Reporting and Investigating Intervals for Cost Variances

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I. Introduction

Analytical cost control models which have appeared to date in the literature focus on a subset of the elements of the cost control process.(1) In particular, the major efforts have been directed toward the determination of an optimal policy for process investigation, the determination of an optimal reporting schedule, or the decision implementation effect of altering the performance evaluation scheme. An exception to this generalization is the work of Stallman (1971, 1972), who provides a heuristic discussion as to how an optimal combination of preventive and corrective action can be determined.

The objectives of this study are (1) to develop a cost control model to be used for studying the control process and (2) to understand the interacting nature of decisions concerning the reporting interval and the timing of the investigations.

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(1) See, for example, Dyckman (1969), Kaplan (1969, 1975), Hannum (1974), Gonedes (1971), Hinomoto (1971), and Demski (1969, 1970, 1971).

The major benefits of this study are provided by insights into the design of effective cost control policies under a variety of environmental conditions, and into the potential consequences of using non-optimal policies. The general concepts developed in this study would be applicable to a variety of processes.

To develop an integrated cost control that is comprehensive in the sense that it embraces all significant aspects of the control process, it is necessary to describe certain mathematical relationships among the elements in the control model as shown in Figure 1. Such a description enables us to determine the dependent relationships among the interacting variables in the model, and to identify the critical variables which have a significant impact on the efficiency and effectiveness of the process under consideration. As a first step toward this ultimate goal, this study critically examines the relationship between the process intervention decision and the variance reporting interval decision.

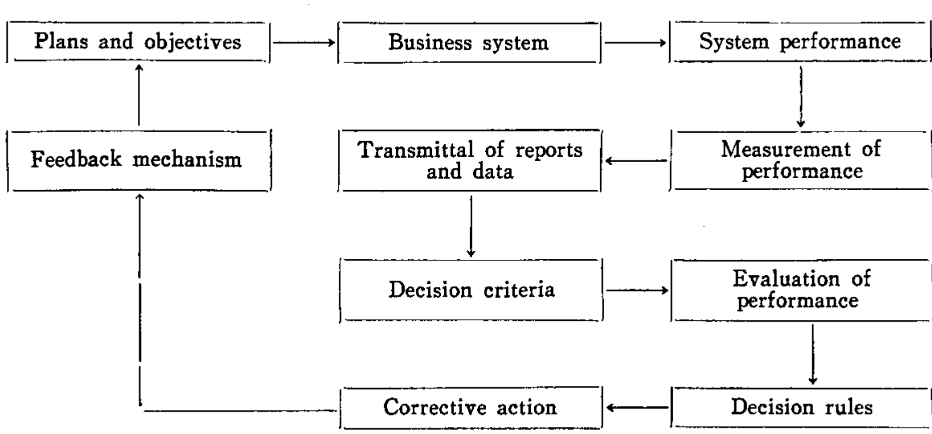


Fig. 1. Flow of Control Process

II. General Formulation of the Problem

A survey of the existing literature reveals that a few studies advocate a non-constant reporting interval.(2) In general, one would speculate that a best reporting interval would decrease with the increasing degradation of the process.

(2) For example, see Hannum (1974), Parretta (1974, 1975).

Of course, the optimal reporting interval also depends on the cost of preparing a report. Thus it is necessary to construct a general model that is directed toward answering the following questions:

(a) Given a variance report when should an investigation of the variance be pursued and (b) what is the optimal length of the next reporting interval, subject to the condition that we minimize the total reporting, investigation, and inefficiency costs over the entire planning horizon N ($N=1, 2, \cdots$)?

Following the development of a general model of the problem, the following situations are analyzed:

The condition of the process is classified as either in-control or out-of-control, and the transition from one state to the alternative state may occur at any point in time.

The performance of a process under consideration is measured in terms of operating cost over the planning horizon of the controlled unit, where there are $1, 2, \ldots, N$ periods in the planning horizon. Since the control objective is to minimize the expected costs which are successively generated in the multistage process, it is appropriate to use a dynamic programming approach. It is assumed that a variance report is prepared only at discrete intervals (but not at fixed intervals) of time, and that a decision to investigate a variance is made immediately upon the receipt of the variance report on the basis of the process condition. Furthermore, it is assumed that the report does not convey perfect information as to the current condition of the process.

For the case where the condition of the process can be represented by a single variable, the stochastic behavior of the process may be described by

$$\theta(n+1)=\theta(n)+\xi \tag{1}$$

where $\theta(n+1)$ =condition of the process at time $n+1$

$\theta(n)$ =condition of the process at time n

ξ =random variable having some known probability density function $f(\xi)$

If, on the other hand, the condition of the process is represented by multiple discrete states, the above behavior may be described by

$$\theta_i(n+1)=\theta_i(n)f(\xi) \tag{2}$$

where $\theta_i(n+1)$ =ith state of the process at time $n+1$

$\theta_i(n)$ =ith state of the process at time n

$f(\xi)$ =transition probability density function

As an example, assume that the process is dichotomized into "in-control" (θ_1) and "out-of-control" (θ_2) states, and that the transition probability from θ_1 to θ_2 and the one from θ_2 to θ_1 are $1-\alpha$ and $1-\beta$, respectively. Then

$$\theta_1(n+1)=\theta_1(n)\times\alpha+\theta_2(1-\beta)$$

$$\theta_2(n+1)=\theta_1(n)\times(1-\alpha)+\theta_2(n)\times\beta \tag{3}$$

However, as a general formulation of the problem, the subscript of θ is omitted in the remaining part of this section.

Since the report conveys imperfect information as to the current state of the process, it is necessary to generate a posterior probability distribution over the states given the reported variance.

For simplicity of presentation, let

R=cost of preparing a single report

I=cost of investigation and correction

K=number of periods until the next report is prepared

t=number of periods elapsed since the last correction of the process

$\theta(t)$=current state of the process with the posterior probability density $f(\theta(t);t)$

Z=state of the process immediately after the correction with the probability density $f(Z)$

$\theta(K)$=state of the process at the time the next report is prepared with the probability density $f(\theta(K);Z,K)$, assuming that the process is corrected at the beginning of the current period

$\theta(t+K)$=state of the process at the time the next report is prepared with the probability density $f(\theta(t+K);\theta(t),K)$, assuming that the process is not corrected at the beginning of the current period

$C(\theta(t), K)$ = operating cost of the process over the next K periods given $\theta(t)$

$V_N((\theta), (t))$ = minimum expected cost that can be achieved starting from an initial estimate of $f(\theta(t); t)$ and following an optimal policy for the next N periods, $N \geq K$.

When the variance report is received one of two possible actions may be taken, i.e., investigate, or do nothing.

Suppose the decision is not to investigate the reported variance and to prepare the next report at the end of Kth period. The expected cost to be incurred over the next K periods is then the sum of the reporting cost and the expected operating cost of the process over the K periods

$$R + \int C(\theta(\mathrm{t}), \; K) \; f(\theta(t); t) d\theta \tag{4}$$

The expected cost during $N-K$ periods is the weighted average of costs resulting from a state $\theta(t+K)$ with the corresponding probability being used as a weight:

$$\int V_{N-K}[\theta(t+K)] f(\theta(t+K); \theta(t), K) d\theta \tag{5}$$

Note that the expected cost varies, depending on the values of K. But we want to find a particular K which results in the minimum expected cost. Thus, the expected cost of a policy which incorporates an optimal reporting interval over the N periods is given by:

$$V_N(\theta(t)) = R + \min_K \left\{ \int C(\theta(t), K) f(\theta(t); t) d\theta + \right.$$
$$\left. \int V_{N-K}[\theta(t+K)] f(\theta(t+K); \theta(t), K) d\theta \right\} \tag{6}$$

On the other hand, if the reported variance is investigated, and the next report is prepared at the end of the Kth period, then the expected cost to be incurred over the first K periods is

$$R + I(\theta) + \int C(Z, K) f(Z) dZ \tag{7}$$

The expected cost over the remaining $N-K$ periods is

$$\int V_{N-K}[\theta(K)] f(\theta(K); Z, K) d\theta \tag{8}$$

In this case, if K is the optimal reporting interval, the minimum expected cost over the N periods is given by

$$V_N(\theta(t))=R+I(\theta)+\min_K\left\{\int C(Z,K)f(Z)dZ+\right.$$
$$\left.\int V_{N-K}[\theta(K)]f(\theta(K);Z,K)d\theta\right\} \tag{9}$$

The optimal decision rule results from minimizing expected cost as follows:

$$V_N(\theta(t))=\min\begin{cases} R+\min\limits_K\left\{\int C(\theta(t);K)d\theta+\right. \\ \qquad \left.\int V_{N-K}[\theta(t+K)\}f(\theta(t+K);\theta(t),K)d\theta\right\} \\ R+I(\theta)+\min\limits_K\left\{\int C(Z,K)f(Z)dZ+\right. \\ \qquad \left.\int V_{N-K}[\theta(K)]f(\theta(K);Z,K)d\theta\right\} \end{cases} \tag{10}$$

III. General Description of the Solution Algorithms

This section describes a solution approach to the discrete case. The next section will discuss computer algorithms to implement the control model for the discrete case. The general framework introduced here follows Bertsekas' discussion (1976), but with some modifications and clarifications. An outline of the solution approach is followed by the results of three numerical analyses carried out on the computer in the next section. The purpose of such analyses is to ascertain the validity of the computer program. The solution procedures for the third example are provided in detail to demonstrate the application of dynamic programming algorithms.

Suppose the decision maker makes the following observations:

$$z_0=h_0(x_0, v_0)$$

$$z_t=h_t(x_t,\ u_{t-1},\ v_t),\quad t=1,2,\cdots,N-1$$

A random observation disturbance v_t[3] is characterized by the given probability measures $P_{v_0}(\cdot\,|\,x_0)$ and $P_{v_t}(\cdot\,|\,x_t, u_{t-1})$. The initial state x_0 is also random and characterized by a given probability measure P_{x_0}. The probability

(3) v_t may be viewed as a measurement process which does not yield perfect state information.

measure $P_{w_t}(\cdot\,|\,x_t, u_t)$ of input disturbance w_t is given and may depend explicitly on x_t and u_t. Let I_t denote the information available to the decision maker at time t, and call it the information vector. Then

$$I_0 = z_0$$

$$I_t = (z_0, z_1, \cdots, z_t, u_0, u_1, \cdots, u_{t-1}), \quad t=1, 2, \cdots, N-1$$

The problem is to find an admissible control policy $\pi = (\mu_0, \mu_1, \cdots, \mu_{N-1})$ that minimizes the cost functional

$$V = \sum_{x_0, w_t, v_t} \{g_N(x_N) + \sum_t g_t(x_t, \mu_t(I_t), w_t)\}, \quad t=0, 1, \cdots, N-1 \tag{11}$$

subject to the system equation

$$x_{t+1} = f_t(x_t, \mu_t(I_t), w_t), \text{(4)} \quad t=0, 1, \cdots, N-1 \tag{12}$$

and the observation equation

$$z_0 = h_0(x_0, v_0)$$

$$z_t = h_t(x_t, \mu_{t-1}(I_{t-1}), v_t), \quad t=1, 2, \cdots, N-1 \tag{13}$$

Once a control policy $\pi = (\mu_0, \mu_1, \cdots, \mu_{N-1})$ is adopted, the following sequence of events may be conceived (Bertsekas, 1976, pp. 114-115). At stage 0:

1. The initial state x_0 is generated according to the given probability measure P_{x_0}.
2. The observation disturbance v_0 is generated according to the probability measure $P_{v_0}(\cdot\,|\,x_0)$.
3. The decision maker observes $z_0 = h_0(x_0, v_0)$ and applies $u_0 = \mu_0(I_0)$, where $I_0 = z_0$.
4. The input disturbance w_0 is generated according to the probability measure $P_{w_0}(\cdot\,|\,w_0, \mu_0(I_0))$.
5. The cost $g_0(x_0, \mu_0(I_0), w_0)$ is incurred.
6. The next state x_1 is generated according to the system equation $x_1 = f_0(x_0, \mu_0(I_0), w_0)$.

At stage t:

1. The observation disturbance v_t is generated according to the probability

(4) Notation $\mu_t(I_t)$ indicates that each function μ_t maps the information vector I_t into the control space.

measure $P_{v_t}(\cdot\,|\,x_t, u_{t-1})$.

2. The decision maker observes $z_t = h_t(x_t, \mu_{t-1}(I_{t-1}), v_t)$ and applies $u_t = \mu_t(I_t)$, where $I_t = (z_0, \cdots, z_t, u_0, \cdots, u_{t-1})$.
3. The input disturbance w_t is generated according to the probability measure $P_{w_t}(\cdot\,|\,x_t, \mu_t(I_t))$.
4. The cost $g_t(x_t, \mu_t(I_t), w_t)$ is incurred and added to previous costs.
5. The next state x_{t+1} is generated according to the system equation $x_{t+1} = f_t(x_t, \mu_t(I_t), w_t)$.

At the last stage $(N-1)$:

1. The observation disturbance v_{N-1} is generated according to $P_{v_{N-1}}(\cdot\,|\,x_{N-1}, \mu_{N-2})$.
2. The decision maker observes $z_{N-1} = h_{N-1}(x_{N-1}, \mu_{N-2}(I_{N-2}), v_{N-1})$ and applies $u_{N-1} = \mu_{N-1}(I_{N-1})$, where $I_{N-1} = (z_0, \cdots, z_{N-1}, u_0, \cdots, u_{N-2})$.
3. The input disturbance w_{N-1} is generated according to the probability measure $P_{w_{N-1}}(\cdot\,|\,x_{N-1}, \mu_{N-1}(I_{N-1}))$.
4. The cost $g_{N-1}(x_{N-1}, \mu_{N-1}(I_{N-1}), w_{N-1})$ is incurred and added to previous costs.
5. The final state x_N is generated according to the system equation $x_N = f_{N-1}(x_{N-1}, \mu_{N-1}(I_{N-1}), w_{N-1})$.
6. The terminal cost $g_N(x_N)$ is incurred and added to previous costs.

The problem with the above approach is that at each state, the optimal control policy must be determined for all possible values of the information vector (i.e., for every sequence of observations made and controls employed up to time t), thereby causing a dimensional problem. By representing the state variable in terms of a sufficient statistic instead of an information vector, the dimensionality can be reduced. A widely-used sufficient statistic is the conditional probability measure of the state x_t, given the information vector I_t. This conditional probability $P_{x_t|I_t}$ summarizes all the information necessary for control purposes at time t. In the case where the knowledge of the state is imperfect, the decision maker can be viewed as controlling the probabilistic state $P_{x_t|I_t}$ to minimize the expected future cost conditioned on the information I_t available.

The sufficient statistic $P_{x_t|I_t}$ is generated recursively in time and can be viewed as the state of a controlled system. Using Bayes' rule, $P_{x_t|I_t}$ can be written as

$$P_{x_{t+1}|I_{t+1}}=\phi_t(P_{x_t|I_t}, u_t, z_{t+1}), \quad t=0, 1, \cdots, N-2$$

where ϕ_t may be considered as a stage transformation process which expresses each component of the output state as a function of the input state, decisions, and random variable. The above equation indicates that $P_{x_{t+1}|I_t}$ can be expressed in terms of $P_{x_t|I_t}$ for all possible combinations of u_t and z_{t+1}. If the state space for each time t is a finite set $(x^1, \cdots, x^n)$, then $P_{x_{t+1}|I_{t+1}}$ can be recursively generated as follows:

$$\begin{aligned} P_{x_{t+1}|I_{t+1}} &= P(x_{t+1}=i\,|\,I_{t+1}) \\ &= \frac{P(x_{t+1}=i\,|\,I_t, u_t, z_{t+1}=j)}{P(z_{t+1}=j\,|\,I_t, u_t)} \\ &= \frac{P(z_{t+1}=j\,|\,u_t, x_{t+1}=i)\,P(x_{t+1}=i\,|\,I_t, u_t)}{\sum_i P(z_{t+1}=j\,|\,u_t, x_{t+1}=i)\,P(x_{t+1}=i\,|\,I_t, u_t)} \end{aligned} \tag{14}$$

where $P(x_{t+1}=i\,|\,I_t, u_t)=\sum_l P(x_{t+1}=i\,|\,x_t=l, u_t)\,P(x_t=l\,|\,I_t)$.

The dynamic programming algorithm now can be written in terms of the sufficient statistic $P_{x_t|I_t}$ as follows:

$$\begin{aligned} V_{N-1}(P_{x_{N-1}|N-1}) = \min_{u_{N-1}, x_{N-1}, w_{N-1}} \{E \; &\{g_N(x_N)+g_{N-1} \\ &(x_{N-1}, u_{N-1}, w_{N-1})\,|\,I_{N-1}, u_{N-1}\}\} \end{aligned} \tag{15}$$

$$\begin{aligned} V_t(P_{x_t|I_t}) = \min_{u_t, x_t, w_t, z_{t+1}} \{E \; &\{g_t(x_t, u_t, w_t) \\ &+ V_{t+1}[\phi_t(P_{x_t|I_t}, u_t, z_{t+1}]I_t, u_t\}\} \end{aligned} \tag{16}$$

Note that in the above algorithm the expectation is conditioned on I_t and u_t. Equations (15) and (16) indicate that the optimal controller can be separated into two parts: (1) an estimator which uses u_{t-1}, z_t to generate new conditional probability $P_{x_t|I_t}$, and (2) a controller u_t* that generates optimal inputs as a function of $P_{x_t|I_t}$.

The above algorithm yields a control policy of the form:

$$u_t^* = u_t^*(P_{x_t \mid I_t}), \quad t = 0, 1, \cdots, N-1$$

The optimal value of the problem is given by:

$$V_0^* = \underset{z_0}{E}\{V_0(P_{x_0 \mid I_0})\} \tag{17}$$

Equation (17) makes it necessary to generate the probability measure $P_{z_0}(\cdot \mid x_0)$. Alternatively, the optimal value of the problem can be obtained simply by extending one more period backward, say $t=-1$, and then considering the optimal decision at $t=-1$ to be "do nothing." In this setting, the computation of backward optimization of equation (16) at $t=-1$ automatically yields the value of V_0^* in equation (17).

In the special case, where both $g_t(x_t, u_t)$ and arguments of all random variables are countable, equation (16) can be written as:

$$\begin{aligned} V_t(P_{x_t \mid I_t}) &= \min_{\mu_t} \underset{x_t}{E}\{g_t(x_t, u_t) \mid I_t, u_t\} + \underset{z_t}{E}\{V_{t+1}(P_{x_{t+1} \mid I_{t+1}}) \mid I_t, u_t\}\} \\ &= \min_{u_t}\{\sum_i g_t(x_t = i, u_t) P(x_t = i \mid I_t) + \sum_j \\ &\quad V_{t+1}[\phi_t(P_{x_t \mid I_t, \mu_t, z_{t+1}=j}) P(z_{t+1}=j \mid I_t, \mu_t)]\} \end{aligned} \tag{18}$$

The probability $P(z_{t+1}=j \mid I_t, u_t)$ in the above equation is given by:

$$P(z_{t+1}=j \mid I_t, u_t) = \sum_l P(z_{t+1}=j \mid x_t = l, u_t) P(x_t = l \mid z_t) \tag{19}$$

IV. Computer Program

The numerical analysis was performed using a FORTRAN program. Figure 2 provides a description of the program logic flow. This program has been tested, using three sets of test data. The first example was drawn from Bertsekas (1976, pp. 115, 126–128), the second example from Kaplan (1969), and the third example was contributed for purposes of the present study. All three examples deal with two-state cases (in-control and out-of-control states). Assuming periodic reporting, the first two examples cover only the "operating decision" to investigate or not to investigate the process. With a conjecture that the periodic reporting is not necessarily optimal, the contributed example

Start
↓
Initialize the parameter values
↓
→Iterate over the periods, starting with one period to go, then the immediately preceding period
↓
→Iterate over the values of the state variable
↓
→Iterate over the values of the decision variable
↓
Use the transformation equation
↓
Compute the value of recursive function
↓
Find a decision that minimizes the value of recursive function
↓
Proceed to the next state variable
↓
Save the optimal decision and minimum value of recursive function
↓
End

⟨Fig. 2⟩ General Logic Flow of Computer Program to Compute Optimal Policies

superimposes the "reporting decision" on the operating decision. To demonstrate the solution approach to the conceptual framework discussed above, the solution process for the last example, which covers a relatively longer planning horizon, is discussed in detail.

1. Bertsekas' Example

Bertsekas (1976) provides the following example to illustrate the nature of the sequential optimization problem with imperfect state information:

A machine can be in one of two states: a good state or a bad state. If the machine is operated for one unit of time, it stays in good state with probability 2/3 provided it started in a good state; and it stays in a bad state with probability 1 if it started in a bad state. The machine is operated for a total of three units of time and starts in a good state. At the end of the first and second unit of time the machine is inspected and there are two possible inspection outcomes: a probably good state or a probably bad state. If the machine is in a good state the inspection outcome is a probably good state with probability 3/4; if the machine is in a bad state, the inspection outcome is a probably bad state with probability 3/4.

After each inspection one of two possible actions can be taken:

Action C: Continue operation of the machine.

Action S: Stop the machine, do a complete and accurate inspection, and if the machine is in a bad state bring it back to a good state.

There is a cost of 2 units for using a machine in a bad state for one time unit and zero cost for using a machine in a good state for one time unit. There is also a cost of 1 unit for taking action S.

The problem is to determine the policy that minimizes the expected costs over the three periods.

In this example, a periodic inspection interval is assumed. The choice of possible actions is restricted to a choice of whether or not the machine should be stopped. We call this class of optimization problems "operating decision" problems. The decision choices concerning the determination of an inspection interval or a reporting interval may be labelled a "reporting decision" problem. The last example in this section will incorporate the operation decision with the reporting decision.

Bertsekas' analytical solutions to the above problem are summarized below.

Period to Go	*Decision Policy*
1	stop if $q \leq 1/2$
	continue if $q > 1/2$
2	stop if $q \leq 5/8$
	continue if $q > 5/8$

where q represents the posterior probability that the machine is in a good state. The minimum expected cost for the process over the three periods is 1.222. A numerical analysis, using this example, was carried out on a computer. As indicated in the discussion in equation (7), the numerical analysis was performed over three time periods, fixing the decision at the beginning of the process at Action C(continue). The range of the state variable was divided into 200 equal intervals. The computer provided solutions identical to Bertsekas', except that the expected cost was 1.214. The relative error is approximately 0.7%. The error is due to a combination of computer rounding error and finite sub-division of a state variable.

2. Kaplan's Example

Kaplan's (1969) example is also intended to illustrate how the optimal operating decision should be made. His example deserves special attention. Although his paper is widely referenced, his solution approach contains some *mistakes.*[5] In this section. (1) an alternative way of reformulating his example within the general framework discussed earlier is provided; (2) some conceptual mistakes made in his paper are pointed out; (3) suggestions as to how to correct them are made; and (4) the results of the implementation of his example on computer are reported.

He considers the following situation:

> A reporting segment of a firm, when in control, reports zero deviation 80 percent of that time, and a positive (unfavorable) deviation of three units 20 percent of the time. If this segment is out of control, it will always report a positive deviation of three units. An out-of-control situation when discovered can be corrected in a short time, and the cost of an investigation is one unit. The probability that the division operates in an in-control mode during a period, given that it entered that period in control, is .95. The discount factor for a single period is .98. The problem for management is to decide if it pays to investigate after receiving a cost report of three units.

Using discrete dynamic programming, he obtains the optimal decision policies and resulting costs as summarized in Table 1.

Table 1. Kaplan's Solution to his Example

Periods to Go	Decision Rule		Mnimum Cost
1	investigate if	$q \leq .533$	1.72
	do not inspect if	$q > .533$	$3-2.4q$
2	investigate if	$q \leq .675$	2.537
	do not inspect if	$q > .675$	$\begin{cases} 4.685-3.184q & \text{if } .675<q\leq.865 \\ 5.94-4.63q & \text{if } .865<q\leq.95 \end{cases}$
3	invetsigate if	$q \leq .654$	3.4
	do not inspect if	$q > .654$	$\begin{cases} 5.48-3.18q & \text{if } .654<q\leq.925 \\ 7.59-5.46q & \text{if } .925<q\leq.95 \end{cases}$
4	investigate if	$q^* \leq .658$	45.22
	do not inspect if	$q > .658$	$\begin{cases} 47.31-3.18q & \text{if } .658<q\leq.919 \\ 49.37-5.42q & \text{if } .919<q\leq.95 \end{cases}$

For the infinite period case, discounting future costs guarantees that the critical value (break-even value) q^* converges. As noted in Table 1, q^* con-

(5) It should be pointed out that these mistakes have not been corrected in his later paper (1975).

verges to 0.658. Then the optimal decision policy is to investigate if $q \leq 0.658$. Observe that

$$P(X_{t+1}=\theta_1 | Z_{t+1}=3)=\frac{.2q_K}{1-.8q_K}$$

Starting from an initial value of $q_K=0.95$, the probability that the division is in control after reporting 3 units of deviation in two consecutive periods is 0.43. The optimal rule, then, is "to inspect and take corrective action immediately, after two consecutive high cost outputs have been reported."

Since it was impossible to run an infinite period case on the computer, it was decided to see whether the computer output would agree with the figures in Table 1 for three periods, and whether the critical value q^* would converge to 0.658. In the process of testing the computer program, hand calculations were carried out because of this author's suspicion that Kaplan *incorrectly solved* his example.

For convenience of presentation, let:

$\mu_K=N$: do not investigate the division

$\mu_K=I$: investigate the division

$x_K=\theta_1$: the division is in control

$x_K=\theta_2$: the division is out of control

q_K : posterior probability that the division is in control

g : probability of the process remaining in control for one period

α : discount factor

$z_K=\$0$: the division reports a deviation of zero unit

$z_K=\$3$: the division reports a deviation of three units

$P(\$0)$: probability that a report indicating zero deviation is to be received

$P(\$3)$: probability that a report indicating three deviations is to be received

With some modifications of equation (4), the estimates of sufficient statistic q_{K+1} is given by:

$$q_{K+1}=\begin{cases} .95 & \text{if } \mu_K=N \text{ and } z_{K+1}=\$\,0 \\ \dfrac{.19_K}{1-.8q_K} & \text{if } \mu_K=N \text{ and } z_{K+1}=\$\,3 \\ .95 & \text{if } \mu_K=I \text{ and } z_{K+1}=\$\,0 \\ .752 & \text{if } \mu_K=I \text{ and } z_{K+1}=\$\,3 \end{cases}$$

One can immediately observe that once a zero deviation is reported, the posterior probability of the division's state is constant at 0.95 regardless of whether the division is corrected in the previous period. This observation is *not intuitively appearing*. One would expect that q_{K+1}, following the implementation of corrective action, would be higher than it otherwise would be. Some clarifying comments are in order.

The Kaplan example assumes that "the probability that the division operates in an in-control mode during a period, given that it entered that period in control, is 0.95." Because of the resulting implication that transitions occur after costs are reported and after an investigation is made, equasion (14) needs to be modified. For the first two cases (cases where $\mu_K=N$), the computation of q_{K+1} is carried out as if there were no transition from one state to another state. Call the resulting figure a "pseudo-posterior." The real posterior q_{K+1} is the product of the pseudo-posterior and transition probability g. For example, for the case in which $\mu_K=N$ and $z_{K+1}=\$\,3$,

$$q_{K+1}=g\times\frac{P(z_{K+1}=\$\,3)\,|\,x_{K+1}=\theta_1)\times q_K}{P(z_{K+1}=\$\,3)\,|\,x_{K+1}=\theta_1)\times q_K+P(z_{K+1}=\$\,3)\,|\,x_{K+1}=\theta_2)\times(1-q_K)}$$

$$=.95\times\frac{.2q_K}{.2q_K+(1-q_K)}$$

$$=\frac{.19q_K}{1-.8q_K}$$

On the other hand, for the last two cases (cases where $\mu_K=I$), Kaplan's solutions force one to compute q_{K+1} using equation (14) except to the extent that the result of equation (14) is again multiplied by the transition probability g. For example, for the case where $\mu_K=1$ and $z_{K+1}=\$\,3$,

$$q_{K+1}=g\times\frac{P(z_{K+1}=\$\,3\,|\,x_{K+1}=\theta_1)\times(g)}{P(z_{K+1}=\$\,3\,|\,x_{K+1}=\theta_1)\times(g)+P(z_{K+1}=\$\,3\,|\,x_{K+1}=\theta_2)\times(1-g)}$$

$$=.95\times.\frac{2\times.95}{.2\times.95+.05}$$

$$=.752$$

Note that the notation of g, which appeared in parenthesis, is used as a prior probability that the division is in control. However, due to Kaplan's explicit assumption that an investigation forces the division back to the in-control state, the above probability 0.752 is *wrong*. The implication of his approach is that the transition occurs twice within the same period, if and when an investigation is taken, which is clearly *unjustifiable*. In the event that an investigation has taken place, the correct posterior probabilities would be:

$$q_{K+1}=\begin{cases}\dfrac{P(z_{K+1}=\$0\mid x_{K+1}=\theta_1)\times g}{P(z_{K+1}=\$0\mid x_{K+1}=\theta_1)\times g+P(z_{K+1}=\$0\mid x_{K+1}=\theta_2)+(1-g)}\\[2ex]\dfrac{P(z_{K+1}=\$3\mid x_{K+1}=\theta_1)\times g}{P(z_{K+1}=\$3\mid x_{K+1}=\theta_1)\times g+P(z_{K+1}=\$3\mid x_{K+1}=\theta_2)\times(1-g)}\end{cases}$$

$$=\begin{cases}1\\0.792\end{cases}$$

Since the main purpose of using the Kaplan example is to check the validity of the computer program, his "wrong" probabilities are used in the subsequent analysis.

With the use of equation (19), the observation probabilities may be expressed in terms of q_K as:

$$P(z_{K+1}=\$0\mid\mu_K=N)=0.76q_K$$

$$P(z_{K+1}=\$3\mid\mu_K=N)=1-0.76q_K$$

$$P(z_{K+1}=\$0\mid\mu_K=I)=0.76$$

$$P(z_{K+1}=\$3\mid\mu_K=I)=0.24$$

As an example, consider $P(z_{K+1}=\$3\mid\mu_K=N)$.

$$P(z_{K+1}=\$3\mid I_{K+1},\mu_{K+1})=\sum_l P(z_{K+1}=\$3\mid x_K=l,\mu_K=N)\ \ P(x_K=l\mid z_K)$$

$$=P(x_K=\theta_1\mid z_K)[P(z_{K+1}=\$3\mid x_K=\theta_1,\mu_K=N)]+P(x_K=\theta_2\mid z_K)$$
$$[P(z_{K+1}=\$3\mid x_K=\theta_2,\mu_K=N)]$$

$$=P(x_K=\theta_1\mid z_K)[P(z_{K+1}=\$3\mid x_{K+1}=\theta_1)P(x_{K+1}=\theta_1\mid x_K=\theta_1,\mu_K=N)$$
$$+P(z_{K+1}=\$3\mid x_{K+1}=\theta_2)P(x_{K+1}=\theta_2\mid x_K=\theta_1,\mu_K=N)]$$

$$+P(x_K=\theta_1|z_K)[P(z_{K+1}=\$3|x_{K+1}=\theta_1)P(x_{K+1}=\theta_1|x_K=\theta_2,\mu_K=N)$$
$$+P(z_{K+1}=\$3|x_{K+1}=\theta_2)P(x_{K+1}=\theta_2|x_K=\theta_2,\mu_K=N)]$$
$$=q_K[(0.2)(0.95)+(1)(0.05)]+(1-q_K)[(0.2)(0)+(1)(1)]$$
$$=1-0.76q_K$$

If, in fact, Kaplan's final solution is correct, then one *must* assume that the cost resulting from the combination of state and decision (CPC) is given by:

$CPC(x_K=\theta_1$ and $\mu_K=N)=\$0.6$

$CPC(x_K=\theta_1$ and $\mu_K=I)=\$1.72$

$CPC(x_K=\theta_2$ and $\mu_K=N)=\$3$

$CPC(x_K=\theta_2$ and $\mu_K=I)=\$1.72$

The procedure by which the above cost figures are obtained, and the question of whether these figures are correct, are discussed below. Assuming that the above figures are correct for the time being, equations (15) and (16) yield the following dynamic programming algorithms:

$$V_{N-1}(q_{N-1})=\min\begin{cases}0.6q_{N-1}+3(1-q_{N-1})\\1.72q_{N-1}+1.72(1-q_{N-1})\end{cases}$$

$$V_K(q_K)=\min\begin{cases}3-2.4q_K+\alpha[(0.76q_K)V_{K+1}(q_{K+1})+(1-0.76q_K)V_{K+1}(q_{K+1})]\\1.72+\alpha[0.76V_{K+1}(q_{K+1})+0.24V(q_{K+1})]\end{cases}$$

When the above algorithm is carried out in a backward manner, solutions identical to Kaplan's shown in Table 1 are generated. This study also reports that the computer program, which is slightly modified to take into account the timing of transition occurrence and the data for the CPC matrix, provides the same results. Having divided the state variable into 200 equal intervals, a numerical analysis ($N=15$) was carried out on the computer. The behavior of the critical value q^* was observed as follows:

Period to Go	q^*
1	0.530
2	0.680
3	0.655
4 and thereafter	0.660

Recall that the probability of the process being in control is treated as a state variable, and that the expected cost is a function of the state variable. However, the state variable is a real number from zero to one. Thus, the computation of the expected cost on the computer makes it necessary to sub-divide the value of the state variable. The entire range of the state variable is divided into 200 equal intervals (i.e., an increment of 0.005 starting with zero). Then the expected cost is computed for each of the 201 values of the state variable. Therefore, it is not possible to generate Kaplan's steady-state value of 0.658. However, the computed value of 0.660 is the best that one can obtain in this setting of finite sub-division.

The elements of the above CPC matrix are the sum of the expected cost associated with a particular state and cost of investigation, if any. Then it follows that:

$$CPC(x_K=0_1 \text{ and } \mu_K=N)=(0.8)(\$0)+(0.2)(\$3)=\$0.6$$

$$CPC(x_K=0_2 \text{ and } \mu_K=N)=(9)(\$0)+(1)(\$3)=\$3$$

However, Kaplan might follow the following approach for the other two elements:

$$CPC(x_K=\theta_1 \text{ and } \mu_K=I)=(0.76)(\$0)+(0.24)(\$3)+\$1=\$1.72$$

$$CPC(x_K=\theta_2 \text{ and } \mu_K=I)=(0.76)(\$0)+(0.24)(\$3)+\$1=\$1.72$$

Note that $P(\$0)=0.76$ and $P(\$3)=0.24$.

This computation incorrectly implies that the probability of reporting \$3 deviation after an investigation is higher than one in the absence of corrective action.

3. Example of the Reporting Interval and the Timing of an Investigation

The decision alternatives in the preceding two examples are restricted to the choice of the operating decision in a two-state case (in-control and out-of-control states). For simplicity of computation, the two-state case is retained in this contributed example. However, it adopts the argument that periodic reporting is not necessarily optimal, nor should it be followed by convention. Relaxing the assumption of periodic reporting makes it necessary to consider the operating decision and the reporting decision simultaneously. This situation may be envisioned as indicated in the next paragraph.

At the beginning of period K, a controller estimates posterior probabilities of the states. He then chooses, by making use of posterior probabilities, one of the following four decision alternatives open to him:

Action $\mu_K=1$ Do not investigate the process at the beginning of period K, but prepare a variance report at the end of the period.

Action $\mu_K=2$ Investigate the process at the beginning of period K, and prepare a variance report at the end of the period.

Action $\mu_K=3$ Do not investigate the process at the beginning of period K, and do not prepare a variance report at the end of the period.

Action $\mu_K=4$ Investigate the process at the beginning of period K, but do not prepare a variance report at the end of the period.

At the beginning of the next period, $K+1$, the controller again estimates the posterior probabilities of the states following the results of the previous decision made. In the event that the variance report is obtained at the end of period K, the estimates of posterior probabilities reflect the information contained in the report. If, on the other hand, a report is not prepared at the end of period K, the estimation of posterior probabilities is done by incorporating prior probabilities with the presumption of transition probability from one state to another state. The controller then makes a decision based on these posterior probabilities. The choice of optimal decision policies is demonstrated by means of an example.

Consider a process that can be in one of two states, "in-control" or "out-of-control." It is assumed that once the process is in the out-of-control state, it remains in that state until it is investigated and corrected. The time the process remains in-control before going out-of-control state is assumed to be an exponentially distributed random variable with mean $1/\lambda$ (i.e., the average rate of occurrence of a shift is λ per unit of time). At the end of a period, a variance report may be prepared at a cost of R, or one may not be prepared at all. The variance report conveys imperfect information as to the state of the underlying process. Upon receipt of a variance report, the manager may investigate the process and correct it to the in-control state at a cost of I. The inefficiency cost to operate the process in the out-of-control state for a unit of time is OC_2; that is, OC_2 is an opportunity cost resulting from an out-of-control state for a unit of time. The planning horizon is N periods, with the length of one period of h units of time. For simplicity of computation, it is assumed that the process is in the in-control state at the beginning of the planning horizon, and that the shift from one state to another state occurs at the end of a period, but just before a variance report is prepared. If the underlying process is an in-control state, the variance report indicates the in-control state with the probability of $1-\alpha$; if the underlying process is an out-of-control state, the variance report indicates the out-of-control state with the probability of $1-\beta$. The problem is to determine how long the optimal reporting interval should be, when the investigation decision is made that minimizes the expected total costs over the entire planning horizon N.

To carry out the calculations to find the optimal policies it is necessary to assign values for the process parameters and cost coefficients. For input data, let:

$$N = 10$$
$$G = 0.8$$
$$1-\alpha = 1-\beta = 0.9$$
$$R = \$15$$
$$I = \$30$$
$$OC = \$100$$

Notice that, for ease of calculations, it is assumed that a shift from one state to another state occurs at the end of a period. When this assumption is made, the values of an opportunity cost per period (OC) and transition probability (G) are to be simply assigned. In the absence of this assumption, OC and G

should be computed incorporating h, λ, and ATI (θ_i), where ATI stands for an average time interval.

To characterize the structure of the dynamic programming problem at each state K, $K=0, 1, \cdots, N-1$, it is necessary to specify the following six elements.

1. Input state variable $= q_K$
2. Output state variable $= q_{K+1}$
3. Decision variable $= \mu_K$
4. Random variable: (see below)
5. Stage return: (see below)
6. Stage transformation: (see below)

For convenience of presentation, define:

q_K	posterior probability that the process is in-control
$x_K=\theta_1$	the process is in-control
$x_K=\theta_2$	the process is out-of-control
$z_K=\theta_1$	report signals an in-control state
$z_K=\theta_2$	report signals an out-of-control state
$P(z_K=\theta_i \mid x_K=\theta_j)$	conditional probability that the report signals a state θ_i, when the process is in state θ_j
$P(z_{K+1}=\theta_i \mid \mu_K=j)$	conditional probability that the report signals a state θ_i, given a decision j made in the period K
$\mathrm{CPC}(x_K=\theta_i, \mu_K=j)$	cost resulting from a combination of state θ_i and decision j

Random Variable. Since both $1-\alpha$ and $1-\beta$ are 0.9, which may be interpreted as a 90 percent accuracy of reporting signal, the probability distribution of a measurement (actual observation) z_K is given by:

$$P(z_K=\theta_1 \mid x_K=\theta_1) = .9$$

$$P(z_K=\theta_2 \mid x_K=\theta_1) = .1$$

$$P(z_K=\theta_1 \mid x_K=\theta_2) = .1$$

$$P(z_K=\theta_2 \mid x_K=\theta_2) = .9$$

In addition, the probabilities $P(z_{K+1}=\theta_i \mid I_K, \mu_K)$ are given by:

$$P(z_{K+1}=\theta_1 | I_K, \mu_K=1) = .1 + .64q_K$$
$$P(z_{K+1}=\theta_2 | I_K, \mu_K=1) = .9 - .64q_K$$
$$P(z_{K+1}=\theta_1 | I_K, \mu_K=2) = .74$$
$$P(z_{K+1}=\theta_2 | I_K, \mu_K=2) = .26$$

Calculations of the above probabilities are carried out by making use of equation (17). Recall that for a decision, $\mu_K=3$ or $\mu_K=4$, no variance report is prepared at the end of period K, thereby making $P(z_{K+1}=\theta_i | I_K, \mu_K)$ nonexistent.

Stage Return. The cost functional resulting from a sequence of states θ_i and decisions μ_K is given by:

$$V_N = \min_{\mu_K} \underset{x_K}{E}\{\sum_{K=0}^{N-1} \mathrm{CPC}(x_K, \mu_K)\}$$

where $\mathrm{CPC}(x_K=\theta_1, \mu_K=1) = R = \$\,15$

$\mathrm{CPC}(x_K=\theta_2, \mu_K=1) = R + OC = \$\,115$

$\mathrm{CPC}(x_K=\theta_1, \mu_K=2) = R + I(\theta_2) = \$\,45$

$\mathrm{CPC}(x_K=\theta_2, \mu_K=2) = R + I(\theta_2) = \$\,45$

$\mathrm{CPC}(x_K=\theta_1, \mu_K=3) = \$\,0$

$\mathrm{CPC}(x_K=\theta_2, \mu_K=3) = OC = \$\,100$

$\mathrm{CPC}(x_K=\theta_1, \mu_K=4) = I(\theta_2) = \$\,30$

$\mathrm{CPC}(x_K=\theta_2, \mu_K=4) = I(\theta_2) = \$\,30$

Thus, the return at stage K can be expressed as:

$$\mathrm{RET}_K = \begin{cases} 15q_K + 115(1-q_K) & \text{if } \mu_K=1 \\ 45q_K + \ 45(1-q_K) & \text{if } \mu_K=2 \\ 0 \ \ + 100(1-q_K) & \text{if } \mu_K=3 \\ 30q_K + \ 30(1-q_K) & \text{if } \mu_K=4 \end{cases}$$

Stage Transformation. The stage transformation, expressing each component of the output state as a function of the input state and decision, represents the probability measure $P(x_{K+1}=\theta_i | I_{K+1})$. The calculation of these probabilities is carried out by making use of equation (14). The results are given by

$$q_{K+1}=\begin{cases} \dfrac{.72q_K}{.1+.64q_K} & \text{if } z_{K+1}=\theta_1 \text{ and } \mu_K=1 \\ \dfrac{.08q_K}{.9-.64q_K} & \text{if } z_{K+1}=\theta_2 \text{ and } \mu_K=1 \\ .973 & \text{if } z_{K+1}=\theta_1 \text{ and } \mu_K=2 \\ .308 & \text{if } z_{K+1}=\theta_2 \text{ and } \mu_K=2 \\ .8q_K & \text{if } \mu_K=3 \\ .8 & \text{if } \mu_K=4 \end{cases}$$

Now, algorithms (15) and (18) may be written in terms of the state variable and the stage transformation as follows:

$$V_{N-1}(q_{N-1})=\min_{\mu_{N-1}}\begin{cases} 115-100q_{N-1} & \text{if } \mu_{N-1}=1 \\ 45 & \text{if } \mu_{N-1}=2 \\ 100-100q_{N-1} & \text{if } \mu_{N-1}=3 \\ 30 & \text{if } \mu_{N-1}=4 \end{cases}$$

$$V_K(q_K)=\min_{\mu_K}\begin{cases} 115-100q_K+(.1+.64q_K)\,V_{K+1}\left(\dfrac{.72q_K}{.1+.64q_K}\right) \\ \qquad +(.9-.64q_K)\,V_{K+1}\left(\dfrac{.08q_K}{.9-.64q_K}\right) \\ 45+.74V_{K+1}(.973)+.26V_{K+1}(.308) \\ 100-100q_K+V_{K+1}(.8q_K) \\ 30+V_{K+1}(.8) \end{cases}$$

The minimum expected total costs and optimal decision policies can now be found by solving the above recursive relations.

The calculations are basically done in two facets. The first facet is to solve the above functions in a backward manner. For a given value of K, it is necessary to determine the range of the state variable within which the value of a particular function is the lowest among the values of the above four functions. Then the tables of μ_K and $V_K(q_K)$ within the range are saved for future calculations. The second facet is to trace the optimal decision policies and minimum expected total costs in a forward manner.

The results of the first facet are summarized in Table 2. A computerized numerical analysis for this example was run; Table 2 also reports the results of this analysis. The results of the second facet are reported in Fig. 3.

The critical value of the state variable (the point at which the values of two functions are identical) is sensitive to an error in function value which, in turn, results from the finite sub-division of the state variable; it is conceivable that the computer output would show erroneous decisions around some value of the state variable. To minimize such an error, the function values are computed by incrementing the value of the state variable by 0.002. In other

〈Table 2〉 Solution to the Example

	Hand Calculations			Computer Output	
Period to Go	State Variable	Decision	Cost	State Variable	Decision
1	$q<.7$	4	30	$q\leq.698$	4
	$.7\leq q$	3	100—100q	$.7<q$	3
2	$q\leq.8$	4	50	$q\leq.8$	4
	$.8<q\leq.809$	3	130—100q	$.802<q\leq.804$	3
	$.809<q\leq.909$	1	152—127.2q	$.806<q\leq.906$	1
				$q=.908$	3*
				$q=.910$	1**
	$.909<q$	3	200—180q	$.912\leq q$	3
3	$q\leq.736$	2	76.4	$q\leq.736$	2
	$.736<q\leq.893$	3	150—100q	$.738\leq q\leq.882$	3
	$.893<q$	1	180—133.6q	$.884\leq q$	1
4	$q\leq.764$	4	100	$q\leq.762$	4
	$.764<q\leq.848$	3	176.4—100q	$.764\leq q\leq.840$	3
	$.848<q\leq.963$	1	201.76—129.89q	$.862\leq q\leq.966$	1
	$.963<q$	3	250—180q	$.968\leq q$	3
5	$q\leq.736$	2 or 4	126.4	$q\leq.736$	2 or 4
	$.736<q\leq.887$	3	200—100q	$.738\leq q\leq.876$	3
	$.887<q\leq.926$	1	225.18—128.39q	$.878\leq q\leq.996$	1
	$.929<q\leq1.0$	1	230—133.6q	$q=.998$	3
	$q=1$	1 or 3	96.4	$q=1.0$	1 or 3
6, 8, 10	Identical to the results with 4 periods to go except cost.				
7, 9	Identical to the results with 5 periods to go except cost.				

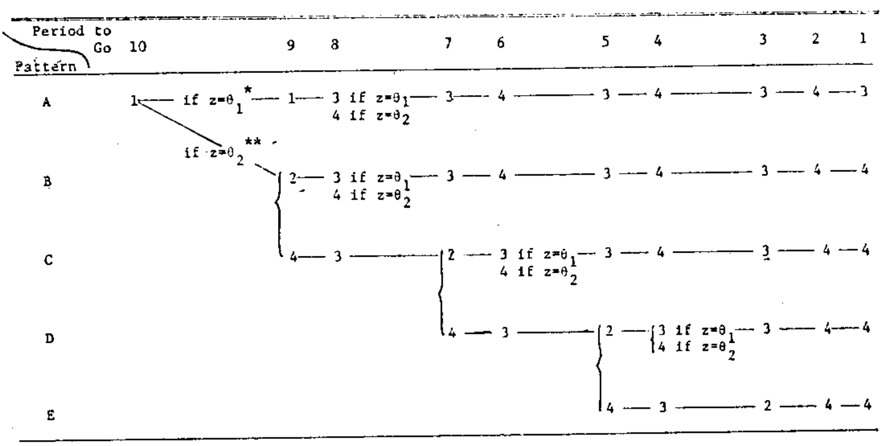
* When q=0.908, the decisions should be 1 as opposed to the computer output of 3. This error is due to the discretization of state variable.

** The decision should be 3 rather than 1 because of the error explained above.

words, the state variable in divided into 500 equal intervals. A comparison of the analytical and numerical results reveals that the minimum expected costs of the two results differ by no more than 0. 3 out of a total expected cost of $ 220. As expected, the comparison also shows some erroneous decisions in the computer output. Had a finer division been used, this problem would have been alleviated. However, even the 500 sub-division takes a substantial amount of CPU time on a computer.

This example reveals an interesting aspect of the reporting policy. Table 2 shows a switch in decisions from no report ($\mu_K=3$) to report ($\mu_K=1$) as q_K (transition probability) increases, which may not be obvious at the first glance. Although some insight into the optimal decision policies may be gained by examining Fig. 3, an intuitive argument is also helpful.

Suppose that there is a range of q_K within which an intervention of the process is not economically justified at the beginning of period K. Consider the case where the process is around the lower end of this range. Stated differently, it may be said that the present condition of the process is not bad enough to justify an immediate intervention, but is sufficiently bad to trigger an investigation in the following period. Then a value of q_{K+1}, which is



*$z=\theta_1$ designates that the variance report indicates an in-control state.

**$z=\theta_2$ desgiantes that the variance report indicates an out-of-control state.

Fig. 3. Sequence of Optimal Decisions

estimated in the light of the information contained in the report obtained at the end of period K, would suggest an investigation at the beginning of period $K+1$. Perhaps another value of q_{K+1}, which incorporates a controller's prior expectation about the deterioration rate of the process, might also signal an investigation at the beginning of period $K+1$. This being the case, a decision to suppress a report at the end of period K (i.e., $\mu_K=3$) would yield the minimum expected cost.

Next, consider the case where the process is in the higher end of this range. Here, the process is not bad enough to justify an immediate investigation, and its current condition may or may not require an investigation in the following period. In this case, it may be advisable to make the intervention decision after gathering more information about the process condition at the end of period K(i.e., $\mu_K=1$). The above argument explains why it is sometimes best to switch from $\mu_K=3$ to $\mu_K=1$ as q_K increases.

As shown in Table 2, with five periods to go, both decisions $u_5=2$ and $\mu_5=4$ result in the same minimum cost. Given that the decision $\mu_5=2$ incurs a reporting cost, this conclusion may not be obvious at the first glance. The table also shows that with three periods to go, the optimal decision is $\mu_3=2$ (if $q \leq 0.736$) even though this decision is not indicated during the early periods of the planning horizon. One may wonder why a report should be prepared when the planning horizon approaches the terminal point, rather than at an early period. These perplexing phenomena can be explained by examining the sequence of optimal decisions to follow.

Suppose that the process begins with an in-control state, and that the controller has decided to prepare a report at the end of the first period. He estimates the probability that the process is in-control, using the state transformation functions. On the basis of this estimate of posterior probability, he makes a decision. The optimal path to be pursued at each period is shown in Figure 3. Note that the subscript in Figure 3 represents the number of periods to go. Some comments on Figure 3 are in order.

First, consider the case in which the report signals an in-control state. With nine periods to go, the optimal decision is not to investigate the process, but to prepare another report at the end of the period. Then, depending on the reporting signal the process is either investigated or not investigated in the period. From that period on, the process is investigated in every other period. But no report is due in this time span.

Next, consider the case where the report signals an out-of-control state with nine periods remaining. In this case, the optimal decision is either an investigation, followed by another report, or just an investigation (i.e., either $\mu_9=2$ or $\mu_9=4$). If the controller decides to investigate the process and prepare a report at the end of the period (i.e., the adoption of $\mu_9=2$), the subsequent paths of optimal decision follow the one described in the preceding paragraph. If, on the other hand, the controller decides to investigate the process and not to prepare a report at the end of the period (i.e., implementation of $\mu_9=4$), his optimal decision in the next period is to do nothing (i.e., $\mu_8=3$). However, after one more period elapses, he again has to choose between $\mu_7=2$ and $\mu_7=4$. As shown in Figure 3, the adoption of $\mu_7=4$ leads to another problem of choice (selection of either $\mu_5=2$ or $\mu_5=4$) two periods later. If $\mu_5=2$ is chosen, his decisions in the subsequent periods are restricted to the operating decision. If he chooses $\mu_5=4$, he must generate a report two periods later (i.e., $\mu_3=2$).

In this example, $\mu_{10}=1$ suggests that one more report may or may not be prepared depending on the information contained in the first report. If the first report signals an in-control state, no report needs to be prepared for the remaining periods. An indication of an out-of-control state on the first report requires that another report be prepared. However, the timing of the report is not fixed: the controller is free to choose the reporting time.

This example assumes that the reporting cost is \$ 15. Following the sequence of optimal decisions, the controller will incur the expected total minimum cost of \$ 220 for the ten periods. If the reporting cost were \$ 13, the report would be prepared in every period except the last two periods. The resulting expected minimum cost is only $207. This is because additional reporting costs are offset by the savings in opportunity costs. If, on the other hand, it were $17, no report would be prepared in any period. The expected total minimum cost is $222. These observations, made on a very limited scale, support the argument that the reporting interval is contingent upon the reporting cost. Earlier in this study, it is also argued that the reporting interval also depends on the condition of the process.

This example assumes that the probability of the process staying in the in-control state is 0.8. Assuming the reporting cost to be $15, but merely varying the transition probability from 0.8 to 0.7, another computerized numerical analysis was run. The result shows that the report would be prepared in every period except the very last period. The expected total minimum cost is increased to $258.68. An intuitive explanation for the change of reporting interval would be that a fast deterioration of the process results in a need for more frequent feedback on the process.

V. Conclusion

As stated earlier, the objective of this study is to construct a cost control model to be used for studying the control process. The model developed is an extension of the Kaplan (1969) development, and is used to examine issues relating to the reporting interval as well as to the timing of investigations.

This study demonstrates the interwound nature of the reporting decision and the investigation decision. It also provides a way to assess the economic consequences of using a non-optimal policy.

The prevailing current practice is to prepare a report every period. In the event that the reporting cost is small relative to other costs such as opportunity costs and investigation costs, and that the reporting accuracy is high, the results of this study provide theoretical support for such a practice. If, however, the above conditions are not met, then the variance report, whose sole purpose is to provide information useful for the investigation decision should not be prepared for every period. In the light of the fact that the proper timing of an investigation is also affected by the reporting interval chosen, it must be concluded that the investigation policy based on the assumption of a constant report in every period is not necessarily optimal.

Of course, the above conclusions should not be accepted in every case. It is possible that the length of a reporting period, as now practiced, may not coincide with the one that is envisioned in this study. This study assumes only one shift from an in-control to an out-of-control state during one period. In other words, in the two-state case, an in-control state may switch to an out-of-control state during one period. In the instance of the multistate case, either a shift from an in-control state to any one of the out-of-control states or a shift from a less severe out-of-control state to one of more severity takes place during one period. In effect, the length of a period is defined in terms of the number of switches, rather than a calendar breakdown such as weekly, bi-weekly, monthly, periods. To the extent that such a switch-governed period agrees with the conventional accounting period, it is suggested that the prevailing current reporting interval is too short.

The optimal reporting policy involving no-report might be disconcerting to some accountants, especially those who believe that a report conveys valuable information. If the optimal reporting policy calls for an irregular reporting interval, then the use of a non-optimal policy involving a report in every period does not appear to have serious consequences in terms of opportunity loss.

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