A STUDY OF OPTIMUM REPLACEMENT POLICY

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Ⅰ. INTRODUCTION

The problems concerned with replacement of equipment, parts and

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components, and personnel can generally be viewed in two ways, depending on the life pattern of the above items. The first is a kind of replacement which has resulted from deterioration of items in use(Category I) and the second is a kind of replacement which has resulted from instantaneous and complete failure of items in use(Category II).[1]

Category I replacement deals mainly with such alternative measures as whether to make continuous use of outworn and inefficient equipment without replacing it or introduce more efficient equipment by sacrificing the replacement cost. Category II replacement can in turn be divided into two parts; (1) the failure of any one item can result in the failure of the entire system and (2) with the increase in the number of failures in a system, the overall efficiency of the system will accordingly drop.

In the former case each faulty item should be replaced at the time of failure but it is really in the latter case that the alternative problem arises of whether to keep the inefficient equipment or to replace the failures at the sacrifice of replacement cost. In this thesis only the latter case in Category II replacement will be analyzed and studied in some detail.

Although failures are continuous, it is likely that failure data would be collected and recorded within discrete time intervals. But in practice it is measured at discrete time period. Thus, most replacement models are constructed under the assumption that failures occur instantaneously only at the end of a time period. Unit(or individual) Replacement System and Group Replacement System are good examples of this. Generally accepted optimum replacement policy is known to follow one or a mixture of these two extremes, Unit Replacement System and Group Replacement System.[2]

On the other hand, an assumption is made in such a way that failure

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(1) Fabrycky, W.J., Torgersen, P.E., *Operations Economy*, Englewood Cliffs: Prentice-Hall, Inc., 1966. p. 155.

(2) Churchman, C.W., Ackoff, R.L., Arnoff, E.L., *Introduction to Operations Research*, New York: John-Wiley & Sons, Inc., 1957. pp. 478—479.

of items occurs continuously, failures in a certain period will be set aside until the time of replacement at the end of the period. In any case, because of replacement intervals, the accumulation of failures for a certain period is inescapable. And it is necessary to make an economic choice between inefficiency cost and replacement. If inefficiency cost is dealt with as a sort of replacement cost, then replacement problems can be reduced to minimize the total replacement cost including inefficiency costs.

The production efficiency in the system which has only all "live" items differs greatly from the system where only part of the items have failed while part of the items are still operating. Also, frequent replacement of faulty items for better production efficiency will render high replacement cost.

Thus, our problem is to determine *how frequently* and *how many* replacements should be made to obtain the minimum replacement cost. To do this, let's think of a possible model which can answer the question of how many and how frequently replacements should be made. We shall call one such model "Part Replacement Model", which replaces only the amount at the time when failures are accumulated up to the point. The process for finding out the optimum replacement policy under "Part Replacement Model" is discussed in Chapter V.

There is a strong analogy between a probabilistic inventory model and our part replacement model. The former has two random variables, the procurement quantity and the number of periods per cycle fixed. The latter also has two random variables, the size of replacement and the replacement interval. Furthermore, both of them are for problems concerning minimum cost expedition.

We have several ways to solve probabilistic inventory problems: Fixed—Interval Systems, Fixed—Quantity Systems and (s,S) Control Systems.[3] I

(3) Magee, J.F., Boodman, D.M., *Production Planning and Inventory Control*, New York: Mc Graw-Hill, Inc., 1967. pp. 119—132.

thought that these techniques could also be applied to finding out an optimum replacement policy.

In this paper, the techniques of Fixed-Interval Systems, Fixed-Quantity Systems and (s,S) Control Systems are employed for optimum replacement policy, and examples are given in order to illustrate the methods.

II. ANALYSIS OF FAILURE DATA

1. Conditional Probability of Faliure

Replacement models for items that fail require the use of probabilistic concepts and statistics of failure data. We should be able to analyse the given data to solve the replacement problem. Some definitions and terms are introduced as follows.[4]

Definition 1

The reliability of a component at time t, say $R(t)$, is defined as $R(t)=Pr(T>t)$, where T is the life length of the component. $R(t)$ is called the reliability function.

The definition given here simply says that the reliability of a component equals the probability that the component does not fail during the interval $[0, t]$. In terms of the probability density function of T, say f, we have $R(t)=\int_t^\infty f(s)ds$. In terms of the cumulative distribution function of T, say F, we have $R(t)=1-Pr(T\leqq t)=1-F(t)$.

In addition to the reliability function R, another function plays an important role in describing the failure characteristics of an item.

Definition 2

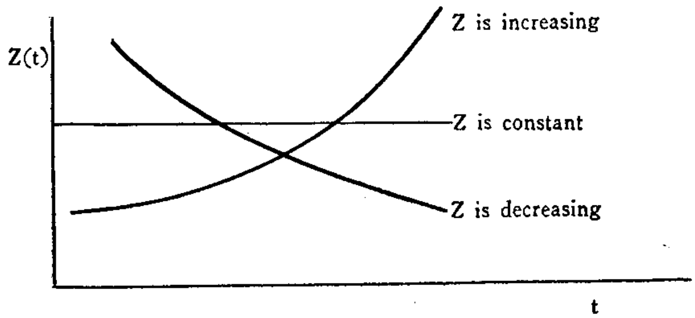
The failure rate Z associated with the random variable T is given by $Z(t)=f(t)/1-F(t)=f(t)/R(t)$, defined for $F(t)<1$.

(4) Meyer, P.L., *Introductory Probability and Statistical Applications*, Addison-Wesley, Inc., 1966. pp. 207—216.

In order to interpret $Z(t)$, consider the conditional probability $Pr(t \leqq T \leqq t+\Delta t) = \Delta t f(\xi)/R(t)$, where $t \leqq \xi \leqq t+\Delta t$. For small Δt and supposing that f is continuous at 0^+, the right term is approximately equal to $\Delta t Z(t)$. Thus, simply, $\Delta t Z(t)$ represents the proportion of items that will fail between t and $t+\Delta t$, among those items which are still functioning at time t.

The pdf of T, uniquely determines the failure rate Z. Conversely, the failure rate Z uniquely determines the pdf f. Failure rate Z is an increasing, decreasing or constant function of t according to the life pattern of the equipment. (Cf. figure 2.1)

〈2.1 **Figure**〉



There are many types of Components whose failure behavior may be represented by an increasing failure rate. The most typical pattern of these is showed in the case of normal distribution. But it is known that "infant mortality" is one of the cases which is represented by a decreasing failure rate.

We very interestingly have a third case. The time for failure T has an exponential distribution if and only if it has constant failure rate. Applying the definition of conditional probability, we find that, for small Δt, $Pr(t \leqq T \leqq t+\Delta t / T > t) = 1-\exp.(-\alpha\Delta t) = \alpha\Delta t$, if $Z(t)=\alpha$(constant). Hence, this is independent of t, depending only on Δt.

It is in this sense that we may say that an exponential failure law implies that the probability of failure is independent of past history. That is, as long as the item is still functioning, it is "as good as new". For example, it is quite reasonable to suppose that a fuse or a jeweled bearing is "as good as new" while it is still functioning.

However, when exponential failure law is assumed, past history does have an effect on an item's performance. And in this case, we must consider the data which is available in age-probability relations. As an example, table 2.1 is given here.(5)

〈Table 2.1〉

PERIOD (A)	SURVIVORS (B)	FAILURES (C)	P(t) (D)	Pc(t) (E)
0	1,000	—	—	—
1	900	100	0.10	100/1,000
2	750	150	0.15	150/900
3	500	250	0.25	250/750
4	200	300	0.30	300/500
5	0	200	0.20	200/200

Column (B) of the table gives the number of items functioning properly at the end of each time period. Column (C) gives the number of items which failed within each time period. The probability of an item failing within each time period is given in column (D).

The probability that a bulb, having survived to an age $t-1$, will fail during the interval $t-1$ to t can be defined as the conditional probability of failure. These conditional probabilities are given in column (*E*). This example is based on empirical data. If there is good reason to believe that failure data conform to a known theoretical distribution, then the entries in column (B) could be found by calculation.

2. Computation of the Number of Failures

Definition 3

An n-component age distribution A is a row vector written as $(a_0, a_1, a_2, \cdots\cdots, a_{n-1})$. The $i-th$ component a_i, $i=0, 1, 2, \cdots\cdots, n-1$, are assumed to

(5) Sasieni, M., Yaspan, A., Friedman, L., *Operations Research:methods and problems*, New york: John-Wiley & Sons, Inc., 1959. pp. 108—109, Example 4.

be the number of items whose age is i. Thus, a_0 is the number of new items just replaced.

Definition 4

The transition matrix for a Markov Chain is the matrix P with entries p_{ij}. "p_{ij}" means the probability that an item in state i will be in state j at next time period. If a problem has n different states, transition matrix for the problem will be an $n \times n$ matrix.(6)

In the replacement problem, conditional probability p_{ij} is defined as follows; (1) $p_{ij}=0$, if $j \neq i+1$ (2) $p_{ii}=0$, for all i except $i=0$.(7)

The entries p_{i0} of the first column in a transition matrix indicate the probabilities that an item in state i is replaced at once, because of failure at the next time period.

If we use the transition matrix $P=(p_{ij})$, $p_{i0} \neq 0$, in calculation, then it means that we replace failures at every period. By induction, we can get an equality $A_t=A_{t-1}P$. Assume that transition matrix P_0 with its first column entries, p_{i0}, vanished, we can get an age distribution of the next period by the equation $A_t={}_{t-1}P_0$ even if we do not replace the failures each period.

Theorem 1

There exists a steady age distribution A_n such that $A_n=A_nP$, for some n, in replacement problem. If we do not replace the failures each period, A_t converges to the null vector 0.(8)

Theorem 2

The first component $a^t{}_0$ of A_t means the number of failures at the end of period t. The first component $a^n{}_0$ of the steady age distribution A_n is the mean of the numbers of failures per period. The reciprocal of $a^n{}_0$ is the

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(6) Kemeny, J.G., Snell, J.L., *Finite Markov Chains*, New York: D. Van Nostrand, Inc., 1960. pp. 24—26.

(7) Churchman, C.W., *op. cit.*, p.500.

(8) Churchman, C.W., *op. cit.*, p.501 An illustrative example is given in Chapter V (p.59) in this thesis.

mean life length of the item.[9]

Ⅲ. COST FACTORS RELEVANT TO REPLACEMENT

1. Costs Depending on the Number of Replacements

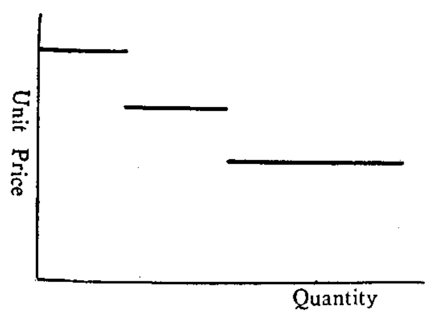
A. Purchasing Costs

When items are purchased, quantity discounts may be allowed, i.e., the unit price of the item may be adjusted depending on the quantity purchased. Quantity discounts are conventionally quoted in terms of price breaks or brackets, volume limits within which fixed unit prices apply, or in terms of a discount schedule, a statement of percentage allowance granted on orders or quantities of given amounts or over.

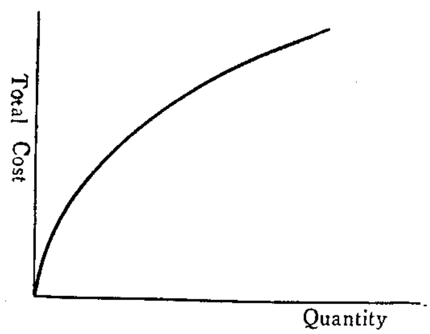
The differences in cost may result from reduction of paper work or machine set up incident to an order, differences in manufacturing method, economies in shipping and packing, or even administrative or selling economies. Figure 3.1 and 3.2 illustrate a typical unit price-quantity relation from a discount schedule and the corresponding total cost-quantity relation.

If such costs in personnel replacement as those of announcement, testing,

〈Figure 3.1〉
Unit Price-Quantity Relation



〈Figure 3.2〉
Total Cost-Quantity Relation



(9) Clough, D.J., *Concepts in Management Science*, Englewood Cliffs: Prentice-Hall, Inc., 1963. p. 229.

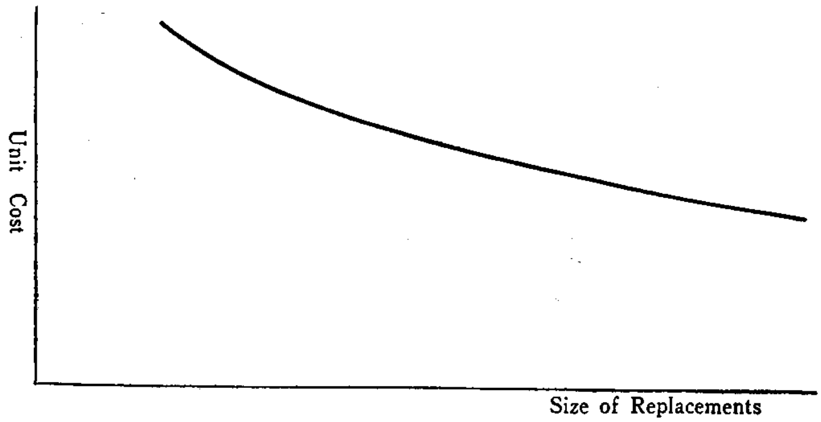
and physical examination necessary for recruiting, selecting, and placing personnel are considered as purchasing cost, the problems or theory concerning equipment replacement can directly be applicable to those in personnel replacement.

B. Replacing Cost

Since equipment replacement is indispensable for normal production activities, the cost for equipment replacement is an important factor in overall production cost. Here the replacing cost, as a part of replacement costs, includes all costs for removing failures and installing new items. Important elements in this cost are the cost for workers who participate in replacement operation and the cost incurred from suspending activities of the company (Cf. Figure 3.3).

Especially if replacement and installation make necessary specialized techniques and relatively long time periods, the replacing cost should be considered as important. If suspension of operations will take place at every replacement, then economy can be achieved by minimizing the number of replacement operations.

⟨Figure 3.3⟩ Replacing Cost



Replacing cost will accordingly be increased when the replacement is troublesome and necessitates a long period of work. It follows then for minimum replacing cost to choose the one which minimizes the frequency of replacements and maximizes the size of replacements (the number of items replaced at one replacement). Certainly, if the replacing cost incurred from suspension of operations is not significant, replacing cost in general will be increased in proportion to the number of items per one replacement.

Replacing cost in personnel replacement, unlike equipment replacement, includes such costs as necessary for training personnel recruited to fill vacancies. Though the training cost will differ depending on the methods of training such as orientation, vestibule school training, and programmed training,(10) it will generally be dependent upon the number of replaced personnel and particularly the amount of recruitment in a certain period.

2. Costs Resulting from Retention of Failures.

A. Maintenance Cost

Maintenance cost is usually an important element of factory overhead cost. As maintenance cost indicates the cost of maintaining items in use or "live items", it follows that maintenance cost increases in direct proportion to the quantity of "live items". Accordingly, if failures are not retained and if instead replaced at the time of failure, there will be only those items which are always operating. In that case maintenance cost remains constant. And maintenance cost will not be relevant to the discussion of replacement problem. This is also the case with the Unit Replacement model as well as with the Group Replacement model.

But in a model in which retention costs of failures over a certain period is significant, the maintenance cost of existing survivors should not be regarded for it is usually affected by or dependent upon a replacement policy. While both purchasing cost and replacing cost occur only at the time of

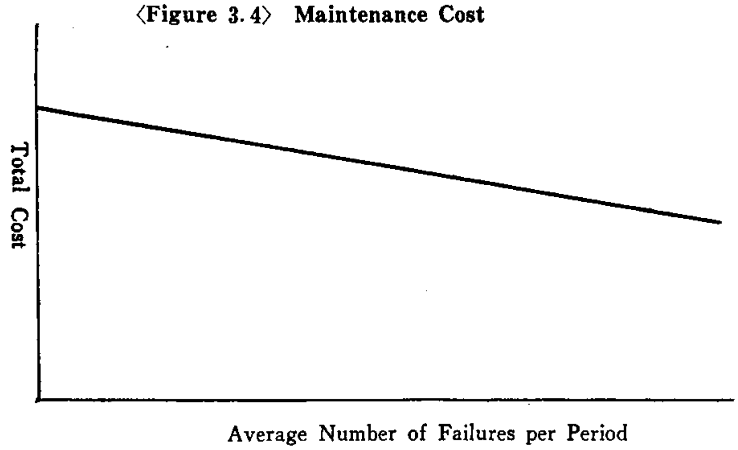
(10) Yoder, D., *Personnel Management and Industrial Relations*, Englewood Cliffs: Prentice-Hall, Inc., 1962. pp. 394—402.

replacement, maintenance cost will always occur for a certain period even when items are not used or functioning.

The calculation of maintenance cost for a period is as follows: (Maintenance cost per period) = (the average quantity of survivors for a period) × (Maintenance unit cost per period).

In personnel replacement, the cost corresponding to or comparable to maintenance cost is *Arbeits haupt Kosten* such as wages, salary and bonus. Since the wages for employees are proportional to the number of employees, the total wages will accordingly decrease as the number of vacancies increase.

〈Figure 3.4〉 Maintenance Cost



In personnel replacement the total cost of wages taken for recruiting needed personnel to fill current vacancies is what we are interested in and we can regard it as the replacing cost in personnel replacement.

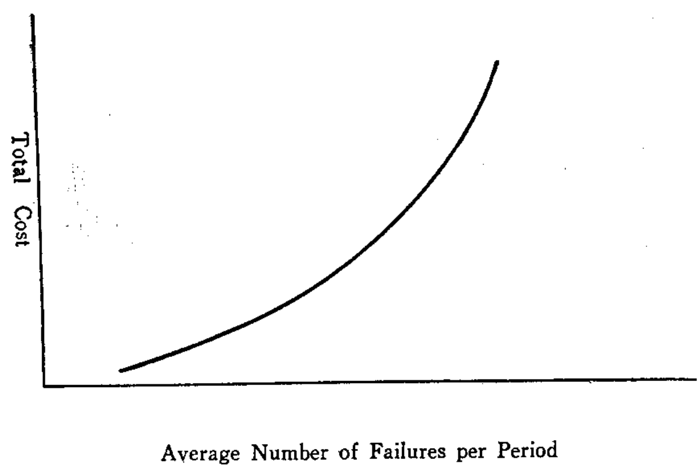
B. Inefficiency Cost

Along with the cost resulting from suspending operations, inefficiency cost, though not shown in general financial statements, is a functional cost which should be included as important when regarding replacement policy. Penalty cost in inventory control is calculated as adding additional surplus to satisfy a shortage by assumming that when a backlog is present, the shortage to fully meet the demand is met by overseas import or delivery from remote distances.

On the other hand, in replacement, the estimated amount of reduction in revenue is added to the amount of failed items kept over a period for the estimated reduction in revenue is from sluggish production caused by retaining failures over a period. If the number of failures increases, inefficiency will accordingly appear and finally the system as a whole will stop functioning. We do not in the least want this worst case to come about.

Figure 3.5 shows the relationship between the average number of failures retained and the total inefficiency cost.

〈Figure 3.5〉 **Inefficiency Cost**



As in the case of maintenance cost, if immediate replacement for failures in order to fully operate existing items is made possible, inefficiency cost will not be significant and thus can be ignored. This is also true in Unit Replacement as well as in Group Replacement. If the occurrence of failures is continuous over time, replacing must be carried out at discrete intervals.

In the light of the above fact, we should not disregard inefficiency cost but take it into consideration. The calculating formula for this cost is derived from the average number retained over a period multiplied by the unit inefficiency cost over a period. The average number retained over a period comes to a half of the number of failures in a period under the assumption that the number of failures are equally distributed over a small period.

3. Other Costs

The preceding four factors have been thought pertinent in dealing with replacement problems but the following also hold as the occasion demands.

A. Inspection Cost

B. Clerical Cost

C. Capital Cost

D. Miscellaneous Costs

Ⅳ. SYSTEMATIC CONTROL OF REPLACEMENT

1. Unit Replacement System (UR) (11)

A. Assumptions

(1) Failures occur instantaneously only at the end of a time period.

(2) Units are replaced as they fail.

(3) In a Unit Replacement System, only purchasing cost and replacing cost are considered as cost factors relevant to replacement.

B. Objective Function

(1) Notations $U(t)$ = the cost of UR per period

x_i = the number of failures made at the end of the i-th period.

$c(x)$ = cost of UR per unit

(11) Churchman, C.W., *op. cit.*, p. 478 and p. 491.

(2) The cost of UR per period may be expressed as

$$U(t)=\sum_{i=1}^{t} c(x)x_i/t.$$

If $c(x)=c$ (constant), then $U(t)=c\sum_{i=1}^{t} x_i/t=cx'$, where x' is the average number of failures per period.

2. Group Replacement System (GR) (12)

A. Assumptions

(1) Failures occur instantaneously only at the end of a time period.

(2) All units, both survivors and failures, are replaced as one or more of them fail.

(3) Only purchasing cost and replacing cost are considered as cost factors relevant to replacement.

B. Objective Function

(1) Notations

$G(t)$=the cost of GR per period

N=total number of components in a system

$c(N)$=cost of GR per unit

(2) If we replace all the components n times for t periods, the cost of GR per period may be expressed as follows:

$$G(t)=c(N)n\ N/t,\ n\leqq t.$$

3. Unit and Group Replacement System (UGR) (13)

A. Assumptions

(1) Failures occur instantaneously only at the end of a time period.

(2) Units are replaced as they fail but, in addition, all units will be replaced at a specified interval.

(3) Only purchasing cost and replacing cost are considered as cost factors relevant to replacement.

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(12) Churchman, C.W., *op. cit.*, p. 478 and pp. 491—492.
(13) Fabrycky, W. J., *op. cit.*, p. 186.

B. Objective Function

Let $K(t)$ be total replacement cost per period through t periods after the previous group replacement. The total cost per period will be the cost of UR per period plus the cost of GR per period, or $K(t)=U(t)+G(t)$.

The cost of UR per period may be expressed as $U(t)=\sum_{i=1}^{t-1} c(x)x_i/t$, where the summation of x_i is over $t-1$ periods, to allow for the replacement of failures in period t as part of the group.

The cost of GR per period may be expressed as

$$G(t)=c(N)N/t.$$

Therefore, the total cost per period with UGR may be expressed as follows:

$$K(t)=\sum_{i=1}^{t-1} c(x)x_i/t+c(N)N/t.$$

4. Part Replacement System (PR)

A. Assumptions

(1) Failure of items occurs continuously.

(2) Accumulative number of failures is to be replaced at a specified period according to replacement policy.

(3) Cost factors relevant to replacement are

a. purchasing cost

b. replacing cost

c. maintenance cost

d. inefficiency cost

e. miscellaneous costs

B. Objective Function

Total cost of replacement under PR system is determined by a replacement policy; the policy is determined by the time to replace and the number of failures to handle. Hence, the objective function is to follow the process of policy-making. Chapter V will show you the cost function under a PR system.

5. The Point of Problem Suggested by an Example

Suppose that a group of 10,000 light bulbs is installed and at the end of t time periods the number of bulbs surviving is some function of t. Failure data is given in Table 4.1.

〈Table 4.1〉 **Analysis of light bulb failure data**[14]

Period (A)	Survivors (B)	Failures (C)	$P(t)$ (D)	$Pc(t)$ (E)	$Ps(t)$ (F)
0	10,000	—	—	—	1.00
1	9,000	1,000	0.10	1/10	0.90
2	7,000	2,000	0.20	2/9	0.70
3	4,000	3,000	0.30	3/7	0.40
4	2,000	2,000	0.20	1/2	0.20
5	500	1,500	0.15	3/4	0.05
6	0	500	0.05	1	1.00
7	0	0	0.00	—	—

$P(t)$=(failures during the time period t)/10,000. $Pc(t)$=the conditional probability that a bulb, having survived to an age $t-1$, will fail during the interval $t-1$ to t and computed as $Pc(t)$=(failures during the time period t)/(the number of survivors through time $t-1$). $Ps(t)$=the probability of survival to an age t and computed as (the number of survivors through time t)/10,000

This example is based on empirical data. In an actual analysis of failure data, more precision would be obtained by increasing the number of time periods and dividing the survivor data finely.

To calculate the number of failures, we have to make a transition matrix P and an age distribution.

The Markov Chain in this case has 7 states—0, 1, 2, 3, 4, 5, 6—and hence, matrix P could be made as follows:

(14) Fabrycky, W. J., *op. cit.*, p. 182. (The example and its solution)

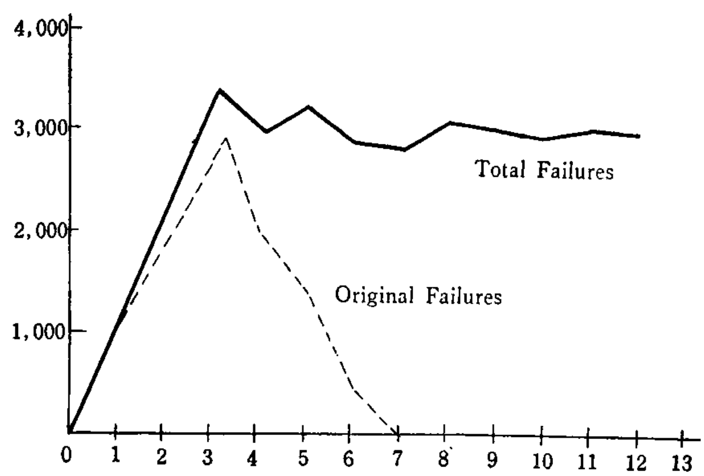
$$P = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1/10 & 9/10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2/9 & 0 & 7/9 & 0 & 0 & 0 & 0 \\ 2 & 3/7 & 0 & 0 & 4/7 & 0 & 0 & 0 \\ 3 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 4 & 3/4 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Let A_i be the age distribution at the end of the i-th period, then

$$\begin{array}{lllllllll} A_0 = (& 10,000, & 0, & 0, & 0, & 0, & 0, & 0) & \\ A_1 = (& 1,000, & 9,000, & 0, & 0, & 0, & 0, & 0) & = A_0P \\ A_2 = (& 2,100, & 900, & 7,000, & 0, & 0, & 0, & 0) & = A_1P \\ A_3 = (& 3,410, & 1,890, & 700, & 4,000, & 0, & 0, & 0) & = A_2P \\ A_4 = (& 3,061, & 3,069, & 1,470, & 400, & 2,000, & 0, & 0) & = A_3P \\ \vdots & & & & \vdots & & & & \vdots \end{array}$$

The number of failures at the end of each period is the first component of an age distribution corresponding to the period. Figure 4.1 illustrates that the number of replacements required per period oscillates until a steady-state condition is achieved.

⟨Figure 4.1⟩ Total failures per time period and failures per time period of the original units



A. Solution by Unit Replacement Policy

The average life of the units under consideration is given by the expression $\sum_{t=1}^{n} t\,P(t)$. For example, the average life of the light bulbs is $1(0.10)+2(0.20)+3(0.30)+4(0.20)+5(0.15)+6(0.05)=3.25$ periods. Hence, the number of replacements required per time period in the steady-state condition is 10,000/3.25 or 3,080. This value agrees with that illustrated by Figure 4.1.

If the cost of replacing one unit is $0.10, the cost per period would be $0.10(3,080) or $308. This period cost should be compared with the policy of group replacement.

B. Solution by Unit and Group Replacement Policy

Assume that units are replaced as they fail but, in addition all units will be replaced at a specified interval. And assume that the population of 10,000 bulbs has a group replacement cost of $0.05 per unit. As before, assume that the cost of unit replacement per unit is $0.10.

The total cost per period for various group replacement intervals is calculated from $K(t)=0.1\sum_{i=1}^{t-1} x_i/t+0.05(10,000)/t$.

⟨Table 4.2⟩ Solution by Unit and Group Replacement

Period t	$\sum_{i=1}^{t-1} x_i$	$0.1\sum_{i=1}^{t-1} x_i/t$	$(0.05)(10,000)/t$	$K(t)$
1	0	0	500	500
2	1000	50	250	300
3	3100	103	167	270
4	6510	163	125	288
5	9571	191.4	100	291.4

A group replacement interval of three periods will result in a minimum total cost per period. Since this cost is $270, a saving of $308 less $270, or $38 per period, will result if the policy of group replacement is implemented.

C. The Point of Problem

The above solution is based on two important assumptions:

(1) Replacement cost $0.10 per unit is independent of the size of replacements.

(2) A number of failures, even more than 3,000 items, fail instantaneously at the end of each period.

Under these assumptions, replacement model becomes very simple but it is far from the practical realities. It is desired to build a model that is more practical. When items are purchased, quantity discounts may be allowed, i.e., the unit price of the item may be adjusted depending on the quantity purchased. Generally, the bigger the size of replacements, the cheaper the cost of replacing one unit.

A number of failures fail continuously and we should consider the inefficiency of the organization caused by the retention of the cumulated failures. For small time period, we may assume that the average number of retention per period is calculated as follows;

(the number of failures per period)/2.

The calculating formula for inefficiency cost is derived from the average number of retention over a period multiplied by unit inefficiency cost over a period.

If some cost data are added here in this problem (Cf. Table 4.3), the solution by Unit and Group Replacement could be showed as Table 4.4 with a difference from Table 4.2.

〈Table 4.3〉 Cost data for replacement

Number of Failures	Purchasing cost & Replacing cost	Average No. of Failures	Inefficiency cost
1—1000	$0.10	1—1000	$0.02
1001—2000	$0.08	1001—2000	$0.08
2001—3000	$0.07	2001—3000	$0.10
3001—4000	$0.06	3001—4000	$0.20
more than 4000	$0.05	more than 4000	$0.80

〈Table 4.4〉 **Another solution by the new data**

Period t (A)	No. of failures (B)	Cost of UR per period (C)	Cost of GR per t (D)	Inefficiency cost per t (E)	$K(t)$ (F)
1	0	0	500	$\frac{(1000)(0.02)}{2}$	510
2	1000	$\frac{(1000)(0.1)}{2}$	250	$\frac{(10+0.08\times 2100/2)}{2}$	297
3	2100	$\frac{0.1\times 1000+0.07\times 2100}{3}$	167	$\frac{(94+0.08\times 3410/2)}{3}$	326.1
4	3410	$\frac{100+147+0.06\times 3410}{4}$	125	$\frac{(230.4+\frac{0.08\times 3061}{2})}{4}$	326.1
5	3061	$\frac{451.6+3061\times 0.06}{5}$	100	$\frac{(352.8+\frac{3318\times 0.08}{2})}{5}$	324.2
6	3318	………	……	………	……

Since the corresponding failure at the end of each period converges to a stable number by the existance of a steady age distribution, the replacing cost in column (C) and the inefficiency cost in column (E) of Table 4.4 slowly approach stability.

Table 4.4 shows us that the second period is the best time for group replacement. The purpose of this table lies not in getting the best solution but in suggesting that the solution from Table 4.2 is not practical due to the over simplified assumptions.

Ⅴ. PROCESS FOR PART REPLACEMENT POLICY

1. Workable Policies

Two random variables are important in the replacement process as in the stochastic inventory process. If we let one variable r be the size of replacements(15) and the other variable I be the replacement interval, then we may have a two-variable function $C=f(r, I)$ where C is total replacement cost per unit period.

(15) The term "the number of replacements" is somewhat ambiguous since it can refer either to the size of replacements (the number of items replaced at one replacement) or to the frequency of replacements. In this paper, this distinction should not be confused.

If $\partial C/\partial r$, $\partial C/\partial I$ exist, it would be useful to examine the followings:[16]

$$\begin{aligned} \partial C/\partial r &\leqq 0 \\ \partial C/\partial I &\leqq 0 \\ r &\geqq 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots (1) \\ I &\geqq 0 \end{aligned}$$

If equations (1) holds, we can get the values of $r=r^*$ and $I=I^*$ which minimize C. But we have no way to find out whether function f has the partial derivatives with respect to r and I, and moreover, we are expected to determine the exact pattern of function f. Accordingly, no analytical solution of the problem using calculus can be possible.

We now turn to the stochastic inventory process to build an adequate replacement model for this problem. We have seen that the two variable function $f(r,I)$ holds true. From this we have two different approaches to analyse the problem. One is to determine a policy by examining the changing value of I when variable r is kept fixed. The other is to examine the changes in value of r when variable I is kept constant.

To put it simply, (1) Fixed-Quantity System (2) Fixed-Interval System are the two possible approaches to finding the minimum cost policy.

When the number of cumulative failures increases to a certain fixed level, the failures are immediately replaced in a Fixed-Quantity System. In this system, replacement intervals are hardly expected to be constant as we are only interested in the value r. That is, the length of replacement interval I is a variable. In this system the times of replacing cumulative failures are determined by or dependent upon the level we fix r. The selection of a constant value r will also affect the total cost per perod.

In a replacement problem unlike inventory, we can hardly expect the Fixed-Quantity System to actually exist. The number of failures is in practice calculated at discrete periods in the case of given failure data. But in reality,

(16) Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Inc., 1964. pp. 185–190.

failures occur not discretely but continuously. Since failures are conceived in terms of continuous processes, the exact time at which the cumulated failures reach a fixed value of r can not be checked.

We can find out the exact time at which cumulative failures reach a fixed value of r by reviewing continuously. And it is meant by finely splitting the length of a period of the given data. But the method of splitting the length of a period has two important demerits: First, it is uneconomical because of inspection costs and because of too much computer time for calculation of the number of failures in each subperiod.

Second, if we plan to replace failures not by calculation based on failure data but by practical inspection, no future prediction will be made possible. In fact, if we plan to calculate the failures continuously, the dimension of the transition matrix we use becomes enormously high.

But the use of given failure data helps eventually to reduce the determination time when the cumulative failures reach a fixed value of r at discrete intervals. That is, discrete approximations to the continuous distribution of life spans are applied. Some compromises between the fixed-interval system and the fixed-quantity rules are possible and often useful, not only as practical operating systems, but also as models for the examination of certain replacement systems characteristics.

An excellent example of this is found in (s, S) system. In this thesis, the two approaches to determining optimum replacement policy will be modified as follows: (1) (s, S) Control System (2) Fixed-Interval System.

2. (s, S) Control Systems Approach

The ordering rule for an (s, S) inventory control system can be very simply stated: If fewer than s units are available order enough to bring stock up to a level S; otherwise, do not order. In application to a fixed-interval system, the rules operate as follows:

(i) Choose two inventory levels S and s, S larger than s.

(17) Magee, J.F., *op. cit.*, pp. 136—137.

(ii) At each review period, compare the available inventory h with S and s.

(iii) If h lies between S and s, place no order. If h is at or below the level s, place an order for an amount equal to S-h.

This consideration of the inventory model also holds for the replacement model. That is, with given set-intervals of examining the quantity of survivors, immediate replacement must be made if cumulated failures exceed r, and no additional supply is necessary until the next period if cumulated failures are less than r.

Determination of a minimum cost policy by (s, S) system presupposes adequate solutions of the following three points:

(i) Determination of the value of r. (What value of r contributes to finding minimum cost policy?)

(ii) Determination of the length of period. (At what interval should the examination of failures be carried out?)

(iii) What is the cost functions for finding the total replacement cost per period in (s, S) system, especially in the case of short-term policy and of long-term policy?

We will begin with a simplified method of finding minimum cost policy in such a way that we first set several possible policies and then try to find out the total cost per period for each policy and compare them one another.

What are then the possible policies and how do we find them?

A. Criteria for Finding the Fixed Value r

(1) Based on the given quantity-cost data, pick out several quantities where total cost per unit is minimized.

(2) Consider the various limitations set for overall systems, especially when top management limits the quantity of failures on hand and adjusts the quantity suitable for service level.

(3) Consider the average number of failures per period in UR. We have

a stable age distribution when replacement is made at every period. (Cf. Chapter II) Then, we can get the convergence of failures to a fixed value a_0. If we select an r such that $r \leqq a_0$, failures per period are more than r. And then we must replace them at every period as in Unit Replacement. If $r > a_0$, the length of replacement cycle will be more then 2. Accordingly, there is no need to choose several values of r such that $r < a_0$. Instead, it will be much simpler if we choose only one value of r as in UR and later compare it with r's where $r > a_0$.

B. Determination of Optimum Length of an Interval

There are some differences in meaning between the length of an age-period on the given failure data and the length of review period. The frequency of review can be determined by examining the effect of cost factors.

It is better to have shorter review periods for lower inefficiency cost. On the other hand, it is better to have longer review periods either for lower inspection cost or for replacing cost.

There is a case where the subdivision of measuring period is desirable. Let's suppose the average number of failures per period a_0 is sufficiently big so that selection of r's where $a_0 < r$ becomes difficult. In this case, an alternative to the solution is directed at determining the value of r where $a_0 > r$.

As a result, the number of failures per period will be lower than the lower control limit $s(=S\text{-}r)$. The implication from the above observation demands that the replacement be made at every set period. This method of subdividing the length of an age-period on given failure data will yield to more elaborate analysis of the problem.

It is, however, not an easy problem to find the probabilistic age distribution in terms of more subdivided periods. For this problem, the following assumptions may be made.

(1) If data is given in theoretical probability distributions such as normal distribution, binomial distribution, and Poisson distribution, the probability

value for the corresponding period can be found by calculation.

(2) Whether data is given in the form of theoretical probability distributions or obtained by other empirical methods, each initial probability of failure for the initial intervals of a given period can be so adjusted as to allow uniform distribution of the probability for the more subdivided interval of the adjusted period.

To have a clear picture of this, the following example is given. Suppose that the following is a table for failure probability distribution of an item whose maximum span of life is 4 when its age is taken in two months.

PERIOD t	PROBABILITY $P(t)$
0	—
1	0.10
2	0.20
3	0.40
4	0.30

Then, transition matrix P is made as follows:

$$P = \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 2/9 & 0 & 7/9 & 0 \\ 4/7 & 0 & 0 & 3/7 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let the stable age distribution be $A_n=(a_0, a_1, a_2, a_3)$ such that $A_nP=A_n$, and $\sum_{i=0}^{3} a_i=1,000$.

We have 5 linear equations relating 4 variables. They can be written:

$$a_0+a_1+a_2+a_3=1,000 \quad \cdots\cdots\cdots\cdots(1)$$

$$(1/10)a_0+(2/9)a_1+(4/7)a_2+a_3=a_0 \quad \cdots\cdots\cdots\cdots(2)$$

$$(9/10)a_0=a_1 \quad \cdots\cdots\cdots\cdots(3)$$

$$(7/9)a_1=a_2 \quad \cdots\cdots\cdots\cdots(4)$$

$$(3/7)a_2=a_3 \quad \cdots\cdots\cdots\cdots(5)$$

Not all of these equations are independent since equation (2) can be obtained from equations (3) and (4) and (5). The above set of linear equations can be solved and the value of a_0, the number of failures per period, is thus calculated.

$$a_0=345$$

If top management does not allow for the failures on hand to exceed 300, the only possible determination of the value of r is limited by the maximum number of 300. And hence, it is inescapable to make a conclusion such that the best policy is "to replace failures at every end of the period". But it will be too hasty to have the above solution as the optimum policy we are seeking.

Because in this system, the number of failures on hand at all times exceeds 300 even though replacement is carried on at every end of the period. That is, allowance for failures per period is not to exceed 300 but the number of failures is always over 300. This method does not seem to lead to an adequate solution.

To improve the above solution, for example, we may divide the initial age period into two and set half of the initial age period as the new age period. And then, the maximum span of life is increased to the age of 8 when its age is taken in one month. The following is a table for the failure probability distribution of the above item whose maximum span of life is 8.

INITIAL PERIOD t	DIVIDED PERIOD t'	PROBABILITY $P(t)$	PROBABILITY $P(t')$
0	0	—	—
—	1	—	0.05
1	2	0.10	0.05
—	3	—	0.10
2	4	0.20	0.10
—	5	—	0.20
3	6	0.40	0.20
—	7	—	0.15
4	8	0.30	0.15

Transition matrix P' for the above table will be made as follows:

$$P'=\begin{pmatrix} 5/100 & 95/100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5/95 & 0 & 90/95 & 0 & 0 & 0 & 0 & 0 \\ 1/9 & 0 & 0 & 8/9 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & 0 & 7/8 & 0 & 0 & 0 \\ 2/7 & 0 & 0 & 0 & 0 & 5/7 & 0 & 0 \\ 2/5 & 0 & 0 & 0 & 0 & 0 & 3/5 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let A_n' be the stable age distribution of 1×8 matrix for this problem. The value of a_o' of the first element in A_n' yields 189 which is obviously less than 300. From this, we can find some possibilities of more refined replacement models.

The further subdivision of the periods is made, the higher the exponent of transition matrix will be but calculation of the value becomes more difficult. With the help of a computer, subdivision into 1/2, 1/3, 1/4, ……, etc. of the initial periods will be easily achieved.

C. Computation of Replacement Cost

(1) Short-term policy (finite period)

Of several possible policies, the choice of optimum policy is made, after we have calculated costs according to the policies. Since we are dealing with relatively short-term policies, not much time will be spent calculating replacement cost.

〈Table 5.1〉 Computation of Replacement Cost

Period t	Failures	Cumulative failures	Purchasing & Replacing cost	Average No. of failures in a period	Inefficiency cost	Maintenance cost	Total Repl. cost
1	x_1	x_1	0	$F_1=x_1/2$	$c_4(F)F_1$	$c_5(S)S_1$	
2	x_2	x_1+x_2	0	$F_2=x_1+x_2/2$	$c_4(F)F_2$	$c_5(S)S_2$	
3	x_3	$x_1+x_2+x_3$	$c_1(x)\sum_{i=1}^{3}x_i$	$F_3=x_1+x_2+x_3/2$	c_4F_3	c_5S_3	
4	x_4	x_4	0	$F_4=x_4/2$	c_4F_4	c_5S_4	
5	x_5	x_4+x_5	$c_1\sum_{i=4}^{5}x_i$	$F_5=x_4+x_5/2$	c_4F_5	c_5S_5	

The above chart illustrates how to compute the total replacement cost for a given policy that is to replace at the 3rd period and at the 5th period. (Cf. Table 5.1)

Notes: (i) c_1, $c_1(x)$=Purchasing cost and replacing cost per unit.

(ii) c_4, $c_4\ (F)$=Inefficiency cost per unit per period.

(iii) c_5, $c_5\ (S)$=Maintenance cost per unit per period.

(iv) Average number of failures in a period, F_i, was calculated under the assumption that items fail in direct proportion to time in a period. Hence, F_i is calculated by adding the initial number of failures of the t-th period with a half of the number of items failed in the t-th period.

(v) $S_i = N - F_i$; Average number of survivors in a period.

N= Total number of components in a system.

To determine the minimum cost policy, each replacement cost for different policies should be properly compared and for this purpose, it is desired to determine the replacement cost per period for each policy. That is, total replacement cost of the above table will be divided by t, the number of periods. Generally, if we replace the failures at t-th period and at s-th period during the first s periods ($t<s$) then, total replacement cost during the first s periods is equal to the sum (a)+(b)+(c)+(a)′+(b)′+(c)′ given below.

(a) Purchasing cost and replacing cost during the first t periods:

$c(x)(x_1+x_2+\cdots\cdots+x_t)$

(b) Maintenance cost during the first t periods:

$$(N-x_1/2)c(S)+[N-(x_1+x_2/2)]c(S)+\cdots+\left[N-\left(\sum_{i=1}^{t-1}x_i+x_t/2\right)\right]c(S)$$

(c) Inefficiency cost during the first t periods:

$$(x_1/2)c(F)+(x_1+x_2/2)c(F)+\cdots+\left(\sum_{i=1}^{t-1}x_i+x_t/2\right)c(F)$$

(a)′ Purchasing cost and replacing cost during a time interval from the $(t+1)$-th period to the s-th period:

$c(x)(x_{t+1}+x_{t+2}+\cdots\cdots+x_s)$

(b)′ Maintenance cost during a time interval from the $(t+1)$-th period to the s-th period:

$$(N-x_{t+1}/2)c(S)+[N-(x_{t+1}+x_{t+2}/2)]c(S)+\cdots+\left[N-\left(\sum_{i=t+1}^{s-1}x_i+x_s/2\right)\right]c(S)$$

(c)′ Inefficiency cost during a time interval from the $(t+1)$-th period to the $s-$th period:

$$(x_{t+1}/2)c(F)+(x_{t+1}+x_{t+2}/2)c(F)+\cdots\cdots+\left(\sum_{i=t+1}^{s-1}x_i+x_s/2\right)c(F)$$

What we are interested in is to choose a policy which minimizes $\frac{(a)+(b)+(c)+(a)'+(b)'+(c)'}{s}$ for each policy.

(2) Determination of long-term policy

In the foregoing, we have discussed an optimum policy for short-term replacement which can be best determined by comparing the (s, S) replacement policies with each other. In this section, we are dealing with a long-term replacement policy which necessitates an entirely different approach. Unlike the short-term policy, calculating the replacement period for a (s, S) policy is a tedious job and it requires a great amount of time because repeated trials are necessary until the replacement cost approaches a limited value. To solve this difficulty, the simulation process is introduced here: a technique of calculating the mean number of periods per replacement interval (cycle).

The age distribution changes in accordance with certain policies established by the decision maker. Assume the values of s and S, and that we apply the (s, S) policy to a given replacement problem. We can get the number of periods per replacement cycle for the given policy as in Table 5. 2. Refer to Table 5. 2 and Table 5. 3 in order to get a more complete understanding of the simulation scheme.

The data in Table 5. 2 is drawn from the record of a firm which is adopting(s, S) policy. The age distribution for the next period is obtained by multiplying the initial age distribution by transition matrix P_0 where the element of the first column, i.e., P_{i0}, where $i=0, 1, 2, \cdots\cdots, \; n-1$ is taken as zero. This means failed items are retained without immediate replacement. As the period passes, the number of survivors will be less than the already set value of s. Then immediate replacement is made and the size of replacement is filled in the column of the first component of the age distribution. After replacing the accumulated failures, we again operate matrix P_0 on the age distributions.

〈Table 5.2〉 The Number of Periods per Cycle under a (s, S) Policy

Cycle	Period	Initial survivors	Failures	Age distribution						Final stock
				a_0	a_1	a_2	……	a_{n-2}	a_{n-1}	

Table 5.3 is an abridged cycle-by-cycle summary of the simulated replacement process performed on a digital computer. Column (A) gives the cycle number. Column (B) gives the number of periods in the cycle, designated P_x, since it is a random variable. Column (C) gives the running average, P_m, of the individual values in column(B).

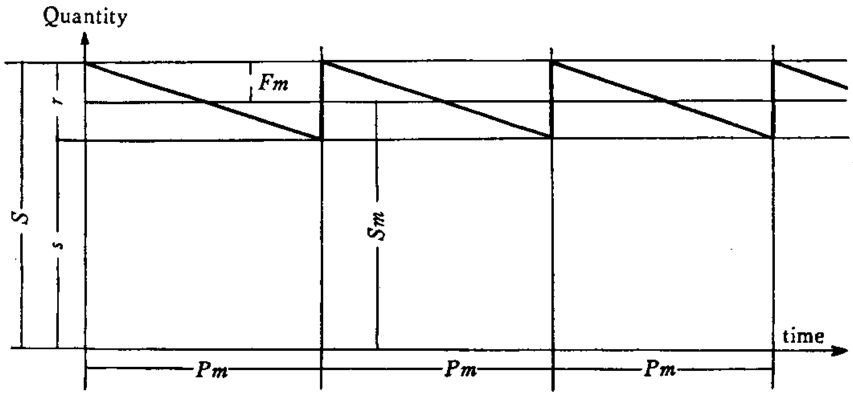
〈Table 5.3〉 Limiting Values of P_x, S_x, and F_x

Cycle (A)	Number of periods P_x (B)	Average no. of periods per cycle P_m (C)	Total no. of survivors S_x (D)	Average no. of survivors S_m (E)	Total no. of failures F_x (F)	Average no. of failures (G) F_m

The relative stability of the mean values may be noted by comparing the terminal cycles with the initial cycles in Table 5.3. Continuing the simulation for more times would contribute further to their stability. Table 5.3

is a simplified version of the results obtained from Table 5.2. From the results obtained from Table 5.2, we can make a simplified graph where the value P_m of the average number of periods per a replacement cycle has already been determined and P_m corresponds to the initial fixed value of r in (s,S) policy. If we adopt the notation used in (s,S) inventory system, we have $s=S-r$ where S is the total number of items in a system.

⟨Figure 5.1⟩



In the above graph, P_m is the average number of periods per cycle (replacement interval) and is obtained by hundreds of trials of (s,S) system-simulation. S_m is the average number of survivors and F_m is the average number of failures on hand.

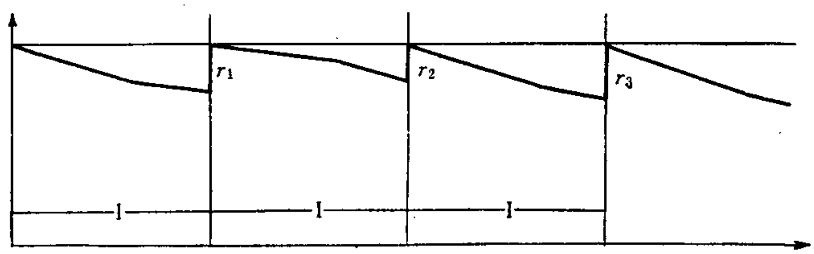
Total replacement cost per period is expressed as follows, C/P_m and $C=c(x)r+c(F)F_m+c(S)S_m$.

So far we have applied simulation technique to determining optimum replacement policy. The simulation model we have achieved is believed effective for the long term replacement policy.

3. Fixed-Interval Systems Approach

Periodic replacing systems are popular and are frequently used, particularly where it is convenient to examine items on a fixed time cycle. Look at the survivors on a fixed frequency, and replace the failures that are cumulated since the last replacement, as shown in Figure 5.2.

⟨Figure 5.2⟩ Balances Under a Fixed-Interval System



As shown in the above, each corresponding value of the number of replacing items ($r_1, r_2, r_3, \cdots\cdots r_t, \cdots$) is determined by a certain regular interval I. As has been shown in the case of (s,S) control system, of several workable assumed values of I, one value of $I=I^*$ is empirically set such that it can generate the minimum replacing cost.

To in practice determine the minimum cost policy by fixed interval system presupposes the solution of the following two problems. First, what is the exact value of I to generate the minimum cost policy (the choice of the value I)? Second, what is the best model for calculating the replacing cost by this system (calculation of the replacement cost)?

A. Choice of the Interval I

As the name indicates, the fixed-interval system is a system where the replacement cost for each interval is determined by the length of replacing interval. Accordingly, the replacing intervals should be chosen such that they can generate the minimum replacêment cost and so that the calculation of failures at each interval is possible. The factors affecting the choice of the length of interval I will generally be viewed in two ways: the effect of age-length based on the given data and the effect of cost factors.

(1) The effect of age-length on choosing the interval I

In determining the interval I, it is convenient to set it either as integer-multiples of the age-length or as equal-division of the age-length. This provides an easier way for calculating the number of failures per interval with transition matrices. That is

$$I=\begin{cases} uL, & \text{if } I\geqq L \\ L/v, & \text{if } I\leqq L, \end{cases}$$ where u,v are positive integers and L is the given age-length.

$I=uL$ means that an interval consists of u periods. Then the number of failures per interval can be calculated by accumulating the number of failures per period within a given interval.

In the case of $I=L/v$, the number of failures can be determined by dividing equally a given period into subperiods as shown in the foregoing section; 2. B. Then it becomes necessary to remodel the matrix.

If we try to apply the equal-division of age-length to replacement, however, the given data are usually not sufficient enough for precise and clear-cut analysis; consequently, the initial age-length is assumed to have been set longer than what is relevant for our analysis.

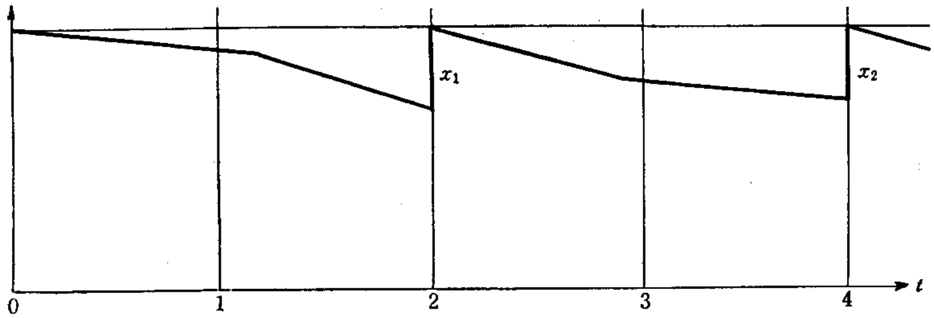
(2) The effect of cost factors

It is better to have shorter replacement intervals for lower inefficiency cost. On the other hand, it is better to have longer replacement intervals either for lower inspection cost or replacing cost.

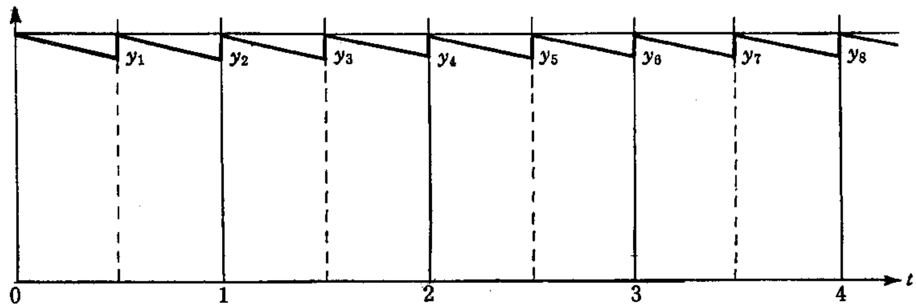
Let's study the following figures to compare the two approaches. The below shows a four-period division of age-length from $t=0$ to $t=4$. The top figure is for the case of $I=2L$ while the lower is for the case of $I=L/2$.

In the case of $I=2L$, x_1 and x_2 amounts of replacement and made at $t=2$ and $t=4$, respectively. In the case of $I=L/2$, 8 times replacement $y_1, y_2, \cdots\cdots, y_8$ are made from $t=0$ to $t=4$. As clearly shown in the figure below, $x_i > y_j$ and the average retentions of failures in the case of $I=2L$ is much larger than those in the case of $I=L/2$. It follows that ineffіcency cost in

〈Figure 5.3〉



〈Figure 5.4〉



the former is higher than that in the latter.

On the other hand, the number of replacements (or inspections) is larger by the ratio 2 : 8 in the latter case. That is, the overall inspection cost occurring just before the actual replacement and the replacement cost are 4 times higher in the latter case than in the former.

It is tiresome and uneconomical to replace failures too frequently and continuously in order to have lower inefficiency cost. On the contrary, if the interval is set long enough to have lower inspection and replacing cost, the value n in the case of $I=nL$ becomes larger. The problem in this case is that all or most items in a system may stop operating, which is a very undesirable consequence. Then it becomes a matter of balance in choosing the value I so that after careful consideration of various cost factors involved, several relevant values of I should be chosen from the many possible values.

B. Calculation of Replacement Cost

To determine the minimum cost policy, each replacement cost for difference policies should be properly compared and for this purpose, replacement cost for the initial age-period is set as a basis for the comparison. As discussed in Chapter III, Cost Factors, the replacement cost is divided into two parts: the cost occurring at each replacement and the cost occurring due to the retention of failures for a period.

The costs occurring at every replacement such as purchasing cost, replacing cost, and inspection cost vary according to the number of replacements and the number of replaced items.

The costs occurring due to the prolonged retention of failures are inefficiency cost and the cost holding "live" items. These costs vary according to the size of the calculation period.

That is, in the case of $I=uL$, the average number of failures on hand for an interval is calculated as

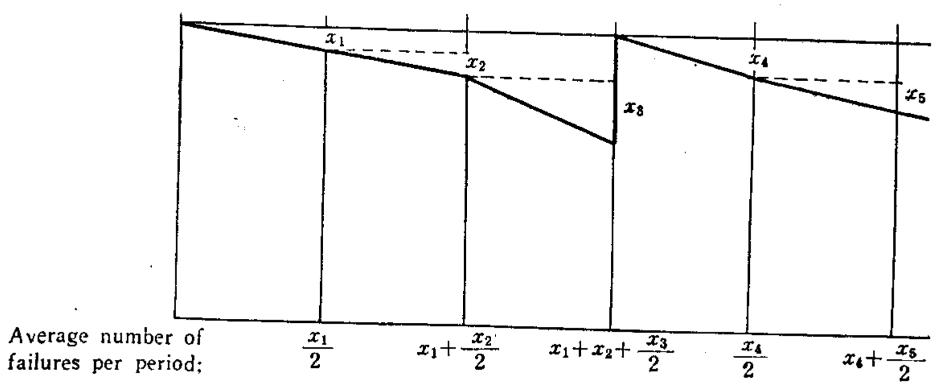
$$\frac{1}{u}\left[\left(\frac{x_1}{2}\right)+\left(x_1+\frac{x_2}{2}\right)+\left(x_1+x_2+\frac{x_3}{2}\right)+\cdots\cdots+\left(\sum_{i=1}^{u-1}x_i+\frac{x_u}{2}\right)\right]$$

where x_1, x_2, x_3, ……, x_u are the number of failures of the 1st, 2nd, ……, the u-th period within an interval.

If the data on inefficiency cost are made for age-length, the total inefficiency cost can be calculated by adding each inefficiency cost for each period. Refer to the following figure in the case of $I=3L$.

If the interval is taken by dividing equally the age period i. e., in the case of $I=L/v$, it is not as easy to calculate the number of failures per interval(also refer to the foregoing section (s, S) control system). In other words, the probability distribution for failures for each period is first changed into the probability distribution for failures in more divided intervals

〈Figure 5.5〉



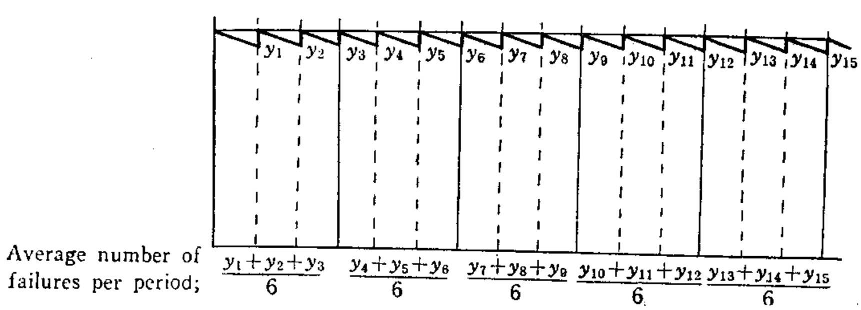
and next, the transition matrix is remodeled. Then this remodeled matrix is used for calculating the number of failures for each interval.

If the number of failures is set as $y_1, y_2, \cdots\cdots, ,y_v$, for the 1st, the 2nd, ……, the v-th interval, the average number of failures on hand for the age period of L is calculated as

$$(y_1+y_2+\cdots\cdots+y_v)/2v.$$

If the data on inefficiency cost are made for one period, the total inefficiency cost can be calculated by adding each inefficiency cost for each period. The following figure is for the case of $I=L/3$.

〈Figure 5.6〉



(1) Notations

$c_1(x)$=purchasing cost per unit

$c_2(x)$=replacing cost per unit

c_3=inspection cost for one inspection

$c_4(F)$=inefficiency cost per unit per period

$c_5(S)$=maintenance cost per unit per period

$c_4(F)'$=inefficiency cost per unit for one replacement interval

$c_5(S)'$=maintenance cost per unit for one replacement interval

r_i=the number of replacement at the i-th interval (In the case of $I=uL$, r is the number of cumulative failures)

F=average number of failures per period

$S=N-F$, where N is the total number of components in a system.

F'=average number of failures over one replacement interval

$S'=N-F'$

(2) The cost related to replacement (pattern—1 replacement cost)

Let the number of replaced items for each replacement be $r_1, r_2, \cdots\cdots, r_t$, when the replacement is made $1, 2, \cdots\cdots, t$ times for z periods in which the total costs are assumed.

(a) Purchasing Cost$=\sum_{i=1}^{t} c_1(x) r_i$ ……………………………1*

(b) Replacing Cost$=\sum_{i=1}^{t} c_2(x) r_i$ ……………………………2*

(c) Inspection Cost$=t\ c_3$ ……………………………………3*

(3) The costs incurred from retention of failed items (pattern-2)

(a) Inefficiency Cost$=\sum_{i=1}^{t} c_4(F)' F_i'$ ……………4*

$$F_i' = \begin{cases} \dfrac{1}{u}\left(\left(\dfrac{x_{(i-1)u+1}}{2}\right) + \left(x_{(i-1)u+1} + \dfrac{x_{(i-1)u+2}}{2}\right) + \cdots + \left(\sum_{j=(i-1)u+1}^{iu-1} x_j + \dfrac{x_{iu}}{2}\right)\right) \\ \qquad \text{if } I=uL \text{ and } u \text{ is a positive integer.} \\ x'=y, \text{ if } I=L/v \text{ and } v \text{ is a positive integer.} \end{cases}$$

On the other hand, if we have not F_i' but F_i, then the total inefficiency cost for z periods is expressed as follows:

(i) $z=ut$ (in the case of $I \geqq L$)

total inefficiency cost for z periods

$$=\left[c_4(F)\left(\frac{x_1}{2}\right)+c_4(F)\left(x_1+\frac{x_2}{2}\right)+\cdots+c_4(F)\left(\Sigma x_i+\frac{x_u}{2}\right)\right]$$

$$+\left[c_4(F)\left(\frac{x_{u+1}}{2}\right)+\cdots+c_4(F)\left(\sum_{i=u+1}^{2u-1}x_i+\frac{x_{2u}}{2}\right)\right]+\cdots+\left[c_4(F)\frac{x_{(t-1)u+1}}{2}\right.$$

$$\left.+\cdots+c_4(F)\left(\sum_{i=(t-1)u+1}^{tu-1}x_i+\frac{x_{tu}}{2}\right)\right]\cdots 4^*—1$$

$c_4(F)$ in the above equation is not a constant number but a function of a variable F.

(ii) $z=t/v$ (in the case of $I\leqq L$)

total inefficiency cost for z periods

$$=\left(\frac{y_1+y_2+\cdots+y_v}{2v}\right)c_4(F)+\left(\frac{y_{v+1}+y_{v+2}+\cdots+y_{2v}}{2v}\right)c_4(F)+\cdots\cdots$$

$$+\left(\frac{y_{(z-1)v+1}+\cdots+y_{zv}}{2v}\right)c_4(F)\cdots\cdots\cdots\cdots\cdots\cdots 4^*—2$$

(b) Maintenance Cost$=\sum_{i=1}^{t}c_5(S)'S_i'$, $\cdots\cdots\cdots\cdots\cdots\cdots 5^*$

where $S_i'=N-F_i'$.

On the other hand, if we have not S_i' but S_i, then the total maintenance cost for z periods is expressed as follows:

(i) $z=ut$ (in the case of $I\geqq L$)

total maintenance cost for z periods

$$=\left[\left(N-\frac{x_1}{2}\right)c_5(S)+\left\{N-\left(x_1+\frac{x_2}{2}\right)\right\}c_5(S)+\cdots\right.$$

$$\left.+\left\{N-\left(\sum_{i=1}^{u-1}x_i+\frac{x_u}{2}\right)\right\}c_5(S)\right]+\cdots+\left[\left(N-\frac{x_{(t-1)u+1}}{2}\right)c_5(S)+\cdots\right.$$

$$\left.+\left\{N-\left(\sum_{i=(t-1)u+1}^{tu-1}x_i+\frac{x_{tu}}{2}\right)\right\}c_5(S)\right]\cdots\cdots\cdots\cdots\cdots 5^*—1$$

(ii) $z=t/v$ (in the case of $I\leqq L$)

total maintenance cost for z periods

$$=\left(N-\frac{y_1+y_2+\cdots+y_v}{2v}\right)c_5(S)+\cdots+\left(N-\frac{y_{(z-1)v+1}+\cdots+y_{zv}}{2v}\right)c_5(S)\cdots 5^*—2$$

(4) The total replacement cost for z periods: C

$$C=1^*+2^*+3^*+4^*+5^* \text{ or}$$

$$C=1^*+2^*+3^*+4^*-1(\text{or } 4^*-2)+5^*-1(\text{or } 5^*-2).$$

(5) The total replacement cost per period

The total replacement cost per period is obtained by calculating C/z.

Ⅴ. APPLICATIONS

1. Equipment Replacement Problem

An Illustrative Example

Let's suppose a company is equipped with 1,000 drills of the same quality and the bits of drills deteriorating below a certain quality should be replaced by new ones.

In most cases, the maximum age-length of the bits approximates twelve months and the failure data for the bits conforms to a normal distribution. With a given data on the given cost, we can determine the optimum replacement policy for a fixed-interval system so that we can calculate the minimum replacement cost for the following one year period. In this case, if the interval is assumed to be set as three months, two months, and one month, then which one is the most appropriate solution for us?

〈Table 6.1〉 Quantity-Total Inefficiency Cost Relations

Average number of failures per month	Total inefficiency cost per month
1— 50	$ 1,000
51—100	$ 1,300
101—150	$ 1,800
151—200	$ 2,500
201—300	$ 4,000
301—400	$ 7,000
401—500	$10,000
more than 501	$20,000

〈Table 6.2〉 **Purchasing Cost and Replacing Cost (Quantity-Unit Cost Relations)**

The size of replacement	Purchasing cost	Replacing cost
1— 50	\$ 20	\$ 20
51—100	\$ 16	\$ 14
101—150	\$ 14	\$ 11
151—200	\$ 12	\$ 8
201—300	\$ 10	\$ 5
301—400	\$ 9	\$ 5
401—500	\$ 8	\$ 4
more than 501	\$ 7	\$ 4

Solution

A. One-month Interval

Transition matrix P for a Markov Chain of 12 states in the case of normal distribution is calculated as below.

$$P=\begin{pmatrix}
\frac{6}{1000} & \frac{994}{1000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{17}{994} & 0 & \frac{977}{994} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{44}{977} & 0 & 0 & \frac{933}{977} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{92}{933} & 0 & 0 & 0 & \frac{841}{933} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{150}{841} & 0 & 0 & 0 & 0 & \frac{691}{841} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{191}{691} & 0 & 0 & 0 & 0 & 0 & \frac{500}{691} & 0 & 0 & 0 & 0 & 0 \\
\frac{191}{500} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{309}{500} & 0 & 0 & 0 & 0 \\
\frac{150}{309} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{159}{309} & 0 & 0 & 0 \\
\frac{92}{159} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{67}{159} & 0 & 0 \\
\frac{44}{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23}{67} & 0 \\
\frac{17}{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{23} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

〈Table 6.3〉 **Total Replacement Cost in the Case of One-month Interval**

Period	Failures	Age Distribution												Size of Replacement	Purchasing & Replacing Cost	Average No. of Failures	Ineffi. Cost	Total Cost
		a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}					
1	6	6	994	0	0	0	0	0	0	0	0	0	0	6	$ 240	3	$ 1,000	
2	17	17	6	977	0	0	0	0	0	0	0	0	0	17	680	9	1,000	
3	44	44	17	6	933	0	0	0	0	0	0	0	0	44	1,760	22	1,000	
4	92	92	44	17	6	841	0	0	0	0	0	0	0	92	2,760	46	1,000	
5	154	154	91	43	16	5	691	0	0	0	0	0	0	154	3,080	77	1,300	
6	197	197	153	89	41	15	5	500	0	0	0	0	0	197	3,940	99	1,300	
7	207	207	196	150	85	37	12	4	309	0	0	0	0	207	3,105	104	1,800	
8	181	181	206	193	143	77	30	9	2	159	0	0	0	181	3,620	91	1,300	
9	145	145	180	203	184	129	63	22	6	1	67	0	0	145	3,625	73	1,300	
10	126	126	144	177	194	166	106	47	14	3	0	23	0	126	3,150	63	1,300	
11	134	134	125	141	169	175	136	77	29	7	1	0	6	134	3,350	67	1,300	
12	148	148	133	123	135	152	144	99	48	15	3	0	0	148	3,700	74	1,300	
TOTAL															$33,010		$14,900	$47,910

Let's assume the initial age distribution is $A_0=(1000,0,0,\cdots\cdots, 0)$, then the number of replacements per interval and their cost are calculated as in the above chart-Table 6.3 where the age distribution of a given period is determined by multiplying P by the age distribution of the preceding period.

B. Two-month Interval

The following transition matrix is a P_0 matrix. The age distribution of the final period of a certain year can be given by multiplying P_0 by the final age distribution of the preceding year. The amount of shortage a_0 is calculated such that the total number of replacement reaches 1,000. (Cf. Table 6.4)

$$P_0=\begin{pmatrix}
0 & \frac{994}{1000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{977}{994} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{933}{977} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{841}{933} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{691}{841} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{500}{691} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{309}{500} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{159}{309} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{67}{159} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23}{67} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

〈Table 6.4〉 **Total Replacement Cost in the Case of Two-month Interval**

Period	Failures	Age Distribution												Size of Replacement	Purchasing & Replacing Cost	Average Number of Failures	Ineffi. Cost	Total Cost
		a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}					
1	6	0	994	0	0	0	0	0	0	0	0	0	0	0	$ 0	3	$ 1,000	
2	17	23	0	977	0	0	0	0	0	0	0	0	0	23	920	15	1,000	
3	44	0	23	0	933	0	0	0	0	0	0	0	0	0	0	22	1,000	
4	92	136	0	23	0	841	0	0	0	0	0	0	0	136	3,400	90	1,300	
5	152	0	135	0	22	0	691	0	0	0	0	0	0	0	0	76	1,300	
6	195	347	0	133	0	20	0	500	0	0	0	0	0	347	4,858	250	4,000	
7	203	0	345	0	127	0	16	0	309	0	0	0	0	0	0	102	1,800	
8	172	375	0	339	0	115	0	12	0	159	0	0	0	375	5,250	289	4,000	
9	134	0	375	0	324	0	95	0	7	0	67	0	0	0	0	67	1,300	
10	111	245	0	367	0	292	0	69	0	4	0	23	0	245	3,675	190	2,500	
11	114	0	244	0	351	0	240	0	43	0	2	0	6	0	0	57	1,300	
12	133	247	0	240	0	316	0	174	0	22	0	1	0	247	3,705	181	2,500	
TOTAL															$ 21,808		$ 23,000	$ 44,808

〈Table 6.5〉 **Total Replacement Cost in the Case of Three-month Interval**

Period	Failures	Age Distribution												Size of Replacement	Purchasing & Replacing Cost	Average Number of Failures	Ineffi. Cost	Total Cost
		a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}					
1	6	0	994	0	0	0	0	0	0	0	0	0	0	0	$ 0	3	$ 1,000	
2	17	0	0	977	0	0	0	0	0	0	0	0	0	0	0	15	1,000	
3	44	67	0	0	933	0	0	0	0	0	0	0	0	67	2,010	45	1,000	
4	92	0	67	0	0	841	0	0	0	0	0	0	0	0	0	46	1,000	
5	152	0	0	65	0	0	691	0	0	0	0	0	0	0	0	168	2,500	
6	193	437	0	0	63	0	0	500	0	0	0	0	0	437	5,244	341	7,000	
7	201	0	434	0	0	56	0	0	309	0	0	0	0	0	0	101	1,800	
8	167	0	0	427	0	0	46	0	0	159	0	0	0	0	0	285	4,000	
9	123	491	0	0	408	0	0	34	0	0	67	0	0	491	5,892	430	10,000	
10	100	0	488	0	0	368	0	0	21	0	0	23	0	0	0	50	1,000	
11	101	0	0	480	0	0	302	0	0	11	0	0	6	0	0	151	2,500	
12	118	319	0	0	458	0	0	218	0	0	5	0	0	319	4,466	260	4,000	
TOTAL															$17,612		$36,800	$54,412

C. Three-month Interval

As in the case of the two-month interval, the age distribution of a given period can be given by calculating P_0 matrix. That is, the total number of replacement per three-month period should reach 1,600. (Cf. Table 6.5)

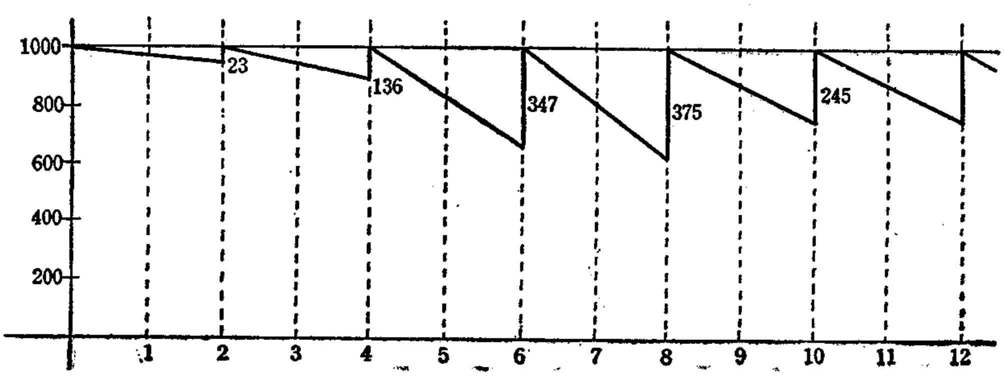
The following table results from the comparison of the three different intervals. From this, we can easily understand that the best policy for the fixed-interval system is the two-month interval system. The longer the length of interval is set, the inefficiency cost increases though the purchasing cost and replacing cost decrease.

〈Table 6.6〉 **Costs for the three policies**

Length of Interval	Number of replacing per year	Size of Repl. per year	Purchasing & Replacing cost	Average no. of failures per month	Ineffi. cost	Total cost
one-month	12	1,451	33,010	61	14,900	47,910
two-month	6	1,126	21,808	112	23,000	44,808 ←
three-month	4	1,314	17,612	158	36,800	54,412

The second case of our optimal policy problem is depicted as in the following graph.

〈Figure 6.1〉



2. **Staffing Problem**

Illustrative Example

Suppose an airline company with 500 hostesses has a personnel policy which requires that for every applicant the qualification for the initial application is limited to only single persons of under 23 years of age. The maximum age limitation for the job is 31 years of age. Under this condition, the transfer rates of hostesses approximates 1/5 and the necessary costs spent for recruiting those shows in the following figure.

In case that absentee rates become high, depending on its degree, a number of problems result from bad service due to the charge of route, additional payments for the additional work load, the reduction of the usual flight schedules, and the overwork or exhaustion of the remaining hostesses.

Suppose we have 500 new hostesses under 23 years of age, let's try to determine the optimum replacement policy (or personnel recruitment policy) for the following 5 years.

〈Table 6.7〉 Cost Data

Number of vacancies	Cost for staffing (Recruitment, selection & training)	Total Inefficiency cost per year
1—30	$ 2,100	$ 100,000
31—60	$ 1,680	$ 200,000
61—100	$ 1,370	$ 300,000
101—150	$ 1,100	$ 600,000
151—200	$ 950	$ 1,500,000
201—250	$ 850	$ 5,000,000
more than 250	$ 650	$ 10,000,000

* Salary for one year= $6,000

〈Table〉 6.8 Failure Data

Period	Age	Survivors	Vacancies	Conditional Probability $Pc(t)$
0	—	100	—	—
1	23	80	20	1/5
2	24	64	16	1/5
3	25	51	13	1/5
4	26	41	10	1/5
5	27	33	8	1/5
6	28	26	7	1/5
7	29	21	5	1/5
8	30	17	4	1/5
9	31	14	3	1/5
10	32	0	14	1/5

Of the three workable policies, that is, (400, 500) system, (350, 500) system, and (300, 500) system, which is the best policy?

Solution

Transition matrix P and age distribution A_i are calculated as follows:

$$P=\begin{pmatrix} 1/5 & 4/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 4/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 4/5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 4/5 & 0 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 & 4/5 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 & 0 & 4/5 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 4/5 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/5 & 0 \\ 1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/5 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} A_0 &= (500,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\ A_1 &= (100,\ 400,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\ A_2 &= (100,\ 80,\ 320,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\ A_3 &= (100,\ 80,\ 64,\ 256,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\ A_4 &= (100,\ 80,\ 64,\ 51,\ 205,\ 0,\ 0,\ 0,\ 0,\ 0) \\ A_5 &= (100,\ 80,\ 64,\ 51,\ 41,\ 164,\ 0,\ 0,\ 0,\ 0) \\ &\ \vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}$$

If the replacement of personnel is made at the end of every year, the total needed number of personnel is 100. With a given transfer rate of

〈Table 6.9〉 **Total replacement cost for 5 years under (400, 500) policy**

Year t	Failures x_t	Cumulative x	Policy	Cost for staffing	Average failures	Inefficiency cost	Salary
1	100	100	Replace	137,000	50	200,000	2,700,000
2	100	100	Replace	137,000	50	200,000	2,700,000
3	100	100	Replace	137,000	50	200,000	2,700,000
4	100	100	Replace	137,000	50	200,000	2,700,000
5	100	100	Replace	137,000	50	200,000	2,700,000
			Total	685,000		1,000,000	13,500,000

Total replacement cost for 5 years = $15,185,000

1/5, we can find that the group replacement policy is not relevant. The reason is that group replacement policy is entirely against our existing personnel policy in a democratic society because the purpose of group replacement lies in preventing the massive transfer of personnel at a single time.

⟨Table 6.10⟩ Total replacement cost for 5 years under (350, 500) policy

Year t	Failures x_t	Cumulative x	Policy	Cost for staffing	Average failures	Inefficiency cost	Salary
1	100	100		0	50	200,000	2,700,000
2	80	180	Replace	171,000	140	600,000	2,160,999
3	100	100		0	50	200,000	2,700,000
4	80	180	Replace	171,000	140	600,000	2,160,000
5	100	100		0	50	200,000	2,700,000
			Total	342,000		1,800,000	12,420,000

Total replacement cost for 5 years= **$14,562,000.**

⟨Table 6.11⟩ Total replacement cost for 5 years under (300, 500) policy

Year t	Failures x_t	Cumulative x	Policy	Cost for staffing	Average failures	Inefficiency cost	Salary
1	100	100		0	50	200,000	2,700,000
2	80	180		0	140	600,000	2,160,000
3	64	244	Replace	207,400	212	5,000,000	1,728,000
4	100	100		0	50	200,000	2,700,000
5	80	180		0	140	600,000	2,160,000
			Total	207,400		6,600,000	11,448,000

Total replacement cost for 5 years= **$18,255,400.**

The following table is the results of comparing the three different (s, S) policies.

⟨Table 6.12⟩ Costs for the three policies

Policy	Number of Replacing	Cost for Staffing	Inefficiency Cost	Total Salary	Total Cost
(400, 500)	5	$ 685,000	$ 1,000,000	$ 13,500,000	$ 15,185,000
(350, 500)	2	342,000	1,800,000	12,420,000	14,562,000←
(300, 500)	1	207,400	6,600,000	11,448,000	18,255,400

From the above table, we can easily find out that the best policy of the given(s,S) policies is (350, 500) control system.

Ⅶ. CONCLUSIONS

When we try to find out optimum replacement policy under(s,S) control system, we are faced with problems which must be solved first. Initially, we should set several values of the length of interval between reviews and the number and values of replacement point s's. Next, we should try to answer the question whether the length of replacing cycle P_x converges or not when we calculate the replacement cost for long term policy.

In this thesis, the writer believes the first problem has been dealt with vigorously and in some detail. On the other hand, the second problem has been touched rather loosely without proof for the question whether P_x converges with P_m or not. It is believed that the proof would be achieved if the given data was calculated with the use of a computer; hence, the writer admits that further study on this problem remains to be explored.

Part replacement model can be dealt with only when we consider the inefficiency resulting from the retention of failures; hence, any comparison between part replacement and group replacement which is based on different assumptions is considered irrelevant or meaningless. If the data on inefficiency cost, quantity discount schedule, and inspection cost were available, a better replacing policy can be achieved by the use of more applicable model, part replacement model. Where such data is not available, the replacement policy should rely upon simpler models, unit replacement model and group replacement model.

In using part replacement model, we can compare and evaluate the two systems for optimum policy, (s,S) control system and fixed-interval system. In conclusion, the optimum replacement policy under part replacement model is achieved by choosing the one which renders the minimum replacement cost between the cost obtained from (s,S) control system and the cost for fixed-interval system.