The Product Matrices and New Gain Formulas

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Abstract—New matrices, the product matrix P, the voltage product matrix P^v, and the current product matrix P^i, are defined at first. And new gain formulas T^v_{qp} for the voltage gain and T^i_{qp} for the current gain are presented on the basis of the product matrices. Although the formulas are topological ones, the procedures for their evaluations are quite systematic, completely indirect and purely numerical, and no sign rule is required. So the new formulas are suitable for computer-aided symbolic network analysis without restriction on symbols, in number or type.

I. Introduction

With Mason's gain formula based on the signal flow graph [1], there have been many topological formulas, some of which are now being used for computer-aided symbolic network analysis. Since symbolic network functions have various advantages [2], many papers were recently devoted to the computer-aided algorithms for their generation [3], among which the signal-flow-graph method, the tree enumeration method and the parameter extraction method are typical.

The purpose of this paper is to present new matrices named the product matrices after the product graph introduced by Barbay *et al.* in 1972 [4], and to present new gain formulas based on the product matrices, which have some advantages over the existing methods as a new method of computer-aided symbolic network analysis.

The product matrices and the gain formulas are presented in Section II for the reciprocal network, and in Section III for the nonreciprocal network. Several properties of the new gain formulas are discussed in Section IV.

II. Reciprocal Network

Let N^r be a reciprocal network with n nodes $N_1,\cdots,N_n$, b branches $B_1,\cdots,B_b$ and s sources (v voltage sources $V_{s1},\cdots,V_{sv}$ and i current sources $I_{s(v+1)},\cdots,I_{ss}$). And we define some notations as follows.

Definition 1: Notations G, G^0, G^c, and G^p

i) G is the *graph* of N^r with tree T and cotree C.

ii) G^0 is the *cold graph* [5] of G, where the cold graph means the graph with all the sources removed.

iii) G^c is the *compact signal flow graph* of G.

iv) G^p is the *product graph* [4] of G.

Let $T[C]$ of G be composed of the v voltage [i current] source branches and $r(=n-1)$ tree [$\mu(=b-n+1)$ cotree] branches $B_1,\cdots,B_r$ [$B_{r+1},\cdots,B_{r+\mu}$]. Then we have G^c with the r tree voltage nodes $V_1,\cdots,V_r$, the μ cotree current nodes $I_{r+1},\cdots,I_{r+\mu}$, the r tree branch impedances $Z_1,\cdots,Z_r$ and the μ cotree branch admittances $Y_{r+1},\cdots,Y_{r+\mu}$. And we have G^p with the r tree impedance nodes $Z_1,\cdots,Z_r$ and the μ cotree admittance nodes $Y_{r+1},\cdots,Y_{r+\mu}$.

Definition 2: Primitive Product Matrix P^0

We define the *primitive product matrix* $P^0=[p_{ij}]$, $i=1,2,\cdots,r$ and $j=r+1,\cdots,r+\mu$, by $p_{ij}=1$ if there exists an edge between the tree impedance node Z_i and the cotree admittance node Y_j in G^p; and $p_{ij}=0$, otherwise.[1] We give the names Z_i and Y_j to the ith row and the jth column of P^0.

Theorem 1

Let Q_L and B_T be submatrices of the fundamental cutset matrix $Q_f=[I_rQ_L]$ and the fundamental circuit matrix $B_f=[B_TI_\mu]$, respectively, of the nonoriented graph G^0. Then we have

$$P^0=Q_L=B_T^t.$$

Proof: G^p can be drawn directly from G^0, not through G^c, with the same cutsetting method as that for Q_f [4]. And there exists a one-to-one correspondence between the edges of G^p and the elements of P^0. Thus the theorem follows.

We now find a new way connecting the signal flow graph with the cutset or circuit matrix, through the product graph G^p and the product matrix P^0. And P^0 can be obtained directly from G^0, although P^0 is defined on the basis of G^p and named after G^p.

Definition 3: Product Matrix P

We define $P=Q_L(=-B_T^t)$ as the *product matrix* of the oriented graph G^0. And we give the names Z_i and Y_j to the ith row and the jth column of P.

An illustration of N^r, G, G^0, G^p, and P is given in Fig. 1.

Among the b branches of G, let the branch of source admittance Y_s [source impedance Z_s] for the voltage

[1] In defining P^0 (equally, as we do in defining P, P^v, or P^i), we denote by $j(=r+g)$ the gth column of P^0, for $g=1,2,\cdots,\mu$. Thus we have the $r\times\mu$ matrix P^0 whose column numbers are $r+1$ through $r+\mu$ instead of 1 through μ. This somewhat anomalous denotation is used for the sake of convenience in notation, for the gth column of P^0 always has the name Y_{r+g}, i.e., Y_j. And we use the term *jth column* to denote the column with column number j and column name Y_j, that is, the gth column hereafter.

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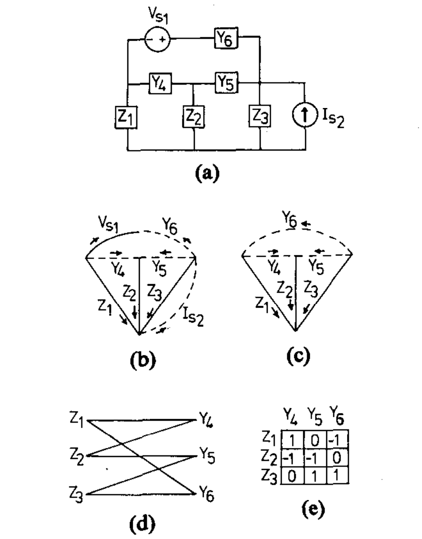

Fig. 1. (a) N^r. (b) G. (c) G^0. (d) G^P. (e) P.

source $V_{sp}(1 \leqslant p \leqslant v)$ [current source $I_{sp}(v+1 \leqslant p \leqslant s)$] be in $C[T]$, and the measuring branch of voltage V_q [current I_q] with branch impedance Z_q [admittance Y_q] be in $T[C]$ of G.

Definition 4: Substituted Product Matrix P^s

The *substituted product matrix* P^s is obtained from P as follows:

i) Substitute 0 for every element of the row named as $Z_q[Z_s]$ and for every element of the column named as $Y_s[Y_q]$ of P except the element at the intersecting position.

ii) For the element at the intersecting position.

a) Substitute $+1$, if the tree branch [cotree branch] of the voltage source V_{sp} [current source I_{sp}] has the same arrow orientation as the cotree branch [tree branch] of the source admittance Y_s [source impedance Z_s] in G, when cutsetting.

b) Substitute -1, otherwise.

We denote by $P_k[P_k^s]$ or $P_k(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)$ $[P_k^s(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)]$ the $k\times k$ submatrix of $P[P^s]$ consisting of the elements intersected by the i_1th, i_2th, $\cdots$, i_kth rows named as $Z_{i_1}, Z_{i_2}, \cdots, Z_{i_k}$, and the j_1th, j_2th, $\cdots$, j_kth columns named as $Y_{j_1}, Y_{j_2}, \cdots, Y_{j_k}$, in $P[P^s]$.

Theorem 2

Let $T_{qp}^v[T_{qp}^i]$ denote the voltage [current] transmission (or gain) from the voltage source V_{sp} with source admittance Y_s [the current source I_{sp} with source impedance Z_s] to the voltage V_q across [the current i_q through] the measuring branch in the reciprocal network N^r. Then

$$T_{qp}^v = T_{qp}^i = \frac{-\sum_{k=1}^{h} \sum_{m(q,s)} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k)(\det P_k^s) \right]}{1 + \sum_{k=1}^{h} \sum_{m} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k)^2 \right]} \tag{1}$$

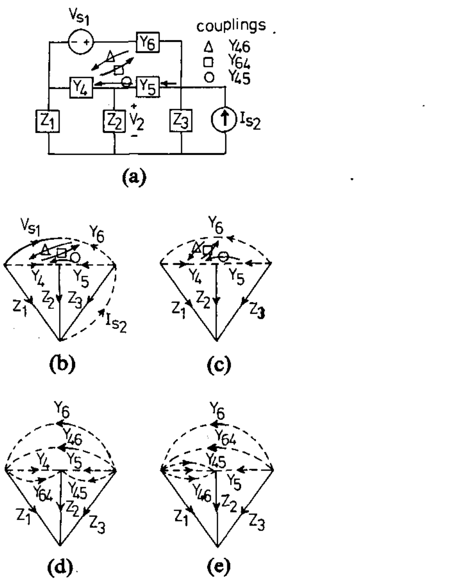

Fig. 2. (a) N^n. (b) G. (c) G^0. (d) G^v. (e) G^i.

where

$$i_m = i_{m-1}+1, i_{m-1}+2, \cdots, r-k+m,$$

$$j_m = j_{m-1}+1, j_{m-1}+2, \cdots, r+\mu-k+m,$$

$$i_0 = 0, \quad j_0 = r, \quad \text{and} \quad h = \min(r,\mu).$$

In (1), Σ_m means the summation of all the possible i_m's and j_m's; $\Sigma_{m(q,s)}$, the summation of all the possible i_m's and j_m's including i_q in i_m's and j_s in j_m's for every possible selection; and $\prod_{m=1}^{k} Z_{i_m} Y_{j_m}$, the product of the k impedances and the k admittances of the $k\times k$ submatrix P_k or P_k^s. The subscripts q and s of i_q and j_s are borrowed from the measuring branch immittance Z_q (or Y_q) and the source immittance Z_s (or Y_s) to denote the corresponding rows and columns.

Proof: Refer to [6] and the Appendix.

III. Nonreciprocal Network

Let N^n be a nonreciprocal network with n nodes $N_1, \cdots, N_n$, b branches $B_1, \cdots, B_b$, s sources (v voltage sources $V_{s1}, \cdots, V_{sv}$ and i current sources $I_{s(v+1)}, \cdots, I_{ss}$) and c couplings.

Definition 5: Notations G, G^0, G^v and G^i

i) G is the *graph* of N^n with tree T and cotree C.

ii) G^0 is the *cold graph* of G.

iii) G_v is the *voltage graph* [7] of G^0.

iv) G^i is the *current graph* [7] of G^0.

Let $T[C]$ of G be composed of the v voltage [i current] source branches and r tree [μ cotree] branches $B_1, \cdots, B_r[B_{r+1}, \cdots, B_{r+\mu}]$. And let all the coupled elements be in C. Then we have G^v and G^i with the r tree impedances $Z_1, \cdots, Z_r$ and the $\mu+c$ cotree admittances $Y_{r+1}, \cdots, Y_{r+\mu}$ and $Y_{c1}, \cdots, Y_{cc}$.

An illustration of N^n, G, G^0, G^v, and G^i is given in Fig. 2.

Definition 6: Voltage Product Matrix P^v [Current Product Matrix P^i]

The *voltage* [*current*] *product matrix* $P^v[P^i]$ is defined as the product matrix of the voltage [current] graph $G^v[G^i]$. In P^v or P^i, we give the name Z_i to the ith row; Y_j to the jth column, when $r+1 \leqslant j \leqslant r+\mu = b$; and Y_{ct} to the jth column, when $b+1 \leqslant j \leqslant b+c, j=b+t$.

Let the conditions of the branches of sources and the branches of measurement be as in Section II.

Definition 7: Substituted Voltage Product Matrix P^{vs} [Substituted Current Product Matrix P^{is}]

The *substituted voltage* [*current*] *product matrix* $P^{vs}[P^{is}]$ is defined as the substituted product matrix of the voltage [current] product matrix $P^v[P^i]$.

We denote by $P_k^v[P_k^{vs}: P_k^i: P_k^{is}]$ or $P_k^v(i_1,i_2,\cdots,i_k; j_1,j_2,\cdots,j_k)$ $[P_k^{vs}(i_1,i_2,\cdots,i_k; j_1,j_2,\cdots,j_k): P_k^i(i_1,i_2,\cdots,i_k; j_1,j_2,\cdots,j_k): P_k^{is}(i_1,i_2,\cdots,i_k; j_1,j_2,\cdots,j_k)]$ the $k \times k$ submatrix of $P^v[P^{vs}: P^i: P^{is}]$, consisting of the elements intersected by the i_1th, i_2th, $\cdots$, i_kth rows and the j_1th, j_2th, $\cdots$, j_kth columns of $P^v[P^{vs}: P^i: P^{is}]$.

Theorem 3

The voltage transmission (or gain) T_{qp}^v and the current transmission T_{qp}^i of the nonreciprocal network N^n are

$$T_{qp}^v = \frac{-\sum_{k=1}^{h} \sum_{m(q,s)} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k^{vs})(\det P_k^i) \right]}{1 + \sum_{k=1}^{h} \sum_{m} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k^{v})(\det P_k^i) \right]} \qquad 2(a)$$

$$T_{qp}^i = \frac{-\sum_{k=1}^{h} \sum_{m(q,s)} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k^{v})(\det P_k^{is}) \right]}{1 + \sum_{k=1}^{h} \sum_{m} \left[\left(\prod_{m=1}^{k} Z_{i_m} Y_{j_m} \right) (\det P_k^{v})(\det P_k^i) \right]} \qquad 2(b)$$

where

$$i_m = i_{m-1}+1, i_{m-1}+2, \cdots, r-k+m,$$
$$j_m = j_{m-1}+1, j_{m-1}+2, \cdots, r+\mu+c-k+m,$$
$$i_0 = 0, \quad j_0 = r, \quad h = \min(r, \mu+c),$$

and all the other notations are as in Section II.

Proof: Refer to the Appendix.

Theorem 4

If all the couplings of N^n are removed, then

$$P^v = P^i = P$$

and (2) reduces to (1).

Proof: Since there is no coupling, we have $G^v = G^i = G$. This completes the proof.

Example

We want to find the voltage transmission $T_{21}^v (= V_2/V_{s1})$ and the current transmission $T_{52}^i (= I_5/I_{s2})$ of the nonreciprocal network in Fig. 2.

The product matrices P^v, P^i, P^{vs}, and P^{is} for them are as shown in Table I. All the possible combinations for

TABLE I

	Y_4	Y_5	Y_6	Y_{45}	Y_{46}	Y_{64}
Z_1	1	0	−1	0	−1	1
Z_2	−1	−1	0	−1	0	−1
Z_3	0	1	1	1	1	0

P^v

	Y_4	Y_5	Y_6	Y_{45}	Y_{46}	Y_{64}
Z_1	1	0	−1	1	1	−1
Z_2	−1	−1	0	−1	−1	0
Z_3	0	1	1	0	0	1

P^i

	Y_4	Y_5	Y_6	Y_{45}	Y_{46}	Y_{64}
Z_1	1	0	0	0	−1	1
Z_2	0	0	+1	0	0	0
Z_3	0	1	0	1	1	0

$P^{vs}(T_{21}^v)$

	Y_4	Y_5	Y_6	Y_{45}	Y_{46}	Y_{64}
Z_1	1	0	−1	1	1	−1
Z_2	−1	0	0	−1	−1	0
Z_3	0	−1	0	0	0	0

$P^{is}(T_{52}^i)$

TABLE II

$\binom{i_1}{j_1}$	$P_1^v\binom{i_1}{j_1}$	$P_1^i\binom{i_1}{j_1}$	$\det P_1^v$	$\det P_1^i$	product
$\binom{1}{4}$	1	1	1	1	Z_1Y_4
$\binom{1}{5}$	0	0	0	0	0
$\binom{1}{6}$	−1	−1	−1	−1	Z_1Y_6
$\binom{1}{45}$	0	1	0	1	0
$\binom{1}{46}$	−1	1	−1	1	$-Z_1Y_{46}$
$\binom{1}{64}$	1	−1	1	−1	$-Z_1Y_{64}$
$\binom{2}{4}$	−1	−1	−1	−1	Z_2Y_4
$\binom{2}{5}$	−1	−1	−1	−1	Z_2Y_5
$\binom{2}{6}$	0	0	0	0	0
$\binom{2}{45}$	−1	−1	−1	−1	Z_2Y_{45}
$\binom{2}{46}$	0	−1	0	−1	0
$\binom{2}{64}$	−1	0	−1	0	0
$\binom{3}{4}$	0	0	0	0	0
$\binom{3}{5}$	1	1	1	1	Z_3Y_5
$\binom{3}{6}$	1	1	1	1	Z_3Y_6
$\binom{3}{45}$	1	0	1	0	0
$\binom{3}{46}$	1	0	1	0	0
$\binom{3}{64}$	0	1	0	1	0

TABLE III

$\begin{pmatrix}i_1&i_2\\j_1&j_2\end{pmatrix}$	$P_2^v\begin{pmatrix}i_1&i_2\\j_1&j_2\end{pmatrix}$	$P_2^i\begin{pmatrix}i_1&i_2\\j_1&j_2\end{pmatrix}$	$\det P_2^v$	$\det P_2^i$	product
$\begin{pmatrix}1&2\\4&5\end{pmatrix}$	$\begin{matrix}1&0\\-1&-1\end{matrix}$	$\begin{matrix}1&0\\-1&-1\end{matrix}$	-1	-1	$Z_1Z_2Y_4Y_5$
$\begin{pmatrix}1&2\\4&6\end{pmatrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	-1	-1	$Z_1Z_2Y_4Y_6$
$\begin{pmatrix}1&2\\4&45\end{pmatrix}$	$\begin{matrix}1&0\\-1&-1\end{matrix}$	$\begin{matrix}1&1\\-1&-1\end{matrix}$	-1	0	0*
$\begin{pmatrix}1&2\\4&46\end{pmatrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	$\begin{matrix}1&1\\-1&-1\end{matrix}$	-1	0	0*
$\begin{pmatrix}1&2\\4&64\end{pmatrix}$	$\begin{matrix}1&1\\-1&-1\end{matrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	0	-1	0*
$\begin{pmatrix}1&2\\5&6\end{pmatrix}$	$\begin{matrix}0&-1\\-1&0\end{matrix}$	$\begin{matrix}0&-1\\-1&0\end{matrix}$	-1	-1	$Z_1Z_2Y_5Y_6$
$\begin{pmatrix}1&2\\5&45\end{pmatrix}$	$\begin{matrix}0&0\\-1&-1\end{matrix}$	$\begin{matrix}0&1\\-1&-1\end{matrix}$	0	1	0*
$\begin{pmatrix}1&2\\5&46\end{pmatrix}$	$\begin{matrix}0&-1\\-1&0\end{matrix}$	$\begin{matrix}0&1\\-1&-1\end{matrix}$	-1	1	$-Z_1Z_2Y_5Y_{46}$
$\begin{pmatrix}1&2\\5&64\end{pmatrix}$	$\begin{matrix}0&1\\-1&-1\end{matrix}$	$\begin{matrix}0&-1\\-1&0\end{matrix}$	1	-1	$-Z_1Z_2Y_5Y_{64}$
$\begin{pmatrix}1&2\\6&45\end{pmatrix}$	$\begin{matrix}-1&0\\0&-1\end{matrix}$	$\begin{matrix}-1&1\\0&-1\end{matrix}$	1	1	$Z_1Z_2Y_6Y_{45}$
$\begin{pmatrix}1&2\\6&46\end{pmatrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	$\begin{matrix}-1&1\\0&-1\end{matrix}$	0	1	0*
$\begin{pmatrix}1&2\\6&64\end{pmatrix}$	$\begin{matrix}-1&1\\0&-1\end{matrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	1	0	0*
$\begin{pmatrix}1&2\\45&46\end{pmatrix}$	$\begin{matrix}0&-1\\-1&0\end{matrix}$	$\begin{matrix}1&1\\-1&-1\end{matrix}$	-1	0	0*
$\begin{pmatrix}1&2\\45&64\end{pmatrix}$	$\begin{matrix}0&1\\-1&-1\end{matrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	1	-1	$-Z_1Z_2Y_{45}Y_{64}$
$\begin{pmatrix}1&2\\46&64\end{pmatrix}$	$\begin{matrix}-1&1\\0&-1\end{matrix}$	$\begin{matrix}1&-1\\-1&0\end{matrix}$	1	-1	$-Z_1Z_2Y_{46}Y_{64}$
$\begin{pmatrix}1&3\\4&5\end{pmatrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	1	1	$Z_1Z_3Y_4Y_5$
$\begin{pmatrix}1&3\\4&6\end{pmatrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	1	1	$Z_1Z_3Y_4Y_6$
$\begin{pmatrix}1&3\\4&45\end{pmatrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}1&1\\0&0\end{matrix}$	1	0	0*
$\begin{pmatrix}1&3\\4&46\end{pmatrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	$\begin{matrix}1&1\\0&0\end{matrix}$	1	0	0*
$\begin{pmatrix}1&3\\4&64\end{pmatrix}$	$\begin{matrix}1&1\\0&0\end{matrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	0	1	0*
$\begin{pmatrix}1&3\\5&6\end{pmatrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	1	1	$Z_1Z_3Y_5Y_6$
$\begin{pmatrix}1&3\\5&45\end{pmatrix}$	$\begin{matrix}0&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	0	-1	0*
$\begin{pmatrix}1&3\\5&46\end{pmatrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	1	-1	$-Z_1Z_3Y_5Y_{46}$
$\begin{pmatrix}1&3\\5&64\end{pmatrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	-1	1	$-Z_1Z_3Y_5Y_{64}$
$\begin{pmatrix}1&3\\6&45\end{pmatrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	$\begin{matrix}-1&1\\1&0\end{matrix}$	-1	-1	$-Z_1Z_3Y_6Y_{45}$
$\begin{pmatrix}1&3\\6&46\end{pmatrix}$	$\begin{matrix}-1&-1\\1&1\end{matrix}$	$\begin{matrix}-1&1\\1&0\end{matrix}$	0	-1	0*
$\begin{pmatrix}1&3\\6&64\end{pmatrix}$	$\begin{matrix}-1&1\\1&0\end{matrix}$	$\begin{matrix}-1&-1\\1&1\end{matrix}$	-1	0	0*
$\begin{pmatrix}1&3\\45&46\end{pmatrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	$\begin{matrix}1&1\\0&0\end{matrix}$	1	0	0*
$\begin{pmatrix}1&3\\45&64\end{pmatrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	-1	1	$-Z_1Z_3Y_{45}Y_{64}$
$\begin{pmatrix}1&3\\46&64\end{pmatrix}$	$\begin{matrix}-1&1\\1&0\end{matrix}$	$\begin{matrix}1&-1\\0&1\end{matrix}$	-1	1	$-Z_1Z_3Y_{46}Y_{64}$
$\begin{pmatrix}2&3\\4&5\end{pmatrix}$	$\begin{matrix}-1&-1\\0&1\end{matrix}$	$\begin{matrix}-1&-1\\0&1\end{matrix}$	-1	-1	$Z_2Z_3Y_4Y_5$
$\begin{pmatrix}2&3\\4&6\end{pmatrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	-1	-1	$Z_2Z_3Y_4Y_6$
$\begin{pmatrix}2&3\\4&45\end{pmatrix}$	$\begin{matrix}-1&-1\\0&1\end{matrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	-1	0	0*
$\begin{pmatrix}2&3\\4&46\end{pmatrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	-1	0	0*
$\begin{pmatrix}2&3\\4&64\end{pmatrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	0	-1	0*
$\begin{pmatrix}2&3\\5&6\end{pmatrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	-1	-1	$Z_2Z_3Y_5Y_6$
$\begin{pmatrix}2&3\\5&45\end{pmatrix}$	$\begin{matrix}-1&-1\\1&1\end{matrix}$	$\begin{matrix}-1&-1\\1&0\end{matrix}$	0	1	0*
$\begin{pmatrix}2&3\\5&46\end{pmatrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	$\begin{matrix}-1&-1\\1&0\end{matrix}$	-1	1	$-Z_2Z_3Y_5Y_{46}$
$\begin{pmatrix}2&3\\5&64\end{pmatrix}$	$\begin{matrix}-1&-1\\1&0\end{matrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	1	-1	$-Z_2Z_3Y_5Y_{64}$
$\begin{pmatrix}2&3\\6&45\end{pmatrix}$	$\begin{matrix}0&-1\\1&1\end{matrix}$	$\begin{matrix}0&-1\\1&0\end{matrix}$	1	1	$Z_2Z_3Y_6Y_{45}$
$\begin{pmatrix}2&3\\6&46\end{pmatrix}$	$\begin{matrix}0&0\\1&1\end{matrix}$	$\begin{matrix}0&-1\\1&0\end{matrix}$	0	1	0*
$\begin{pmatrix}2&3\\6&64\end{pmatrix}$	$\begin{matrix}0&-1\\1&0\end{matrix}$	$\begin{matrix}0&0\\1&1\end{matrix}$	1	0	0*
$\begin{pmatrix}2&3\\45&46\end{pmatrix}$	$\begin{matrix}-1&0\\1&1\end{matrix}$	$\begin{matrix}-1&-1\\0&0\end{matrix}$	-1	0	0*
$\begin{pmatrix}2&3\\45&64\end{pmatrix}$	$\begin{matrix}-1&-1\\1&0\end{matrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	1	-1	$-Z_2Z_3Y_{45}Y_{64}$
$\begin{pmatrix}2&3\\46&64\end{pmatrix}$	$\begin{matrix}0&-1\\1&0\end{matrix}$	$\begin{matrix}-1&0\\0&1\end{matrix}$	1	-1	$-Z_2Z_3Y_{46}Y_{64}$

evaluation of those transmissions are given in Table II– Table VI.[2] And we get

$$T_{21}^{v}=\frac{-\sum_{k=1}^{3}\sum_{m(2,6)}\left[\left(\prod_{m=1}^{k}Z_{i_m}Y_{j_m}\right)(\det P_k^{vs})(\det P_k^{i})\right]}{1+\sum_{k=1}^{3}\sum_{m}\left[\left(\prod_{m=1}^{k}Z_{i_m}Y_{j_m}\right)(\det P_k^{v})(\det P_k^{i})\right]}=\frac{Z_1Z_2Y_4Y_6-Z_1Z_2Y_6Y_{46}-Z_2Z_3Y_5Y_6-Z_2Z_3Y_6Y_{45}-Z_2Z_3Y_6Y_{46}}{\Delta}$$

$$T_{52}^{i}=\frac{-\sum_{k=1}^{3}\sum_{m(3,5)}\left[\left(\prod_{m=1}^{k}Z_{i_m}Y_{j_m}\right)(\det P_k^{v})(\det P_k^{is})\right]}{1+\sum_{k=1}^{3}\sum_{m}\left[\left(\prod_{m=1}^{k}Z_{i_m}Y_{j_m}\right)(\det P_k^{v})(\det P_k^{i})\right]}.$$

$$=\frac{Z_3Y_5+Z_1Z_3Y_4Y_5+Z_1Z_3Y_5Y_6-Z_1Z_3Y_5Y_{46}-Z_1Z_3Y_5Y_{64}+Z_2Z_3Y_4Y_5-Z_2Z_3Y_5Y_{46}}{\Delta}$$

[2]But it is not necessary to evaluate all the combinations in these tables. Since every column in $P^v[P^i]$ named Y_j, $Y_{aj}[Y_{ja}]$, or $Y_{a'j}[Y_{ja'}]$ has the same entries, for $j=r+1,\cdots,r+\mu$, $a=r+1,\cdots,r+\mu$, $a'=r+1,\cdots,r+\mu$, all the combinations including any two of these result in zero, and can be eliminated in advance. These columns are due to the parallel cotree branches with the same originating nodes and the same terminating nodes in G^v or G^i. All the asterisked zeros (0*) in the tables correspond to these preeliminable combinations.

TABLE IV

$\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$P_3^v\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$P_3^i\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$\det P_3^v$	$\det P_3^i$	product	$\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$P_3^v\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$P_3^i\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$	$\det P_3^v$	$\det P_3^i$	product
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	0	0	0	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 45 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 45 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 46 \end{pmatrix}$	$\begin{matrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 46 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 64 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 45 & 46 \end{pmatrix}$	$\begin{matrix} 0 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 45 \end{pmatrix}$	$\begin{matrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 45 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 46 \end{pmatrix}$	$\begin{matrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 46 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	0	0	0
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 64 \end{pmatrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 45 & 46 \end{pmatrix}$	$\begin{matrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 45 & 46 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 45 & 64 \end{pmatrix}$	$\begin{matrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 45 & 64 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 46 & 64 \end{pmatrix}$	$\begin{matrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	0	0	0*
$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 46 & 64 \end{pmatrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$	0	0	0*	$\begin{pmatrix} 1 & 2 & 3 \\ 45 & 46 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$	0	0	0*

TABLE V

k	$\begin{pmatrix} i_m\text{'s} \\ j_m\text{'s} \end{pmatrix}$	P_k^{vs}	P_k^i	$\det P_k^{vs}$	$\det P_k^i$	product.	k	$\begin{pmatrix} i_m\text{'s} \\ j_m\text{'s} \end{pmatrix}$	P_k^{vs}	P_k^i	$\det P_k^{vs}$	$\det P_k^i$	product
1	$\begin{pmatrix} 2 \\ 6 \end{pmatrix}$	1	0	1	0	0	3	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	−1	0	0
2	$\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -1 \\ -1 & 0 \end{matrix}$	1	−1	$-Z_1Z_2Y_4Y_6$		$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 45 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	1	0	0*
	$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$	$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$	0	−1	0		$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 46 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix}$	1	0	0*
	$\begin{pmatrix} 1 & 2 \\ 6 & 45 \end{pmatrix}$	$\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}$	0	1	0		$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 64 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$	0	0	0*
	$\begin{pmatrix} 1 & 2 \\ 6 & 46 \end{pmatrix}$	$\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}$	1	1	$Z_1Z_2Y_6Y_{46}$		$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 45 \end{pmatrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	0	0	0
	$\begin{pmatrix} 1 & 2 \\ 6 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} -1 & -1 \\ 0 & 0 \end{matrix}$	−1	0	0*		$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 46 \end{pmatrix}$	$\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	1	0	0
	$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$	$\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$	$\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}$	0	−1	0		$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{matrix}$	−1	0	0*
	$\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 0 \\ 1 & 1 \end{matrix}$	−1	−1	$Z_2Z_3Y_5Y_6$		$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 45 & 46 \end{pmatrix}$	$\begin{matrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{matrix}$	1	0	0*
	$\begin{pmatrix} 2 & 3 \\ 6 & 45 \end{pmatrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$	1	1	$Z_2Z_3Y_6Y_{45}$		$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 45 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	−1	0	0*
	$\begin{pmatrix} 2 & 3 \\ 6 & 46 \end{pmatrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$	1	1	$Z_2Z_3Y_6Y_{46}$		$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 46 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{matrix}$	−1	0	0*
	$\begin{pmatrix} 2 & 3 \\ 6 & 64 \end{pmatrix}$	$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}$	0	0	0*							

TABLE VI

k	$\begin{pmatrix} i_m\text{'s} \\ j_m\text{'s} \end{pmatrix}$	P_k^v	P_k^{is}	$\det P_k^v$	$\det P_k^{is}$	product
1	$\begin{pmatrix} 3 \\ 5 \end{pmatrix}$	1	-1	1	−1	$-Z_3Y_5$
2	$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$	1	−1	$-Z_1Z_3Y_4Y_5$
	$\begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}$	$\begin{matrix} 0 & -1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$	1	−1	$-Z_1Z_3Y_5Y_6$
	$\begin{pmatrix} 1 & 3 \\ 5 & 45 \end{pmatrix}$	$\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$	0	1	0*
	$\begin{pmatrix} 1 & 3 \\ 5 & 46 \end{pmatrix}$	$\begin{matrix} 0 & -1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$	1	1	$Z_1Z_3Y_5Y_{46}$
	$\begin{pmatrix} 1 & 3 \\ 5 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$	−1	−1	$Z_1Z_3Y_5Y_{64}$
	$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$	$\begin{matrix} -1 & -1 \\ 0 & 1 \end{matrix}$	$\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$	−1	1	$-Z_2Z_3Y_4Y_5$
	$\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$	$\begin{matrix} -1 & 0 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix}$	−1	0	0
	$\begin{pmatrix} 2 & 3 \\ 5 & 45 \end{pmatrix}$	$\begin{matrix} -1 & -1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$	0	−1	0*
	$\begin{pmatrix} 2 & 3 \\ 5 & 46 \end{pmatrix}$	$\begin{matrix} -1 & 0 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$	−1	−1	$Z_2Z_3Y_5Y_{46}$
	$\begin{pmatrix} 2 & 3 \\ 5 & 64 \end{pmatrix}$	$\begin{matrix} -1 & -1 \\ 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix}$	1	0	0
3	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix}$	0	−1	0
	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 45 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{matrix}$	0	0	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 46 \end{pmatrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{matrix}$	0	0	0
	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 64 \end{pmatrix}$	$\begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix}$	$\begin{matrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix}$	0	−1	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 45 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{matrix}$	0	−1	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 46 \end{pmatrix}$	$\begin{matrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{matrix}$	0	−1	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix}$	0	0	0
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 45 & 46 \end{pmatrix}$	$\begin{matrix} 0 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & 0 \end{matrix}$	0	0	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 45 & 64 \end{pmatrix}$	$\begin{matrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{matrix}$	0	1	0*
	$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 46 & 64 \end{pmatrix}$	$\begin{matrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{matrix}$	0	1	0

where

$$\begin{aligned}\Delta = 1 &+ (Z_1Y_4 + Z_1Y_6 - Z_1Y_{46} - Z_1Y_{64} + Z_2Y_4 \\ &+ Z_2Y_5 + Z_2Y_{45} + Z_3Y_5 + Z_3Y_6) \\ &+ (Z_1Z_2Y_4Y_5 + Z_1Z_2Y_4Y_6 + Z_1Z_2Y_5Y_6 \\ &- Z_1Z_2Y_5Y_{46} - Z_1Z_2Y_5Y_{64} + Z_1Z_2Y_6Y_{45} \\ &- Z_1Z_2Y_{45}Y_{64} - Z_1Z_2Y_{46}Y_{64} + Z_1Z_3Y_4Y_5 \\ &+ Z_1Z_3Y_4Y_6 + Z_1Z_3Y_5Y_6 - Z_1Z_3Y_5Y_{46} \\ &- Z_1Z_3Y_5Y_{64} + Z_1Z_3Y_6Y_{45} - Z_1Z_3Y_{45}Y_{64} \\ &- Z_1Z_3Y_{46}Y_{64} + Z_2Z_3Y_4Y_5 + Z_2Z_3Y_4Y_6 \\ &+ Z_2Z_3Y_5Y_6 - Z_2Z_3Y_5Y_{46} - Z_2Z_3Y_5Y_{64} \\ &+ Z_2Z_3Y_6Y_{45} - Z_2Z_3Y_{45}Y_{64} - Z_2Z_3Y_{46}Y_{64}).\end{aligned}$$

IV. Discussion

Four types of transfer functions can be set up from the gain formulas based on the product matrices (the PM formulas, in abbreviation). The voltage gain function is expressed as T_{qp}^v; the current gain function, as T_{qp}^i; the impedance function, as $Z_q \cdot T_{qp}^i$; and the admittance function, as $Y_q \cdot T_{qp}^v$.

The PM formulas are based on the topological method. But once the product matrices are determined from the voltage and current graphs, the remainder depends only on the matrices, and neither the flow graphs (Mason's graph, Coates' graph, etc.,) nor the product graphs are required for their evaluations.

The PM formulas do not hold the terms that can be canceled, while other topological formulas using signal-flow-graph method or tree enumeration method require to cancel during their evaluations. With the PM formulas, canceling terms are extracted during the evaluation of determinants, and no extra attention is required for canceling terms.

With the PM formulas, no special sign rule is required. The signs are automatically determined during the composition of product matrices.

With the PM formulas, the evaluating procedure is quite systematic in comparison with those of other topological formulas. Once the product matrices and the substituted product matrices are obtained from a given graph, the only procedure that remains is to evaluate their determinants systematically, one by one, two by two, etc. On the other hand, other topological formulas need more complicated procedures for memorizing and sorting of the topological informations.

Furthermore, with the PM formulas, the evaluating procedure is completely indirect and purely numerical, that is, symbols do not join in the evaluation directly, as is the case with the parameter extraction method. But with the parameter extraction method, we need other procedures to extract the symbols out of the matrices in addition to the determinant-evaluating procedure, and the number of symbols involved is restricted.

The calculation required is only to evaluate the determinants of the matrices composed of only ± 1 and 0, and the evaluated results are also either ± 1 or 0. If the result is ± 1, we write all the immittances of corresponding matrix with the appropriate sign; and if the result is 0, we write 0 or nothing: In such a way the resultant symbolic network function is obtained. Therefore, there exists no restriction on the symbols, in number or type, for the evaluation of symbolic network functions.

In the determinant-evaluating procedure many of the zero determinant terms can be preeliminated without determinant evaluation, as indicated in the footnote 2. We find from Table II through Table VI that 56 out of 70 zero determinant terms (80 percents) are preeliminable.

These properties all point to the fact that the PM formulas are quite suitable for computer-aided generation of the symbolic network functions. Since the evaluation by the PM formulas is systematic and numerical, neither the complicated path-finding algorithm of the signal-flow-graph method, nor the tree-enumerating algorithm of the tree enumeration method is required. And since the evaluation is indirect, as mentioned above, there exists no restriction on the symbols, which weakens the parameter extraction method. In these respects, the PM formula will render a computer algorithm less complicated but more powerful, for symbolic network analysis, than the existing ones using signal-flow-graph method, tree enumeration method or parameter extraction method.

At present, however, computer-aided symbolic network analysis by the PM formulas is theoretical, and further work is required for the practical algorithm that reduces the computing time increasing almost exponentially with the number of nodes or branches.

V. Conclusion

The concept of the primitive product matrix is derived from the product graph of the reciprocal network at first, and then, it is extended to the product matrix. With the introduction of the product graph and the product matrix, the signal flow graph finds a returning way back to the original cutset or circuit matrix. The voltage product matrix and the current product matrix are defined as the product matrices of the voltage graph and the current graph respectively of the nonreciprocal network.

The PM formulas, the new gain formulas based on the product matrices, are presented. With the PM formulas, topological formulas as they are, only product matrices are sufficient for their evaluations, and cancelling terms are extracted during their evaluations. Their evaluating procedures are quite systematic, completely indirect, and purely numerical. And sign rules are not required for their evaluations. In these respects, the PM formulas may be called semitopological formulas, and are suitable for computer-aided symbolic network analysis with some advantages over the existing methods. But further work is required for the practical algorithm with the reduction of computing time.

Appendix
Proof of the Theorem 3

We denote by G^c the compact signal flow graph of G. Then, in G^c, we have the r tree voltage nodes $V_1, \cdots, V_r$, the μ cotree current nodes $I_{r+1}, \cdots, I_{r+\mu}$, the r tree branch impedances $Z_1, \cdots, Z_r$, the μ cotree branch admittances $Y_{r+1}, \cdots, Y_{r+\mu}$ and the c coupling admittances $Y_{c1}, \cdots, Y_{cc}$.

Let G^s be the partitioned signal flow graph of G^c with every current node of G^c partitioned as

$$I_j = I_{jj} + \sum_c I_{jc} \tag{3}$$

where I_{jj} refers to the current due to the voltage and i_{jc} is due to the coupling. Then we have r voltage nodes and $\mu + c$ current nodes in G^s, and every edge incoming to node V_i has weight $\pm Z_i$; to node I_j, weight $\pm Y_j$; and to node I_{jc}, weight $\pm Y_{jc}$.

From G^s, we get

$$V_i = \sum_j \pm Z_i I_{jj} + \sum_j \pm Z_i I_{jc} + \sum_j \pm Z_i I_{sj}$$
$$\text{(for every } i \text{ and some of the } j\text{'s)} \qquad \text{4(a)}$$

$$I_{jj} = \sum_i \pm Y_j V_i + \sum_i \pm Y_j V_{si}$$
$$I_{jc} = \sum_i \pm Y_{jc} V_i$$
$$\text{(for every } j \text{ and some of the } i\text{'s)} \qquad \text{4(b)}$$

where some of the j's [some of the i's] for $V_i[I_j]$ are those whose corresponding cotree branches B_j's [tree branches B_i's] are in the same cutset [circuit] as the tree branch B_i [cotree branch B_j] in $G^i[G^v]$, and the current [voltage] source for $V_i[I_{jj}]$ is that whose source impedance [admittance] is $Z_i[Y_j]$. And all the signs follow the sign rules for the cutset and circuit matrices.

Rewriting (4) in matrix form, we get

$$\begin{bmatrix} V_1 \\ \vdots \\ V_r \\ I_{r+1} \\ \vdots \\ I_{r+\mu} \\ I_{c1} \\ \vdots \\ I_{cc} \end{bmatrix} = \left[\begin{array}{c|c} 0 & -Z \\ \hline Y & 0 \end{array}\right] \begin{bmatrix} V_1 \\ \vdots \\ V_r \\ I_{r+1} \\ \vdots \\ I_{r+\mu} \\ I_{c1} \\ \vdots \\ I_{cc} \end{bmatrix} + \left[\begin{array}{c|c} 0 & -Z_s \\ \hline Y_s & 0 \end{array}\right] \begin{bmatrix} V_{s1} \\ \vdots \\ V_{sv} \\ I_{s(v+1)} \\ \vdots \\ I_{ss} \end{bmatrix} \tag{5}$$

where $I_{r+1}, \cdots, I_{r+\mu}$ refers to $I_{(r+1)(r+1)}, \cdots, I_{(r+\mu)(r+\mu)}$, i.e., to all I_{jj}'s, and $I_{c1}, \cdots, I_{cc}$ refers to all the currents due to

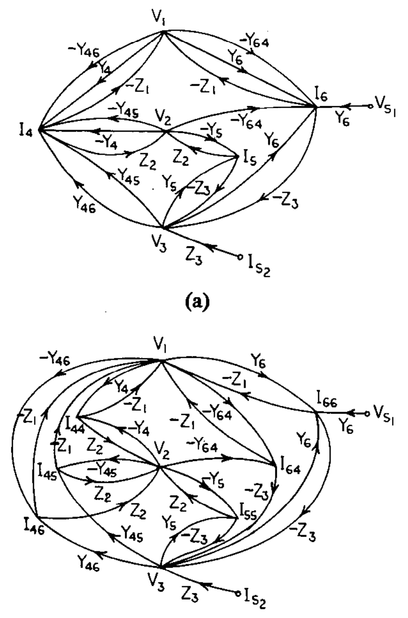

Fig. 3. (a) G^c. (b) G^s.

couplings, i.e., to all I_{jc}'s. And in (5)

$-Z$: $r\times(\mu+c)$ impedance matrix,[3]
Y: $(\mu+c)\times r$ admittance matrix,
$-Z_s$: $r\times i$ impedance matrix,
Y_s: $(\mu+c)\times v$ admittance matrix,

and each row of $-Z$(or $-Z_s$) and Y(or Y_s) consists of $\pm Z_i$ or 0, and $\pm Y_j$ or 0, respectively, for $i=1,2,\cdots,r$ and $j=r+1,\cdots,r+\mu$. And for $j=b+1,\cdots,b+c,j=b+t$, each row of Y consists of coupling element Y_{ct}, while each row of Y_s consists of only 0.

As an illustration, G^c and G^s of the G in Fig. 2 are shown in Fig. 3.

Simplifying (5), we have

$$X=AX+BX \tag{6}$$

where

$$X=[V_1,\cdots,V_r,I_{r+1},\cdots,I_{r+\mu},I_{c1},\cdots,I_{cc}]^t, \qquad (b+c)\times 1,$$

$$X_s=[V_{s1},\cdots,V_{sv},I_{s(v+1)},\cdots,I_{ss}]^t, \qquad s\times 1,$$

$$A=\begin{bmatrix}0 & -Z\\ Y & 0\end{bmatrix}, \qquad (b+c)\times(b+c),$$

$$B=\begin{bmatrix}0 & -Z_s\\ Y_s & 0\end{bmatrix}, \qquad (b+c)\times s.$$

From (6), we easily get

$$T_{qp}=\frac{\det[I-A]_{qp}}{\det[I-A]}=\frac{\Delta'}{\Delta} \tag{7}$$

[3]In G^c or G^s, every first-order loop consisting of the tree branch Z_i and the cotree branch Y_j of G has the impedance edge Z_i and the admittance edge Y_j, for $i=1,2,\cdots,r$ and $j=r+1,\cdots,r+\mu$. And the signs of Z_i and Y_j are always opposite to each other. The minus sign of $-Z$ and $-Z_s$ indicates this relationship.

where T_{qp} is T^v_{qp} or T^i_{qp}, Δ is the determinant of the matrix $[I-A]$ and Δ' is the determinant of the matrix $[I-A]_{qp}$, which is obtained from the matrix $[I-A]$ with the qth column replaced by the pth column of B.

To complete the proof, it is left to calculate Δ and Δ'.

A. The Denominator Δ

$$\Delta=\det\begin{bmatrix}I_r & Z\\ -Y & I_{\mu+c}\end{bmatrix} \tag{8}$$

where I_r and $I_{\mu+c}$ are the identity matrices of order r and $\mu+c$.

Let Z_I and Y_I denote, respectively, the Z and Y with every nonzero element replaced by unity, and with no sign exchange. Then, from (4), it turns out that $Z_I[Y_I]$ is the same as $Q_L[B_T]$ of the current [voltage] graph $G^i[G^v]$. And, from the definitions of product matrices, P^i and P^v are equal to $Q_L(=-B^t_T)$ of G^i and G^v. Thus we get

$$Z_I=P^i \tag{9(a)}$$

$$-Y^t_I=P^v. \tag{9(b)}$$

By determinant calculation of (8) and (9), we have

$$\Delta=\sum_{k=1}^{h}\sum_{m}\left[\left(\prod_{m=1}^{k}Z_{i_m}Y_{j_m}\right)\{\det P^v_k(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\}\cdot\{\det P^i_k(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\}\right] \tag{10}$$

where Σ_m means the summation of all i_m's and j_m's, for

$$i_m=i_{m-1}+1,i_{m-1}+2,\cdots,r-k+m,$$
$$j_m=j_{m-1}+1,j_{m-1}+2,\cdots,r+\mu+c-k+m,$$
$$i_0=0,\quad j_0=r,\quad\text{and}\quad h=\min(r,\mu+c).$$

B. The Numerator Δ'

We can derive the numerator by the closed system method [8]. It is well known that if a feedback path of weight $-F$ is drawn from the voltage node V_q to the voltage source node V_{sp}, in case of the voltage transmission T^v_{qp}, the graph determinant Δ_c of the closed system is

$$\Delta_c = \Delta + F \cdot \Delta'. \tag{11}$$

Let the source impedance of V_{sp} be Z_s, and we denote by P^{vF} the voltage product matrix with feedback path $-F$, that is, P^v with element $e-F$ instead of $e(=1,-1,$ or 0) at the position intersected by the row i_q and the column j_s. Then we have

$$\begin{aligned}\Delta_c = 1 + \sum_{k=1}^{h}\sum_{m}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \left\{\det P_k^{vF}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &\cdot \left\{\det P_k^{i}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg]\\ = 1 + \sum_{k=1}^{h}\sum_{m}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \Big[\left\{\det P_k^{v}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &- F\cdot\left\{\det P_k^{vs}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Big]\\ &\cdot \left\{\det P_k^{i}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg]\\ = 1 + \sum_{k=1}^{h}\sum_{m}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \left\{\det P_k^{v}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &\cdot \left\{\det P_k^{i}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg]\\ - F\cdot\sum_{k=1}^{h}\sum_{m(q,s)}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \left\{\det P_k^{vs}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &\cdot \left\{\det P_k^{i}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg].\end{aligned} \tag{12}$$

Therefore, the numerator Δ' of T^v_{qp} is

$$\begin{aligned}\Delta' = -\sum_{k=1}^{h}\sum_{m(q,s)}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \left\{\det P_k^{vs}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &\cdot \left\{\det P_k^{i}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg].\end{aligned} \tag{13}$$

For the current transmission T^i_{qp}, from the current source I_{sp} with source admittance Y_s to the current node I_q, we get the numerator Δ' in the same way as for the voltage transmission.

$$\begin{aligned}\Delta' = -\sum_{k=1}^{h}\sum_{m(q,s)}\Bigg[\left(\prod_{m=1}^{k} Z_{i_m}Y_{j_m}\right) &\cdot \left\{\det P_k^{v}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\\ &\cdot \left\{\det P_k^{is}(i_1,i_2,\cdots,i_k;j_1,j_2,\cdots,j_k)\right\}\Bigg].\end{aligned} \tag{14}$$

Hence we have (2).

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