The Vehicle Routing Problem

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1. Introduction

The routing of a fleet of vehicles is an area of both theoretical and practical importance. From a practical point of view, government and industry could save many million dollars by routing vehicles efficiently. Its practical applications encompass a wide variety of activities such as school bus routing, railway fleet routing, delivery of mail, and dispatching of delivery truck for customer goods. There are numerous variations due to the various underlying assumptions associated with the problem. Despite these varieties, the essential components of the problem are a fleet of vehicles and a set of customers with known demands.

To formalize the routing problem we provide notation and give below a formulation of the vehicle routing problem.

Parameters

$V=\{1, \ldots, K\}$, set of vehicles

$I=\{1, \ldots, N\}$, set of customers

$I'=I\cup\{0\}$, index 0 denotes the depot

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q_k : capacity of vehicle k

f_k : fixed cost of using vehicle k

d_i : demand of customer i

p_{ik} : fixed cost of serving customer i by vehicle k

L : restriction on tour length for each vehicle

Variables

$$y_k = \begin{cases} 1, & \text{if vehicle } k \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ik} = \begin{cases} 1, & \text{if customer } i \text{ is assigned to vehicle } k \\ 0, & \text{otherwise} \end{cases}$$

$T_k(X)$ = length of shortest tour which vehicle k travels to serve the customers assigned to it

In this paper we consider a variant of the vehicle routing problem where there is a restriction on the length of each tour travelled by vehicles. With the above parameters and variables the formulation of the vehicle routing problem is as follows.

The Vehicle Routing Problem is:

$$Z_{IP} = \mathrm{Min} \sum_{i=1}^{N} \sum_{k=1}^{K} p_{ik} x_{ik} + \sum_{k=1}^{K} f_k y_k$$

subject to

$$\sum_{k=1}^{K} x_{ik} = \begin{cases} 1, & i \in I \\ \sum_{k=1}^{K} y_k, & i = 0 \end{cases} \quad (1)$$

$$0 \leq x_{ik} \leq y_k \leq 1, \quad i \in I, \quad k \in V \quad (2)$$

$$x_{ik}, \; y_k, \; \textit{integral}, \quad i \in I, \quad k \in V \quad (3)$$

$$\sum_{i=1}^{N} d_i x_{ik} \leq q_k y_k, \quad k \in V \quad (4)$$

$$T_k(X) \leq L, \quad k \in V \quad (5)$$

Expression (1), (2), (3) are constraints of the simple plant location problem and state that each route starts and finishes at the central depot, and that every customer is served by some vehicle. Expression (4) states that the sum of demands of customers assigned to each vehicle is within its capacity limits.

Expression (5) states that the tour length of each vehicle must be less than a pre-specified restriction. Once the clustering variables y_k and x_{ik} are determined, for given k, $T_k(X)$ is an optimal value of the traveling salesman problem over customers assigned to vehicle k and customer 0 (depot).

The above formulation can 'be basically viewed as a composite of two-well known combinatorial problems-traveling salesman and simple plant location problems. It is important to distinguish the clustering decision which is represented by x_{ik} and y_k variables from the routing decision associated with $T_k(X)$ variables. Once the complicating clustering decision has been made, the problem reduces to a relatively easy traveling salesman problem for each vehicle.

In the next section we propose a decomposition algorithm which iterates between solving (VRP) as a master problem to determine clustering variabels x_{ik} and y_k and solving traveling salesman problem as a sub-problem to determine the actual vehicle route to serve the customers.

2. Decomposition Method

The constraint set (5) can be expressed as a set of linear constraints which is similar to the Benders cut.

Let $X=\{x_{11}, \ldots, x_{1K}, \ldots, x_{N1}, \ldots, x_{NK}\}$, $Y=\{y_1, \ldots, y_K\}$.

$$z_{ij}^k=\begin{cases} 1, & \text{if vehicle } k \text{ travels from customer } i \text{ to } j \\ 0, & \text{otherwise} \end{cases}$$

c_{ij} : distance of traveling directly from customer i to customer j

$T_k(X)$ can be defined mathematically as:

$$T_k(X)=\text{Min}\sum_{i=0}^{N}\sum_{j=0}^{N}c_{ij}z_{ij}^k$$

subject to

$$\sum_{j=0}^{N}z_{ij}^k=2x_{ik}, \quad i\in I' \tag{6}$$

$$\sum_{(i,j)\in S\times S}z_{ij}^k\leq|S|-1, \quad \begin{cases} S\subseteq I \\ 2\leq|S|\leq|I'|-1 \end{cases} \tag{7}$$

$$0 \leq z^k_{ij} \leq x_{ik} \quad i, j \in I' \tag{8}$$

$$z^k_{ij}, binary, \quad i, j \in I' \tag{9}$$

By replacing expression (9) with cutting planes, that is, by relaxing the integrality condition on z^k_{ij} and adding cutting planes to enforce integrality, we obtain an equivalent linear program which we denote by $(\text{TSP})_k$ as follows:

$$T_k(X) = \text{Min} \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} z^k_{ij}$$

subject to

$$\sum_{j=0}^{N} z^k_{ij} = 2x_{ik}, \quad i \in I' \tag{10}$$

$$0 \leq z^k_{ij} \leq x_{ik} \quad i, j \in I' \tag{11}$$

$$AR \leq B \tag{12}$$

where expression (12) *includes sub-tour elimination constraints and cutting planes to enforce integrality.*

Let U, V, W be dual variables associated with constraints (10), (11), (12) respectively. Any dual feasible solution to $(\text{TSP})_k$ satisfies

$$\sum_{i=0}^{N} (2u_i - \sum_{j=0}^{N} V_{ij}) x_{ik} - WB \leq L,$$

since the objective value of the dual problem to $(\text{TSP})_k$ is a lower bound on the optimum of $(\text{TSP})_k$, which itself is at most L. Let $(U^t, W^t, V^t)_{t=1,\ldots,T}$ be the set of all extreme solutions of dual problem to $(\text{TSP})_k$.

Proposition 1:

Tour $T_k(X)$ has a length of at most L if and only if

$$\sum_{i=0}^{N} (2u^t_i - \sum_{j=0}^{N} v^t_{ij}) x_{ik} - W^t B \leq L, \quad t = 1, \ldots, T \tag{13}$$

or equivalently,

$$\sum_{i=0}^{N} I^t_i x_{ik} \leq L^t, \quad t = 1, \ldots, T \tag{14}$$

Proof: immediate

Hence re-formulation of vehicle routing, which we denote by (VRP), is (VRP):

$$Z = \text{Min}\sum_{i=1}^{N}\sum_{k=1}^{K} P_{ik}x_{ik} + \sum_{k=1}^{K} f_k y_k$$

subject to (1), (2), (3), (4), (14)

Note that the expression (14) is summed not over $i \in I$ but over $i \in I'$. This can be easily resolved, since $\sum_{k=1}^{K} x_{OK} = \sum_{k=1}^{K} y_k$ and the fact that $y_k = 1$ means $x_{0k} = 1$. Hence the expression (14) can be re-written as

$$\sum_{i=1}^{N} I_i^t x_{ix} + I_0^t y_k \leq L^t, \quad t = 1, \ldots, T$$

Since the expression (14) can be regarded as a kind of Benders cut, the above reformulation offers an elegant method of iteratively constructing a solution.

We use an algorithm based on a decomposition method. The algorithm iterates between solving (VRP) as a master problem to determine clustering variables X and Y and solving $(\text{TSP})_k$ as a sub-problem to determine the actual vehicle route through customers determined by clustering variables and customer 0 (depot). Any sub-problem whose optimal $T_k(X)$ is greater than L generates Benders' type cut of expression (14). A detailed description of the solution method is given in the following section.

2.1. Sub-problem

Let $X = \{x_{11}, \ldots, x_{1K}, \ldots, x_{N1} \ldots, x_{NK}\}$, $Y = \{y_1, \ldots, y_K\}$.

Given clustering variables X' and Y', consider a sub-problem, for given k, which gives the length of tour $T_k(X')$.

$$T_k(X') = \text{Min}\sum_{i \in V_k}\sum_{j \in V_k} c_{ij} z_{ij}^k$$

subject to

$$\sum_{j \in V_k} z_{ij}^k = 2x_{ik}(X'), \quad i \in V_k$$

$$\sum_{(i,j) \in S \times S} z_{ij}^k \leq |S| - 1, \quad \begin{cases} i, j \in V_k \\ 2 \leq |S| \leq |V_k| - 1 \end{cases}$$

$$0 \leq z_{ij}^k \leq x_{ik}(X'), \quad i, j \in V_K$$

$$z_{ij}^k, \ \textit{binary}, \quad i, j \in V_k$$

where $V_k = \{i : x_{ik}=1\} \cup \{0\}$

We solve this sub-problem as proposed by Miliotis, for example, by Gomory's method of integer form. In other words, we relax the integrality condition on z^k_{ij} and add Gomory's cuts when needed to achieve integrality. Hence the equivalent formulation which we denote by $(\text{TSP})_k$ is as follows.

$(\text{TSP})_k$:

$$T_k(X') = \text{Min} \sum_{i \in V_k} \sum_{j \in V_k} c_{ij} z^k_{ij}$$

subject to

$$\sum_{j \in V_k} z^k_{ij} = 2x_{ik}(X'), \quad i \in V_k$$

$$0 \leq z^k_{ij} \leq x_{ik}(X') \quad i, j \in V_k$$

$$AR(X') \leq B$$

where $V_k = \{i : x_{ik}=1\} \cup \{0\}$, *and*

expression (8) *includes sub-tour elimination constraints and cutting planes to enforce integrality.*

We need a cutting plane algorithm to solve the sub-problem $(\text{TSP})_k$. Bellmore and Nemhauser [1] surveyed early research on the cutting plane approach to the traveling salesman problem. Grotschel [7], Grotschel and Padberg [6], Padberg and Hong [12], Padberg and Rao [11], and Miliotis [9, 10] have made an impressive success recently. According to Fisher and Jaikumar [3], A. Land reported that the Miliotis algorithm sloved 100 city problem relatively comfortably. The Miliotis algorithm, based on Gomory's method of integer forms, is a dual based method which deals with constraints (12) implicitly, and generates as they are violated.

We find that the algorithm goes well with our requirement, because we can terminate the algorithm whenever the objective value of relaxed sub-problem exceeds L, and still get a valid cut of expression (14).

2.2. Master Problem

We solve the master problem (VRP) which is actually a relaxed master problem by the Lagrangian relaxation method. The idea of Lagrangian method is that many difficult problems can be viewed as a relatively easy problem complicated by a set of side constraints. Lagrangian relaxation has applied to many difficult problems and made great success during the 1970s, since Held and Karp [8] devised a dramatically successful algorithm for the traveling salesman problem using subgradient optimization. Existing applications of the Lagrangian relaxation method are surveyed in Fisher [4, 5] By dualizing constraint sets (1), (4), (14), we have a Lagrangian problem.

$(LR_{\alpha,\beta,\gamma})$

$$Z_D(\alpha,\beta,\gamma)=\mathrm{Min}\sum_{i=1}^{N}\sum_{k=1}^{K}p_{ik}x_{ik}+\sum_{k=1}^{N}f_k y_k+\sum_{i=1}^{N}\alpha_i\Big(\sum_{k=1}^{K}x_{ik}-1\Big)+\sum_{k=1}^{K}\beta_k\Big(\sum_{i=1}^{N}d_i x_{ik}-q_k y_k\Big)$$

$$+\sum_{t\in T}\sum_{k=1}^{K}\gamma_k^t\Big(\sum_{i=0}^{N}I_i^t x_{ik}-L^t\Big)$$

$$=\mathrm{Min}\sum_{k=1}^{K}\Big[\sum_{i=0}^{N}\Big(p_{ik}+\alpha_i+d_i\beta_k+\sum_{t\in T}\gamma_k^t I_i^t\Big)x_{ik}+\Big(f_k-q_k\beta_k+\sum_{t\in T}\gamma_k^t I_0^t\Big)y_k\Big]$$

$$-\sum_{i=1}^{N}\alpha_i-\sum_{k=1}^{K}\sum_{t\in T}L^t\gamma_k^t$$

subject to (2), (3)

It is simple to solve $(LR_{\alpha,\beta,\gamma})$ optimally to determine $Z_D(\alpha,\beta,\gamma)$ for fixed Lagrangian multipliers.

Let $a_{ik}=p_{ik}+\alpha_i+d_i\beta_k+\sum_{t\in T}I_i^t\gamma_k^t,\quad i\in I,\quad k\in K$

The VUB constraints (2) and the objective of Lagrangian problem imply that

$$x_{ik}=\begin{cases}y_k, & \text{if } a_{ik}\leq 0\\ 0, & \text{otherwise}\end{cases}$$

Hence, defining

$$\bar{c}_k=\sum_{i=1}^{N}\mathrm{Min}(0,a_{ik})+f_k-q_k\beta_k+\sum_{t\in T}I_0^t\gamma_0^t$$

then, optimal y_k must solve

$$\mathrm{Min}\sum_{k=1}^{K}\bar{c}_k y_k$$

subject to

y_k binary

which is a trivial problem.

The above solution procedure is very similar to that of Cornuejols et al [2] for the k-median problem. The best choice for Lagrangian multipliers would be an optimal solution to the Lagrangian dual problem, $Z_D(\alpha, \beta, \gamma)$.

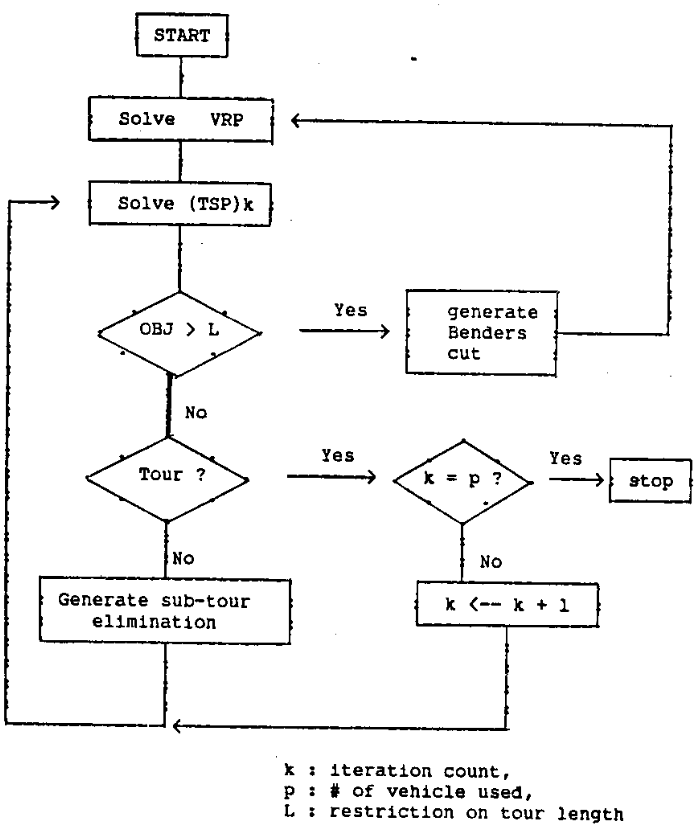
$$Z_D = \underset{\alpha, \beta, \gamma}{\text{Max}}\ Z_D(\alpha, \beta, \gamma)$$

We use the subgradient method to determine multipliers α, β, γ.

2. 3. Interaction between Master Problem and Subproblem

We initialize the above procedure by solving master problem. After obtaining

Fig. 1 : Overall Flowchart for the VRP



values of X and Y variables, we solve a traveling salesman problem for each vehicle. Any subproblem whose optimal value is greater than L generates cut of expression (10). Note that if $T_k(X)$ is greater than L, we should add cut of expression (10) to master problem (VRP) for all $k \in K$. Suppose we we add cut for only $k=k_0$, then at later iteration we might have same sub-problem for $k=k'$. This is especially true at the early stage of iteration and master problem has a solution with few y_k being positive. One other thing to note is that when we solve sub-problem, $(TSP)_k$, as noted at section 3, we can terminate sub-problem whenever value is greater than L. The overall flowchart for our algorithm is given in Fig. 1.

3. Example

In practical situation, there are lots of variations in Vehicle Routing Problem. In this section we will illustrate two examples. First example has operating cost and fixed cost. In second example, there is no operating cost.

The following example illustrates the procedure described in section 2 with $N=4$, $K=3$. The distance matrix (C_{ij}), cost matrix (P_{ik}) are given below. The limit on length of each route is 10.

Table 1 : Shortest Distance Matrix for Example 1

	0	1	2	3	4	
0	—	4	2	1	4	
1	4	—	3	2	3	
2	2	3	—	2	4	"O" refers to the depot.
3	1	2	2	—	5	
4	4	3	3	5	—	

By solving the master problem (VRP), we obtain the clustering variables, $Y=(0,1,0)$, $x_{12}=x_{22}=x_{32}=x_{42}=1$ and other X are zero with cost of 21. Since the value of $(TSP)_2$ is greater than 10, we generate cut (14) from dual variables.

$$3x_{1k}+3x_{2k}+x_{3k}+3x_{4k}+y_k \leq 10, \quad k \in K. \tag{15}$$

At iteration 2, the master problem with cut (15) produces a solution with $Y=(0, 1, 1)$, $x_{12}=x_{23}=x_{33}=x_{42}=1$ and other X are zero. The cost is 24. The solution of the sub-problem $(TSP)_k$ are:

Table 2: Other Data for Example 1

		1	2	3	demand	
	1	10	3	12	3	
	2	7	6	1	4	(P_{tk})
	3	1	3	1	5	
	4	2	4	3	3	
Fixed cost:		10	5	10		
Capacity:		15	15	15		

Route for vehicle 2 : 0—1—4, $(TSP)_2=11$

Route for vehicle 3 : 0—2—3, $(TSP)_3=5$

Since $(TSP)_2$ is greater than 10, the route for vehicle 2 generates cut (16).

Table 3: Shortest Distance Matrix for Example 2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	2	4	4	5	5	6	6	4	5	5	4	2	2	6	5
1	2	0	1	3	4	6	7	7	6	8	7	6	4	4	5	2
2	4	1	0	2	4	6	7	8	7	9	9	8	6	5	5	6
3	4	3	2	0	1	5	6	7	7	9	9	9	7	6	8	8
4	5	4	4	1	0	4	5	6	6	9	9	9	7	6	9	9
5	5	6	6	5	4	0	1	2	3	6	7	7	6	4	11	10
6	6	7	7	6	5	1	0	1	3	6	7	7	7	4	11	11
7	6	7	8	7	6	2	1	0	2	5	6	6	6	4	12	11
8	4	6	7	7	6	3	3	2	0	3	4	4	4	2	10	9
9	5	8	9	9	9	6	6	5	3	0	1	2	4	4	10	8
10	5	7	9	9	9	7	7	6	4	1	0	1	3	4	9	7
11	4	6	8	9	9	7	7	6	4	2	1	0	2	3	8	6
12	2	4	6	7	7	6	7	6	4	4	3	2	0	2	6	4
13	2	4	5	6	6	4	4	4	2	4	4	3	2	0	8	6
14	6	5	5	8	9	11	11	12	10	10	9	8	6	8	0	2
15	5	5	6	8	9	10	11	11	9	8	7	6	4	6	2	0

Table 4: The Optimal Solution to Example 2

Vehicle	customer	length of tour	Route
1	1, 2, 3, 4	11	0-1-2-3-4-0
2	9, 10, 11	11	0-9-10-11-0
3	14, 15	13	0-15-15-0
4	5, 6, 7, 8, 12	15	0-5-6-7-8-13-12-0

$$3x_{1k}-2x_{3k}+4x_{4k}+4y_k\leq 10,\quad k\in K. \tag{16}$$

At iteration 3 with cuts (15) and (16), a solution generated by the master problem is: $Y=(0, 1, 1)$, $x_{12}=x_{23}=x_{32}=x_{43}=1$ and other X are zero. The cost is 25. The length of tour of vehicle 2, 3 is 10, 9 respectively. So the algorithm terminates.

In the following example, $N=15$, $M=5$ and the restriction on tour length is 15.

For above example, our algorithm generates 14 cuts to produce a solution and the solution is provided in table 4.

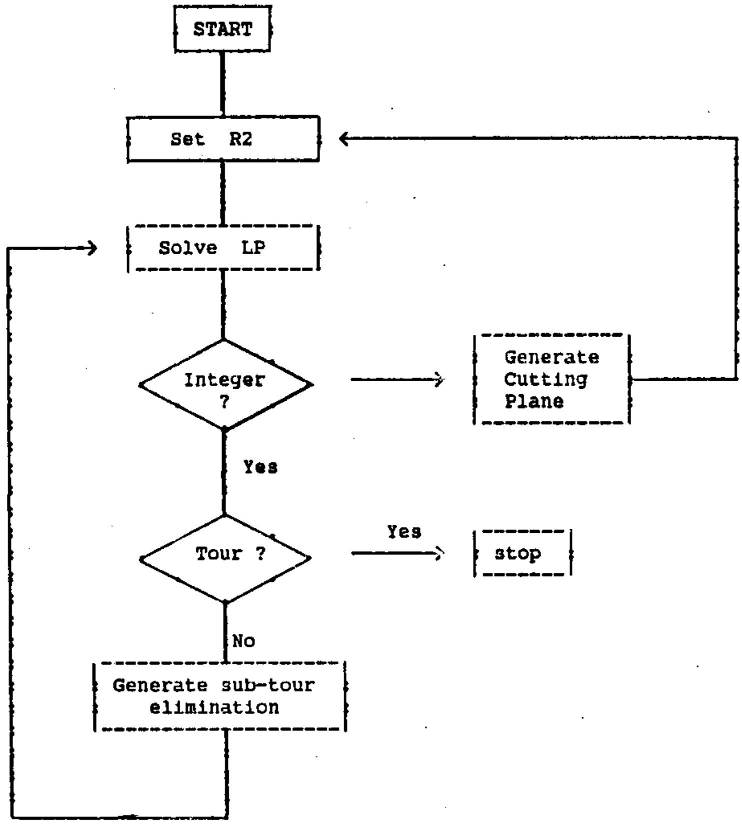
4. Computer Implementation

A computer program is developed for our algorithm based on fast algorithms

Table 5: Computional Results on the VRP

#of customers	#of cuts generated	#of nehicles used	CPU Time (seconds) on DEC-20
10	4	2	0.330
10	5	3	0.716
10	6	4	1.018
10	7	4	1.001
15	2	2	0.671
15	9	3	0.989
15	14	4	1.354
20	15	4	1.332
20	22	5	2.379
30	30	5	2.790
40	12	3	2.598
40	30	5	3.291
50	14	3	5.756
50	45	5	8.721

Fig. 2: The Traveling Salesman Flowchart



for the simple location problem and the traveling salesman problem. The computer algorithm incorporate sub-gradient method to find the Lagrangian dual and Miliotis' method to find the traveling salesman optimum. The overall flowchart for our algorithm and flowchart for sub-problem are given in Fig. 1, Fig. 2 respectively. The computational results are given in Table 5.

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