RISK-POOLING EFFECTS OF DYNAMIC ROUTING IN A MULTI-ECHELON DISTRIBUTION SYSTEM

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Ⅰ. Introduction

Logistics is a very important component of the economy and includes a wide variety of managerial activities. In the U.S., at the level of individual firms, distribution costs represent 10 to 30% of the total costs of goods sold (Robeson and Copacino, 1994). Because of this importance, a vast body of research has appeared in the area of logistics. However, most of this research has focused on optimizing the individual functions of the logistics system such as transportation, inventory allocation, location, etc.. This paper integrates three logistical functions: system inventory replenishment, delivery routing, and inventory allocation.

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We examine distribution policies for managing a one-warehouse N-retailer system facing stochastic demand and operating in a periodic-review mode. In the specific system examined, the warehouse places a system-replenishment order with an outside supplier every m periods, and receives it after a fixed leadtime. At that time, a delivery vehicle starts from the warehouse with the system-replenishment quantity (the warehouse holds no inventory.), visits each retailer once (and only once), allocating its inventory to the retailers along its delivery route. The distribution policy specifies: (1) *the system-replenishment policy*; that is, how much the warehouse will order from its outside supplier; (2) *the routing policy*; that is, the sequence in which the retailers are visited; and (3) *the inventory-allocation policy*; that is, how to allocate the system-replenishment quantity among the retailers.

We are interested in two kinds of delivery routing: fixed and dynamic. With *fixed routing*, the delivery vehicle visits each retailer once along a predetermined route that does not change over time. With *dynamic routing*, the delivery vehicle travels along a route that is determined sequentially. In particular, just before the delivery vehicle leaves the warehouse or any retailer, a decision rule is used to decide which retailer to visit next, based on the inventory status of the subsystem of retailers not yet visited. We also examine two different types of inventory allocation: *static and dynamic*. *Static allocations* are determined at the moment the delivery vehicle leaves the warehouse, based on the system inventory status at that instant. *Dynamic allocations* are determined sequentially upon arrival of the vehicle at each retailer, based on the inventory status of that retailer and the subsystem of retailers not yet visited. The distribution policies developed in this paper incorporate both dynamic delivery route and inventory allocation.

Kumar et al. (1995) examine the benefit of using dynamic allocation instead of static allocation while traveling a fixed delivery route in a one-warehouse N-retailer distribution system. Based on a numerical study, they conclude that dynamic allocations can yield significantly lower holding and backordering costs per

replenishment cycle than static allocations. Our research extends Kumar et al.'s study. We are concerned with the following questions: First, assuming that routing and allocation decisions are made dynamically, what decision rules (routing, allocation and system-replenishment) should be used to do so? Second, does dynamic routing combined with dynamic allocation improve system performance?

As might be expected, dynamic routing induces variability in the time between inventory allocations (i.e., the replenishment cycle) at each retailer, which, in turn, makes analysis of dynamic routing correspondingly more complex. In order to minimize this extra complexity we focus our analysis on a symmetric system. "Symmetric" means that all the retailers experience identical costs and face identical probability distributions of demand, and that it takes the delivery vehicle the same number of time periods, a, to reach any of the N retailers from the warehouse and the same number of time periods, b, to travel between any two retailers.[1)]

The additional assumptions and objective of the model are as follows: If retailer inventory is not sufficient to meet demand, shortages are backordered. We assume that these shortages occur only at the end of each replenishment cycle. Purchasing, inventory-holding, and backorder costs are linear. We assume perfect information about retailer net inventories at the beginning of each period. We seek a distribution policy that minimizes the sum of total expected discounted system purchasing, inventory-holding, and backorder costs over the infinite number of cycles.

Our major results are: In the N-retailer symmetric case: (1) LIF (go to the

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1) One can find distribution systems which can be viewed as symmetric or at least close to symmetric. For example, when Wal-Mart designs its distribution system, it locates the regional distribution centers (RDCs) first, and then, builds retail stores around them. This system is not exactly symmetric, but transportation times between the RDC and retail stores or those between retail stores are not very different. Also, if transaction time (i.e., time for unloading) dominates total leadtime (= transportation time + transaction time) between retail stores, the assumption of equal leadtimes between retail stores may be a good approximation.

retailer with the Least Inventory First) is the optimal routing policy; and (2) under the "allocation assumption", which allows negative allocations to retailers, a myopic policy is optimal. In the two-retailer case: (1) Equal allocation (replenish both retailers up to the same inventory position) always satisfies the first-order condition for optimality, but is not necessarily optimal; (2) LIF plus fixed-route allocation provides most of the total benefits from dynamic routing and dynamic allocation; and (3) the better of equal and fixed-route allocation is near-optimal.

Ⅱ. Related Research

There have been many articles on replenishment and allocation policies for multi-echelon distribution systems. Graves (1996) provides a brief review of the works on the multi-echelon distribution systems with both deterministic and stochastic demand. Research directly related to ours is as follows:

Federgruen and Zipkin (1984b), Anily and Fefergruen (1990, 1993), and Gallego and Simchi-Levi (1990) integrate inventory decisions with routing considerations. In particular, Federgruen and Zipkin (1984b) analyze a combined vehicle-routing and inventory-allocation problem with stochastic demand. In their model, both allocation and routing are static; that is, once determined, the route for each vehicle and allocation for each location are fixed. Their objective is to determine a joint route-allocation strategy that minimizes the sum of expected inventory cost and transportation cost for the entire system. In their model the interdependence between routing and inventory allocation arises from the fact that while the optimal allocation may prescribe a positive allocation to some particular retailer, the cost of routing the vehicle through that retailer may exceed the savings achieved by that allocation. Another source of interdependence is the vehicle capacities. Overall savings accruing from the joint consideration of the inventory-allocation and routing decisions, of 5-6% is reported. Anily and Federgruen (1990) study the dynamic vehicle-routing and inventory problem in one-warehouse multiple-retailer

systems when demand is deterministic.

Most dynamic-routing research focuses on dynamic vehicle-routing problem (VRP), wherein, as in our model, delivery routes are determined dynamically based on real-time information. Its application areas include fleet management (Powell, 1986), traffic assignment (Fiesz, et al., 1989), air traffic control (Vranas, et al., 1993). See Bertsimas and Simchi-Levi (1996) for a complete review of VRP. What differentiates our work from dynamic VRP is that dynamic VRP dynamically decides a set of customers served by a specific route, equivalently, a specific vehicle, while dynamic routing in our problem dynamically decides a sequence in which a given set of retailers are visited.

Kumar, Schwarz, and Ward (1995) examine static and dynamic policies for replenishing and allocating inventories amongst N retailers located along a static-delivery route. Their major analytical results, under the appropriate dynamic (static) allocation assumption, are: (1) optimal allocations under each policy involve bringing each retailer's "normalized-inventory" to a corresponding "normalized" system inventory; (2) optimal system replenishments employ base-stock policies; (3) the minimum expected cost per cycle of the dynamic (static) policy can be derived from an equivalent dynamic (static) "composite retailer". Given this, they prove that the "risk-pooling incentive", a simple measure of the benefit from adopting dynamic allocation policies, is always positive. Simulation tests confirm that dynamic-allocation policies yield lower costs than static policies, regardless of whether or not their respective allocation assumptions are valid. The magnitude of the cost savings, however, is sensitive to some system parameters.

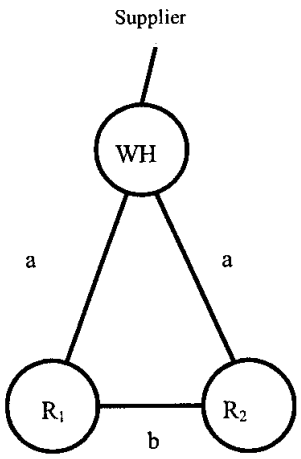
This paper is organized as follows; In Section 3 and 4, we describe the two-retailer symmetric system and prove the optimality of LIF routing. In Sections 5 and 6, we formulate the dynamic routing and allocation problem as a dynamic program and prove the optimality of a myopic policy. Section 7 derives some important properties of the optimal myopic allocation policy in the two-retailer case, and Section 8 develops heuristic allocation policies. Section 9

compares the computer-simulated performance of the optimal myopic allocation policy with the heuristic allocation policies, both to measure the cost-reduction effect of dynamic routing and to test the effectiveness of the heuristic policies. Finally, Section 10 summarizes the paper and suggests possible extensions.

Ⅲ. A One-Warehouse N-Retailer Symmetric System

The one-warehouse N-retailer symmetric system studied here has one warehouse and N identical retailers; i.e., all the retailers have the same demand distribution, and the same per-unit inventory-holding and backorder costs. Each retailer faces i.i.d. periodic demand. Furthermore, it takes the delivery vehicle the same number of time periods, a, to reach any of the N retailers to/from the warehouse and the same number of time periods, b, to travel between any two retailers. Let R_i represent retailer i, $i=1,\ldots,N$. Figure 1 shows the system when $N=2$.

Figure 1 The Symmetric System When $N=2$



The warehouse places a system-replenishment order every m periods, which arrives after a fixed leadtime L. The first system-replenishment order is placed

at time 0. Correspondingly, the tth system-replenishment order will be placed at time$(t-1)m$. After L periods, each system-replenishment order is delivered to the warehouse, a routing decision is made (i.e., which retailer to go to first), and the vehicle begins the route. Given a specific route, define the jth retailer as the one visited j^{th} on that route. The vehicle arrives at the first retailer a periods after the first routing decision, allocates part or all its inventory, and, under a dynamic-routing policy, decides which retailer to go to next. After b periods, the vehicle arrives at the second retailer, makes the second allocation decision, and the third routing decision. The vehicle repeats these allocation and routing decisions every b periods until it visits the next-to-the-last retailer on the route, and makes the $(N-1)^{st}$ allocation and N^{th} routing decision. In effect, all remaining inventory is allocated to the Nth retailer at this time, although it is delivered b periods later. The same sequence of decisions are repeated every m periods. We assume $m \geq (N-1)b$, which guarantees that retailer-replenishments do not cross, that is, the tth replenishment to R_i is delivered before (or at the same time as) the $(t+1)^{st}$ replenishment.

For the tth system replenishment, define C_{it}, $i=1,\ldots, N$ as the set of continuous time periods between the tth and $(t+1)^{st}$ visits to R_i, and define $\{C_{1t},\ldots, C_{Nt}\}$ as the tth *allocation cycle*.[2] Under a fixed-route policy (Kumar, Schwarz, Ward, 1995), each C_{it} contains exactly m periods for all t values. However, under dynamic routing, the number of time periods in each C_{it} and which particular periods are first and last depend on the tth and $(t+1)^{st}$ routing decisions. More generally, the number of periods in any given C_{it} can take on $(2N-1)$ possible values (i.e., $(m-(N-1)b)$, $(m-(N-2)b),\ldots,$ $(m-b)$, m, $(m+b),\ldots,$ or $(m+(N-1)b)$), depending on the tth and $(t+1)^{st}$ routes.

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2) For completeness, define $\{C_{10},\ldots,C_{N0}\}$ as the *zeroth allocation cycle*, where $C_{i0}=\{1,\ldots,a+(j-1)b\}$ if R_i is the jth retailer on the first route. That is, the zeroth allocation cycle for the jth retailer ends $(a+(j-1)b)$ periods after the first routing decision.

Ⅳ. The Optimality of Least-Inventory-First (LIF) Routing

We first derive the optimal routing policy. Routing affects total expected discounted costs of the tth allocation cycle for any retailer by determining which time period will be the last time period in that retailer's current allocation cycle. In the following lemma, we show that for any given allocation decision, there exists a routing decision which minimizes total expected costs of the current cycle without changing the costs of the subsequent cycles. Define the least-inventory-first routing policy (LIF) as the policy under which the delivery vehicle goes first to the retailer with the smallest current inventory position.

LEMMA 1: LIF is the optimal routing policy for the infinite-horizon problem.

Proof: [See Appendix 1].

Intuitively, Lemma 1 makes sense since at the time of any routing decision, the retailers differ only in their inventories. Since LIF is optimal in the infinite-horizon problem, we will henceforth limit all routing decisions to be LIF. The optimality of LIF routing can be generalized for any $N>2$ retailer symmetric systems. See Park (1997) for the proof.

V. DP Formulation

Our goal is to select a distribution policy that minimizes the sum of expected discounted total costs (= purchasing+inventory-holding+backorder costs) over the infinite number of allocation cycles. Note that inventory-holding and backorder costs at each retailer for any allocation cycle depend: (1) on the amount of inventory at that retailer in the first period of its allocation cycle immediately after allocation, and (2) on the demand during each time period in the allocation cycle.

In this section, we formulate the infinite-horizon problem given LIF as a

dynamic program (DP). For simplicity of the presentation, we will formulate the case when $L=0$ and $m \geq a+(N-2)b$.[3] All of the results derived below, in particular, the optimality of a 'LIF' routing policy and myopic allocation still hold in a general case.

Notation

c = Purchasing cost per unit

h = Inventory-holding cost per unit per period for units held either at any retailer or on the delivery vehicle.

p = Backorder cost per unit per period.

α = Discounting factor

i = Index for the retailers, i=1,,N.

I_j = Set of the retailers not yet visited at the instant of the jth routing decision on a route (Note that $I_1=\{1,\ldots,N\}$ and $I_j = I_{(j-1)} - \{[j-1]\}$).

(j) = Index for the jth retailer on a route.

t = Index for the allocation cycles (or the system replenishments).

x_{it} = Net inventory at Ri at the instant before the tth system-replenishment decision.

$\overline{X}_t$ = $(x_{1t},\ldots,x_{Nt})$

q_t = t^{th} system-replenishment quantity.

z_{it} = t^{th} allocation to R_i

$\overline{Z}_t$ = $(z_{1t},\ldots,z_{Nt})$

δ = Single-period random demand with mean μ and standard deviation σ.

δ_i^k = k-period random demand at R_i with probability density function $\Phi^k(.)$ and cumulative distribution function $\Phi^k(.)$.

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3) When $m<a+(N-2)b$, since at the time of tth system-replenishment decision, the $(t-1)^{st}$ route is not completed, we have to define additional state variables for the dynamic program that represent the subsystem of the retailers to be visited on the $(t-1)^{st}$ route and the amount of inventory left on the vehicle, which makes the presentation more difficult to understand.

$\bar{\delta}^k =$ Vector of δ_i^k's, that is, $(\delta_1^k, \ldots, \delta_N^k)$.

$\Delta^k = \sum_{i=1}^{N} \delta_i^k$, system demand over k periods.

Define $f_t(\bar{X}_t)$o be total expected discounted costs at the time of the tth system-replenishment decision given the state variables $\bar{X}_t$ given LIF routing policy. Also define $L^k(v)$ to be total k-period expected discounted inventory-holding and backorder costs at a retailer given net inventory v at the beginning of the first period and assuming no additional delivery. Let $v_{[j]t}$, $j=1,\ldots,N-1$ be net inventory of the jth retailer on the tth route at the instant after its tth allocation, and let $v_{[N]t}$ be the inventory position at the Nth retailer at the instant after the $(N-1)^{st}$ allocation decision. Also let $y_t(\equiv \sum_{i=1}^{N} x_{it} + q_t)$ be the system inventory position at the instant after the tth system-replenishment decision. After some rearrangement of cost terms (see Appendix 2 for details), one obtains

$$f_t(\bar{X}_t) = \min_{q_t \geq 0}\{c(y_t) + E[\min_{z_{[1]t}}\{H_1(z_{[1]t}) + \alpha^a L^{(N-2)b}(v_{[1]t}) + E[\min_{z_{[2]t}}\{H_2(z_{[2]t}) + \alpha^{a+b} L^{(N-3)b}(v_{[2]t})$$
$$+ E[\ldots E[\min_{z_{[N-1]t}}\{H_{N-1}(z_{[N-1]t}) + H_N(z_{[N]t}) + G(\overline{IP}_t)\}]\}]\cdots]\} + \alpha^m E[f_{t+1}(\bar{X}_t + \bar{Z}_t - \bar{\delta}^m)]\} \quad (1)$$

Subject to: $\sum_{i=1}^{N} z_{it} = q_t$ (2), $z_{it} \geq 0, i = 1,\ldots N$ (3), and $[j] \in I_j, j = 1,\ldots,N$ (4)

In the above formulation, $c(y_t)$ is the purchasing cost allocated to the tth allocation cycle, $H_j(v_{[j]t})$ the on-vehicle holding cost allocated to the jth retailer on the tth route, and $G(.)$ is the total expected discounted costs of the N retailers during the remaining periods of the tth allocation cycle at the instant after the $(N-1)^{st}$ routing decision. See Appendix 2 for details. Feasible allocations must satisfy the following constraints: (2) requires that the sum of the allocations to the N retailers on the tth route must equal the tth system-replenishment quantity q_t, (3) represent non-negativity constraints; i.e., all allocations must be non-negative, and (4) means that the delivery vehicle visits each retailer once and only once per allocation cycle.

DP is not easily solved. Note that in the fixed-route case, since each retailer's allocation cycle has a fixed number of time periods, it is known at the time of each retailer's current allocation when that retailer will receive its next allocation. In contrast, under dynamic routing, each retailer's allocation cycle has a yet-to-be-determined number of periods. Therefore, at the time of its current allocation it is not known when each retailer will receive its next allocation. Instead, this depends on the next route, which will be determined by the allocations and the demand realizations at all the retailers between the current allocation decision and the next routing decision. Hence, to compute the expected costs of the tth allocation cycle expectations are taken with respect to not only demand realizations but also the next route.

Another complication is that the probability distribution of each retailer's total demand during its allocation cycle is generally not a standard distribution. To illustrate: in the fixed-route case analysis by Kumar et al. (1995), where retailer demand each period is a normally distributed, the distribution of each retailer's total demand during its allocation cycle is also normal. However, under dynamic routing, even though retailer demand each period follows a normal distribution, the distribution of each retailer's total demand during its allocation cycle is not normal because, as noted above, each retailer's total demand during its allocation cycle will depend upon the demand realizations at all the retailers between the current allocation decision and the next routing decision. Because of this dependency, the convolution of all the period demands in the allocation cycle is not normal. We will discuss how this dependency affects the analysis of the allocation problem in Section 7.

VI. The Myopic Problem

In this section, we prove that if the non-negativity constraints (3) are relaxed, then a myopic policy is optimal in the infinite-horizon problem. The *myopic problem* (MP) is the problem in which the system-replenishment and

allocation decisions during the tth allocation cycle are chosen to minimize the sum of total expected purchasing, inventory-holding, and backorder costs during each separate allocation cycle without regard to their impact on expected costs in subsequent allocation cycles. MP is identical to (1) except that $f_{t+1} \equiv 0$. Define M_t as minimum total expected discounted costs of the tth allocation cycle. For $\forall t$,

$$\text{MP:} \quad M_t(\overline{X}_t) = \min_{q_t \geq 0}\{c(y_t) + Q(\overline{X}_t, q_t)\} \tag{5}$$

Subject to: (2), (3), and (4).

where y_t is the order-up-to level at the time of the tth system replenishment decision and

$$Q(\overline{X}_t, q_t) = E[\min_{z_{[1]t}}\{H_1(z_{[1]t}) + \alpha^a L^{(N-2)b}(v_{[1]t}) + E[\min_{z_{[2]t}}\{H_2(z_{[2]t}) + \alpha^{a+b} L^{(N-3)b}(v_{[2]t})$$
$$+ E[\ldots E[\min_{z_{[N-1]t}}\{H_{N-1}(z_{[N-1]t}) + H_N(z_{[N]t}) + G(\overline{IP}_t)\}]\}]\cdots]$$

By introducing an approximation $\tilde{f}_t$ to the function f_t in (1), by relaxing the non-negativity constraints (3), it is straightforward to prove that for $\tilde{f}_t$, myopic allocation is optimal. See Kumar, et al. (1995), Federgruen and Zipkin (1984a), etc. for similar proofs. Relaxing (3), in effect, permits costless and instantaneous transshipments between retailers whenever the optimization calls for it, thereby decoupling adjacent allocation cycles.

<u>LEMMA 2</u>: When the non-negativity constraints, (3), are relaxed, MP solves DP.

Although the allocation assumption appears to be a strong one - and impossible to implement - in our computational tests it is seldom invoked. For example, in the simulation study to be presented in Section 9, negative allocations were called for on average less than 1.25% of time (5% at most).

Ⅶ. The Myopic Allocation Problem (MAP) for N=2 Retailers

In this section, we derive some important properties of the optimal myopic allocation policy in the two-retailer case. Note that in the two-retailer case, only one allocation decision is made in each allocation cycle; i.e., the amount allocated to the first retailer in effect determines the amount allocated to the second retailer.

In this and following sections, for simplicity of the presentation, we assume that there is no cost discounting, i.e., $\alpha = 1$. Let v be the system net inventory at the time of the allocation decision. Let v_i be inventory position at R_i at the instant after the allocation decision and $\bar{v} = (v_1, v_2)$, where $v_2 = v - v_1$. In the two-retailer case, MAP is defined as

$$\text{MAP:} \quad \hat{M}(v) = \min_{v_1}\{C(\bar{v})\} \ . \tag{6}$$

where $C(\bar{v}) = H_1(v_1) + H_2(v_2 - b\mu) + G(\bar{v})$.

Under LIF, which retailer will be visited first on the $(t+1)^{st}$ allocation cycle depends on the retailer inventory positions at the instant after the tth allocation decision less each retailers' demands in the subsequent $(m-a)$ periods when the $(t+1)^{st}$ routing decision will be made. Define $p_i(\bar{v}, u_i)$ as the probability that R_i will be the first retailer on the $(t+1)^{st}$ route given $\bar{v}$ and u_i, the demand at Ri between the tth allocation decision and the $(t+1)^{st}$ routing decision. Under LIF, R_1 (R_2) will be visited first on the $(t+1)^{st}$ route if and only if R_1 (R_2) has the smaller inventory position at the time of the $(t+1)^{st}$ routing decision; that is, if and only if $v_1 - u_1 < v_2 - u_2$ ($v_2 - u_2 < v_1 - u_1$), which is equivalent to $u_2 < u_1 + v - 2v_1$ ($u_1 < u_2 + v - 2v_2$). Therefore, $p_i(\bar{v}, u_i) = \Phi^{m-a}(u_i + v - 2v_i)$. Without loss of generality, assume that R_1 is the first retailer on the tth route. Then,

$$C(\bar{v}) = H_1(v_1) + H_2(v_2 - b\mu) + L^{m-a}(v_1) + E[L^{m-a-b}(v_2 - \delta^b)]$$
$$+ \sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[\begin{array}{c} p_i(\bar{v},\delta)(L^a(v_i - \delta) - (v_i - m\mu)ah) \\ + (1 - p_i(\bar{v},\delta))(L^{a+b}(v_i - \delta) - (v_i - (m+b)\mu)(a+b)h) \end{array} \right] \phi^{m-a}(\delta) d\delta \qquad (7)$$

The first and second terms in (7) are the on-vehicle inventory-holding costs charged to R_1 and R_2, respectively. The third term is the total expected inventory-holding and backorder costs at R_1 during the first $(m-a)$ periods of the tth allocation cycle. The fourth term is that at R_2 during the first $(m-a-b)$ periods (the tth allocation cycle of R_2 starts b periods after the allocation decision). The last term represents the sum of total expected costs during the remaining periods of the tth allocation cycle and the credit (when there are leftovers) or charge (when there are backorders) at the end of the allocation cycle according to the cost reallocation scheme explained in Appendix 2.

In order to simplify the problem, we assume that for each retailer, backorders occur only in the last period of its allocation cycle. Jnsson and Silver (1987) demonstrate that the last-period-backorder assumption is appropriate for systems with high service levels. Under the last-period-backorders assumption,

$$L^k(x) = \sum_{i=1}^{k-1} (x - i\mu) + l^k(x)$$
$$= \frac{k(k-1)h\mu}{2} + (x - k\mu)(k-1)h + l^k(x) \qquad (8)$$

Correspondingly, $C(.)$ function in (7) becomes as follows:

(1) If $m \leq a$

If $m \leq a$, then at the time of the allocation decision, the $(t+1)^{st}$ route is already known or decided simultaneously. Correspondingly, the P_i's in (7) will be known at the time of allocation. In particular, if R_1 is the first retailer on the $(t+1)^{st}$ route, then $p_1 = 1$ and $p_2 = 0$, for $\forall \delta$, and if R1 is the second retailer on the (t+1)st route, =0 and =1, for . Therefore, MAP becomes the fixed-route allocation problem in Kumar et al. (1995). In other words, if, $m \leq a$, $C(\bar{v})$ is known to be a convex function of v_1 (Kumar et al., 1995).

(2) If $m > a$

If $m > a$, under the last-period-backorder assumption,

$$C(\bar{v}) = (2a+b)m\mu h + m(m-1)\mu h + (v-(2m+b)\mu)(m-1)h$$
$$+\sum_{i=1}^{2}\int_{-\infty}^{\infty}(p_i(\bar{v},\delta)l^a(v_i-\delta)+(1-p_i(\bar{v},\delta))l^{a+b}(v_i-\delta))\phi(\delta)d\delta \tag{9}$$

(see Appendix 3 for details).

Expression (9) is quite different from that in the corresponding static-route case because at the time of the tth allocation decision, the $(t+1)^{st}$ route is probabilistic. One consequence of this is that $C(.)$ in (9) is not necessarily convex on the interval $[-\infty,\infty]$. In particular, despite the fact that for any given δ, both $l^a(.)$ and $l^{a+b}(.)$ are convex, the products $p_i(.)\cdot l^a(.)$ and $p_i(.)\cdot l^{a+b}(.)$ are not necessarily convex. Indeed, it is possible to construct parameterizations where $C(.)$ isn't convex.

This possible non-convexity complicates the determination of the optimal allocation, thereby requiring a numerical search to find it. However, this search is simplified somewhat because $C(.)$ is symmetric with respect to $v_1=\frac{v}{2}$; that is, one gets the exactly same function by plugging $(v_1, v-v_1)$ or $(v-v_1, v_1)$ into (9). Hence: (1) one only needs to search the first half-interval $\left[-\infty, \frac{v}{2}\right]$ for an optimal allocation; and, more importantly; (2) equal allocation (i.e., $v_1=v_2=\frac{v}{2}$) will be always either a local minimum or a local maximum. In other words, if equal allocation is the only local minimum on the first half-interval, then equal allocation is the global minimum. [4]

Although we were not able to establish $C(.)$'s convexity on $[-\infty, \infty]$ nor its

4) Note further that if $C(.)$ is unimodal on the first half-interval, then only one local minimum will exist on the first half-interval. This local minimum would, of course, be the global minimum. However, if $C(.)$ isn't unimodal on the first half-interval, then $C(.)$ may have multiple local minima and maxima there. Although we could not prove its unimodality, $C(.)$ was observed to be unimodal on the first half-interval in all 128 parameter sets used in our computational study; that is, $C(.)$ had a unique global minimum.

unimodality on the half-interval $\left[-\infty, \frac{v}{2}\right]$, we have been able to derive the following intuitively-appealing first-order (i.e., necessary) condition for an optimal allocation policy: Let $P_i(\bar{v})$ be the probability that there will be no leftovers at R_i at the end of the tth allocation cycle. Let v_i^*, $i=1,2$, be the v_i's which solve MAP and $\bar{v}^* = (v_1^*, v_2^*)$.

LEMMA 3: Under the last-period-backorder assumption, if $C(\bar{v})$ is continuous and differentiable, then

$$P_1(\bar{v}^*) = P_2(\bar{v}^*) \tag{10}$$

Proof: [See Appendix 4].

Condition (10) says that in an optimal allocation, both retailers have the same probability of no leftovers, or, equivalently, the same stock-out probability. Since equal allocation satisfies condition (10), then equal allocation will be the optimal allocation provided $C(.)$ is unimodal. However, since is not necessarily unimodal, equal allocation may not yield the global minimum. It could, in fact yield a local maximum of $C(.)$. In order to identify all allocations satisfying (10), we employ a linear search procedure.

As noted above, $v_1^* \in [-\infty, \frac{v}{2}]$. Lemma 4 further narrows the interval of search to $[v_f, \frac{v}{2}]$, where v_f is an optimal inventory position at R_1 at the instant after the allocation decision when the next route is fixed as $R_1 \rightarrow R_2$ (which means that the delivery vehicle visits R_1 first).

LEMMA 4

$$v_f \leq v_1^* \leq \frac{v}{2}$$

Proof: [See Appendix 5].

Ⅷ. Heuristic Allocation Policies for N=2 Retailers

This section and the next develop heuristic allocation policies under the assumption that retailer demand each period follows a normal distribution with mean μ and standard deviation σ. All of the heuristics, of course, employ the least-inventory-first (LIF) routing.

As noted above, one of the difficulties in finding an optimal allocation is that total demand during the allocation cycle (i.e., cycle demand) at each retailer isn't normal, even though retailer demand each period is normal. Let d_i, $i=1,2$, be demand at R_i between the tth allocation decision and the end of the tth cycle for the retailer. Although d_i doesn't follow a normal distribution, Heuristic 1 assumes that the d_i's are normally distributed with mean $\mu_1=(m+\bar{p}_1 b)\mu$, $\mu_2=(m+\bar{p}_2 b)\mu$ and standard distribution σ_1, σ_2, where $\bar{p}_i$ is the probability that R_i is the second retailer on the $(t+1)^{st}$ route. Note that $\bar{p}_i=1-\Phi^0(v-2v_i)$, where $\Phi^0(.)$ is the cumulative distribution function of $N(0,\sigma\sqrt{2(m-a)})$.

Correspondingly, the following allocation satisfies (10):

$$\frac{v_1-\mu_1}{\sigma_1}=\frac{v_2-\mu_2}{\sigma_2}=\frac{v-(\mu_1+\mu_2)}{\sigma_1+\sigma_2} \tag{11}$$

Calculating the σ_i^2's, unfortunately, requires a numerical evaluation. Instead, we estimate σ_i^2 using $s_i^2=(m+\bar{p}_i\cdot b)\sigma^2$. To summarize: under Heuristic 1, the allocation satisfies

$$\frac{v_1-\mu_1}{s_1}=\frac{v_2-\mu_2}{s_2}=\frac{v-(\mu_1+\mu_2)}{s_1+s_2}\text{, where } s_i=\sigma\sqrt{m+\bar{p}_i\cdot b}\ . \tag{12}$$

Fixed-Route Allocation (Heuristic 2) allocates as if the next route were fixed as $R_1 \rightarrow R_2$ (but LIF is used for actual routing). In the fixed-route allocation, v is allocated in the following way (Kumar et al., 1995):

$$v_1 = m \cdot \mu + \frac{\sqrt{m}(v-(2m+b)\mu)}{\sqrt{m}+\sqrt{m+b}} \text{ and } v_2 = (m+b) \cdot \mu + \frac{\sqrt{m+b}(v-(2m+b)\mu)}{\sqrt{m}+\sqrt{m+b}}.$$

Equal Allocation (Heuristic 3) allocates exactly $\frac{v}{2}$ to each retailer.

Heuristic 4 uses the better of the fixed-route allocation and the equal allocation. More specifically, for a given set of model parameters, the expected costs of both the fixed-route and the equal allocation are estimated by computer simulation (each allocation policy is used throughout the entire simulation run), and the one with smaller total expected costs is chosen.

These four heuristics are compared through computer simulation in Section 9.

Ⅸ. Numerical Study of the Two-Retailer Symmetric Case

In this section, we compare the computer-simulated performance of the optimal myopic allocation policy and the four heuristic allocation policies both to measure the risk-pooling benefits of dynamic routing and to test the effectiveness of the heuristics. In testing these policies, Kumar et al.'s method is used to compute system-replenishment quantities. In other words, the warehouse makes system-replenishment decisions as if the next route were fixed and the same as the current route. In this case, under the allocation assumption, the optimal system-replenishment policy is a base-stock policy. The same base-stock policy is used for the other five distribution policies. Correspondingly, a total of seven distribution policies (i.e., combined routing and allocation policies), as defined in Table 1, were evaluated. D1 is Kumar et al.'s: The delivery route is fixed, and allocation is the optimal dynamic allocation given this fixed route (Kumar et al., 1995). D2 is MAP: LIF (least-inventory-first) is used for routing, while allocation is the optimal dynamic allocation given LIF. The amount allocated is determined by computer search of all points satisfying the first-order condition. For D3-D6, the routing policy is LIF but the allocation policy is one of the heuristics.

Table 1 Six Distribution Policies

Distribution	Routing	Allocation
D1	Fixed	Optimal for the fixed-route case
D2	LIF	Optimal for the dynamic-routing case
D3	LIF	Normal approximation
D4	LIF	Fixed-route allocation
D5	LIF	Equal allocation
D6	LIF	Better of fixed-route or equal

A total of 128 different system parameter sets were simulated with m, μ, and h fixed at 4, 100, and 1, respectively. The remaining four system parameter values were varied as follows: a=0,1,2,3, b=1,2,3,4, σ=20,50,70,100, and ρ=10,15. The simulation estimated total expected costs per cycle, total expected inventory-holding costs per cycle, total expected backorder costs per cycle, and the probability that negative allocation is prescribed by MAP for each case. Given each parameter set, every distribution policy was simulated for 300,200 allocation cycles. Since the purpose of the simulation is to compare the performances of different distribution polices, every distribution policy was simulated with the same initial random-number seed; that is, every distribution policy was simulated using the same demand realizations. The first 200 cycles were used to eliminate the effect of any initial conditions and next 300,000 allocation cycles were used to obtain the point estimates of the four statistics reported below. Note that although negative allocations are allowed in MAP, negative allocations were not allowed in the simulation; that is, whenever MAP prescribed a negative allocation to R_i, no units were allocated to R_i, and all the units on the vehicle were allocated to the other retailer. Moreover, despite the end-of-cycle-backorders assumption, backorder costs were charged each period whenever there were outstanding backorders (not just at the end of allocation cycle). Under D2, v_i^* is determined by computer search of v_i's satisfying the first-order condition (10) over the interval. The simulation results are summarized in Table 2 through 4.

Table 2 % Reduction in Estimated Total Costs per Cycle(D1 vs. D2)

	average	maximum
all 128 sets	1.9%	12.0%
when $\sigma=20$	0.04	0.5
when $\sigma=50$	1.2	7.0
when $\sigma=70$	2.2	9.2
when $\sigma=100$	4.2	12.0

Table 3 % Reduction in Estimated Inventory-Holding Costs per Cycle(D1 vs. D2)

	average	maximum
all 128 sets	1.2%	1.7%
when $\sigma=20$	0.0	0.05
when $\sigma=50$	0.1	0.8
when $\sigma=70$	0.2	1.2
when $\sigma=100$	0.5	1.7

Table 3 % Reduction in Estimated Backorder Costs per Cycle(D1 vs. D2)

	average	maximum
all 128 sets	6.2%	35.6%
when $\sigma=20$	0.4	5.3
when $\sigma=50$	5.1	34.5
when $\sigma=70$	7.7	34.5
when $\sigma=100$	11.4	35.6

Fixed vs. Dynamic Routing (D1 vs. D2)

The simulation results indicate a significant benefit from combining LIF with the optimal dynamic allocation for some parameter sets. Let HC_j and BC_j be the estimated inventory-holding and backorder costs per cycle of distribution policy j, respectively, and $TC_j = HC_j + BC_j$. Table 2 displays $\left[\frac{TC_1 - TC_2}{TC_1}\right] \times 100$, the % reduction in estimated total costs per cycle by using D2 instead of D1. Table 3 and 4 show the % reduction in estimated inventory-holding and estimated backorder costs per cycle, respectively. The % reduction in estimated total costs per cycle was 1.9% on average and had the largest value (12.0 %) in the case when $a=0$, $b=2$, $\sigma=100$, and $p=10$. The % reduction in estimated inventory-holding costs per cycle was 0.2% on average and had a maximum

value of 1.7%. Estimated backorder costs per cycle were reduced 6.2% on average and a maximum value of 35.6%. Note that the reduction in expected inventory-holding costs per cycle was very small, probably because inventory-holding cost is mainly determined by system-replenishment decisions, which are identical in all six distribution policies simulated. The % reduction in estimated total costs per cycle was much smaller than that in estimated backorder costs per cycle because the portion of estimated inventory-holding costs in estimated total costs per cycle was larger than that of expected backorder costs (more than 70% on average).

These reductions (in all three categories) were greater for cases with small a (and maximized when $a=0$). This can be interpreted as follows: Under dynamic routing, the $(t+1)^{st}$ route will be determined $(m-a)$ periods after the tth allocation decision. Since the risk pooling is over these $(m-a)$ periods, the smaller a is (i.e., the later the next route is determined), the more risk pooling there will be from dynamic routing.

The effect of b on the benefits from dynamic routing was not monotonic like that of a. Instead, the benefits from dynamic routing were observed to be largest for intermediate values of b: e.g., at $b=1$ in some cases and at $b=2$ in the other cases. We interpret this as follows: as b increases, it affects the benefits from dynamic routing in two different ways. First, when an imbalance in retailers' inventories is observed at the time of routing (i.e., LIF prescribes a route change), then, without dynamic routing, we wait longer to fix this imbalance in the case with large b. Therefore, the benefits from dynamic routing will be greater for large b than for small b. In other words, the benefits will be small when b is small. In particular, when $b=0$, there will be no benefits from using dynamic routing because the tth allocation cycle ends at the same time for both retailers, regardless of which is visited first on the $(t+1)^{st}$ route. Second, as b increases, a route change will be less likely (i.e., the benefits from dynamic routing will be smaller): As b increases, with the other parameters fixed, at optimum, the difference between allocations to R_2 and R_1

will increase, and result in fewer route changes. When we consider only the second effect, the benefits of dynamic routing are small for large b. Indeed, when b is large enough, the optimal allocation to the first retailer is close to its fixed-route allocation v_f, since the probability of changing the route will be very close to zero. We believe that what we observe is due to a combination of these two effects.

Heuristic Allocation Policies

The simulation results indicate that four heuristic allocation policies (Heuristic 1, 2, and 4 (with $\pi_c = 0.1$)) performed very close to optimal. The % difference in estimated total costs per cycle ($= \left[\frac{TC_i - TC_2}{TC_2}\right] \times 100$) were .02%, .48%, and .00% on average, and at most .25%, 4.18%, and .35% for Heuristic 1, 2, and 4, respectively. The % difference in estimated backorder costs per cycle ($= \left[\frac{BC_i - BC_2}{BC_2}\right] \times 100$) were .06%, 1.93%, and .00% on average, and at most .69%, 18.83%, and 2.07% for Heuristic 1, 2, and 4, respectively. These small differences suggest that a simple heuristic allocation policy can be used to achieve a near-optimal performance when combined with LIF routing. Heuristic 1 performed very well for all the cases simulated.

We did t-tests to check the validity of our claim that the heuristic allocations performed near optimally. We used the four different hypotheses on the % difference in estimated total costs per cycle between the optimal and heuristic allocation policies. The 4 different null hypotheses used were that the % difference is greater than 0, 1, 2, and 5%, respectively.[5] When Heuristic 1, or 4 was used, we rejected the null hypothesis that the difference is greater than

5) The same simulation runs which were used to obtain point estimates were used to compute t-statistics for the hypotheses: 300,000 allocation cycles (excluding the first 200 allocation cycles to eliminate the effect of initial condition) were equally divided into ten 30,000 allocation cycles to obtain 10 estimates of total expected costs per cycle. We checked the autocorrelations among the 10 estimates using Durbin-Watson test with 99 % confidence interval and found no significant autocorrelations in any parameter set.

1% for all the parameter sets. When Heuristic 2 was used, we rejected the null hypothesis that the difference is greater than 2% in about 87% of parameter sets and rejected the null hypothesis that the difference is greater than 5% in all parameter sets.

The simulation results suggest that LIF alone explains more than two-thirds of total benefits from dynamic routing; that is, using LIF with fixed-route allocation (D4) explains on average about 70% of the difference in estimated total costs per cycle between D1 and D2. D4 also explains on average about 70% of the difference in estimated backorder costs per cycle between D1 and D2.

Fixed-route allocation worked best in the cases when $\sigma = 20$, and equal allocation worked well in the cases when $\sigma = 100$, but did badly in the cases with smaller σ's. We can interpret this as follows: As σ increases with everything else fixed, the probability of changing the routes will increase, that is, optimal allocations will move away from fixed-route allocation to equal allocation. Therefore, in the cases with small σ, optimal allocations are close to the fixed-route allocation, whereas they are close to the equal allocation in the cases with large σ. Heuristic 4 (the better of fixed-route allocation and equal allocation) performed very well for all the values of σ.

X. Concluding Remarks

In the above, we have analytically shown that LIF is the optimal routing policy, and that under the allocation assumption, a myopic policy is optimal. We have also derived the first-order optimality condition of myopic allocation and the interval that always includes the optimal allocation to the first etailer. We have empirically demonstrated that dynamic routing yields lower expected costs than static routing, regardless of whether or not their respective allocation assumptions are valid. However, our numerical study also indicates that the magnitude of the cost savings depends on some system parameters;

i.e., in the medium-to-large demand-variance cases, it is significant, but in the small demand-variance cases, it is very small. Also, static-route heuristics for allocation has been shown to be very effective.

From a management perspective our results suggest that using LIF alone (without optimizing allocation decisions) explains most of benefit from dynamic routing. Finally, the increased costs of operating dynamic-routing policy must be considered, since such a policy involves building more rigid information system. This is because under dynamic routing, retailer net inventories (or demand) must be assessed and transferred up to N-1 times per cycle. This transfer must be made to the warehouse, which must then communicate it to the vehicles along the route, which must then compute the allocations and make the routing decisions. Consequently, dynamic policies require a more sophisticated information and decision infrastructure.

We close with brief comments about possible extension and future research. First, our analysis can be easily extended to the case with fixed-ordering cost at the warehouse. The difference is in the form of the system-replenishment policy. Second, from the previous discussion, we can see that optimizing the distribution policy in the N> 2 retailer case is impractical. Hence, we had better focus on developing a heuristic that results in the good lower bound on the true benefit of dynamic routing under the optimal policy. Finally, we can develop effective heuristics for the non-symmetric case.

Appendix 1

We prove (Lemma 1) in this appendix.

Proof:

Note that given any fixed sequence of system-replenishment quantities and retailer allocations over time, each retailer's net inventory $(a+b)$ periods after the tth routing decision is invariant to that routing decision. Hence, the expected discounted costs in the interval between $(a+b)$ periods after the tth

routing decision and the $(t+1)^{st}$ routing decision are fixed. Define the *single-cycle routing problem* (SCRP) as the problem in which the tth route is chosen to minimize the sum of total expected discounted inventory-holding and backorder costs between the tth routing decision and $(a+b)$ periods after that routing decision given retailer inventory positions at the time of the tth routing decision. Let s_1 and s_2 be the inventory position at R_1 and R_2 at the moment of the tth routing decision, respectively. If R_1 is visited first, then R_1's current allocation cycle will end a periods later and that of $R_2(a+b)$ periods later. We can do the opposite by going to R_2 first, and to R_1 later. Let TC_i be total expected discounted costs during the $(a+b)$ periods following the routing decision when R_i is visited first. Let $l^k(v)$ be expected inventory-holding and backorder costs at the end of the kth period given net inventory v at the beginning of the first period and assuming no additional delivery. Also let z_i, $i=1,2$ be the tth allocation to R_i. Assume that on-vehicle inventory-holding cost is allocated to the retailer to whom a unit is allocated. Then,

$$TC_1 = (\sum_{k=1}^{a}\alpha^k l^k(s_1) + \sum_{k=1}^{a}\alpha^k z_1 h + \sum_{k=a+1}^{a+b}\alpha^k l^k(s_1+z_1) + \sum_{k=1}^{a+b}\alpha^k l^k(s_2) + \sum_{k=1}^{a+b}\alpha^k z_2 h) ,$$

and $$TC_2 = (\sum_{k=1}^{a+b}\alpha^k l^k(s_1) + \sum_{k=1}^{a+b}\alpha^k z_1 h + \sum_{k=1}^{a}\alpha^k l^k(s_2) + \sum_{k=1}^{a}\alpha^k z_2 h + \sum_{k=a+1}^{a+b}\alpha^k l^k(s_2+z_2)) .$$

After some rearrangement, $TC_2 - TC_1$ is

$$\sum_{k=a+1}^{a+b}\alpha^k((l^k(s_1)+z_1h-l^k(s_1+z_1))-(l^k(s_2)+z_2h-l^k(s_2+z_2))) . \qquad \text{(A1-1)}$$

Assume $s_1 < s_2$; that is, at the moment of the tth routing decision R_2 has more inventory than R_1. Then, (A1-1)>0, since by the convexity of $l^k(.)$, it is easily shown that,

$$(l^k(s_1)+z_1h-l^k(s_1+z_1)) \geq (l^k(s_2)+z_2h-l^k(s_2+z_2)) \text{ for } k=a+1,\ldots, a+b$$

Hence, LIF minimizes total expected discounted costs of the SCRP.

Now, we prove the optimality of LIF routing in the infinite-horizon problem.

Let (O,A,R) be an arbitrarily chosen distribution policy (= a joint system-replenishment (O), allocation (A), and routing (R) policy) over the infinite horizon. Suppose that for some demand realization, (O,A,R) does not follow LIF. Specifically, suppose that the tth route does not follow LIF, given the set of demand realizations D to that point in time. Let (O,A,R') be the distribution policy which makes the same system-replenishment, allocation, and routing decisions as (O,A,R) except that (O,A,R') follows LIF in the tth routing decision. As noted above, given D, total expected discounted costs of the SCRP will be reduced by using (O,A,R'), while total expected discounted costs in the other periods remain the same. Therefore, the total expected discounted costs of (O,A,R) are greater than those of (O,A,R').

Appendix 2

Derivation of (1):

Define $\bar{f}(\overline{X}_t)$ to be total expected discounted costs at the time of the tth system-replenishment decision given the state variables $\overline{X}_t$ and LIF. Let $v_{[j]t}$, $j=1,\ldots,N$ be inventory position of the jth retailer on the tth route at the instant after the tth allocation decision. Then,

$$\bar{f}_t(\overline{X}_t)=\min_{q_t\geq 0}\{cq_t+E[\min_{z_{[1]t}}\{H_1(z_{[1]t})+\alpha^a L^{(N-2)b}(v_{[1]t})+E[\min_{z_{[2]t}}\{H_2(z_{[2]t})+\alpha^{a+b}L^{(N-3)b}(v_{[2]t})$$
$$+E[\ldots E[\min_{z_{[N-1]t}}\{H_{N-1}(z_{[N-1]t})+H_N(z_{[N]t})+\alpha^{a+(N-2)b}g(\overline{IP}_t)\}]\}]\cdots]\}+\alpha^m E[f_{t+1}(\overline{X}_t+\overline{Z}_t-\bar{\delta}^m)]\} \tag{A2-1}$$

Subject to: (2), (3), and (4)

At the instant after the jth allocation decision, the on-vehicle inventory-holding cost $H_j(z_{[j]t})\,(\equiv h\frac{\alpha(1-\alpha^{a+(j-1)b})}{1-\alpha}z_{it})$ and the expected discounted total costs at the jth retailer between the jth and $(N-1)^{st}$ allocation decisions, are charged to the jth retailer. At the instant after the $(N-1)^{st}$ allocation decision, when all the allocations z_{it}'s, $i=1,\ldots,N$, are known, we can write the

expression for $g(\overline{IP}_t)$, the total expected discounted costs over all the retailers during the remaining periods of the tth allocation cycle, where $\overline{IP}_t$ is the vector of retailer inventory positions at the instant after the next-to-last $(=(N-1)^{st})$ allocation decision.

By reallocating some cost terms, we redefine the dynamic programming problem. Let $w_{[j]t}$ be net inventory of the jth retailer on the $(t+1)$st route at the end of the tth allocation cycle. We will redefine $f_t(.)$ by rearranging terms in DP. Since $H_j(.)$ is a linear function, it is easily shown that:

$$\begin{aligned} H_j(z_{[j]t}) &= H_j(v_{[j]t}) - H_j(v_{[j]t} - z_{[j]t}), \text{ for } j = 1,\ldots,N-1 \\ &= H_j(v_{[j]t} - b\mu) - H_j(v_{[j]t} - b\mu - z_{[j]t}), \text{ for } j = N, \text{ for } t = 1,\ldots,T. \end{aligned} \tag{A2-2}$$

Note that by definition, $w_{[j](t-1)} = (v_{[j]t} - z_{[j]t})$, for $j=1,\ldots,N-1$, and, $w_{[j](t-1)} = (v_{[j]t} - \delta^b - z_{[j]t})$, for $j=$N. Let $K_t(\overline{X}_t)$ be the expected sum of the second terms on the right-hand side of (A2-2) for $j=1,\ldots,N$. Then,

$$K_t(\overline{X}_t) = E[-H_1(w_{[1](t-1)}) + E[-H_2(w_{[2](t-1)}) + \ldots + E[-H_N(w_{[N](t-1)})]\ldots]].$$

Since $K_t(\overline{X}_t)$ is not a function of the allocation decisions during the tth allocation cycle, we then reallocate $K_t(\overline{X}_t)$ in DP without changing its solution: we subtract $K_t(\overline{X}_t)$ from the expected costs of the tth allocation cycle and add it to the expected costs of the $(t-1)^{st}$ allocation cycle. We can reallocate the purchasing costs in the same way. By reallocating $\sum_{j=1}^{N} -cw_{[j](t-1)}$ from the tth to the $(t-1)^{st}$ allocation cycle for $\forall t$, the expected discounted purchasing cost charged to the tth allocation cycle is $cq_t - E[\sum_{j=1}^{N} -cw_{[j](t-1)}] + \alpha^m E[\sum_{j=1}^{N} -cw_{[j]t}]$ Since, $\sum_{j=1}^{N} w_{[j]t} = \sum_{i=1}^{N} w_{[j](t-1)} + q_t - \eta$ where η is the system demand during the tth allocation cycle, $E[\sum_{i=1}^{N} w_{i(t-1)}] = \sum_{i=1}^{N} x_{it} - \mu(Na + \frac{N(N-1)}{2}b)$ and, the expected purchasing cost in the tth allocation cycle after the reallocation is equal to $(1-\alpha^m)cy_{ft} + \alpha^m cNm\mu - (1-\alpha^m)\mu(Na + \frac{N(N-1)}{2}b)$, where $y_t = q_t + \sum_{i=1}^{N} x_{it}$. Denote this purchasing cost as $c(y_t)$.

Under the reallocation scheme we can redefine $f_t(.)$ as follows: Define

$$G(\overline{IP}_t) = \alpha^{a+(N-2)b} g(\overline{IP}_t) + \alpha^m K_{t+1}(\overline{X}_{t+1}) \tag{A2-3}$$

Then, (A2-1) is equivalent to:

$$f_t(\overline{X}_t) = \min_{q_t \geq 0}\{c(y_t) + E[\min_{z_{[1]t}}\{H_1(z_{[1]t}) + \alpha^a L^{(N-2)b}(v_{[1]t}) + E[\min_{z_{[2]t}}\{H_2(z_{[2]t}) + \alpha^{a+b} L^{(N-3)b}(v_{[2]t})$$
$$+ E[\ldots E[\min_{z_{[N-1]t}}\{H_{N-1}(z_{[N-1]t}) + H_N(z_{[N]t}) + G(\overline{IP}_t)\}]\}]\cdots]\} + \alpha^m E[f_{t+1}(\overline{X}_t + \overline{Z}_t - \overline{\delta}^m)]\}$$

Subject to: (2), (3), and (4).

Appendix 3

<u>The $C(.)$ function when $m > a$:</u>

When $m > a$,

$$C(\overline{v}) = v_1 ah + (v_2 - b\mu)(a+b)h + \int_{-\infty}^{\infty}\{p_1(\overline{v},\delta)[\frac{m(m-1)h\mu}{2} + (v_1 - \delta - a\mu)(m-1)h + l^a(v_1 - \delta) - (v_1 - \delta - a\mu)ah]$$
$$+(1-p_1(\overline{v},\delta))[\frac{(m+b)(m+b-1)h\mu}{2} + (v_1 - \delta - (a+b)\mu)(m+b-1)h + l^{a+b}(v_1 - \delta) - (v_1 - \delta - (a+b)\mu)(a+b)h]\}\phi^{m-a}(\delta)d\delta$$
$$+\int_{-\infty}^{\infty}\{p_2(\overline{v},\delta)[\frac{(m-b)(m-b-1)h\mu}{2} + (v_2 - \delta - a\mu)(m-b-1)h + l^a(v_2 - \delta) - (v_2 - \delta - a\mu)ah]$$
$$+(1-p_2(\overline{v},\delta))[\frac{m(m-1)h\mu}{2} + (v_2 - \delta - (a+b)\mu)(m-1)h + l^{a+b}(v_2 - \delta) - (v_2 - \delta - (a+b)\mu)(a+b)h]\}\phi^{m-a}(\delta)d\delta \tag{A3-1}$$

Let $\overline{p}_i = E[p_i(\overline{v},\delta)]$. Then,

$$C(\overline{v}) = v_1 ah + (v_2 - b\mu)(a+b)h + \overline{p}_1\frac{m(m-1)h\mu}{2} + (1-\overline{p}_1)\frac{(m+b)(m+b-1)h\mu}{2}$$
$$+(v_1 - m\mu)(m-a-1)h - (1-\overline{p}_1)b(m-a-1)\mu h + \int_{-\infty}^{\infty}\{p_1(\overline{v},\delta)l^a(v_1 - \delta) + (1-p_1(\overline{v},\delta))l^{a+b}(v_1 - \delta)\}\phi^{m-a}(\delta)d\delta$$
$$+\overline{p}_2\frac{(m-b)(m-b-1)h\mu}{2} + (1-\overline{p}_2)\frac{m(m-1)h\mu}{2} + (v_2 - m\mu)(m-a-b-1)h - (1-\overline{p}_2)b(m-a-b-1)\mu h$$
$$+\int_{-\infty}^{\infty}\{p_2(\overline{v},\delta)l^a(v_2 - \delta) + (1-p_2(\overline{v},\delta))l^{a+b}(v_2 - \delta)\}\phi^{m-a}(\delta)d\delta \tag{A3-2}$$

After some simplifications and using $\overline{p}_1 = 1 - \overline{p}_2$,

$$C(\overline{v}) = (2a+b)m\mu h + m(m-1)\mu h + (v - (2m+b)\mu)(m-1)h$$
$$+\int_{-\infty}^{\infty}\{p_1(\overline{v},\delta)l^a(v_1 - \delta) + (1-p_1(\overline{v},\delta))l^{a+b}(v_1 - \delta)\}\phi^{m-a}(\delta)d\delta$$
$$+\int_{-\infty}^{\infty}\{p_2(\overline{v},\delta)l^a(v_2 - \delta) + (1-p_2(\overline{v},\delta))l^{a+b}(v_2 - \delta)\}\phi^{m-a}(\delta)d\delta$$

$$= (2a+b)m\mu h + m(m-1)\mu h + (v - (2m+b)\mu)(m-1)h$$
$$+\sum_{i=1}^{2}\int_{-\infty}^{\infty}(p_i(\overline{v},\delta)l^a(v_i - \delta) + (1-p_i(\overline{v},\delta))l^{a+b}(v_i - \delta))\phi(\delta)d\delta$$

Appendix 4

We prove (Lemma 3) in this appendix.

Proof:

(i) If $m \leq a$ - see Kumar et al. (1995).

(ii) If $m > a$

Under the last-period-backorders assumption, by taking the first derivative of $C(.)$ with respect to v_1, we get

$$\begin{aligned}
\frac{\partial C(\bar{v})}{\partial v_1} = \int_{-\infty}^{\infty} \{ & -\Phi^{m-a}(\delta - v_1 + v_2)(1-\Phi^a(v_1-\delta)) \\
& -2\phi^{m-a}(\delta - v_1 + v_2)\int_{v_1-\delta}^{\infty}(\eta-(v_1-\delta))\phi^a(\eta)d\eta \\
& -(1-\Phi^{m-a}(\delta - v_1 + v_2))(1-\Phi^{a+b}(v_1-\delta)) \\
& +2\phi^{m-a}(\delta - v_1 + v_2)\int_{v_1-\delta}^{\infty}(\eta-(v_1-\delta))\phi^{a+b}(\eta)d\eta \qquad \text{(A4-1)} \\
& +\Phi^{m-a}(\delta + v_1 - v_2)(1-\Phi^a(v_2-\delta)) \\
& +2\phi^{m-a}(\delta + v_1 - v_2)\int_{v_2-\delta}^{\infty}(\eta-(v_2-\delta))\phi^a(\eta)d\eta \\
& +(1-\Phi^{m-a}(\delta + v_1 - v_2))(1-\Phi^{a+b}(v_2-\delta)) \\
& -2\phi^{m-a}(\delta + v_1 - v_2)\int_{v_2-\delta}^{\infty}(\eta-(v_2-\delta))\phi^{a+b}(\eta)d\eta\}\phi^{m-a}(\delta)d\delta
\end{aligned}$$

After applying the change of variables ($\delta = \delta' - v_1 + v_2$) for the sixth and eighth lines in (A4-1), they will be canceled out with the second and fourth lines. Since $p_1(\bar{v},\delta) = \Phi^{m-a}(\delta - v_1 + v_2)$ and $p_2(\bar{v},\delta) = \Phi^{m-a}(\delta + v_1 - v_2)$, we get

$$\begin{aligned}
& \int_{-\infty}^{\infty}\{(1-\Phi^a(v_1^*-\delta))p_1(\bar{v}^*,\delta)+(1-\Phi^{a+b}(v_1^*-\delta))(1-p_1(\bar{v}^*,\delta))\}\phi^{m-a}(\delta)d\delta \qquad \text{(A4-2)} \\
& = \int_{-\infty}^{\infty}\{(1-\Phi^a(v_2^*-\delta))p_2(\bar{v}^*,\delta)+(1-\Phi^{a+b}(v_2^*-\delta))(1-p_2(\bar{v}^*,\delta))\}\phi^{m-a}(\delta)d\delta
\end{aligned}$$

In (A4-2), $(1-\Phi^a(v_1^*-\delta))$ is the probability that the demand at R_1 between the $(t+1)^{\text{st}}$ routing decision and the end of the tth allocation cycle for R_1 (a periods after the routing decision) is greater than $(v_1^*-\delta)$, the net inventory at

R_1 at the instant of the $(t+1)^{st}$ routing decision. $(1-\Phi^{a+b}(v_1^*-\delta))$, $(1-\Phi^{a}(v_2^*-\delta))$, $(1-\Phi^{a+b}(v_2^*-\delta))$ can be interpreted similarly. The left-hand side and the right-hand side of (A5-2) are $P_1(\bar{v}^*)$ and $P_2(\bar{v}^*)$, respectively. This proves the Lemma.

Appendix 5

We prove (Lemma 4) in this appendix.

Proof:

(i) $v_1^* \le \frac{v}{2}$

Because of the symmetry of optimal allocations, there always exists v_1^* which is less than or equal to $\frac{v}{2}$.

(ii) $v_1^* \ge v_f$

We prove this by contradiction. Suppose that $\hat{v}_1 < v_f$ and $\hat{v}_1$ is an optimal allocation to R_1. Let $\hat{v}=(\hat{v}_1, v-\hat{v}_1)$. In the fixed-route case, the total expected costs function is convex and its value goes to infinity as v1 goes to infinity or minus infinity. Since its first derivative is - $P_1(\bar{v})$ + $P_2(\bar{v})$ like in the dynamic-route case, $-P_1(\hat{v})+P_2(\hat{v}) < 0$ when the fixed route is used; that is, $P_1(\hat{v}) > P_2(\hat{v})$ in the fixed-route case. Compared to the fixed-route case, under dynamic routing, $P_1(\hat{v})$ will increase and $P_2(\hat{v})$ will decrease: When LIF prescribes no route change, the probability of stockout at the end of the allocation cycle will remain the same at both retailers, but when LIF prescribes a route change, that probability at $R_1(R_2)$ will increase (decrease). Therefore, under dynamic routing, $P_1(\hat{v}) > P_2(\hat{v})$. $\hat{v}_1$ can not be optimal since it does not satisfy the first-order condition.

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