서울大學校 師大論叢 第34輯(1987. 6. 30)

Group Rings Isomorphic to an Integral Group Ring With Periodic Group Basis

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Ⅰ. Introduction

Let G be a group and R be a ring with unity. The mapping $\varepsilon : R[G] \to R$ defined by $\varepsilon(\Sigma a_x x) = \Sigma a_x$ is an R-homomorphism. This mapping ε is called the augmentation mapping of $R[G]$. The kernel of this augmentation mapping is called the augmentation ideal of $R[G]$ and we denote it by $\omega(R[G])$. That is,

$$\omega(R[G]) = \{\Sigma a_x x \in R[G] \mid \Sigma a_x = 0\}.$$

Now if $H \lhd G$, there is a mapping $\rho_H : R[G] \to R[G/H]$ given by $\rho_H(\Sigma a_x x) = \Sigma a_x \bar{x}$ where $\bar{x} = xH$ is the image of x in G/H. This mapping is an R-algebra homomorphism, which is called the natural homomorphism. The kernel of this homomorphism of ρ_H is an ideal of $R[G]$ which is equal to $\omega(R[H])R[G]$.

Let $\{g_i\}$ be a subset of G, H be the subgroup it generates and K be a field. Then,

$$\Sigma(g_i - 1)K[G] = \omega(K[H])K[G],$$

$$\Sigma K[G](g_i - 1) = K[G]\omega(K[H]).$$

Furthermore if H is infinite then left and right annihilators of $\omega(K[G])$ are zero. That is, $l(\omega(K[H])) = r(\omega(K[H])) = 0$. If H is finite and $\hat{H}$ denotes the sum of the elements of H in $K[G]$ then,

$$l(\omega(K[H])) = K[G]\hat{H},\ \ r(\omega(K[H])) = \hat{H}K[G],$$

$$l(\hat{H}) = K[G]\omega(K[H]),\ \ r(\hat{H}) = \omega(K[H])K[G] \ \ [3;\ p.68].$$

Let C be a finite conjugacy class of G and let $\hat{C}$ be the sum in $R[G]$ of finitely many elements of C. We call these group ring elements $\hat{C}$ the finite class sums. If $x \in G$ and R is commutative ring then we let C_x be the class containing x. Then the center, $C(R[G])$, of $R[G]$ has as an R-basis the set of all finite class sums. Furtheremore, the structure constant c_{xyz} defined by $\hat{C}_x\hat{C}_y = c_{xyz}\hat{C}_z$ are given by

$$c_{xyz} = \text{the number of ordered pairs } (x', y')$$

with $x' \in C_x, y' \in C_y$ and $x'y'=z$ [3; pp. 113-114].

Now for a group G, we define

$$\Delta=\Delta(G)=\{x\in G \mid |G:C_G(x)|<\infty\},$$

$$\Delta^+=\Delta^+(G)=\{x\in\Delta \mid o(x)<\infty\}.$$

Then Δ and Δ^+ are characteristic subgroups of G and Δ^+ is a locally finite subgroup of G [3; p. 116]. Clearly if R is commutative $C(R[G])\subseteq R[\Delta(G)]$.

We now define a map $tr: R[G]\to R$, called the trace mapping, by $tr(\Sigma a_x x)=a_1$ (coefficient of identity of G). This mapping is clearly a R-module homomorphism and for all elements $\alpha=\Sigma a_x x$ and $\beta=\Sigma b_x x$ of $R[x]$ we have $tr\alpha\beta=\Sigma a_x b_{x^{-1}}=\Sigma b_x a_{x^{-1}}=tr\beta\alpha$ if R is commutative.

Let us define a mapping $*: R[G]\to R[G]$ by $(\Sigma a_x x)^*=\Sigma a_x x^{-1}$. Then this mapping is an antiautomorphism of $R[G]$ with order 2.

Suppose the integral group rings $\boldsymbol{Z}[G]$ and $\boldsymbol{Z}[H]$ are isomorphic.
Then

$$\boldsymbol{Q}[G]\cong\boldsymbol{Z}^{-1}\cdot\boldsymbol{Z}[G]\cong\boldsymbol{Z}^{-1}\cdot\boldsymbol{Z}[H]\cong\boldsymbol{Q}[H]$$

and we conclude that $K[G]\cong K[H]$ for all fields K of characteristic 0 [4].
Moreover

$$GF(p)[G]\cong\boldsymbol{Z}[G]/p\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]/p\boldsymbol{Z}[H]\cong GF(p)[H]$$

and $K[G]\cong K[H]$ for all fields K of characteristic p [4]. In otherwords, $\boldsymbol{Z}[G]$ determines $K[G]$ for all fields K, and therefore the integral group ring must hold more information.

The results listed below are a number of properties of G and $\boldsymbol{Z}[G]$ that are determined just by the ring structure of $\boldsymbol{Z}[G]$.

Let G be a finite group, and suppose that we are given the ring $\boldsymbol{Z}[G]$ along with its augmentation mapping $\varepsilon: \boldsymbol{Z}[G]\to\boldsymbol{Z}$. Then the following objects are determined.

(1) The elements of the center $Z(G)$ of G.

(2) The trace mapping $tr: \boldsymbol{Z}[G]\to\boldsymbol{Z}$.

(3) $\omega(\boldsymbol{Z}[G])$ and $\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G]$, hence the isomorphism class of G/G' where G' is the commutator subgroup of G.

(4) The isomorphism class of G/G'' where G'' is the second derived subgroup of G.

(5) The mapping $*: C(\boldsymbol{Z}[G])\to C(\boldsymbol{Z}[G])$.

(6) The class sums in $\boldsymbol{Z}[G]$.

(7) The lattice of normal subgroups of G.

The part (1) is due to Berman [1] and part (6) to Glauberman. Glauberman's original proof appeared in [4]. The part (4) is due to Whitcomb [5] and Jackson [2], and the other parts are proved in [3; pp. 665–669].

The purpose of this paper is to generalize the above properties in the case when G is periodic, and to prove some related theorems.

II. Preliminary lemmas

The following lemmas will be used in the proofs of the main theorems.

LEMMA 1. *Let G be a finitely generated abelian group and let g be an element of G with $g\neq 1$. Then there exists a subgroup N with $g\notin N$ and G/N is cyclic.*

Proof. By the assumption there exists a finite subset $\{a_1, \cdots, a_n\}$ of G such that $a_i\neq 1$ for all i and $G=\langle a_1\rangle\langle a_2\rangle\cdots\langle a_n\rangle$ is a direct product. Then g is represented by $g=g_1g_2\cdots g_n$, $g_i\in\langle a_i\rangle$, uniquely. Since $g\neq 1$, $g_k\neq 1$ for some k. Now let $N=\langle a_1\rangle\cdots\langle a_{k-1}\rangle\langle a_{k+1}\rangle\cdots\langle a_n\rangle$, then $g_k\notin N$. Hence $g\notin N$ and $G/N\cong\langle a_k\rangle$ is cyclic.

LEMMA 2. *Let G be a finitely generated group and $\{N_\lambda\}$ be*

$$\{N_\lambda\}=\{N\lhd G\,|\,G/N \text{ is cyclic}\}.$$

Then $G'=[G,G]=\bigcap N_\lambda$.

Proof. Clearly we have $G'\subseteq\bigcap N_\lambda$. Conversely suppose $g\in G-G'$. Then $\bar{g}\neq\bar{1}$ in $G/G'=\bar{G}$. Since $\bar{G}$ is finitely generated abelian group, there exists a subgroup $\bar{N}$ with $\bar{g}\notin\bar{N}$ and $\bar{G}/\bar{N}$ cyclic by the above lemma. Let N be the inverse image of $\bar{N}$ in G. Then $N\lhd G$, $G/N\cong\bar{G}/\bar{N}$ and hence G/N is cyclic. Also since $\bar{g}\notin\bar{N}$, we have $g\notin N$. Thus if $g\notin G'$ there exists a normal subgroup N of G with $g\notin N$ and G/N cyclic, i.e. $g\notin\bigcap N_\lambda$. Therefore $G'=\bigcap N_\lambda$.

LEMMA 3. *Let G be a group, K be a field. Suppose α is an algebraic element of $K[G]$ over K with minimal polynomial $f(\zeta)\in K[\zeta]$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the distinct roots of $f(\zeta)$ in some algebraic closure of K.*

(1) *If char $K=0$, then there exist rational numbers $r_1, r_2, \cdots, r_n$ satisfying $0<r_i$ and $r_1+r_2+\cdots+r_n=1$ with*

$$tr\ \alpha=r_1\lambda_1+r_2\lambda_2+\cdots+r_n\lambda_n.$$

(2) *If char $K=p\neq 0$ and if either G is p'-group or $f(\zeta)$ is separable, then there exist $r_1, r_2, \cdots, r_n\in GF(p)$ with $r_1+r_2+\cdots+r_n=1$ and*

$$tr\ \alpha=r_1\lambda_1+r_2\lambda_2+\cdots+r_n\lambda_n.$$

Proof. The proof can be found in [3; pp. 50-51].

LEMMA 4. *Let α be an element of an integral group ring $\boldsymbol{Z}[G]$ with $\alpha^n=1$ for some positive integer n and $tr\alpha\neq 0$. Then $\alpha=\pm 1$.*

Proof. Since $\boldsymbol{Q}[G]\cong \boldsymbol{Z}^{-1}\cdot\boldsymbol{Z}[G]\supseteq\boldsymbol{Z}[G]\ni\alpha$, we can see that α is an element of $\boldsymbol{Q}[G]$. Since $\alpha^n=1$, α is algebraic over the rational number field $\boldsymbol{Q}$ and its minimal polynomial $f(\zeta)\in\boldsymbol{Q}[\zeta]$ is a divisor of ζ^n-1 and separable over $\boldsymbol{Q}$. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the roots of $f(\zeta)$in the complex number field, where $m=\deg f(\zeta)$. Then by the above lemma, there exist rational numbers $r_1, r_2, \cdots r_m$ satisfying $0<r_i$ and $r_1+r_2+\cdots+r_m=1$ with

$$tr\alpha=r_1\lambda_1+r_2\lambda_2+\cdots+r_m\lambda_m.$$

Now since $|\lambda_i|=1$ and

$$1\leq|tr\alpha|=|r_1\lambda_1+\cdots+r_m\lambda_m|\leq|r_1\lambda_1|+\cdots+|r_m\lambda_m|$$
$$=r_1+r_2+\cdots+r_m=1,$$

we have

$$1=|tr\alpha|=|r_1\lambda_1+\cdots+r_m\lambda_m|=|r_1\lambda_1|+\cdots+|r_m\lambda_m|,\ \text{i.e. } tr\alpha=\pm 1$$

and there exist positive real numbers $k_1, \cdots, k_{m-1}$ such that

$$r_2\lambda_2=r_1\lambda_1k_1,\ \ r_3\lambda_3=r_1\lambda_1k_2, \cdots, r_m\lambda_m=r_1\lambda_1k_{m-1}.$$

Then $r_i=|r_i\lambda_i|=|r_1\lambda_1k_{i-1}|=r_1k_{i-1}$, that is $k_1, k_2, \cdots, k_{m-1}$ are rational numbers and the equality $\pm 1=tr\alpha=r_1\lambda_1(1+k_1+\cdots+k_{m-1})$ shows that $\lambda_1, \cdots, \lambda_m$ are rational roots of $f(\zeta)$. But the rational nth roots of 1 in the complex number field are 1 and -1 only. So $\deg f(\zeta)=m=2$ or 1. Suppose $\deg f(\zeta)=2$, then $f(\zeta)=\zeta^2-1$ and the roots of $f(\zeta)$ are 1 and -1. Say $\lambda_1=1$ and $\lambda_2=-1$. Then we have

$$r_1\lambda_1+r_2\lambda_2=r_1-r_2=tr\alpha=\pm 1.$$

This contradicts the fact $1>|r_1-r_2|\geq 0$. Therefore the minimal polynomial of α is $\zeta-1$ or $\zeta+1$, hence $\alpha=\pm 1$.

LEMMA 5. *Let $\varepsilon:\boldsymbol{Z}[G]\to\boldsymbol{Z}$ and $\delta:\boldsymbol{Z}[H]\to\boldsymbol{Z}$ be the augmentation mappings of integral group rings. If the ring $\boldsymbol{Z}[G]$ is isomorphic onto $\boldsymbol{Z}[H]$, then there exists a ring isomorphism $\eta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ satisfying $\delta\eta=\varepsilon$. For this isomorphism η we have*

(1) $\eta(\omega(\boldsymbol{Z}[G]))=\omega(\boldsymbol{Z}[H])$.

(2) *The mapping $\bar{\eta}:\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[G])\to\boldsymbol{Z}[H]/\omega(\boldsymbol{Z}[H])$ defined by $\bar{\eta}(\alpha+\omega(\boldsymbol{Z}[G]))=\eta(\alpha)+\omega(\boldsymbol{Z}[H])$ is an isomorphism.*

Proof. Let $\theta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ be an isomorphism. Then $\delta\theta:\boldsymbol{Z}[G]\to\boldsymbol{Z}$ is a $\boldsymbol{Z}$-algebra homomorphism and for all $g\in G$ $\delta\theta(g)$ is a unit in $\boldsymbol{Z}$ and equal to ± 1. Let us denote

$G_0=\{\delta\theta(g)^{-1}g \mid g\in G\}$, then G_0 becomes a multiplicative group in $\boldsymbol{Z}[G]$. If we define $\psi: G\to G_0$ by $\psi(g)=\delta\theta(g)^{-1}g$, this ψ is a group isomorphism that extend to an automorphism $\Phi: \boldsymbol{Z}[G]\to\boldsymbol{Z}[G]=\boldsymbol{Z}[G_0]$ and $\delta\theta(\delta\theta(g)^{-1}g)=1$. In other words, if we replace G by G_0, then $\delta\theta: \boldsymbol{Z}[G_0]\to\boldsymbol{Z}$ becomes the ordinary augmentation mapping of $\boldsymbol{Z}[G_0]$. Now we denote $\eta=\theta\Phi$. Then $\eta: \boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ is an isomorphism and for any $\alpha=\Sigma a_g g\in\boldsymbol{Z}[G]$ we have

$$\begin{aligned}\delta\eta(\alpha)&=\delta\theta\Phi(\alpha)=\delta\theta(\Sigma a_g\psi(g))=\delta\theta(\Sigma a_g\delta\theta(g)^{-1}g)\\&=\Sigma a_g\delta\theta(\delta\theta(g)^{-1}g)=\Sigma a_g=\varepsilon(\alpha).\end{aligned}$$

Therefore $\delta\eta=\varepsilon$.

(1) We have

$$\omega(\boldsymbol{Z}[G])=\ker\varepsilon=\ker\delta\eta=\eta^{-1}(\ker\delta)=\eta^{-1}(\omega(\boldsymbol{Z}[H])),$$

hence $\eta(\omega(\boldsymbol{Z}[G]))=\omega(\boldsymbol{Z}[H])$.

(2) This follows immediately from the above results.

Now let A be an algebra over a field and define $[A, A]$, the commutator subspace of A, to be the subspace spanned by all Lie products $[\alpha, \beta]=\alpha\beta-\beta\alpha$ with $\alpha, \beta\in A$. Then we have

LEMMA 6. *Let A be an algebra over a field of characteristic $p\neq 0$. If $\alpha_1, \alpha_2, \cdots, \alpha_m\in A$ and if $n>0$ is a given integer, set $q=p^n$. Then there exists an element $\beta\in[A, A]$ with*

$$(\alpha_1+\alpha_2+\cdots+\alpha_m)^q=\alpha_1^q+\alpha_2^q+\cdots+\alpha_m^q+\beta.$$

Proof. The proof can be found in [3; pp. 45–46].

LEMMA 7. *Let $\alpha=\Sigma a_x x$ be an element of a group ring $K[G]$ over a field K. Then $\alpha\in[K[G], K[G]]$ if and only if the sum of the coefficients a_x over each conjugacy class of G is zero. Moreover if $\tau: K[G]\to K$ is a linear functional, then τ annihilates $[K[G], K[G]]$ if and only if it is constant on the conjugacy classes of G.*

Proof. The proof can be found in [3; p. 46].

III. Group rings isomorphic to $\boldsymbol{Z}[G]$ with periodic group basis G

In this section we will derive some properties determined just by the ring structure of $\boldsymbol{Z}[G]$ with the periodic group basis G.

THEOREM 1. *Let G be a group,* $\varepsilon: \boldsymbol{Z}[G]\to\boldsymbol{Z}$ *be its augmentation mapping and* $Z(G)_T$ *be the torsion subgroup of the center* $Z(G)$ *of G. Then*

(1) $Z(G)_T=\{u\in C(\boldsymbol{Z}[G])\,|\,u^n=1$ *for some positive integer n and* $\varepsilon(u)=1\}$ *where* $C(\boldsymbol{Z}[G])$ *is the center of the ring* $\boldsymbol{Z}[G]$.

(2) *If H is a group with* $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$, *then there exists an isomorphism* $\eta: \boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ *such that the restriction on* $Z(G)_T$ *induce the group isomorphism* $\eta: Z(G)_T\to Z(H)_T$. *In particular if G and H are periodic then* $Z(G)\cong Z(H)$.

Proof. (1) Clearly each element $z\in Z(G)_T$ is a central unit of $\boldsymbol{Z}[G]$ of finite multiplicative order and with $\varepsilon(z)=1$. Conversely, let u be any central unit of $\boldsymbol{Z}[G]$ of finite multiplicative order and with $\varepsilon(u)=1$. Then the image u^* of u under the antiautomorphism $*: \boldsymbol{Z}[G]\to\boldsymbol{Z}[G]$ defined by $(\Sigma a_x x)^*=\Sigma a_x x^{-1}$ also has this property, and hence, because u and u^* commute, so does $\alpha=uu^*$. Now if β is any element of $\boldsymbol{Z}[G]$, then $tr\beta\beta^*$ is the sum of the squares of the coefficients of β, and hence $tr\beta\beta^*\geq(tr\beta)^2$. In particular, because α is invariant under $*$, it follows by induction that $tr\alpha^{2^n}\geq(tr\alpha)^{2^n}$ for any nonnegative integer n. But $tr\alpha$ is a nonnegative integer and α has finite multiplicative order so that we must have $tr\alpha=1$ and $u=\pm z$ for some $z\in Z(G)$. Because $\varepsilon(u)=1$, this fact follows.

(2) Let $\varepsilon: \boldsymbol{Z}[G]\to\boldsymbol{Z}$ and $\delta: \boldsymbol{Z}[H]\to\boldsymbol{Z}$ be the augmentation mappings of $\boldsymbol{Z}[G]$ and $\boldsymbol{Z}[H]$ respectively. Then by Lemma 5, there exists a ring isomorphism $\eta: \boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ with $\delta\eta=\varepsilon$. Now by the above (1) we have

$$Z(G)_T=\{u\in C(\boldsymbol{Z}[G])\,|\,u^n=1 \text{ for some } n\in\boldsymbol{Z},\ n\geq1 \text{ and } \varepsilon(u)=1\},$$

$$Z(H)_T=\{v\in C(\boldsymbol{Z}[H])\,|\,v^n=1 \text{ for some } n\in\boldsymbol{Z},\ n\geq1 \text{ and } \delta(v)=1\}.$$

For any $u\in Z(G)_T$, $u^n=1$ for some $n\in\boldsymbol{Z}$, $n\geq1$ and $\varepsilon(u)=1$ so $\eta(u)^n=\eta(u^n)=\eta(1)=1$ and $\delta(\eta(u))=\varepsilon(u)=1$, that is $\eta(u)\in Z(H)_T$. Therefore the mapping $\eta: Z(G)_T\to Z(H)_T$ is the well defined group isomorphism. In particular if G and H are periodic $Z(G)_T=Z(G)$ and $Z(H)_T=Z(H)$. Hence $Z(G)\cong Z(H)$.

THEOREM 2. *Let G and H be groups and G′ and H′ be their commutator subgroups respectively. Suppose* $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$. *Then for any ring isomorphism* $\theta: \boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$, *we have*

(1) $\theta(\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G])=\omega(\boldsymbol{Z}[H'])\boldsymbol{Z}[H]$

(2) $\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]/\omega(\boldsymbol{Z}[H'])\boldsymbol{Z}[H]$

(3) $(G/G')_T\cong(H/H')_T$. *In particular, if G and H are periodic, then* $G/G'\cong H/H'$.

Proof. (1) Let I denote the ideal of $\boldsymbol{Z}[G]$ generated by the set A of all Lie products of $\boldsymbol{Z}[G]$, i.e. $A=\{\alpha\beta-\beta\alpha \mid \alpha, \beta \in \boldsymbol{Z}[G]\}$. Because the natural mapping $\boldsymbol{Z}[G] \rightarrow \boldsymbol{Z}[G/G']$ maps $\boldsymbol{Z}[G]$ onto a commutative ring, we see that I is contained in $\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G]$, the kernel of this mapping. Conversely, $\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G]$ is generated by all terms of the form

$$ghg^{-1}h^{-1}-1=(gh-hg)g^{-1}h^{-1} \in I,$$

thus we have equality $I=\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G]$. Similarly, if we denote J to be the ideal of $\boldsymbol{Z}[H]$ generated by the set B of all Lie products of $\boldsymbol{Z}[H]$, then $J=\omega(\boldsymbol{Z}[H'])\boldsymbol{Z}[H]$. Since $\theta: \boldsymbol{Z}[G] \rightarrow \boldsymbol{Z}[H]$ is an isomorphism, $\theta(A)=B$ and hence $\theta(I)=J$, so

$$\theta(\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G])=\omega(\boldsymbol{Z}[H'])\boldsymbol{Z}[H].$$

(2) If we define the mapping $\bar{\theta}: \boldsymbol{Z}[G]/I \rightarrow \boldsymbol{Z}[H]/J$ by $\bar{\theta}(\alpha+I)=\theta(\alpha)+J$, $\bar{\theta}$ is an isomorphism.

(3) Since $\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[G'])\boldsymbol{Z}[G] \cong \boldsymbol{Z}[G/G']$ and $\boldsymbol{Z}[H]/\omega(\boldsymbol{Z}[H'])\boldsymbol{Z}[H] \cong \boldsymbol{Z}[H/H']$, we have $\boldsymbol{Z}[G/G'] \cong \boldsymbol{Z}[H/H']$. So by Theorem 1, $(G/G')_T \cong (H/H')_T$. In particular if G and H are periodic, $G/G' \cong H/H'$.

COROLLARY 1. *Let G and H be two periodic groups and let the following series are the upper central series of G and H respectively.*

$$\{1\}=Z_0(G) \subseteq Z(G)=Z_1(G) \subseteq \cdots \subseteq Z_n(G) \subseteq \cdots,$$
$$\{1\}=Z_0(H) \subseteq Z(H)=Z_1(H) \subseteq \cdots \subseteq Z_n(H) \subseteq \cdots.$$

If $\boldsymbol{Z}[G] \cong \boldsymbol{Z}[H]$, then we have

$$\boldsymbol{Z}[G/Z_n(G)] \cong \boldsymbol{Z}[H/Z_n(H)]$$

and

$$Z_{n-1}(G)/Z_n(G) \cong Z_{n-1}(H)/Z_n(H)$$

for all n.

Proof. By Theorem 1, there exists an isomorphism $\eta: \boldsymbol{Z}[G] \rightarrow \boldsymbol{Z}[H]$ such that the restriction $\eta: Z(G) \rightarrow Z(H)$ is a group isomorphism. So that

$$Z_1(G)/Z_0(G) \cong Z(G) \cong Z(H) \cong Z_1(H)/Z_0(H).$$

Now let $\rho_1: \boldsymbol{Z}[G] \rightarrow \boldsymbol{Z}[G/Z(G)]$ and $\rho_2: \boldsymbol{Z}[H] \rightarrow \boldsymbol{Z}[H/Z(H)]$ be the natural homomorphisms. Then $\ker\rho_1=\omega(\boldsymbol{Z}[Z(G)])\boldsymbol{Z}[G]$ and $\ker\rho_2=\omega(\boldsymbol{Z}[Z(H)])\boldsymbol{Z}[H]$, hence $\eta(\ker \rho_1) \cong \ker \rho_2$.

Furthermore we have

$$\boldsymbol{Z}[G]/\ker\rho_1 \cong \boldsymbol{Z}[G/Z(G)], \quad \boldsymbol{Z}[H]/\ker\rho_2 \cong Z[H/Z(H)].$$

Now if we define $\bar{\eta}: \boldsymbol{Z}[G]/\ker\rho_1 \rightarrow \boldsymbol{Z}[H]/\ker\rho_2$ by $\bar{\eta}(\alpha+\ker\rho_1)=\eta(\alpha)+\ker\rho_2$, $\bar{\eta}$ is an isomorphism. Therefore $\boldsymbol{Z}[G/Z(G)] \cong \boldsymbol{Z}[H/Z(H)]$. Because the groups $G/Z(G)$ and

$H/Z(H)$ are also periodic we have $Z(G/Z(G))\cong Z(H/Z(H))$ by Theorem 1. Hence

$$Z_2(G)/Z_1(G)\cong Z_2(H)/Z_1(H).$$

Inductively assume that $\boldsymbol{Z}[G/Z_{n-1}(G)]\cong\boldsymbol{Z}[H/Z_{n-1}(H)]$. Then by Theorem 1, $Z(G/Z_{n-1}(G))\cong Z(H/Z_{n-1}(H))$ and hence

$$Z_n(G)/Z_{n-1}(G)\cong Z_n(H)/Z_{n-1}(H).$$

Now, by the same way as the above proof, we have

$$\boldsymbol{Z}[(G/Z_{n-1}(G))/Z(G/Z_{n-1}(G))]\cong\boldsymbol{Z}[(H/Z_{n-1}(H))/Z(H/Z_{n-1}(H))],$$

thus $\boldsymbol{Z}[G/Z_n(G)]\cong\boldsymbol{Z}[H/Z_n(H)]$, and hence $Z(G/Z_n(G))\cong Z(H/Z_n(H))$, that is $Z_{n+1}(G)/Z_n(G)\cong Z_{n+1}(H)/Z_n(H)$.

THEOREM 3. *Let G be a periodic group with the trace mapping $tr_G:\boldsymbol{Z}[G]\to\boldsymbol{Z}$, and let U be the $\boldsymbol{Z}$-linear span of the set*

$$A=\{\alpha\in\boldsymbol{Z}[G]\,|\,\alpha^n=1 \text{ for some } n\in\boldsymbol{Z},\ n\geq 1 \text{ and } \alpha\neq\pm 1\}.$$

Then,

(1) *$\boldsymbol{Z}[G]=\boldsymbol{Z}\dot{+}U$ (direct sum of $\boldsymbol{Z}$-modules).*

(2) *If $\pi:\boldsymbol{Z}[G]\to\boldsymbol{Z}$ be the projection of the above direct sum onto the factor $\boldsymbol{Z}$, then ker $tr_G=U$ and $tr_G=\pi$.*

(3) *Let H be any group with $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$ and let $tr_H:\boldsymbol{Z}[H]\to\boldsymbol{Z}$ be the trace mapping of $\boldsymbol{Z}[H]$, then for any ring isomorphism $\theta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$ we have $tr_H\theta=tr_G$.*

Proof. (1) Clearly $\boldsymbol{Z}\cap U=0$. If g is an element of G distinct from identity then $g^n=1$ for some positive integer n and $g\neq\pm 1$ in $\boldsymbol{Z}[G]$. Hence $g\in A$. That is $G-\{1\}\subseteq A$. Let α be an element of $\boldsymbol{Z}[G]$, then $\alpha=\Sigma a_g g=a_1+\sum_{g\neq 1}a_g g$ where $a_1\in\boldsymbol{Z}$ and $\sum_{g\neq 1}a_g g\in U$. Hence $\boldsymbol{Z}[G]=\boldsymbol{Z}\dot{+}U$.

(2) It is clear that ker $tr_G\subseteq U$. Conversely suppose that there exists an element $\alpha\in U-$ker tr_G. Then $\alpha=\Sigma b_i\alpha_i$ for some $b_i\in\boldsymbol{Z}$, $\alpha_i\in A$ and $tr_G\alpha\neq 0$. Now since $0\neq tr_G\alpha=\Sigma b_i tr_G\alpha_i$, there exists an element α_k of A with $tr_G\alpha_k\neq 0$, $\alpha_k{}^n=1$ for some positive integer n and $\alpha_k\neq\pm 1$. But by Lemma 4, $\alpha_k=\pm 1$, thus we have a contradiction. Therefore ker $tr_G=U$, and hence $tr_G=\pi$.

(3) Let V be the $\boldsymbol{Z}$-linear span of the set

$$B=\{\beta\in\boldsymbol{Z}[H]\,|\,\beta^n=1 \text{ for some } n\in\boldsymbol{Z},\ n\geq 1 \text{ and } \beta\neq\pm 1\}.$$

Then clearly $\theta(A)=B$ and $\theta(U)=V$. Clearly $\boldsymbol{Z}\cap V=0$ and

$$\boldsymbol{Z}[H]=\theta(\boldsymbol{Z}[G])=\theta(\boldsymbol{Z}\dot{+}U)=\boldsymbol{Z}\dot{+}\theta(U)=\boldsymbol{Z}\dot{+}V.$$

Hence $\boldsymbol{Z}[H]=\boldsymbol{Z}\dot{+}V$. Now utilizing Lemma 4, we can prove that ker $tr_H\supseteq V$ by the same way in the proof of the above (2). For any $\alpha\in \boldsymbol{Z}[G]$, α is uniquely represented as $\alpha=a+u$, $a\in\boldsymbol{Z}$, $u\in U$ and $tr_G\alpha=a$. Hence

$$tr_H\theta(\alpha)=tr_H\theta(a+u)=tr_H(a+\theta(u))=a+tr_H(\theta(u))=a=tr_G\alpha$$

since $\theta(u)\in\theta(U)=V$ and $V\subseteq\ker\ tr_H$. Therefore $tr_H\theta=tr_G$.

THEOREM 4. *Let G be a periodic group. Let $p_1, p_2, \cdots$ be the infinitly many distnct primes with $p_n\equiv-1$ (mod $n!$) ($n=1, 2, \cdots$). Then,*

(1) *For any element α in the center $C(\boldsymbol{Z}[G])$ there exists a positive integer m and the unique element β in $C(\boldsymbol{Z}[G])$ satisfying*

$$\beta\equiv\alpha^{p_n} \pmod{p_nC(\boldsymbol{Z}[G])}$$

for all $n\geq m$, and we have $\alpha^*=\beta$ where $*: C(\boldsymbol{Z}[G])\to C(\boldsymbol{Z}[G])$ is the mapping given by $(\Sigma a_xx)^*=\Sigma a_xx^{-1}$.

(2) *Let H be a periodic group with $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$. Then for any isomorphism $\theta: \boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$, we have*

$$\theta(\alpha^*)=\theta(\alpha)^*$$

for any element α in $C(\boldsymbol{Z}[G])$.

Proof. (1) Let α be an element of $C(\boldsymbol{Z}[G])$ and let $N=\langle\text{supp }\alpha\rangle$. Since for any $x\in$ suppα the conjugacy class C_x containing x is contained in supp α and for any $g\in G$ $g^{-1}C_xg=C_x$, N is a finitely generated normal subgroup of G contained in $\Delta^+(G)=\Delta(G)$. Furthermore since $\Delta^+(G)$ is a locally finite subgroup of G, N is finite and $C(\boldsymbol{Z}[N])\subseteq C(\boldsymbol{Z}[G])$. Let m be a positive integer with $|N|\,|m!$. We show first that $\alpha^{p_m}=\alpha^*+p_m\delta$ for some $\delta\in C(\boldsymbol{Z}[G])$. Let

$$^-: \boldsymbol{Z}[N]\to\boldsymbol{Z}[N]/p_m\boldsymbol{Z}[N]\cong GF(p_m)[N]$$

denote the natural homomorphism and write $\alpha=\Sigma a_xx$.

Then by Lemma 6 and 7,

$$\bar{\alpha}^{p_m}=\sum_{x\in N}\bar{a}_x{}^{p_m}\bar{x}^{p_m}+\bar{\gamma}$$

where the sum of the coefficients in $\bar{\gamma}$ over each conjugacy class of N is zero. Note that $\bar{a}_x{}^{p_m}=\bar{a}_x$ and that $x^{p_m}=x^{-1}$ because $p_m\equiv-1$ mod $m!$ and $|N|\,|m!$. Thus we have $\bar{\alpha}^{p_m}=\bar{\alpha}^*+\bar{\gamma}$, and if $\gamma\in\boldsymbol{Z}[N]$ is defined by $\alpha^{p_m}=\alpha^*+\gamma$, then we conclude that the sum of the coefficients of γ over each conjugacy class of N is divisible by p_m. Now α^{p_m} and α^* belong to $C(\boldsymbol{Z}[N])\subseteq C(\boldsymbol{Z}[G])$ so that γ is also central, and we can write $\gamma=\Sigma n_i\hat{C}_i$ as an integral linear combination of finite class sums. But then the sum of the coefficients of the $i-$th conjugacy class is $n_i\cdot|C_i|$. Because p_m is prime to $|N|$, p_m does not divide $|C_i|$, thus we must have $p_m|n_i$. Hence $\gamma=p_m\delta$ for some $\delta\in C(\boldsymbol{Z}[N])\subseteq C(\boldsymbol{Z}[G])$, and we have $\alpha^{p_m}=\alpha^*+p_m\delta$.

Now since $|N|\,|n!$ for all integer n such that $n\geq m$, we have

$$\alpha^{p_n}\equiv\alpha^* \bmod p_nC(Z[G])$$

for all $n\geq m$. Now suppose $\alpha^{p_n}\equiv\beta \bmod p_nC(\boldsymbol{Z}[G])$ for some $\beta\in C(\boldsymbol{Z}[G])$ and for all $n\geq m$. Then

$$\alpha^*-\beta\in\bigcap_{n\geq m}p_nC(\boldsymbol{Z}[G])\subseteq\bigcap_{n\geq m}p_n\boldsymbol{Z}[G]=0,$$

thus we have $\alpha^*=\beta$.

(2) Let $\theta:\boldsymbol{Z}[G]\rightarrow\boldsymbol{Z}[H]$ be an isomorphism, then the restriction $\theta:C(\boldsymbol{Z}[G])\rightarrow C(\boldsymbol{Z}[H])$ is also an isomorphism. For any $\alpha\in C(\boldsymbol{Z}[G])$, there exists a positive integer m and there exist the unique $\beta\in C(\boldsymbol{Z}[G])$ and $\delta\in C(\boldsymbol{Z}[H])$ with

$$\beta=\alpha^{p_n}+\gamma_n,\ \gamma_n\in C(\boldsymbol{Z}[G]),$$

$$\delta=\theta(\alpha)^{p_n}+\theta(\gamma_n),\ \ \theta(\gamma_n)\in C(\boldsymbol{Z}[H])$$

for all $n\geq m$. Then $\alpha^*=\beta$ and $\delta=\theta(\alpha)^*$. Therefore since $\delta=\theta(\beta)$, we have $\theta(\alpha)^*=\theta(\alpha^*)$.

THEOREM 5. *Let G be a group, $\mathcal{C}$ be the set of all finite class sums in $\boldsymbol{Z}[G]$, and let $\mathcal{L}$ be the lattice of all finite normal subgroups of G. Define*

$$L=\{\alpha\in C(\boldsymbol{Z}[G])\,|\,\alpha\neq 0 \text{ and } tr(\alpha\alpha^*)=\varepsilon(\alpha)\},$$

$$M=\{\alpha\in L\,|\,\alpha\neq\beta+\gamma \text{ for any } \beta,\gamma\in L\}$$

where $\varepsilon:\boldsymbol{Z}[G]\rightarrow\boldsymbol{Z}$ is the augmentation mapping. Then

(1) $\mathcal{C}=M$

(2) $\mathcal{L}=\{\mathrm{supp}\alpha\,|\,\alpha\in L,\ \alpha^2=n\alpha$ *for some integer* $n\}$

Proof. (1) Let α be an element of L. If $\alpha=\Sigma n_i\hat{C}_i$ is written as an integral linear combination of finite class sums, then $\varepsilon(\alpha)=\Sigma n_i|C_i|$ and $tr(\alpha\alpha^*)=\Sigma n_i^2|C_i|$. Because n_i is an integer, we have $n_i^2\geq n_i$ with equality if and only if $n_i=0$ or 1. Hence.

$$L=\{\alpha\in C(\boldsymbol{Z}[G])\,|\,\alpha\neq 0 \text{ and } \alpha=\Sigma n_i\hat{C}_i \text{ with } n_i=0 \text{ or } 1\}$$

and elements of $\mathcal{C}$ are the atom of L. In other words, $\alpha\in\mathcal{C}$ if and only if $\alpha\in L$ and α can not be written as $\alpha=\beta+\gamma$ with $\beta,\ \gamma\in L$. That is, $\alpha\in\mathcal{C}$ if and only if $\alpha\in M$. Hence $\mathcal{C}=M$.

(2) If $N\in\mathcal{L}$, N is a normal subgroup of G and N is a finite union of some finite comjugacy classes of G. Hence $\hat{N}\in L$ and $(\hat{N})^2=|N|\hat{N}$. Conversely if $\alpha\in L$ with $\alpha^2=n\alpha$ for some integer n, then $(\mathrm{supp}\ \alpha)(\mathrm{supp}\ \alpha)\subseteq\mathrm{supp}\ \alpha$, and hence $\mathrm{supp}\alpha=N$ is a finite normal subgroup of G.

COROLLARY 2. *Let G and H be two periodic groups, $\mathcal{C}_1$ and $\mathcal{C}_2$ be the finite class sums in $\boldsymbol{Z}[G]$ and $\boldsymbol{Z}[H]$ respectively, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the lattices of finite normal subgroups of G and H respectively. Suppose $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$. Then there exists an isomorphism $\eta:\boldsymbol{Z}[G]\rightarrow\boldsymbol{Z}[H]$ satisfying the following facts:*

(1) $\eta(\mathcal{C}_1)=\mathcal{C}_2$

(2) *There exists a lattice isomorphism $f:\mathcal{L}_1\rightarrow\mathcal{L}_2$ given by $f(N)=\eta(N)$ for all $N\in\mathcal{L}_1$.*

(3) *For any $N\in\mathcal{L}_1$, we have $\boldsymbol{Z}[G/N]\cong\boldsymbol{Z}[H/\eta(N)]$.*

Proof. (1) We define

$$L_1=\{\alpha\in C(\boldsymbol{Z}[G])\,|\,\alpha\neq 0,\ tr(\alpha\alpha^*)=\varepsilon(\alpha)\},$$
$$M_1=\{\alpha\in L_1\,|\,\alpha\neq\alpha_1+\alpha_2 \text{ for any } \alpha_1,\alpha_2\in L_1\},$$
$$L_2=\{\beta\in C(\boldsymbol{Z}[H])\,|\,\beta\neq 0,\ tr(\beta\beta^*)=\delta(\beta)\},$$
$$M_2=\{\beta\in L_2\,|\,\beta\neq\beta_1+\beta_2 \text{ for any } \beta_1,\beta_2\in L_2\},$$

where ε and δ are augmentation mappings of $\boldsymbol{Z}[G]$ and $\boldsymbol{Z}[H]$ respectively. Then by Lemma 5, Theorem 3 and Theorem 4, there exists an isomorphism $\eta:\boldsymbol{Z}[G]\rightarrow\boldsymbol{Z}[H]$ with $\delta\eta=\varepsilon$ and $tr(\eta(\alpha))=tr\alpha$ for $\alpha\in\boldsymbol{Z}[G]$, and $\eta(\alpha)^*=\eta(\alpha^*)$ for $\alpha\in C(\boldsymbol{Z}[G])$. Also by Theorem 5, $\mathcal{C}_1=M_1$ and $\mathcal{C}_2=M_2$. Now for the isomorphism η, $\alpha\in C(\boldsymbol{Z}[G])$ if and only if $\eta(\alpha)\in C(\boldsymbol{Z}[H])$, $tr(\alpha\alpha^*)=\varepsilon(\alpha)$ if and only if $tr(\eta(\alpha)\eta(\alpha)^*)=\delta(\eta\alpha)$, and $\alpha\neq\alpha_1+\alpha_2$ if and only if $\eta(\alpha)\neq\eta(\alpha_1)+\eta(\alpha_2)$ for $\alpha,\alpha_1,\alpha_2\in L_1$. Therefore we have $\eta(L_1)=L_2$ and $\eta(M_1)=M_2$, and hence $\eta(\mathcal{C}_1)=\mathcal{C}_2$.

(2) By Theorem 5, we have

$$\mathcal{L}_1=\{\text{supp }\alpha\,|\,\alpha\in L_1 \text{ and } \alpha^2=n\alpha \text{ for some intege } n\},$$
$$\mathcal{L}_2=\{\text{supp }\beta\,|\,\beta\in L_2 \text{ and } \beta^2=n\beta \text{ for some integer } n\}.$$

Singe $\eta(\mathcal{C}_1)=\mathcal{C}_2$ and $\alpha^2=n\alpha$ if and only if $\eta(\alpha)^2=n\eta(\alpha)$, $\eta(\text{supp }\alpha)=\text{supp }\eta(\alpha)$. Hence $\eta(\mathcal{L}_1)=\mathcal{L}_2$. Therefore the mapping $f:\mathcal{L}_1\rightarrow\mathcal{L}_2$ defined by $f(N)=\eta(N)$ is clearly a lattice isomorphism.

(3) Let N be a finite normal subgroup of G and let $\rho_1:\boldsymbol{Z}[G]\rightarrow\boldsymbol{Z}[G/N]$ and $\rho_2:\boldsymbol{Z}[H]\rightarrow\boldsymbol{Z}[H/\eta(N)]$ be the natural homomorphisms. Then

$$\ker\rho_1=\omega(\boldsymbol{Z}[N])\boldsymbol{Z}[G]=r(\hat{N}),$$
$$\ker\rho_2=\omega(\boldsymbol{Z}[\eta(N)])\boldsymbol{Z}[H]=r(\widehat{\eta(N)})=\eta(r(\hat{N})),$$

where $r(\hat{N})$ is the right annihilator of $\hat{N}$ in $\boldsymbol{Z}[G]$. Hence $\eta(\ker\rho_1)=\ker\rho_2$. Since $\boldsymbol{Z}[G]/r[\hat{N})\cong\boldsymbol{Z}[G/N]$, $\boldsymbol{Z}[H]/\eta(r(\hat{N}))\cong\boldsymbol{Z}[H/\eta(N)]$, and the mapping $\bar{\eta}:\boldsymbol{Z}[G]/r(\hat{N})\rightarrow\boldsymbol{Z}[H]/\eta(r(\hat{N}))$ given by $\bar{\eta}(\alpha+r(\hat{N}))=\eta(\alpha)+\eta(r(\hat{N}))$ is an isomorphism, we have $\boldsymbol{Z}[G/N]\cong\boldsymbol{Z}[H]/\eta(N)$.

THEOREM 6. *Let G be a group and let H be a periodic subgroup of G. Then the mapping*

$$\sigma : H \to \omega(\boldsymbol{Z}[H])\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$$

given by $\sigma(h)=h-1+\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$ *is a group homomorphism from H onto the additive abelian group with* $\ker\sigma=H'=[H,H]$, *and hence*

$$H/H' \cong \omega(\boldsymbol{Z}[H])\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G]).$$

Proof. In the following, let $\equiv$ denote congruence modulo $\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$. First, observe that for $x, y \in G$

$$xy-1=(x-1)+(y-1)+(x-1)(y-1).$$

In particular, if $x, y \in H$, then $(x-1)(y-1) \in \omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$ so that we have

$$\sigma(xy) \equiv xy-1 \equiv (x-1)+(y-1) \equiv \sigma(x)+\sigma(y)$$

and σ is indeed a group homomorphism. Next observe that $\omega(\boldsymbol{Z}[H])\boldsymbol{Z}[G]$ is spanned as a $\boldsymbol{Z}$-module by all terms of the form $(h-1)g$ with $h \in H$, $g \in G$. Thus because

$$(h-1)g=(h-1)+(h-1)(g-1) \equiv \sigma(h),$$

we see that σ is onto.

It remains to find the kernel of σ. Because the homomorphism is onto an abelian group, we know at least that $\ker\sigma \supseteq H'$. Let $h \in \ker\sigma$ so that $h-1 \in \omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$. Let $T=\{t_\lambda\}$ be a right transversal for H in G with $T \ni t_0=1$. If $g \in G-H$, then $g=at_\lambda$ for some $a \in H$ and $t_\lambda \in T$ with $\lambda \neq 0$. Because $g-1=(a-1)+a(t_\lambda-1)$ and because $\omega(\boldsymbol{Z}[H])$ is an ideal in $\boldsymbol{Z}[H]$, we have clearly

$$\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])=\omega(\boldsymbol{Z}[H])^2+\sum_{\lambda\neq 0}\omega(\boldsymbol{Z}[H])(t_\lambda-1).$$

Hence we can write

$$h-1=\alpha_0+\sum_{i=1}^{n}\alpha_\lambda(t_\lambda-1)$$

with $\alpha_0 \in \omega(\boldsymbol{Z}[H])^2$ and with $\alpha_\lambda \in \omega(\boldsymbol{Z}[H])$ for $\lambda \neq 0$. But this yields

$$\sum_{\lambda=1}^{n}\alpha_\lambda t_\lambda=(h-1)-\alpha_0+\sum_{\lambda=1}^{n}\alpha_\lambda \in \boldsymbol{Z}[H],$$

thus we can conclude that $\alpha_\lambda=0$ for all $\lambda \neq 0$ and that $h-1=\alpha_0 \in \omega(\boldsymbol{Z}[H])^2$. Hence we can write

$$h-1=\alpha_0=\left(\sum_{i=1}^{r}a_i(x_i-1)\right)\left(\sum_{j=1}^{s}b_j(y_j-1)\right)$$

where $a_i, b_j \in \boldsymbol{Z}$ and $x_i, y_j \in H$. Now let $H_0=\langle h, x_i, y_j \mid 1 \leq i \leq r,\ 1 \leq j \leq s\rangle$. Then H_0 is a finitely generated subgroup of H and $h-1 \in \omega(\boldsymbol{Z}[H_0])^2$. Let N be a normal subgroup of H with H/N cyclic. Since H is periodic, H/N is a finite cyclic group. Then $H_0/N \cap H_0$

is isomorphic to a subgroup of H/N, and hence also finite cyclic group, say $H_0/N\cap H_0=\langle y\rangle$. Let $x=(N\cap H_0)h=y^m$. Then by way of the natural homomorphism

$$Z[H_0]\to Z[H_0/N\cap H_0]=Z[\langle y\rangle] \text{ we see that}$$

$$x-1\in\omega(Z[\langle y\rangle])^2=(y-1)^2Z[\langle y\rangle]$$

so that $x-1=(y-1)^2\beta$ for some $\beta\in Z[\langle y\rangle]$. Because $x=y^m$, this yields

$$(y-1)\{(1+y+\cdots+y^{m-1})-(y-1)\beta\}=0.$$

But the annihilator of $y-1$ in $Z[\langle y\rangle]$ certainly consists of all Z-multiples of $\widehat{\langle y\rangle}$ so that we must have

$$1+y+\cdots+y^{m-1}-(y-1)\beta=k\widehat{\langle y\rangle}$$

for some integer k. Applying the augmentation mapping now yields $m=k\cdot o(y)$ so $o(y)$ divides m and $x=y^m=1$. In other words $h\in N\cap H_0\subseteq N$ and because this is true for all such N with H/N cyclic, we conclude that $h\in H'$ by Lemma 2. Thus $\ker\sigma=H'$, and $H/H'\cong\omega(Z[H])Z[G]/\omega(Z[H])\omega(Z[G])$.

THEOREM 7. *Let G be a periodic group and let $\varepsilon: Z[G]\to Z$ be the augmentation mapping. Set $E=\omega(Z[G])$, $I=\omega(Z[G'])Z[G]$ and set*

$$S=\{\alpha\in Z[G]\,|\,\alpha \text{ is a unit in } Z[G] \text{ with } \varepsilon(\alpha)=1 \text{ and } \alpha+I \text{ has finite multiplicative order in } Z[G]/I\},$$

$$T=\{\alpha\in S\,|\,\alpha\equiv 1 \bmod IE\}.$$

Then,

(1) *The sets S and T are multiplicative groups with $T\triangleleft S$ and $S/T\cong G/G''$, where G'' is the second derived group of G.*

(2) *For any periodic group H, if $Z[G]\cong Z[H]$ then $G/G''\cong H/H''$.*

Proof. (1) Since $Z[G]/I$ is isomorphic onto the commutative ring $Z[G/G']$, the set S and T are clearly multiplicative groups and $T\triangleleft S$.

Because we certainly have $G\subseteq S$, it follows the mapping $\tau: G\to S/T$ given by $\tau(g)=gT$ yields a group homomorphism into S/T. The goal is to show that τ is onto and that $\ker\tau=G''$. First let $g\in\ker\tau$ so that

$$g-1\in IE=\omega(Z[G'])\omega(Z[G])\subseteq I.$$

Because I is the kernel of the natural homomorphism $Z[G]\to Z[G/G']$, we see immediately that $g\in G'$. But then, by Theorem 6 with $H=G'$ we see that

$$g-1\in\omega(Z[G'])\omega(Z[G])$$

if and only if $g\in H'=G''$. Thus $\ker\tau=G''$.

Finally, we show that τ is onto. Let W be a transversal for G' in G. Because I is the kernel of the natural homomorphism $Z[G]\to Z[G/G']$, it follows immediately from Theorem 1, applied to $Z[G/G']$, that $S\subseteq\bigcup_{w\in W}(w+I)$.

Now, in Theorem 6 with $H=G'$, because σ is onto we have

$$I=\bigcup_{h\in G'}(h-1+IE)$$

and hence

$$S\subseteq\bigcup_{w\in W}(w+I)\subseteq\bigcup_{w\in W}\bigcup_{h\in G'}(w+h-1+IE).$$

But

$$w+h-1=hw-(h-1)(w-1)\in hw+IE$$

so that

$$S\subseteq\bigcup_{w\in W}\bigcup_{h\in G'}(hw+IE)=\bigcup_{g\in G}(g+IE).$$

In particular, if $\alpha\in S$, then there exists $g\in G$ with $\alpha\equiv g \bmod IE$. Because $\alpha g^{-1}\in S$ and $\alpha g^{-1}\equiv 1 \bmod IE$, we have $\alpha g^{-1}\in T$ and $\alpha\in gT=\tau(g)$. This shows that τ is onto, and the result follows.

(2) Let $\delta: Z[H]\to Z$ be the augmentation mapping. Set $E_0=\omega(Z[H])$,

$I_0=\omega(Z[H'])Z[H]$ and set

$S_0=\{\beta\in Z[H]\mid\beta$ is a unit in $Z[H]$ with $\delta(\beta)=1$ and

$\beta+I_0$ has finite multiplicative order in $Z[H]/I_0\}$,

$T_0=\{\beta\in S_0\mid\beta\equiv 1 \bmod I_0E_0\}$.

Then by the same way as the above (1), we see that $S_0/T_0\cong H/H''$. Now, by Lemma 5 and Theorem 2, there exists an isomorphism

$$\eta: Z[G]\to Z[H] \text{ with } \delta\eta=\varepsilon,\ \eta(E)=E_0 \text{ and } \eta(I)=I_0.$$

Furthermore the mapping $\bar{\eta}: Z[G]/I\to Z[H]/I_0$ given by $\bar{\eta}(\alpha+I)=\eta(\alpha)+I_0$ is an isomorphism. Thus we easily see that $\eta(S)=S_0$, $\eta(T)=T_0$ and hence the mapping $\theta: S/T\to S_0/T_0$ given by $\theta(\alpha T)=\eta(\alpha)T_0$ is a group isomorphism. Therefore we have $G/G''\cong H/H''$.

COROLLARY 3. *Let G and H be the periodic metabelian groups. If $Z[G]\cong Z[H]$, then $G\cong H$.*

Proof. Suppose that $Z[G]\cong Z[H]$. Then by the foregoing theorem we have $G/G''\cong H/H''$. But since G and H are metabelian G' and H' are abelian, so that $G''=H''=\{1\}$. Hence we have $G\cong H$.

References

Berman, D., *On the equation* $x^m=1$ *in an integral group ring*, Ukrainsk. Matem. Zh. **7** (1955), 253-261.

Jackson, D.A., *The groups of units of the integral group rings of finite metabelian and finite nilpotent groups*, Quart. J. Math. (2) **20** (1969), 319-331.

Passman, D.S., *The algebraic structures of group rings*, John Wiley & Sons, New York, 1977.

Passman, D.S., *Isomorphic groups and group rings*, Pac. J. Math. **15**(1965), 561-583.

Whitcomb, A., *The group ring problem*, Thésis, Univ. of Chicago, 1967.

〈要 約〉

整數環 위의 周期群 基底의 群環에 同型인 群環

金 應 泰

이 論文에서 有限群 G에 대한 整數環 위의 群環 $\boldsymbol{Z}[G]$에 대하여, Berman, Glauberman, Whitcomb, Jackson, Passman 등에 의하여 밝혀진 內容을 周期群 G에 대한 群環 $\boldsymbol{Z}[G]$에 一般化하여 다음과 같은 結果가 성립함을 밝혔다.

周期群 G, H에 대하여, $\boldsymbol{Z}[G]\cong\boldsymbol{Z}[H]$이면 다음이 성립한다.

1. $Z(G)\cong Z(H)$. 단, $Z(G)$, $Z(H)$는 각각 群 G, H의 中心이다.

2. $G/G'\cong H/H'$. 단, G', H'은 각각 G, H의 交換子部分群이다.

3. $\{1\}=Z_0(G)\subseteq Z_1(G)\subseteq\cdots\subseteq Z_n(G)\subseteq\cdots$,
$\{1\}=Z_0(H)\subseteq Z_1(H)\subseteq\cdots\subseteq Z_n(H)\subseteq\cdots$

   을 각각 G, H의 upper central series라 하면, 모든 n에 대하여 $\boldsymbol{Z}[G/Z_n(G)]\cong\boldsymbol{Z}[H/Z_n(H)]$이고 $Z_{n-1}(G)/Z_n(G)\cong Z_{n-1}(H)/Z_n(H)$.

4. tr_G, tr_H를 각각 $\boldsymbol{Z}[G]$, $\boldsymbol{Z}[H]$의 trace map이라 하면, 임의의 同型寫像 $\theta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$에 대하여 $tr_H\theta=tr_G$.

5. $\theta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$를 同型寫像이라 할 때, 임의의 $\alpha\in C(\boldsymbol{Z}[G])$에 대하여 $\theta(\alpha^*)=\theta(\alpha)^*$ 단, $C(\boldsymbol{Z}[G])$는 $\boldsymbol{Z}[G]$의 中心이고 $\alpha=\Sigma a_x x$라 할 때 $\alpha^*=\Sigma a_x x^{-1}$이다.

6. $\mathcal{L}_1, \mathcal{L}_2$를 각각 G, H의 有限正規部分群 전체로 이루어진 束(lattice)이라 할 때, 각 $N\in\mathcal{L}_1$에 대하여 $f(N)=\eta(N)$인 束同型寫像 $f:\mathcal{L}_1\to\mathcal{L}_2$와 環同型寫像 $\eta:\boldsymbol{Z}[G]\to\boldsymbol{Z}[H]$가 존재하고 $\boldsymbol{Z}[G/N]\cong\boldsymbol{Z}[H/\eta(N)]$이다.

7. G를 임의의 群, H를 G의 周期部分群이라 할 때, 群 H/H'은 덧셈군 $\omega(\boldsymbol{Z}[H])\boldsymbol{Z}[G]/\omega(\boldsymbol{Z}[H])\omega(\boldsymbol{Z}[G])$과 同型이다. 단, $\omega(\boldsymbol{Z}[G])$, $\omega(\boldsymbol{Z}[H])$은 각각 $\boldsymbol{Z}[G]$, $\boldsymbol{Z}[H]$의 augmentation ideal이다.