A Sense-making Approach to Proof: Initial Strategies of Students in Traditional and Problem-based Number Theory Courses

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Abstract

This paper reports the results of an exploratory study of the perceptions of and approaches to mathematical proof of undergraduates enrolled in lecture-based and problem-based "transition to proof" courses. While the students in the lecture-based course demonstrated conceptions of proof that reflect those reported in the research literature as insufficient and typical of undergraduates, the students in the problem-based course were found to approach the construction of proofs in ways that demonstrated efforts to make sense of mathematical ideas. This sense-making manifested itself in the ways in which students employed initial strategies, notation, prior knowledge and experiences, and concrete examples in the proof construction process. These results suggest that such a problem-based course may provide opportunities for students to develop conceptions of proof that are more meaningful and robust than does a traditional lecture-based course.

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KEY WORDS: proof, sense-making, proof construction, transition to proof courses

I. Introduction

For many students, mathematics is a subject that is done to them rather than one in which they can explore ideas and think creatively. This perception is not limited to pre-college students; undergraduate mathematics majors and secondary mathematics teachers have been observed exhibiting a view of mathematical proof that is nearly procedural, regarding the construction and writing of a proof as an algorithm to follow rather than a creative process for solving a problem (c.f. Harel & Sowder, 1998a; Knuth, 2002). Research on school mathematics learning suggests that when children are actively engaged in problem-based mathematics lessons, they tend to take ownership of the content, viewing mathematical problem solving as a personally rewarding activity (Chazan, 2000; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human 1997). A growing body of research indicates that participation in a community of learners can be a vital part of students' success in mathematics. From a sociological perspective, learning is regarded as the product of the reflexive relationship between communally developed and shared classroom processes and individual constructive activity (c.f. Cobb & Yackel, 1996). In recent years, researchers have begun to examine mathematics learning that takes place in active classrooms, in which social negotiation of mathematical meaning is commonplace. These types of classroom environments are rare at the undergraduate level, however, and the effects of these on undergraduate mathematics learning are only beginning to be studied (c.f. Stephan & Rasmussen, 2002; Yackel, 2002).

A teaching method often encountered at the undergraduate level that employs mathematical discourse among students is the "modified Moore method"(MMM), named after its progenitor, Robert L. Moore. There are as many instantiations of the MMM as there are proponents of it, but what they all have in common is a *problem-based* approach to teaching mathematics, similar to Cognitively Guided Instruction[1] at the elementary level. In most MMM courses, students are given a carefully constructed list of problems to solve on their own, with little or no direct instruction from the professor[2]. The students then present their solutions to the problems in class and the instructor facilitates a whole-group discussion of the solution. Although its proponents claim that the MMM is ideal for teaching students to think mathematically and construct formal proofs (Mahavier, 1999; Renz, 1999; Jones, 1977; Lewis, 1990; Ingram, 2002), no research has been published on its effectiveness in helping students learn to construct and understand proofs.

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1) The MMM differs from CGI in that content is not structured according to students'developing understanding, but is prescribed in advance by an expert according to the logical structure of the subject. However, whole group discussion of student solutions characterizes both teaching approaches. See Carpenter, Fennema, Franke, Levi, & Empson (1999) for more information on CGI.

2) Depending on the instructor, students may or may not be allowed to use other texts or each other as resources. In the "original" method devised by Moore, students worked entirely individually and were not allowed to talk to each other at all. See the references cited in the text above for more information.

The results reported in this paper are from a pilot study for a larger project currently underway. We were invited to begin an investigation of the MMM by a small group of mathematics faculty and staff at a large state university in the southern United States in the fall of 2002. These individuals strongly believed that the MMM is an effective instructional style for introducing students to formal mathematical proof. We thus began a partnership that has grown to include additional mathematics and mathematics education faculty and graduate students. We believe this situation frames our research in a unique way; it began as a genuine collaborative effort between mathematicians and mathematics education researchers.

In this paper, we answer the following question: In what ways do the approaches to constructing proofs of students in an MMM course differ from those of students in a lecture-based course? In this paper, we will discuss the proof construction strategies of students enrolled in each type of course and will demonstrate that the students enrolled in the problem-based course approached the construction of proofs in ways that demonstrated efforts to make sense of the mathematical ideas. This research contributes to the literature on the learning of mathematical proof by demonstrating qualitative differences in the approach to proof between students enrolled in two differently-taught courses. Research to date has mainly focused on the conceptions and approaches to proof of students enrolled in courses in which direct instruction in proof was the norm; we suggest that a problem-based course such as the one studied provides more opportunities for students to develop conceptions of proof that are personally meaningful. In reporting the results of this study, we assume that what students do when first presented with a statement to prove reveals a great deal about their understanding of proof. We emphasize that these are the results of an exploratory study conducted with a small number of participants, and we do not attempt to make broad generalizations based on them.

II. Background And Theoretical Perspective

Many American universities require mathematics majors to enroll in so-called *transition courses*, in which the students are introduced to mathematical formalism. The first year of coursework in mathematics tends to be more practical than theoretical, and most students have not been exposed to formal proof in their prior coursework. Transition courses serve to help students make the transition to advanced courses whose content is primarily theoretical and proof-based. It is in such courses that most students develop their understandings of formal mathematical proof; hence these courses provide an opportunity for researchers to study students' developing conceptions of proof in mathematics. In this section, we provide an overview of expert views of the nature and role of mathematical proof and of students' difficulties with mathematical proof.

A. The Role of Proof in Mathematics Education

The purpose of proof in mathematics teaching is different from its role in the field of mathematics research, in which its primary role is to demonstrate the validity of propositions and conjectures. Hersh (1993) notes that this role of proof does not necessarily translate to the classroom, as most students are quite easily convinced that a mathematical proposition is true before a proof is given. The role of proof in the mathematics classroom is primarily explanatory; that is, students should ideally view proofs as giving insight into *why* propositions are true or false. Many researchers have made similar distinctions between types of proofs: Hanna (1991) distinguishes between "proofs that explain" and "proofs that prove;" Tall (1999) refers to "logical" and "meaningful" proofs; Weber and Alcock (2004) distinguish between "syntactic" and "semantic" proofs; and Raman's (2003) description of the types of ideas used in proof production emphasizes the difference between "heuristic ideas" and "procedural ideas", with the concept of a "key idea" linking the two. What all these researchers seem to be addressing, in one way or another, are two distinct approaches to mathematical proof: a procedural, logical approach on which the prover's intuition is not necessarily engaged, and an approach relying on the prover's intuitive understanding of the mathematical structure involved[3]. Both approaches can be considered valid, even desirable for students to master, but it remains unclear how students come to develop the sort

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[3] It should be noted that the work of Tall and Hanna centers on written arguments produced by others and presented to the students studied, while the work of Weber, Alcock, and Raman is focused on students' proof construction processes.

of mature conception of and facility with proof demonstrated by professional mathematicians.

B. Students' Conceptions of Mathematical Proof

There has been a great deal of interest in the mathematics education community in studying the difficulties students have with proof, possibly because of an increased emphasis on proof in mathematics curricula. Secondary school students are expected to be able to construct and evaluate conjectures and mathematical arguments and proofs by the end of grade 12 (NCTM, 2000), and the development of an understanding of mathematical proof has long been regarded as one of the benchmarks of a major in mathematics (Tall, 1992). It is clearly vital for a student who intends to pursue graduate study in mathematics to be able to construct, understand, and validate formal mathematical arguments (CBMS, 2001; Selden & Selden, 2003), yet research shows that many students ultimately do not succeed in developing an appreciation for mathematical proof by the end of their undergraduate programs.

Harel and Sowder (1998a, 1998b) developed a framework that characterizes the proof schemes of secondary and undergraduate mathematics students. Some students' proof schemes are *externally-based*, in which a "proof" is constructed by appealing to an authority (such as the text or a teacher), or judged to be correct because it has the proper form or uses the appropriate symbols. Another type of proof scheme is the *empirical* scheme, in which the student's proof strategy is to give several examples that "show" the statement to be true, or offer a generic example that appears to embody all cases, but actually does not. Ideally, undergraduate students will develop *analytic* proof schemes, which are deductive in nature and make use of the axiomatic structure of mathematics. Harel and Sowder found that few of the undergraduates in their study had developed analytic proof schemes.

Knuth (2002) found that secondary mathematics teachers often accepted false "proofs" of algebraic statements as correct if those proofs contained correct algebraic manipulations or were in the correct form. These teachers also indicated they would accept an empirical "proof"(a general example) if it explained why the statement was true. Secondary factors such as containing sufficient detail or use of a familiar proof technique were frequently used to determine the validity of a mathematical argument, rather than the correctness of the argument itself. Knuth notes that preservice teachers have few opportunities to examine and discuss proof with others, and states that "the Moore method of teaching ⋯ provides undergraduates with just such an experience" (p. 400).

A good deal is known about what students struggle with and the difficulties they have when learning to prove, but more must be learned about what students can do and what strategies help them to develop a conception of proof that is more like that of a mathematician. Some researchers have demonstrated that students can develop this perspective under certain circumstances; for example, Harel and Sowder (1998a) describe a teaching experiment in which students were gradually introduced to inductive arguments, resulting in a fairly deep understanding of formal mathematical induction by the end of the course.

Weber (2002) found that undergraduate students often had difficulty constructing valid proofs of group theory propositions, even when they possessed all of the pertinent factual knowledge about the concept and the demonstrated ability to construct simple proofs. Weber referred to this type of knowledge as an *instrumental* understanding of proof. The more experienced graduate students in his study employed strategies demonstrating they had connected, intuitive understandings of the topic, and were very successful provers. Weber concluded that this use of *relational* understanding is a necessary factor for proving, and notes that "even students with a strong logical background can prove very little with definitions, facts, and theorems" alone (p. 9). Weber and Alcock (2004) refine these ideas by describing two distinct approaches to proof production: *syntactic* and *semantic*. Syntactic proof productions are "written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way"; semantic proof productions are those for which "the prover uses instantiation(s)[4)] of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). The authors contend that, while syntactic proof productions are often useful, semantic proof productions are more powerful and mathematically valuable. They further argue that mathematicians are adept at both types of proof productions.

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4) Weber and Alcock (2004) define an instantiation of a mathematical object to be "a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual" (p. 310).

While there is a growing body of work concerning students' attitudes towards proof, schemes for proving, and difficulty producing correct and valid proofs, there is little research focusing on how students *begin* the process of constructing a proof and how teaching strategies and learning experiences affect the development of students' understanding. While mathematics education researchers have articulated what a "good" conception of mathematical proof is, little is known about how such conceptions are achieved.

Ⅲ. Method

Because we could find no published research on the effects of the MMM on students' understanding of mathematics, we decided to model our study after other published research studies that had focused on undergraduates' conceptions of mathematical proof (c.f., Harel & Sowder, 1998a). To this end, we studied the experiences of participants enrolled in one of three sections of a number theory course, employing task-based and phenomenological interviews to gain insight into their conceptions of proof and their perspective on the course.

A. Participants and Setting

Five students enrolled in an undergraduate number theory course at a large state university were interviewed twice during the semester of study. Three of the students were from sections being taught using the MMM, and two were from a section being taught in a traditional lecture style[5]. The instructors of each of the three sections of the course were asked to choose several students from their respective sections whom they considered "above-average" based on homework grades, the results of the first exam, and participation in class or office hours. From those suggested, two students from each section were invited to participate in the study. Five of the invitees ultimately agreed to participate. Each ultimately earned a grade of A or B in the course and all had very similar mathematical backgrounds with regard to coursework and grades earned in earlier courses.

The traditional (lecture-based) section of the course was taught by an associate professor, "Dr. L", who was generally recognized as an excellent lecturer by his colleagues and students. Class meetings of this section of the course primarily consisted of lectures; there was occasional teacher-led discussion of the material presented, but there were no student presentations or group work in this section. The two MMM sections were taught by "Dr. T" and "Dr. N", two full professors in the department of mathematics. The structures of the two MMM sections were nearly identical. Every few weeks, students were given handouts called "course notes"; these contained a list of theorems to be proved, along with necessary definitions and some exploratory problems[6]. These course notes had been developed over a period of

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[5] In this paper, the term "MMM sections"will refer to the sections of the course that were taught using the modified Moore method, and the term "traditional section"will refer to the section taught in a standard lecture format. It should be noted that the instructor of the traditional section was a highly regarded teacher in the mathematics department, as were the instructors of the MMM sections.

[6] See the appendix for a sample from the course notes.

several years by Dr. T and Dr. N, and were designed to guide students through a logical sequence of mathematical propositions and concepts. The students in the MMM sections worked through these notes outside of class and presented their proofs during class meetings. At the beginning of each class, the instructor asked for volunteers to present the next three to six proofs in the problem sequence from the course notes. After each proof was presented, the instructor asked for questions or comments, with the goal of facilitating a discussion of the presented proof. The instructors did not indicate whether the presented proofs were correct or incorrect, but expected the class to come to a consensus.

B. Data Collection

The primary source of data for the results presented in this paper were semi-structured task-based interviews conducted during the semester of the study. Two interviews were conducted with each participant, one in the middle of the semester and one at the end of the semester. In these interviews, the participants were presented with several elementary number theory propositions to prove and were asked to "think aloud" while constructing their proofs. After a proof was constructed, the participants were asked to write down the formal proof, as if it were to be turned in to the instructor. The goal of the interview was to observe the participants as they went through the process of constructing a proof of a proposition they had not seen before. An example of an interview task will be discussed in detail later in this paper.

C. Data Analysis

The transcribed interviews were coded using open and axial coding techniques; we consider our approach to be contained in the broad category of "grounded theory"(Strauss & Corbin, 1996). Preliminary scans of the transcripts of each individual interview yielded a set of themes to examine in greater depth. These themes were initially tied to each individual, and participants' written work from the interviews was used in conjunction with this initial coding of the transcripts to produce a set of descriptions of each individual's proof strategies and conceptions over the two interviews. The themes were merged and eventually developed into categories of codes, which were then used to examine the participants' work on each part of the interviews. In particular, we coded the transcripts for each proposition and carefully examined the participants' strategies and proofs. We then examined the resulting sets of codes for the participants in the MMM sections and the traditional section separately, and the differences between the two groups began to emerge.

D. Participants

All of the students who participated in the study were taking the number theory course to fulfill requirements in their degree programs. Alicia[7] and Eli were both studying

7) All names are pseudonyms. Alicia and Carrie were female, while Bill, Daniel, and Eli were male.

mathematics with the goal of becoming actuaries, and Bill and Daniel were mathematics majors intending to pursue graduate studies in mathematics. Carrie was pursuing a liberal arts degree through the honors program; this course was the first and only upper division mathematics course she took at the university[8]. Bill, Carrie, and Daniel were enrolled in the MMM sections, while Alicia and Eli were enrolled in the traditional section of the course.

The participants also had a wide range of previous experience with mathematical proof prior to the course. Carrie had no university-level proof experience, while Bill had taken a summer session course in which the primary activity was proving theorems. Proof was a new topic for Alicia, Eli, and Daniel, though these students were enrolled in other courses that emphasized proof during the semester of the study.

Ⅳ. Results: Strategies for Constructing Proofs

During the interviews, the participants were given several propositions to prove. Analysis of these interviews revealed four primary differences in the ways the students from the MMM and traditional sections approached these tasks; these differences are summarized in Table 1 below. In the following section, we will

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8) Carrie had previously taken a course known as "Math for Liberal Arts Majors" taught by the professor in whose number theory course she was enrolled. The course focused on mathematics in art and history, topics in elementary number theory, and problem solving. She had enjoyed the course and the professor suggested she take his number theory course as well. It fulfilled a requirement in her degree program.

illustrate these differences using examples from a task given in the first interview: *Prove the product of twin primes is one less than a perfect square.* This is a relatively simple proof task in that it only requires the prover to represent the product algebraically using the definition of twin primes, and then apply elementary algebra to this representation. All but one of the participants were able to prove the proposition. We were not particularly interested in what the participants were or were not able to do; rather, we were interested in the strategies each used during the proof construction process.

Table 1.

	Traditional	MMM
Use of initial strategies	Began searching for proof techniques	Tried to make sense of the statement
Use of notation	Introduced notation appropriate to proof technique chosen	Introduced notation naturally, in context of making meaning
Use of prior knowledge & experiences	Related to other proof strategies based on surface features	Related to other proof strategies based on the concept
Use of concrete examples	Reluctant to work concrete examples (not a proof, so not helpful)	Worked concrete examples to gain insight into main idea

A. Use of Initial Strategies

The students from the traditional section began their proofs by searching their memories for potential proof techniques to apply. Alicia's immediate reaction after reading the twin primes proposition was to list strategies she could use to prove it. She mentioned induction and proof by contradiction, and quickly decided that contradiction was the more appropriate choice. Alicia struggled to try to remember *something* that would help her with this problem - theorems in the book, similar problems she had seen before, etc. She seemed at this point to begin "throwing" strategies at the problem in hope of finding something that would work. This behavior is similar to struggling provers' use of syntactic strategies as described by Weber (2002).

Eli also began by trying to determine the best method for proving the theorem. He mentioned contradiction, a proof strategy he seemed to be familiar with and thought might work for this proof as well, though he gave no rationale for this idea. Eli also initially disregarded the definition of twin primes, only using the fact that these primes have a difference of two when he reached an impasse. None of the MMM students made this error.

In contrast, the MMM students began the process of constructing a proof for a given proposition by trying to *make sense* of the statement. Bill began by symbolizing the problem as he initially read the statement, writing " $ab = \sqrt{n} - 1$, where n is an integer, and a and b are twin primes." He struggled to find a way to write down a representation for a perfect square, writing n in its prime factorization representation $(c_1 c_2 \cdots c_n)^2 - 1$, and stated that what he needed to show was that $ab = (c_1c_2\cdots c_n)^2 - 1$. Bill appeared to be listing relevant properties of primes here in order to make sense of the statement; at this point, he was not yet trying to find a proof. Carrie also tried to make sense ofthe statement as she read it for the first time, saying, "One less than a perfect square is like $(2^2 - 1)$ or $(3^2 - 1)$". She did not spend much time trying to recall facts she knew about primes; rather, she immediately checked the statement against a simple example. Both Bill and Carrie made efforts to understand the statement of the proposition *before* they tried to prove it.

B. Use of Notation

The traditional students introduced notation that appeared to be specific to the proof strategies they were using. Eli rewrote the proposition using mathematical notation, saying, "the first thing I would do is try and bring a mathematical definition, write this out in numbers so I can manipulate it." His initial strategy was to apply algebra to the equation he had written. He noted that there were several ways he could think of to denote a perfect square and that he was not certain what "they want" in this particular case. Eli's use of notation seemed to be focused on what he was *supposed* to do to prove the theorem, rather than used as a tool to symbolize his thinking.

One of Alicia's initial efforts was a contradiction strategy; she wrote the statement to contradict as $P_1P_2 \neq t^2 - 1$. She began manipulating this statement algebraically, and did not realize she was not using the assumption that these were twin primes for several minutes. Alicia's use of notation in this instance was driven by the algebraic manipulation strategy she had decided to employ.

The MMM students tended to introduce notation in logical and natural ways in the context of making sense of the proposition to be proved. For example, Carrie quickly realized that the perfect square mentioned in the proposition was the square of the integer between the twin primes. She immediately began to wonder why this was true, and symbolized her thoughts as she spoke:

> "I mean it makes sense because you know, there's only one number between [the] two primes, and so if you multiply the two primes together, it's gonna equal something close to multiplying the two primes, you know, or the average of the primes. So if x is 4, and then 5 would be $x+1$ and 3 is $x-1$, then that's [product] gonna equal x^2-1."

In her attempt to answer her own question of why, she used algebra very naturally, writing down a symbolic representation of her argument after first explaining it in words. In this instance, Carrie used algebra as a tool to verify and symbolize an argument that she had already constructed and understood.

C. Use of Prior Knowledge and Experiences

As noted above, the students from the traditional section tended to begin the proving process by listing everything they knew that might possibly relate to the proposition. This "listing" of properties and choice of proof strategies appeared to be based on *surface features* of the statement to be proved, rather than on an understanding of the problem or concept. For example, Alicia attempted to recall proofs of statements about primes, hoping she could employ a similar technique to prove the twin primes proposition. After struggling and searching for several minutes with no success, she began manipulating the equation she was trying to prove true, and eventually used the strategy of adding "zero"to her representation of the product of the primes. This quickly led to a direct proof. It was apparent during this process that she was applying every algebraic "trick" she could recall, without any clear rationale for doing so. This strategy was based on her previous experiences with proofs involving equations, and not on any understanding of the proposition. Eli's initial strategy for constructing a proof was contradiction. He said, "What I want to prove is that if this is not true, this doesn't equal a perfect square, then the square root of it would not be an integer." [He had written $P^1P^2+1 \neq a$.] His strategy was to contradict the fact that a was a perfect square; this strategy seemed to be based on previous experiences with proofs of equations, and he made little progress using it.

The MMM students also spent time trying to relate the proposition to previously experienced strategies and problems, but appeared to do so based on the *concepts* involved and their understandings of the problem. After reading through and discussing the meaning of the proposition, Daniel spent some time searching his memory for relevant information about primes, mentioning Mersenne primes and Fermat's "little" theorem. He tried substituting values for p in the expression 2^p-1 in order to see if the result would produce any usable information. When asked why he was doing this, Daniel replied, "Well I'm supposed to be able to substitute something for [one of the primes]." Though this substitution strategy may appear to be relevant only because the proposition was about prime numbers, Daniel's use of it and other general strategies for working with primes seemed to be focused on the underlying concept of the statement, rather than on surface features.

D. Use of Concrete Examples

The differences between the two groups'use of concrete examples was particularly striking. The traditional students rarely made use of examples during the proof construction process, while the MMM students did so frequently. The use of examples appears to be related to a student's proving strategies in two ways: as part of the individual's efforts to *make sense* of the proposition, and as a more general *problem-solving strategy.*

Carrie used concrete examples to make sense of the twin primes proposition immediately after reading it, saying, "One less than a perfect square is like (2^2-1) or (3^2-1)". She then paused to write out a few examples, exploring the problem and looking for a pattern:

> "So like some twin primes that I know are 3 and 5, so 3 times 5 is 15, um, and yeah that's one less that 16. It's probably...hmm, yeah, one less than a perfect square...if it's a prime does mean it doesn't have...I don't know. Its only prime factors

> are one and itself. The product of twin primes, oh, well that would be okay, the number between the twin primes...3,4,5...is gonna make the perfect square that it's one less than. Um, *so why would it be one less?*"

As described above, this use of examples subsequently led Carrie to find a proof. Similarly, Carrie used examples to explore her hypothesis that the proposition could be generalized.

> *Carrie*: Well, I have a question about this because um, it's... it refers specifically to 2 prime numbers and it seems like, like with this proof idea that I've got going on, the numbers wouldn't necessarily have to be primes. It would work for any number if the difference is 2. So I don't know if that means that my logic is wrong or if that's just like, it happens to be written for twin primes.
> *Interviewer*: That's an interesting observation.
> *Carrie*: So say they were 4 and 6 and 5 in the middle, so 5 times 5 is 25, 6 times 4 is 24. Um, so it works with 4 and 6, which aren't prime.

Carrie had noticed that she proved this statement without using the fact that the numbers are primes. She conjectured that the statement is true for all pairs of integers whose difference is two. She immediately worked through an example with composite numbers, and in the interview itself, seemed convinced. Her written proof was of the more general statement, and she noted that it followed for twin primes as well.

Daniel was initially reluctant to use examples, but employed them as a problem-solving tool a last resort. "I know if I do examples it might help, but if I do the examples, I mean, that's only like one solution or a couple of solutions to the whole thing. But I'm supposed to prove [it for] all twin primes." After working through a few examples, Daniel realized that it might be helpful to write one of the twin primes in terms of the other and make a substitution. From there, he quickly solved the problem.

V. Concluding Remarks

A. Overview of Results

Despite the small size of the sample, we found marked differences between the students in the MMM sections and the students in the traditional section in their approaches to the construction of proofs. As detailed above in Table 1, the differences in approach to proof fell into four categories: use of initial strategies, use of notation, use of prior knowledge and experiences, and use of concrete examples. Taken together, these four attributes can be considered examples of ways in which students attempt to make mathematics personally meaningful. The students in the MMM sections were focused on *making sense* when constructing proofs, while the traditional students seemed to be searching for a solution that would be recognized as valid by an external authority.

While the MMM students appeared to want to understand the proofs they were constructing and wanted to write them in such a way as to make them clear and meaningful, the traditional students were more concerned about correctly using the appropriate logical structure, such as induction or contradiction. The MMM students wrote their proofs conversationally, in a similar style as how they spoke and thought. The traditional students, on the other hand, tended to focus much more on the "proper" form and order of their proofs. Their comments emphasized the process of learning how to "get through" the proof and discovering what one was "supposed to do" in that situation.

This emphasis on making sense of mathematics is striking for several reasons. Much of the research on students' conceptions of proof reports that they hold naïve perspectives on the nature and role of proof in mathematics (Harel & Sowder, 1998a; Knuth, 2002). The MMM students, however, viewed proof as a means of making sense of mathematics, as a tool for building understanding, and as a way of communicating results to others (Hanna, 1991; Hersh, 1993; Tall, 1992). In some ways, the MMM students' approach to proof is reminiscent of Weber & Alcock's (2004) *notion of a semantic proof production*: "a proof in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). The MMM students' use of initial strategies, notation, prior experiences, and examples could be considered as such instantiations of mathematical concepts, meaningful ways of thinking about mathematical objects.

A point that should be stressed here is that we do not wish to imply that the MMM students' understandings of proof were necessarily *better* or more sophisticated than those of the students in the lecture-based section. Rather, we are interested in the qualitative differences between the approaches to proof exhibited by the students. Though the participants had varied mathematical interests and backgrounds, the differences between the students in the MMM sections and the students in the traditional section were marked. Why might the MMM course have produced students who approached proof so differently? We are currently investigating this question, but our hypothesis is that the difference is a result of participation in a community of inquiry in which learning was based on solving problems and discussing solutions. The MMM sections placed great importance on discussion and consensus-building, and these were the main vehicle for the introduction of content. Unlike a more traditional "introduction to proof" course, the students learned to prove without being taught specific strategies in advance; that is, there was no "section" covering proof by contradiction or induction, followed by ample opportunity to practice. The students in the MMM course were forced to consider each theorem statement individually and to decide for themselves how to best go about proving it. In addition, presenting mathematics to peers and evaluating peers' work was a major course activity. The students in the MMM course were presented with several proofs every class meeting, and it was the responsibility of the group to determine the validity of each. Sense-making was an important aspect of the course, and a requirement for participation; it ultimately affected the development of the students' conceptions of mathematical proof. Though we did not collect enough data to compare the two classroom environments, it is difficult to escape the conclusion that something very interesting happened in the MMM sections that enabled students to develop their ideas about proof in an unusual way (with respect to the research literature, at least). Based on preliminary analyses of more recently collected data, we hypothesize that classroom communities of inquiry (such as the MMM) encourage students to develop semantic approaches to proof construction.

B. Limitations

These results are drawn from a pilot study for a larger project currently underway; the analysis of the data described in this paper informed the design of the larger study. The small number of participants and exploratory nature of the study limit our ability to make generalizations from these results. The participants were selected to be representative of the population enrolled in each section, but we cannot be certain that they were typical students. The two students enrolled in the traditional section were studying to be actuaries, while two of the three students enrolled in the MMM section were studying pure mathematics. Though it could be argued that these students are different enough as to make a comparison unfeasible, it is the case that their mathematical backgrounds were quite similar: they had taken the same courses up to that point in their degree programs. In addition, the case of Carrie is interesting: she had not studied any mathematics beyond a high school level before this course, yet she was able to develop a relatively sophisticated conception of mathematical proof.

C. Questions for Further Study

As noted above, the results reported in this paper come from a small exploratory study; we plan to conduct a larger-scale comparison in the near future. We are currently analyzing data from an in-depth study of a MMM course in an effort to learn more about the nature of the interactions between the teacher and students and how these affect students' developing conceptions of proof. These results raise questions about the differences between courses taught using problem-based approaches and more teacher-centered styles. This research contributes to the field a view of the ways in which such classroom environments can promote more meaningful approaches to mathematical proof. There appears to be a need for more research investigating how students learn the process of proving in non-traditional classroom environments.

Advocates of the MMM have long claimed that it is successful in helping students learn how to "do" mathematics as mathematicians do. These preliminary results suggest that problem-based teaching strategies such as the MMM may encourage students to take sense-making approaches to mathematical proof in ways that students in traditional courses do not. This sort of personal engagement - taking ownership of the content, viewing mathematical problem solving as a personally rewarding activity - may result in deeper understanding of mathematical ideas and processes.

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APPENDIX

Excerpt from the MMM sections' "Course Notes"

(Reprinted with permission of the instructors)

4. Fermat's Little Theorem and Euler's Theorem

Introduction. Modular arithmetic gives us some examples of algebraic structures that lead to many fundamental ideas in abstract algebra. We begin here by exploring how powers of numbers behave mod n. We will find a structure among numbers mod n that is interesting in its own right, has applications, and leads to central ideas of group theory.

4.1. Question.

For $i = 0, 1, 2, 3, 4, 5$, and 6, find the number in the canonical complete residue system to which 2^i is congruent modulo 7. In other words, compute $2^0, 2^1, 2^2, \ldots, 2^6 \pmod 7$.

4.2. Theorem.

Let a and n be natural numbers with $(a, n) = 1$. Then for any natural number j, $(a^j, n) = 1$.

4.3. Theorem.

Let a and n be natural numbers with $(a, n) = 1$. Then there exist natural numbers i and j with $i \neq j$ such that $a^i \equiv a^j \pmod n$.

4.4. Theorem.

Let a and n be natural numbers with $(a, n) = 1$.

Then there exists a natural number k such that $a^k \equiv 1 \pmod{n}$.

Definition. Let a and n be natural numbers with $(a, n) = 1$. The smallest natural number k such that $a^k \equiv 1 \pmod{n}$ is called the *order of a modulo n* and is denoted $\mathrm{ord}_n\, a$.

4.5. Question.

Choose some relatively prime natural numbers a and n and compute the order of a modulo n. Frame a conjecture concerning how large the order of a modulo n can be, depending on n.

4.6. Theorem.

Let a and n be natural numbers with $(a, n) = 1$ and let $k = \mathrm{ord}_n\, a$. Then the numbers $a^1, a^2, \ldots, a^k$ are pairwise incongruent modulo n.

4.7. Theorem.

Let a and n be natural numbers with $(a, n) = 1$ and let $k = \mathrm{ord}_n\, a$. Then for any natural number m, a^m is congruent to one of the numbers $a^1, a^2, \ldots, a^k$ modulo n.

4.8. Theorem.

Let a and n be natural numbers with $(a, n) = 1$, let $k = \mathrm{ord}_n\, a$, and let m be a natural number. Then $a^m \equiv 1 \pmod{n}$ if and only if $k \mid m$.