A Note on Partial Elasticities of Substitution Between Solar Energy and Conventional Fuels for Residential Heating and Cooling by Combined Energy Systems

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Assume that solar and conventional fuels are treated as inputs in a household energy production or mix for its heating and cooling purposes. Then, the household objective function is given by

$$Min\ C=\sum_{i=1}^{n} X_i P_i \quad (i=1, 2, \ldots, n) \tag{1}$$

subject to

$$Q_E=f(X_1, X_2, \ldots, X_n;\ \bar{K}) \tag{2}$$

where C=total energy costs for residential heating and cooling

X_i=energy inputs levels

P_i=energy prices

Q_E=energy output for a residential heating and cooling

$\bar{K}$=all other inputs treated as given.

Corresponding to the production function (2), we assume there exists a homothetic cost function such that

$$C_E=g(Q_E, P_g, P_0, P_e, P_s) \tag{3}$$

where P_g=price of gas, \$/10^6 BTU

P_o=price of oil, \$/10^6 BTU

P_e=price of electricity, \$/10^6 BTU

P_s=price of solar energy, \$/10^6 BTU

C_E=total energy cost.

C_E is homogeneous of degree one in prices. Shephard's lemma [3] holds for the cost function,

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$$\frac{\partial C_E}{\partial P_\gamma}=X_\gamma \quad (\gamma=g, o, e, s). \tag{4}$$

Let $F=\det\begin{bmatrix} 0, f_1, & \cdots, f_n \\ f_1, f_{11}, & \cdots, f_{1n} \\ \cdots & \cdots \\ f_n, f_{n1}, & \cdots, f_{nn} \end{bmatrix}$, and $f_i=\frac{\partial Q_E}{\partial X_i}$, $f_{ij}=\frac{\partial^2 Q_E}{\partial X_i \partial X_j}$ from equation (2).

Then, Allen's partial elasticity of substitution $\sigma_{K\gamma}$ between two factors of production, say energy K and γ $(K\neq\gamma)$ is defined as

$$\sigma_{K\gamma}=\frac{\sum_{i=1}^{n} X_i f_i}{X_K X_\gamma}(F^{-1})_{\gamma K} \tag{5}$$

where $(F^{-1})_{\gamma K}$ is the γK^{th} element of F^{-1}. From (5), $\sigma_{K\gamma}$ is symmetric; i.e.

$$\sigma_{K\gamma}=\sigma_{\gamma K}, \text{ for all } \gamma\neq K \text{ (see [4])}. \tag{6}$$

However, note that the inversion of the bordered Hessian F is a nonlinear transformation, which may increase the estimation errors of $\sigma_{K\gamma}$. But in the case of the cost function, $\sigma_{K\gamma}=\sigma_{\gamma K}$ can be obtained directly from the parameters of the function as shown below (see equation [14]).

First, we form the augmented objective function from equations (1) and (2):

$$\text{Min } Z=\sum_{i=1}^{n} X_i P_i + \lambda\{Q_E - f(X_1, X_2, \ldots, X_n : \bar{K})\}. \tag{7}$$

Then, the first-order conditions are

$$P_i-\lambda f_i=0 \qquad i=1, \ldots, n \tag{8}$$

$$f(X_1, X_2, \cdots, X_n)-Q_E=0. \tag{9}$$

Taking the total differential of the first order conditions and writing the terms after rearrangement in the matrix form, we have

$$\lambda\begin{bmatrix} 0 & f_1 \ldots f_n \\ f_1 & f_{11}\ldots f_{1n} \\ \vdots & \\ f_n & f_{n1}\ldots f_{nn} \end{bmatrix}\begin{bmatrix} d\lambda/\lambda \\ dX_1 \\ \vdots \\ dX_n \end{bmatrix}=\begin{bmatrix} dQ_E \\ dP_1 \\ \vdots \\ dP_n \end{bmatrix}. \tag{10}$$

By Cramer's rule,

$$\begin{bmatrix} d\lambda/\lambda \\ dX_1 \\ \vdots \\ dX_n \end{bmatrix} = \frac{1}{\lambda} F^{-1} \begin{bmatrix} dQ_E \\ dP_1 \\ \vdots \\ dP_n \end{bmatrix} \tag{11}$$

which implies

$$\frac{\partial X_\gamma}{\partial P_K} = \frac{1}{\lambda}(F^{-1})_{\gamma K}. \tag{12}$$

Substituting $f_i = \frac{P_i}{\lambda}$ from (8) and (12) into (5),

$$\sigma_{K\gamma} = \frac{\Sigma X_i f_i}{X_K X_\gamma} \lambda \frac{\partial X_\gamma}{\partial P_K} = \frac{\Sigma X_i P_i}{X_K X_\gamma} \frac{\partial X_\gamma}{\partial P_K}. \tag{13}$$

But note that the derivative of (4) with respect to P_K gives

$$\frac{\partial^2 C_E}{\partial P_\gamma \partial P_K} = \frac{\partial X_\gamma}{\partial P_K}. \tag{14}$$

Substituting (14) into (13)

$$\sigma_{K\gamma} = \sigma_{\gamma K} = \frac{\Sigma X_i P_i}{X_K X_\gamma} \frac{\partial^2 C_E}{\partial P_\gamma \partial P_K}. \tag{15}$$

It is apparent that (13) can be used to derive elasticities of substitution for given input levels and total costs, if the parameters of specific functional form of a cost function have been estimated. Note that (13) can be rewritten as

$$\sigma_{K\gamma} = \sigma_{\gamma K} = \frac{\eta_{\gamma K}}{\alpha_K} \tag{16}$$

where $\eta_{\gamma K} = \frac{\partial X_\gamma}{\partial P_K} \frac{P_K}{X_\gamma}$ and $\alpha_K = \frac{X_K P_K}{\Sigma X_i P_i}$ which is the share of factor K in total costs.

In this paper, a derivation method has been presented for partial elasticities of substitution between energy inputs in a household optimal energy production or mix for its heating and cooling needs by combined system of solar and conventional energy. A residential household is treated as a production unit of energy for its own consumption. The household responses of energy substitution between solar and conventional fuels are subject to empirical studies if relevant sets of data become available. A somewhat similar approach to the derivation of elasticities of substitution among factors in the case of "translog cost function" has been widely attempted recently for the manufacturing industry (for example, see [1] and [2]).

References

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