Ward's equation in [5]. Without *translation invariance*, the edge universality has been proved in [15]. We remark here that different universality class can arise at singular boundary points, such as cusps or double points. See [6] and references therein.

4.2 Unitary Invariant Matrix: Hermitian case

Universality has been proved for wide classes of Hermitian random matrix. In this section, we will fulfill with results proved for unitary invariant matrix models. We state results proved in [10, 17]. We also refer [1, Chapter 6].

Theorem 4.2.1. *Let p_* be a interior point of a droplet where the density is positive. Then, K_n converges to*

$$K^{sine}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}, \tag{4.5}$$

up to cocycles

Proof. We refer [17] for the proof. □

Next, we state *(soft) edge universality* for unitary invariant matrix model.

Theorem 4.2.2. *Let p_* be a regular right-edge point of a droplet. Then, there is a constant $c > 0$ such that*

$$\frac{1}{(cn^{2/3})} K_n(p_* + \frac{x}{(cn)^{2/3}}, p_* + \frac{y}{(cn)^{2/3}}) \tag{4.6}$$

converges to

$$K^{Airy}(x,y) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}'(x)\operatorname{Ai}(y)}{x-y},$$

up to cocycles, where Ai *is the Airy function.*

Proof. We refer [10] for the proof. □

Chapter 5

Scaling limits for almost-Hermitian random matrix

In this chapter, we introduce almost-hermitian random matrix model (AGUE) and prove main results which are stated in Theorem 5.3.3 and Theorem 5.4.2. The main results show that limiting kernel at the edge of AGUE satisfies the mass-one and Ward's equation. Universality of the limiting kernel is open for AGUE.

In (2.3), elliptic GinUE is interpreted as a sum of two independent GUEs via parameter $\tau \in [0, 1)$. It can be seen as "interpolation" between GUE and GinUE by allowing $\tau \in [0, 1]$ with abusing notation. Especially, we see that $\tau = 0$ corresponds to GinUE, while $\tau = 1$ case corresponds to GUE.

For fixed $\tau \in [0, 1]$, theorems introduced in Chapter 4 are valid, so we cannot observe new type of universality class. In other words, for $\tau \in [0, 1)$, the associated limiting kernels are either (4.3) or (4.4). However, for $\tau = 1$, they are neither of them, but either (4.5) or (4.6).

In Section 5.1, we define almost-Hermitian random matrix (AGUE), and relate its eigenvalue distribution to Boltzmann-Gibbs law. In Section 5.2, we collect results about limiting kernels that arise in AGUE. Finally, in Section 5.3 and 5.4, we prove main results.

5.1 Almost-Hermitian random matrix model

In this section, we define almost-Hermitian random matrix model and relate its measure to Boltzmann-Gibbs measure. We follow [2].

Definition 5.1.1. *A one parameter family of random complex n by n matrix A_c is said to be* almost-Hermitian Gaussian Unitary Ensemble *or* AGUE *with parameter $c > 0$ (resp.* modified almost-Hermitian Gaussian Unitary Ensemble *or* modified AGUE*) if*

$$A_c = \sqrt{1+\tau_n}H_1 + i\sqrt{1-\tau_n}H_2$$

where $\tau_n = 1-2c^2/n$ (resp. $\tau_n = 1-2c^2/n^{1/3}$) and H_1 and H_2 are independent GUEs.

As in the Elliptic GinUE case, the law of eigenvalues of AGUE models can be represented by Boltzmann-Gibbs measure.

Proposition 5.1.2. *Let A_c be AGUE (or modified AGUE) and $\{\zeta_j\}_{j=1}^n$ be corresponding eigenvalues of A_c. Then, the law of $\{\zeta_j\}_{j=1}^n$ is given by* (2.1) *with an external field Q_n given by*

$$Q_n(\zeta) = \frac{1}{1+\tau_n}\xi^2 + \frac{1}{1-\tau_n}\eta^2, \qquad \xi := \operatorname{Re}\zeta,\ \eta := \operatorname{Im}\zeta. \tag{5.1}$$

Proof. This is a consequence of Proposition 2.1.4. □

From (2.6), we observe that the right-most endpoint is given by $1+\tau_n$. Furthermore, the limiting spectrum of AGUE is a real interval [-2,2].

5.2 Limiting kernels for AGUE

In this section, we state scaling limits for AGUE. The bulk scaling limits was derived in [12], and the edge scaling limit was derived in [8].

Theorem 5.2.1. *For AUGE with $p_n = p_* \in (-2,2)$ in* (3.2) *be independent of n. Then, K_n converges locally uniformly to*

$$K_{(c)}^{bulk}(z,w) = G(z,w)\cdot\frac{1}{\sqrt{2\pi}}\int_{-2a_c}^{2a_c} e^{\frac{1}{2}(z-\bar{w}-it)^2}dt, \tag{5.2}$$

up to cocycles, where $G(z, w)$ is the Ginibre kernel in (4.3)*, and*

$$a_c = a_c(p_*) = \frac{c}{2}\sqrt{4 - p_*^2} \cdot \mathbf{1}_{[-2,2]}(p_*).$$

Proof. We refer [2, 12] for the proof. □

It was proved that universality conjecture is valid for (5.2) under additional assumption, called *translation invariance*. It was also showed that the limit satisfies Ward's and mass-one equations. For more details, see [2] and reference therein.

Theorem 5.2.2. *For modified AGUE with $p_n = 2(1 - c^2 n^{-1/3})$ in* (3.2)*, K_n converges locally uniformly to*

$$K_{(c)}^{edge}(z, w) = 4\sqrt{2\pi}c^2 \int_0^\infty f_c(z, t) f_c(\bar{w}, t) dt, \tag{5.3}$$

up to cocycles where

$$f_c(z, t) \coloneqq e^{2c^3(t+z) - (\operatorname{Im} z)^2} \operatorname{Ai}(2c(z + t) + c^4)$$

and Ai *is the Airy function.*

Proof. We refer [2, 8] for the proof. □

5.3 The mass-one equation for AGUE

In this section, we prove that the limiting kernel (5.3) satisfies the mass-one equation. We start with lemmata that is related to elementary properties of Airy function. We will not cover proofs for the lemmata, but briefly comment that these follows from integral representation of Airy function via complex analysis.

Lemma 5.3.1. *For $z, w \in \mathbb{C}$, the following holds:*

$$\operatorname{Ai}(z)\operatorname{Ai}(w) = \frac{1}{2^{1/3}\pi} \int_{\mathbb{R}} \operatorname{Ai}\left[2^{2/3}(\alpha^2 + \frac{z + w}{2})\right] e^{i(z-w)\alpha}\, d\alpha. \tag{5.4}$$

Proof. See [20] and references therein. □

Lemma 5.3.2. *For $x \in \mathbb{R}$,*

$$\int_{\mathbb{R}} e^{x\alpha} \operatorname{Ai}(\alpha)\, d\alpha = e^{x^3/3}. \tag{5.5}$$

Proof. See [20] and references therein. $\square$

Theorem 5.3.3. *The limiting kernel $K_{(c)}^{edge}$ in (5.3) satisfies the mass-one equation. i.e.*

$$\int_{\mathbb{C}} |K_{(c)}^{edge}(z,w)|^2\, dA(w) = K_{(c)}^{edge}(z,z).$$

Proof. Let $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $u = \operatorname{Re} w$ and $v = \operatorname{Im} w$. From (5.3),

$$\begin{aligned}\int_{\mathbb{C}} &|K_{(c)}^{\mathrm{edge}}(z,w)|^2\, dA(w) \\ &= 32c^4 \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty f_c(z,t) f_c(\bar{z},s) f_c(\bar{w},t) f_c(w,s)\, dt\, ds\, du\, dv. \end{aligned}\tag{5.6}$$

By (5.4), we have

$$\begin{aligned} f_c(z,t) f_c(\bar{z},s) &= \frac{e^{2c^3(t+s+2x)-2y^2}}{2^{1/3}\pi} \int_{\mathbb{R}} e^{-4cy\alpha+2c(t-s)\alpha i} g(\alpha, 2x+t+s)\, d\alpha, \\ f_c(\bar{w},t) f_c(w,s) &= \frac{e^{2c^3(t+s+2u)-2v^2}}{2^{1/3}\pi} \int_{\mathbb{R}} e^{-4cv\beta-2c(t-s)\beta i} g(\beta, 2u+t+s)\, d\beta, \end{aligned}$$

where

$$g(\gamma, r) := \operatorname{Ai}(2^{2/3}(\gamma^2 + cr + c^4)).$$

Using these equations, (5.6) equals to

$$\begin{aligned} \frac{32c^4}{2^{2/3}\pi^2} \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} & e^{4c^3(t+s+u+v)-2(y^2+v^2)} e^{-4c(y\alpha+v\beta)} e^{2c(t-s)(\alpha-\beta)i} \\ & \times g(\alpha, 2x+t+s) g(\beta, 2u+t+s)\, du\, dv\, d\alpha\, d\beta\, dt\, ds. \end{aligned}\tag{5.7}$$

We apply (5.5) to the integration with respect to u in (5.7), and deduce

$$\begin{aligned} \frac{8c^3}{2^{1/3}\pi^2} \int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} & e^{2c^3(t+s)-2(y^2+v^2)} e^{-4c(y\alpha+v\beta)} e^{2c(t-s)(\alpha-\beta)i} e^{-2c^2\beta^2} \\ & \times g(\alpha, 2x+t+s)\, dv\, d\beta\, d\alpha\, dt\, ds, \end{aligned}\tag{5.8}$$

and integration with respect to v is elementary calculation, so (5.8) is equal to

$$\frac{4\sqrt{2\pi}c^3}{2^{1/3}\pi^2}\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2c^3(t+s)-2y^2}e^{-4cy\alpha}e^{2c(t-s)(\alpha-\beta)i}g(\alpha,2x+t+s) \times d\beta\, d\alpha\, dt\, ds. \quad (5.9)$$

Note that the integration with respect to β is in form of integral representation of Dirac-delta function, so we express (5.9) by

$$4\sqrt{2\pi}c^2\int_0^\infty\int_{\mathbb{R}} e^{4c^3t-2y^2}e^{-4cy\alpha}g(\alpha,2x+t+s)\, d\alpha\, dt,$$

and then again by equation (5.4),

$$4\sqrt{2\pi}c^2\int_0^\infty e^{4c^3t-2y^2}f_c(z,t)f_c(\bar{z},t)\, dt,$$

which equals to $K^{\mathrm{edge}}_{(c)}(z,z)$. □

5.4 Ward's equation for AGUE

In this section, we prove that the limiting kernel in (5.3) satisfies Ward's equation. We follow basic ideas provided in [5].

The following lemma is a consequence of n-level mass-one equation, and mass-one inequality for the limiting kernel. We recall definitions of B_n and C_n from (3.5), (3.6). Throughout this section, we denote Berezin kernel of (5.3) and its Cauchy transform by B and C as in (4.1), and (4.2).

Lemma 5.4.1. *C_n converges boundedly and locally uniformly to C.*

Proof. Choose any $\epsilon > 0$. From (5.3) or Theorem 4.1.1, we observe that $K^{\mathrm{edge}}_{(c)}(z,z) > 0$ for all $z \in \mathbb{C}$. Furthermore, convergence of K_n to $K^{\mathrm{edge}}_{(c)}$ is locally uniform by Theorem 5.2.2. Thus, there exists $N > 0$ such that if $n > N$, then we have

$$|B_n(z,w) - B(z,w)| < \epsilon^2,$$

for any $|z| < 1/\epsilon$ and $|w| < 2/\epsilon$.

Then,

$$
\begin{aligned}
|C_n(z) - C(z)| &\leq \int_{\mathbb{C}} \left| \frac{B_n(z,w) - B(z,w)}{z-w} \right| dA(w) \\
&= \left(\int_{|z-w|>\frac{1}{\epsilon}} + \int_{|z-w|<\frac{1}{\epsilon}} \right) \left| \frac{B_n(z,w) - B(z,w)}{z-w} \right| dA(w) \\
&\leq \epsilon \int_{|z-w|>\frac{1}{\epsilon}} |B_n(z,w) - B(z,w)| dA(w) \\
&\qquad\qquad + \epsilon^2 \int_{|z-w|<\frac{1}{\epsilon}} \frac{1}{|z-w|} dA(w) \\
&\leq \epsilon \int_{|z-w|>\frac{1}{\epsilon}} |B_n(z,w)| + |B(z,w)| \, dA(w) + 2\epsilon.
\end{aligned}
$$

Finally, note that $B_n(z,w)$ and $B(z,w)$ are positive for any $z, w \in \mathbb{C}$, thus by Theorem 3.3.1 and Theorem 4.1.3,

$$
\int_{\mathbb{C}} B_n(z,w) dA(w) = 1, \qquad \int_{\mathbb{C}} B(z,w) dA(w) \leq 1,
$$

so we deduce

$$
|C_n(z) - C(z)| \leq 4\epsilon,
$$

which is the desired conclusion. $\square$

Theorem 5.4.2. *The correlation kernel $K_{(c)}^{edge}$ in (5.3) satisfies Ward's equation*

$$
\bar{\partial} C = R - 1 - \Delta \log R,
$$

pointwisely.

Proof. By Theorem 3.3.3, the n-level Ward's equation

$$
\bar{\partial} C_n(z) = R_n(z) - 1 - \Delta \log R_n(z) + o(1), \tag{5.10}
$$

holds for AGUE. We first show that $\bar{\partial} C_n$ (*resp.* R_n and $\Delta \log R$) converges to $\bar{\partial} C$ (*resp.* R and $\Delta \log R$) in the sense of distribution.

The convergence of R_n to R is already shown in Theorem 5.2.2. By Theorem 5.4.1, C_n converges to C boundedly and locally uniformly. Thus, for any test function ψ, integration by parts gives

$$\int_{\mathbb{C}} \bar{\partial} C_n(z)\psi(z)dz - \int_{\mathbb{C}} \bar{\partial} C(z)\psi(z)dz = \int_{\mathbb{C}} \big(C_n(z) - C(z)\big)\bar{\partial}\psi dz \to 0,$$

as $n \to \infty$, so $\bar{\partial} C_n$ converges to $\bar{\partial} C$ in the sense of distribution.

We have shown convergence of $\bar{\partial} C_n$ to $\bar{\partial} C$, and it of R_n to R in the sense of distribution. Thus, by (5.10), $\Delta \log R_n$ have to converges to $R - 1 - \bar{\partial} C$ in the sense of distribution. Furthermore, since R_n converges to R locally uniformly, it follows that $\Delta \log R_n$ must converge to $\Delta \log R$ in the sense of distribution. Hence, Ward's equation holds in the sense of distribution.

Finally, by (5.3), R is smooth, so $\Delta \log R$ is smooth. Moreover, by (4.2), C and $\bar{\partial} C$ are smooth. Then, by Weyl's lemma, Ward's equation holds at every point on $\mathbb{C}$. $\square$

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국문초록

이 학위논문에서는 비에르미트 랜덤 행렬, 특히 유니터리 불변성을 가지고 있는 유사-에르미트 랜덤 행렬의 국소 척도 극한을 다룬다. 유사-에르미트 랜덤행렬의 척도 극한은 Bender의 논문 [8]에서 이미 계산되어 알려져 있다. 와드 등식은 척도 극한의 보편성을 보일 때 사용된 등식으로, 적절한 가정을 추가하면, 랜덤 정규 행렬의 경계 척도 극한의 보편성이 유도된다. 동일한 가정하에, 유사-에르미트 행렬의 내부점에 대한 국소 척도 극한의 보편성은 이미 밝혀졌지만, 경계 척도 극한에 대해서는 알려진 바가 거의 없다. 이 학위논문에서는 유사-에르미트 행렬의 경계 척도 극한이 와드 등식과 매스-원 등식을 만족시킴을 증명한다.

주요어휘: 유사-에르미트 랜덤 행렬, 와드 공식, 매스-원 공식
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