# A MATHEMATICAL MODEL FOR AGGREGATE PRODUCTION PLANNING

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For any organization, an effective utilization of available resources is a prerequisite of success. The aggregate production planning is a complex process through which the most efficient utilization of resources is sought within the given constraints imposed by environment and management policies. Thus, the economic significance of an effective production planning for the firm as well as for the nation as a whole has been well recognized.[1]

The most difficult problem in aggregate production planning is the change in demand over time. The fluctuations in demand can be absorbed by adopting one or a combination of the following strategies:

1. The production rate can be altered by changing the work force.
2. The production rate can also be altered by maintaining a constant labor force while introducing overtime or idle time.
3. The production rate may be kept on a constant level and the fluctuations in demand be met by altering the level of subcontracting.
4. The production rate may be kept constant and changes in demand be absorbed by changing the inventory level.

Since each of the above courses of action has certain costs associated with them, the problem faced by management is to make decisions that would bring minimum costs to the organization. This decision problem has been under extensive research, and several alternative methods have since

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(1) M. Anshen, C. Holt, F. Modigliani, J. Muth, and H.A. Simon, "Mathematics for Production Scheduling," *Harvard Business Review* (March-April, 1958).

been suggested. Yet, none of these prepared methods have been favorably accepted by industry despite the fact that they are far superior to the prevailing rule-of-thumb or simple judgment procedures. A good reason for this phenomenon may simply be that industry is not ready to accept the use of mathematical models for production planning. However, the more probable reason seems to be that the proposed models are gross over-simplifications of reality. The most serious weakness of these models is that they are not capable of reflecting preferences or policies of management in the analysis. Consequently, an effective application of these suggested methods is possible only at the expense of organizational policies. The purpose of this paper is to review the fundamental limitations of the existing methods and present a goal programming model which improves the quality of managerial decisions in aggregate production planning.

## The Goal Programming Model

Basically, three mathematical methods have been suggested as possible solutions to aggregate production planning. They are; the transportation methods of linear programming[2], the simplex methods of linear programming[3], and the linear decision rule model[4]. Although all of these methods have been under extensive criticism, the models using the simpler method have been accepted rather favorably. The reason for this seems to be that according to empirical studies the deterministic models have provided quite satisfactory results even under stochastic conditions[5].

(2) E.H. Bowman, "Production Scheduling by the Transportation Method of Linear Programming," *Operations Research,* Vol. 4, No. 1 (February, 1956), pp. 100–103.

(3) See F. Hanssman and S.W. Hess, "A Linear Programming Approach to Production and Employment Scheduling," *Management Technology,* No. 1 (January, 1960), pp. 46–51; R.E. McGarrah, *Production and Logistics Management* (New York: John Wiley & Sons, Inc., 1963).

(4) C. Holt, F. Modigliani, and H.A. Simon, "A Linear Decision Rule for Production and Employment Scheduling," *Management Science,* Vol. 2, No. 1 (October, 1955), pp. 1–30.

(5) See B. Dzielinski, C. Baker, and A. Manne, "Simulation Tests of Lot-Size Programming," Management Science, Vol. 9, No. 2 (January, 1963), pp. 229–253.

It seems that the difficulty associated with the general linear programming is not so much in its inability of closely representing the complex reality. Instead, the difficulty lies in obtaining cost information which is required in the application of this method. For example, it is not easy to determine costs associated with layoff of employees if we consider the loss in employee morale, goodwill with the union, and the public image of the firm. It may be equally difficult to determine correct costs of inventory if we consider opportunity loss associated with capital tie-up in inventory and stockouts.

The most frequently practiced way of handling this kind of problem has been simply to ask management to provide its best estimates of costs. However, it could be even more difficult to select the responsible managers who could actually provide such concrete cost estimates. In this situation, neither the linear programming model nor the heuristic computer method may be of great value to management.

If we assume, however, that management is at least capable of providing an ordinal measure for various costs or goals, we can use this information as the analytical framework for decision making. It is under these conditions that goal programming can be utilized most effectively for aggregate production planning.

Goal programming is a special type of linear programming. This method is capable of handling decision problems which deal with a single goal with multiple subgoals as well as problems with multiple goals with multiple subgoals[(6)]. In the conventional linear programming method, the objective function has to be unidimensional-either to maximize profits(effectiveness) or to minimize costs(sacrifice). This dimensional limitation of the

(6) For a detailed discussion of goal programming, see A. Charnes and W.W. Cooper, *Management Models and Industrial Applications of Linear Programming*(New York:John Wiley & Sons, Inc., 1961); Yuji Ijiri, *Management Goals and Accounting for Control*(Amsterdam: North-Holland Publishing Co., 1965).

objective function has been the major weakness of the simple linear programming.

The goal programming model handles multiple goals in multiple dimensions. Therefore, an arbitrary or often completely subjective conversion of other value measures to costs or profit is no longer necessary. For example, four hours of overtime in assembly line A or stockout of product B by the amount of 5 units does not have to be expressed in terms of estimated costs.

Often, goals set by management can be achieved only at the expense of other goals. Therefore, there is a need to establish a hierarchy of importance among these conflicting goals so that the low order goals are considered only after the higher rank goals are satisfied(or have reached the point beyond which no further improvements are possible. Goal programming enables us to treat multiple goals according to the importance assigned to them. If management provides an ordinal measure of goals involved in production planning in terms of their contributions or importance to the organization, we are able to solve aggregate production planning problems through goal programming.

In addition to the treatment of multiple incommensurable objectives, goal programming has distinct, characteristic difference from the simplex method of linear programming. In goal programming, instead of trying to maximize or minimize the objective criterion directly, we try to minimize deviations between goals and what we can actually achieve within the given constraints. In the simplex method, we call such deviations "slack variables," and they are used only as dummy variables. In goal programming, however, these slack variables are not only real but also are the only ones in the objective function. Our objective, then, is to minimize these deviational variables. When the goal programming model is formulated, the computational algorithm is almost the same as the minimization problem of the simplex method. The only difference is that in goal programming the simplex criterion is a matrix, whereas it is a single row in the ordinary simplex algorithm.

## An Example

Although a very complex production planning problem can be effectively solved by the goal programming model, in this paper a very simple example will be discussed for illustrative purposes. Let us consider the following example[7]:

> A production manager of a firm faces a problem of job allocation between his two assembly line teams, A and B. They are identical with each other except for their rates of processing. Team A produces one unit per hour and Team B one unit for every two hours. The normal daily working hours for both teams are 8 hours. Consequently, the normal daily production is 12 units. However, the production manager wants to achieve production level of 15 units per day. If overtime operation is necessary to attain the given production level, he wants to assign it to Team A and Team B in such a way that overtime operation for Team A beyond 2 hours should be avoided and the sum of total overtime should be as small as possible.
>
> He has some desire to avoid any idle time for both teams. But this is desired only after the above mentioned goals are satisfied. Finally, he is not concerned with any over-achievement in the production level once it passed the target.

In the above example, the production manager is to make a decision which would achieve his goals as closely as possible with the minimum sacrifice. Since overtime is allowed in this example, production may exceed the goal of 15 units. Therefore, the operational capacity can be expressed as:

(7) The example is from: Ijiri, *Ibid.*, pp. 48-49.

$$X_1+0.5X_2-d^+_1+d^-_1=15$$

where, $X_1$ is the number of operation hours in Team A, $X_2$ is the operation hours in Team B, $d^+_1$ is over-achievement in the production level, and $d^-_1$ is the under-acheivement of production in units. In this example, variables $d^-$ represent negative deviations from goals or under-achievement, whereas $d^+$ represents the positive deviation from the goal or over-achievement. It should be noted that $d^-_1$, $d^+_1=0$.

The operational capacity constraints for both teams can be written as:

$$X_1-d^+_2+d^-_2=8$$

$$X_2-d^+_2+d^-_3=8$$

where, $d^+_2$ and $d^+_3$ represent overtime operation in Teams A and B respectively, and $d^-_2$ and $d^-_3$ represent underutilization of operation hours in each team.

Now, the final constraint concerning the overtime operation in Team A can be written as:

$$d^+_2-d^+_{21}+d^-_{21}=2$$

where, $d^+_{21}$ is overtime operation beyond 2 hours in Team A and $d^-_{21}$ is the difference between the actual overtime operation and 2 hours set as our maximum ceiling of overtime in Team A.

In addition to variables and constraints described above, the following "preemptive priority factors" are to be defined:

$P_1$ The highest priority assigned by the production manager to the under-achievement of the production level of 15 units(i.e., $d^-_1$),

$P_2$ The second highest priority factor assigned to the overtime operation of Team A beyond 2 hours(i.e., $d^+_{21}$)

$P_3$ The third priority factor given by the manager to the sum of overtime operation in both teams(i.e., $d^+_2$, $d^+_3$). The overtime of Team B is regarded as twice the costs of overtime in Team A because of its slower rate of processing.

$P_4$ The lowest priority assigned to the underutilization of operation hours in both teams(i.e., $d^-_2$ and $d^-_3$).

Now, the model can be formulated. The objective is the minimization of deviations from goals. The deviant variable associated with the highest pre-emptive priority must be minimized to the fullest possible extent. When no further improvement is possible in the highest goal, then we try to minimize the deviations associated with the next highest priority factor, etc. Thus, the model can be expressed as:

Minimize $P_3d^+_2+2P_3d^+_3+P_2d^+_{21}+P_1d^-_1+P_4d^-_2+P_4d^-_3$

Subject to

$$
\begin{array}{lllllll}
X_1+0.5X_2-d^+_1 & & & & +d^-_1 & & =15 \\
X_1 & -d^+_2 & & & -d^-_2 & & =8 \\
X_2 & & -d^+_3 & & & +d^-_3 & =8 \\
 & d^+_2 & & -d^+_{21} & & +d^-_{21} & =2
\end{array}
$$

$X_1,\ X_2,\ d^+_1,\ d^+_2,\ d^+_3,\ d^+_{21},\ d^-_1,\ d^-_2,\ d^-_3,\ d^-_{21}\geq 0$

**Solution of Goal Programming Problem by the Simplex Method**

| | | | $x_1$ | $x_2$ | $d^+_1$ | $d^+_2$ | $d^+_3$ | $d^+_{21}$ | $d^-_1$ | $d^-_2$ | $d^-_3$ | $d^-_{21}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 0 | 0 | $P_3$ | $2P_3$ | $P_2$ | $P_1$ | $P_4$ | $P_4$ | 0 |
| $P_1$ | $d^-_1$ | 15 | 1 | 1/2 | −1 | | | | 1 | | | |
| $P_4$ | $d^-_2$ | 8 | 1 | | | −1 | | | | 1 | | |
| $P_4$ | $d^-_3$ | 8 | | 1 | | | −1 | | | | 1 | |
| 0 | $d^-_{21}$ | 2 | | | | 1 | | −1 | | | | 1 |
| $Z_j$-$C_j$ | $P_4$ | 16 | 1 | 1 | | −1 | −1 | | | | | |
| | $P_3$ | 0 | | | | −1 | −2 | | | | | |
| | $P_2$ | 0 | | | | | | −1 | | | | |
| | $P_1$ | 15 | 1 | 1/2 | −1 | | | | | | | |
| $P_1$ | $d^-_1$ | 7 | | 1/2 | −1 | 1 | | | 1 | −1 | | |
| 0 | $x_1$ | 8 | 1 | | | −1 | | | | 1 | | |
| $P_4$ | $d_3$ | 8 | | 1 | | | −1 | | | | 1 | |
| 0 | $d^-_{21}$ | 2 | | | | 1 | | −1 | | | | 1 |
| $Z_j$-$C_j$ | $P_4$ | 8 | | 1 | | | −1 | | | −1 | | |
| | $P_3$ | 0 | | | | −1 | −2 | | | | | |
| | $P_2$ | 0 | | | | | | −1 | | | | |
| | $P_1$ | 7 | | 1/2 | −1 | 1 | | | | −1 | | |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_1$ | $d^-_1$ | 5 | | 1/2 | −1 | | | 1 | 1 | −1 | | −1 |
| 0 | $x_1$ | 10 | 1 | | | | | −1 | | 1 | | 1 |
| $P_4$ | $d^-_3$ | 8 | | 1 | | | −1 | | | | 1 | |
| $P_3$ | $d^-_2$ | 2 | | | | 1 | | −1 | | | 1 | |
| $Z_j$-$C_j$ | $P_4$ | 8 | | 1 | | | −1 | | | −1 | | |
| | $P_3$ | 2 | | | | | −2 | −1 | | | | 1 |
| | $P_2$ | 0 | | | | | | −1 | | | | |
| | $P_1$ | 5 | | 1/2 | −1 | | | 1 | | −1 | | −1 |
| $P_2$ | $d^+_{21}$ | 5 | | 1/2 | −1 | | | 1 | 1 | −1 | | −1 |
| 0 | $x_1$ | 15 | 1 | 1/2 | −1 | | | | 1 | | | |
| $P_4$ | $d^-_3$ | 8 | | 1 | | | −1 | | | | 1 | |
| $P_3$ | $d^+_2$ | 7 | | 1/2 | −1 | 1 | | | 1 | −1 | | |
| $Z_j$-$C_j$ | $P_4$ | 8 | | 1 | | | −1 | | | −1 | | |
| | $P_3$ | 7 | | 1/2 | −1 | | −2 | | 1 | −1 | | |
| | $P_2$ | 5 | | 1/2 | −1 | | | | 1 | −1 | | −1 |
| | $P_1$ | 0 | | | | | | | −1 | | | |
| $P_2$ | $d^+_{21}$ | 1 | | | −1 | | 1/2 | 1 | 1 | −1 | −1/2 | −1 |
| 0 | $x_1$ | 11 | 1 | | −1 | | 1/2 | | 1 | | −1/2 | |
| 0 | $x_2$ | 8 | | 1 | | | −1 | | | | | 1 |
| $P_3$ | $d^+_2$ | 3 | | | −1 | 1 | 1/2 | | 1 | −1 | −1/2 | |
| $Z_j$-$C_j$ | $P_4$ | 0 | | | | | | | | −1 | −1 | |
| | $P_3$ | 3 | | | −1 | | −3/2 | | 1 | −1 | −1/2 | |
| | $P_2$ | 1 | | | −1 | | 1/2 | | 1 | −1 | −1/2 | −1 |
| | $P_1$ | 0 | | | | | | | −1 | | | |
| $2P_3$ | $d^+_3$ | 2 | | | −2 | | 1 | 2 | 2 | −2 | −1 | −2 |
| 0 | $x_1$ | 10 | 1 | | | | | −1 | | 1 | | 1 |
| 0 | $x_2$ | 10 | | 1 | −2 | | | 2 | 2 | −2 | | −2 |
| $P_3$ | $d^+_2$ | 2 | | | | 1 | | −1 | | | | 1 |
| $Z_j$-$C_j$ | $P_4$ | 0 | | | | | | | | −1 | −1 | |
| | $P_3$ | 6 | | | −4 | | | 3 | 4 | −4 | −2 | −3 |
| | $P_2$ | 0 | | | | | | −1 | | | | |
| | $P_1$ | 0 | | | | | | | −1 | | | |

The optimal solution can be obtained by the simplex method of linear programming as shown in Table 1. One thing to be noted is that the solution should be directed from the bottom where we have the highest priority factor. Furthermore, when there is a positive number in the lower priority factor it cannot be introduced into the program as long as there is any element other than zero in the same column on the row of a higher preemptive priority factor.

The sixth iteration shows that we cannot satisfy all goals completely, since we could not reduce variable $d^+_2$ and $d^+_3$ to zero. This reflects the everyday problem experienced in business when there are several conflicting goals. Nevertheless, the goal programming model enables us to achieve these goals as closely as possible under the given constraints and the priorities assigned to the conflicting goals.

If the problem is too complex to solve by hand, the computer should be used. However, an effective computer program for goal programming problems is yet to be developed.

## Conclusion

The application of goal programming for business problems has been scarce, primarily because it is a relatively new technique. Thus far, its application has been explored only in the areas of accounting and advertising media planning. However, the application potential of goal programming is virtually unlimited, especially for managerial decision problems.

One immediate area of possible application is explored in this paper i.e., aggregate production planning in which the objective is to absorb demand fluctuations while meeting goals in production, employment, and inventories. The example provided demonstrates that goal programming enables us to achieve conflicting goals as closely as possible under the given circumstances.

Other important areas of potential application of goal programming are in the public sector and educational institutions, where multiple conflicting goals are sought with limited resources. As long as relationships considered in the problem are linear, goal programming provides better solutions than by other linear models.