AN OPTIMAL POLICY FOR ECONOMIC GROWTH*

KI-SOO SUNG**

Leontief's closed economic model is optimized using Pontryagin's maximum principle. The objective functional is maximization of time integrated utility.

Pontryagin's/1/ "Maximum Principle" has been successfully applied to problems of optimal policy for economic growth by several authors. Shumpei Kumon/2/ presented a one-sector model using linear flow and stock functions at the 1966 Far Eastern Meeting of Econometric Society. Kurz/3/ and Stoleru/4/ analysed two-sector growth models where relevant equations were not necessarily linear. Chakravarty/5/ formulated an n-sector model using Leontief matrices of stock and flow coefficients. Superiority of the Maximum Principle over the classical calculus of variations lies in the fact that policy variables are treated endogeneously and regions of policy variables may be closed set, giving rise to endogeneous policy variables (the control functions) which may be discontinuous. Switching control functions were observed in the growth models of Kurz, Stoleru, and Kumon. The Chakravarty model did not produce switching control function and the functions found were insensitive to utility function.

In this paper, Kumon's model is generalized to n-sector model based on the Leontief matrices and Pontryagin's Maximum Principle. The model represents a closed economy in which foreign trades are excluded. Optimal fiscal and investment policy is to be obtained to maximize a welfare utility functional. General solution of the growth model contains switching control functions (policy) which depend explicitly on the utility (welfare) function, in contrast to the Chakravarty model. Utility maximizing optimal solution is obtained using a flat utility function with exponential depreciation. Interpretation of the general solution is made briefly. Result

---

* This paper is a revision of "A General Theory of Optimal Policy for Economic Growth" presented at The Second Far Eastern Meeting of Econometrics Society, Tokyo, Japan, Summer 1967.

** The author, a Ph.D in Mechanical Engineering, Harvard University in 1963 is currently Manager, Electronic Data Processing Department, Korea Institute of Science and Technology.
He is also a visiting lecturer of applied mathematics and systems analysis at the Graduate School of Pulbic Administration, Seoul National University.

shows that the optimal policy is comprised by extreme fiscal policies in each sector of the economy. When there are n sectors in the economy and n policy controls, the number of switches in polic' is found to be at most n for each policy control under an infinite time horizon.

Basic equation: of the model are

$$(1)\ x_i=\sum_j a_{ij} x_j+\sum_j b_{ij}\frac{dx_j}{dt}+c_i,\ x_i>0$$

where X is output vector, C consumption vector, A and B the Leontief matrices of flow and stock. The planning board is to control growth rate of each sector of the economy through the control variables.

$$(2)\ u_i(t)=\frac{dx_i}{dt}/x_i$$

$$\underline{u}_i\leq u_i\leq\bar{u}_i$$

The objective functional is

$$(1)\ Z=\int_o^T\sum_i q_i\ c_i dt$$

where $q_i=q_{0i}\ \exp(-w_i t)$ denotes social value of consumption c_i at time t. Eliminating dx_i/dt and c_i in (3) by (1) and (2), the objective functional Z is expressed in terms of the phase variables $x_i\ (t)$ and control variables $u_i\ (t)$:

$$(4)\ Z=\int_o^T\sum_{ij} q_i\ ((\text{I-A})_{ij}\ x_j-b_{ij}\ u_j\ x_j)dt$$

where I is unit matrix. Time dependence of the control vector U(t) is to be determined in such a way that the welfare functional Z is maximized. The Hamiltonian of Pontryagin is then

$$(5)\ \kappa=-\psi^o\sum_{ij} q_i((I-A)_{ij}x_j-b_{ij}u_jx_j)+\sum_i u_ix_i\psi_i$$

where $\psi^o=$constant≤ 0 and ψ is a nonzero continuous vector function defined by

$$(6)\ \frac{d\psi_i}{dt}=-\frac{\partial\kappa}{\partial x_i}=-u_i\psi_i+\sum_j\psi^o q_j((I-A)_{ji}-b_{ji}u_i)$$

optimal policy for sector i is obtained from (5) as switching function:

$$(7)\ u_i(t)=\frac{\bar{u}_i+\underline{u}_i}{1}+\frac{\bar{u}_i-\underline{u}_i}{2}\text{sign}(\psi_i(t)+\psi^o\sum_j q_j(t)b_{ji})$$

$\psi(t)$ is obtained from (6) as

$$(8)\ \psi_i(t)=\psi_{oi}\ \exp(-u_it)+\psi^o\sum_j\frac{q_{oj}}{u_i-w_j}((I-A)_{ji}-b_{ji}u_i)\ \exp(-w_jt)$$

The constants of integration ψ_{oi} can be determined by using boundary conditions. Output vector is integrated from (2) as

(9) $x_i(t)=x_{oi}\exp(u_i t)$

For boundary conditions having fixed initial point and variable terminal point, i.e.,

(10) $X(_o)=X_o, \qquad X(T)=\text{variable}$

transversality condition leads to $\psi(T)=0$, and therefore $U(T)=\underline{U}$ from (7). Then ψ_{oi} for t near T is fixed and $\psi(t)$ for t near T is determined. Then $U(t)=\underline{U}$ for t near T. Tracking backward from $t=T$, the policy $U(t)=\underline{U}$ is maintained until a switching point is encountered in the sign function of control vector (7). $U(t)$ is redetermined and $\psi(t)$ is retermined using new values for $U(t)$ and ψ_{oi} togeter with continuity of $\psi(t)$. Tracking again backward, new policy $U(t)$ is maintained until a switching point is encountered in the sign function of control vector (7). $U(t)$ is redetermined, etc. Thus $\psi(t)$ and Now $x_i(_o)=x_{oi}$ and $x_i(t)=x_{oi}\exp(u_i t)$ for t near 0. Tracking forward from $t=0$, X_o is redetermined whenever a switching point is encountered in control $U(t)$ to make $X(T)$ continuous. Then $X(t)$ is determined for the whole plan period $0\leq t\leq T$. Terminal structure of the economy $X(T)$ is found.

Next, consider the control function $U(t)$ defined by

(11) $u_i(t)=\dfrac{dx_i}{dt}, \qquad \underline{u}_i\leq u_i\leq \bar{u}_i$

instead of (2). Then (1), (3), (11), and (10) constitute an optimal problem having the following solution:

(12) $Z=\int_0^T\sum_{ij} q_i((I-A)_{ij}x_j-b_{ij}u_j)dt$

(13) $k=\sum_i u_i\psi_i-\psi^o\sum_{ij}(q_j((I-A)_{ji}x_i-b_{ji}u_i))$

$\psi^o=\text{constant}\leq 0$

(14) $\dfrac{d\psi_i}{dt}=-\dfrac{\partial\kappa}{\partial x_i}=\psi^o\sum_j q_j(t)\ (I-A)_{ji}$

(15) $u_i(t)=\dfrac{\bar{u}_i+\underline{u}_i}{2}+\dfrac{\bar{u}_i-\underline{u}_i}{2}\text{sign}\ (\psi_i(t)+\psi^o\sum_j q_j(t)b_{ji})$

(16) $\psi_i(t)=-\psi^o\sum_j\dfrac{(I-A)_{ji}}{w_j}(q_j(t)-q_j(T))$

(17) $x_i(t)=x_{oi}+u_i t$

In this case, $\psi(t)$ is independent of control $U(t)$, but output $X(t)$ depends on control $U(t)$. Therefore X_o should be redetermined whenever a switching occurs in control $U(t)$ so that $X(t)$

is continuous. In the argument of sign function in (15), there are n independent exponential function of time . Therefore the control function $u_i(t)$ has at most n switching points for nfinite time hori on. The control vector $U(t)$ can have at most n x n switching points for infinite time hori on.

REFERENCE

(1) Pontryagin, .S., et al.: The Mathematical Theory of Optimal Processes, New York, Interscience Publishers, 1 62.

(2) Shumpei Ku on: A Domar-Type Model of Planned Economic Growth, University of Tokyo, The First Far Ea tern Meeting of the Econometric Society, 1966.

(3) Mordecai K rz: Optimal Paths of Capital Accumulation Under the Minimum Time Objective, Econometric , January, 1965.

(4) Lionel G. St leru: An Optimal Policy for Economic Growth, Econometrica, April, 1967.

(5) Katsuhiko O ata: State Space Analysis of Control Systems, Prentice Hall, 1967.

(6) S. Chakrava ty, "Optimal Program of Capital Accumulation in a Multi-Sector Economy," Econometrica, Vol. 3, No. 3 (July, 1965).