# Computational Comparison of Two Lagrangian Relaxation for the K-median Problem

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## 1. Introduction

The k-median problem has been widely studied both from the theoretical point of view and for its application. An interesting theoretical development was the successful probabilistic analysis of several heuristics (e.g. Fisher and Hochbaum[7], Papadimitriou[12] ), relaxations (e.g. Ahn et at [2]), and polyhedral study( Ahn[2], Guignard[9]) for this problem.

On the other hand, the literature on the k-median problem abounds in exact algorithms. Most (e.g. Cornuejols et al[4]) are based on the solution of relaxation. The computational experience reported in the literature seems to indicate that this relaxation yields impressively tight bounds compared to what can usually be expected in integer programming. In this paper we perform extensive computational analysis of two Largarangian relaxation for the k-median problem.

Consider a set $Y=\{Y_1, Y_2, \ldots, Y_n\}$ of n points, a positive integer $k \leq n$ and let $c_{ij} \geq 0$ be the distance between $Y_i$ and $Y_j$ for each $1 \leq i \leq n$ and $1 \leq j \leq n$. The k-median problem consists of finding a set $S \in Y$, $|S| = k$, that minimizes $\sum_{i=1}^{n} \mathrm{Min}_{j \in S}\, c_{ij}$ (Here $|S|$ denotes the cardinality of the set S.)

The k-median problem has the following integer programming formulation.

$$Z_{IP} = \mathrm{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad (1)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } I=1,2,\ldots,n \qquad (2)$$

$$\sum_{i=1}^{n} y_j = k \qquad (3)$$

$$-x_{ij} + y_j \geq 0 \qquad \text{for } i,j=1,2,\ldots,n \qquad (4)$$

$$x_{ij} \geq 0 \qquad \text{for } i,j=1,2,\ldots,n \qquad (5)$$

$$y_j \in \{0,1\} \qquad \text{for } j=1,2,\ldots,n \qquad (6)$$

In this formulation $y_j = 1$ if $j \in S$, 0 otherwise and, for $1 \leq i \leq n$, we can set $x_{ij} = 1$ for an index that achieves $\text{Min}_{j \in S}\, c_{ij}$. Most successful exact algorithms reported in the literature are based on Lagrangian relaxation obtained by dualizing either constraint (2) or constraint set (3). In this paper we perform and compare computational experience of two Lagrangian relaxation on 3,900 randomly generated test problems.

## 2. Lagrangian Relaxation

By dualizing assignment constraint set (2) with Lagrangian multipliers $u=\{u_1\ u_2,\ldots u_n\}$, we obtain following Lagrangian relaxation.

(LR1)

$$Z_D(u) = \text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} + \sum_{i=1}^{n} u_i \left( \sum_{j=1}^{n} x_{ij} - 1 \right)$$

$$= \text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} + u_i)\, x_{ij} - \sum_{i=1}^{n} u_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} y_j = k \qquad (3)$$

$$-x_{ij} + y_j \le 0 \qquad \text{for } i,j=1,2,\ldots,n \qquad (4)$$

$$x_{ij} \ge 0 \qquad \text{for } i,j=1,2,\ldots,n \qquad (5)$$

$$y_j \in \{0,1\} \qquad \text{for } j=1,2,\ldots,n \qquad (6)$$

For fixed u's, above problem has the 0-1 VUB(variable upper bound) structure. In order to solve (LR1), observe first that the objective function of the (LR1) and the VUB constraints (4). These two imply that, for each i,

$$x_{ij} = \begin{cases} y_j, & \text{if } c_{ij} + u_i \le 0 \\ 0, & \textit{otherwise} \end{cases}$$

Hence with defining $\overline{c_j} = \sum_{i=1}^{n} \text{Min}(0, c_{ij} + u_i)$, (LR1) is equivalent to

$$\text{Min} \sum_{j=1}^{n} \overline{c_j}\, y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} y_j = k$$

$$y_j \in \{0, 1\} \qquad \text{for } j=1,2,\ldots,n$$

which is a trivial problem. That is, optimal $y_j$'s are

$$y_j = \begin{cases} 1 & \text{for } \textit{the first k smallest } \overline{c_j} \\ 0 & \textit{otherwise} \end{cases}$$

Since the objective is to mininize, clealy the best choice for u would be an optimal solution to the dual problem:

(D1)

$$Z_{D1} = \underset{u}{\text{Max}}\ Z_D(u)$$

By dualizing k-median constraint(3) with a Lagrangian multiplier v, we have second Lagrangian relaxation.

(LR2)

$$Z_D(v) = \text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} + v\,(\sum_{j=1}^{n} y_j - k)$$

$$= \text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} + (\sum_{j=1}^{n} v\, y_j) - vk$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for} \quad i=1,2,\ldots,n \qquad (2)$$

$$-x_{ij} + y_j \leq 0 \qquad \text{for} \quad i,j=1,2,\ldots,n \qquad (4)$$

$$x_{ij} \geq 0 \qquad \text{for} \quad i,j=1,2,\ldots,n \qquad (5)$$

$$y_j \in \{0,1\} \qquad \text{for} \quad j=1,2,\ldots,n \qquad (6)$$

For fixed v, above problem is so-called SPLP(simple plant lacation problem). As is known, SPLP is not an easy problem to solve but admits highly efficient dual based algorithm (Krarup and Pruzan [11] and Erlenkotter[5]). So we adopt Erlenkotter's DUALOC to solve SPLP for given v.

Apparently, the best choice for v would be an optimal solution to the Lagrangian dual problem:

(D2)

$$Z_{D2} = \max_{v} Z_D(v)$$

Since (D1) and (D2) are subdifferentiable, we used subgradient method to solve these Lagrangian duals as proposed by Fisher [6]. Note that $Z_{IP} \geq Z_{D1}$, $Z_{IP} \geq Z_{D2}$ and we say there exists duality gap when $Z_{IP} \neq$ dual value. Because $Z_D(u)$ is not increased by removing the integrality restriction on $y_j$ from the constraints of (LR1), $Z_{D1}=Z_{LP}$ (where $Z_{LP}$ is the objective value of linear program relaxation of k-median problem). Geoffrion [8] calls this integrality property. (LR2) does not have the integrality property, so $Z_{D2} \geq Z_{LP}$. Thus $Z_{D2} \geq Z_{D1}$. That is, the lower bound obtained by (LR2) is tighter than that of by (LR1).

Two properties are crucial in evaluating a relaxation.

(1) the tightness of the bound generated

(2) the amount of computational efforts required to get these bounds.

Usually there is a tradeoff between these two properties in choosing a relaxation. Tighter bound usually requires more computational efforts to get it than loose bound. However It is generally difficult to determine whether a relaxation with tighter bounds but great computational effort will end with better overall computational performance. That is, whenever there exists duality gap we have to resort to branch and bound technique to get an optimal solution. A branch and bound scheme incorporated with tighter bound requires smaller search tree than one with loose bound, i.e., if we spend more computational efforts to get an tighter bound, we could cut off the search tree fast. This is why extensive computational experience is needed to determine which relaxation is better in terms of overall computational performance.

## 3. Computational Experience

In this section, we report our computational experiecne with medium-size k-median problem. This computational experience is based on the solutions of 1,700 random problems with n=50 points and 2,200 random problems with n=100 points.

As mentioned earlier, $Z_{D1}$ and $Z_{D2}$ were obtained by solving Lagrangian dual by subgradient optimization. If it happens that the value of the best known feasible solution equals the value of Lagrangian dual or all the subgradients are equal to 0, subgradient iteration terminates because we found optimum. For most of test problems with no duality gap, the algorithm terminated in less than 100 subgradient iterations because of the stopping criterion. If, after 100 subgradient iterations, there was still a gap between the best feasible solution (an upper bound on $Z_{IP}$) and the best Lagrangian relaxation (a lower bound on $Z_{IP}$), we resorted to branch and bound to find $Z_{IP}$.

The first set of experiment involves unit edge length case with n=50 points. We generated 1,700 random graphs on which the k-median problem is defined. $c_{ij}$ is the minimum number of edges on a path joining $Y_i$ to $Y_j$ for $1 \leq i,j \leq n$, where the minimum is taken over all paths joining $Y_i$ to $Y_j$. Thus c is the shortest distance between $Y_i$ and $Y_j$, assuming that all edges have length one. In this case, when there exists a dominating set, Ahn et al [1] proved $Z_{IP} = Z_{LP}$. Therefore we expected first type of relaxation will do better in computational performance.

The results are summarized at Table 1. At (LR1) about 25.6% of test problems have duality gap and at (LR2) about 20.1% problems have duality gap as indicated by $Z_{IP} = Z_D$ at Table 1.

As was expected, the number of instances with dualty gap is fewer in (LR2) than in (LR1). In (LR2), the number of instances with no duality gap is 1,359, whereas the number of instances with no duality gap is 1,265 in (LR1). However, (LR1) is better in terms of overall computational performance. This could be explained by the fact that

| value of k | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | |
|---|---|---|---|---|---|---|---|---|
| | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time |
| 2 | 97 | 1.322 | 73 | 2.401 | 138 | 2.118 | 32 | 4.010 |
| 3 | 97 | 1.002 | 73 | 2.203 | 141 | 2.565 | 29 | 3.999 |
| 4 | 109 | 1.004 | 61 | 2.129 | 133 | 2.330 | 37 | 3.810 |
| 5 | 127 | 1.231 | 43 | 2.308 | 135 | 2.056 | 35 | 2.978 |
| 6 | 125 | 1.361 | 45 | 3.007 | 141 | 2.305 | 29 | 3.917 |
| 7 | 135 | 1.589 | 35 | 2.566 | 135 | 1.946 | 35 | 3.645 |
| 8 | 139 | 1.809 | 31 | 2.897 | 127 | 2.920 | 43 | 4.712 |
| 9 | 139 | 2.062 | 31 | 2.910 | 121 | 2.643 | 49 | 5.244 |
| 10 | 151 | 2.198 | 19 | 2.998 | 132 | 2.906 | 38 | 4.982 |
| 11 | 146 | 1.725 | 24 | 4.100 | 151 | 2.296 | 19 | 4.111 |
| total | 1265 | | 435 | | 1354 | | 346 | |

(Table 1: Unit Edge Length Graphs)

search through on the search tree does not require much efforts when compared to the efforts of getting Lagrangian dual.

The second set of experiment involves 500 trees with n=100 points. $c_{ij}$ is the number of edges on the unique path from $Y_i$ to $Y_j$. As kolen [10] proved, dual ascent procedure for SPLP defined on a tree always finds optimum without entering into branch and bound phase. With this property and $Z_{IP}=Z_{D2}$, we expected second type relaxation would have computational edge over first type relaxation.

The results are summarized at table 2. At (LR1) about 15.6% of test problems have duality gap and at (LR2) about 7.4% problems have duality gap as indicated by $Z_{IP} = Z_D$ at table 2.

| value of k | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | |
|---|---|---|---|---|---|---|---|---|
| | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time |
| 2 | 47 | 0.819 | 3 | 2.001 | 47 | 0.925 | 3 | 1.578 |
| 3 | 49 | 1.085 | 1 | 10479 | 49 | 0.883 | 1 | 1.210 |
| 4 | 44 | 1.586 | 6 | 2.010 | 46 | 1.099 | 4 | 1.785 |
| 5 | 43 | 1.890 | 7 | 2.121 | 49 | 0.954 | 1 | 1.326 |
| 6 | 41 | 20113 | 9 | 2.731 | 48 | 10.63 | 2 | 1.546 |
| 7 | 40 | 20.57 | 10 | 2.809 | 44 | 1.078 | 6 | 1.979 |
| 8 | 41 | 2.110 | 9 | 2.467 | 45 | 1.398 | 5 | 1.876 |
| 9 | 41 | 2.646 | 9 | 3.118 | 49 | 1.149 | 1 | 1.689 |
| 10 | 38 | 2.668 | 12 | 3.893 | 44 | 1.360 | 6 | 2.764 |
| 11 | 38 | 2.597 | 12 | 3.994 | 42 | 1.650 | 8 | 3.009 |
| total | 422 | | 78 | | 463 | | 37 | |

(Table 2: Trees)

As table 2 indicates, (LR2) has fewer instances with duality gap and has better over all computational performance. This is explained as follows. When the underlining structure on which the k-median problem is defined is a tree (LR2) is SPLP on a tree. Therefore, DUALOC always finds optimum for SPLP without entering into branch and bound phase. Moreover Lagrangian multiplier is only one in (LR2) but the number of multipliers in (LR1) is m.

The third set of experiment involves random edge length case with n=100 points. The edge lengths were computed as follows. The points were assigned random integer coordinates in a square of size 10x10 and the length of an edge was the Euclidian distance between its two endpoints, rounded to the closed integer. $c_{ij}$ was taken to be the length of the shotest path joining $Y_i$ to $Y_j$.

The results of 1,700 test problems are summarized at table 3. At (LR1) about 17.7% problems have duality gap and at (LR2) about 9.1% problems have duality gap.

| value of k | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | | $Z_{IP}=Z_D$ | | $Z_{IP}\neq Z_D$ | |
|---|---|---|---|---|---|---|---|---|
| | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time | # of problems | CPU time |
| 2 | 164 | 1.002 | 6 | 1.103 | 170 | 1.996 | 0 | 2.011 |
| 3 | 154 | 1.345 | 16 | 1.832 | 165 | 2.567 | 5 | 2.742 |
| 4 | 152 | 1.724 | 18 | 2.168 | 165 | 1.844 | 5 | 1.913 |
| 5 | 149 | 1.777 | 21 | 3.125 | 157 | 3.382 | 13 | 3.883 |
| 6 | 138 | 1.983 | 32 | 4.167 | 157 | 2.299 | 13 | 2.732 |
| 7 | 131 | 2.167 | 39 | 5.132 | 161 | 1.865 | 9 | 2.157 |
| 8 | 128 | 2.203 | 42 | 7.851 | 150 | 2.239 | 20 | 4.251 |
| 9 | 125 | 2.421 | 45 | 4.334 | 143 | 2.574 | 27 | 4.330 |
| 10 | 125 | 2.407 | 45 | 5.49 | 141 | 2.475 | 29 | 5.219 |
| 11 | 133 | 2.442 | 37 | 7.096 | 137 | 2.43 | 33 | 5.911 |
| total | 1399 | | 301 | | 1546 | | 154 | |

(Table 3: Random Edge Length Graphs)

As Table 3 indicates (LR1) is better in overall computational performance with $k \leq 6$ but (LR2) is better with $k \geq 7$. In this case we can not conclude which relaxation is better in terms of overall computational performance.

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