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Falt Manifolds with First Betti Number Zero

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For a space X, $b_1(X)$ denotes the first Betti number; that is, the free rank of $H_1(X; Z)$. If a flat manifold M has $b_1(M)=k$, it is known that M has a finite covering of the form $T^k \times N$, where T^k is a k-dimenional torus and N is a $(n-k)$-dimensional flat manifold with $b_1(N)=0$. Therefore, flat manifolds with $b_1=0$ are "building blocks".

Much research has been done in recent years on flat manifolds with $b_1=0$, see[**Sz1**, **Sz2**, **HS**, **HMSS**]. In [**Sz1**], $(2n+1)$-dimensional flat manifolds with holonomy group $(\boldsymbol{Z}_2)^{2n}$ were constructed. These particular exemples are rational homology spheres (and hence, orientable and $b_1=0$). In the even dimensional case, one can use [**Sz2**; Theorem 1.5] to construct $(2n+2)$-dimensional flat manifolds with holonomy group $(\boldsymbol{Z}_2)^{2n+1}$ and with $b_1=0$. These manifolds are necessarily non-orientable.

In this paper, we construct orientable flat manifolds with holonomy group $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ and with $b_1=0$. Note that in dimension 1, 2 and 4, all orientable closed manifolds have $b_1=0$. We prove the following:

Theorem. With the exception of dimensions 1, 2 and 4, there exists an orientable flat manifold whose holonomy group is $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ and whose first Betti number is zero.

The rest of this paper will be occupied by a proof of the theorem. First, we change the problem to a purely group-theoretic one. The bridge is that flat manifolds are exactly the Euclidean space-forms as will be explained below.

Let $E(n)=\boldsymbol{R}^n \infty O(n)$ be the group of isometries of $\boldsymbol{R}^n$. It has the group law

$$(a, A)(b, B) = (a+Ab, AB)$$

and acts on $\boldsymbol{R}^n$ by

$$(a, A)x = a+Ax$$

for all $(a, A) \in E(n)$, $x \in \boldsymbol{R}^n$. We call a and A the translation part and rotation part of (a, A), respectively. The subgroup of orientation-preserving isometries is $E^+(n) = \boldsymbol{R}^n \infty \mathrm{SO}(n)$. A discrete subgroup π of $E(n)$ acting properly discontinuously on $\boldsymbol{R}^n$ with compact quotient is called a crystrallographic group. If in addition, π acts freely (or equivalently, π is torsion-free), then π is called a Bieberbach group. In this case, the quotient $\boldsymbol{R}^n/\pi$ is a manifold with fundamental group π. A Riemannian manifold M is flat if and only if $M=\boldsymbol{R}^n/\pi$ where π is a Bieberbach group.

By the above observation, it is enough to find Bieberbach groups with $b_1=0$ in all dimensions except 1, 2 and 4. Since our groups lie in $E^+(n)$, all the manifolds will be orientable. A complete algebraic characterization of Bieberbach groups is known: A torsion-free abstract group π is a Bieberbach group if and only if it has a free abelian group of finite index. Then there exists a unique maximal normal abelian subgroup $\boldsymbol{Z}^n$. The finite quotient $\Phi=\pi/\boldsymbol{Z}^n$ is called the holonomy group of the space $\boldsymbol{R}^n/\pi$ (after one embeds π into $E(n)$). This group acts freely on the torus $\boldsymbol{R}^n/\boldsymbol{Z}^n$ by isometries. Therefore, every flat manifold is finitely covered by a torus.

The first Betti number of $\boldsymbol{R}^n/\boldsymbol{Z}^n$ is equal to the free rank of the abelianization of π. It is known that it also equals the dimension of $(\boldsymbol{R}^n)^\Phi$, the fixed point set of the action of the holonomy group Φ on $\boldsymbol{R}^n$.

The proof breaks up into two cases:

Case $n=2k+1$ $(k>1)$. Let

$$I=\begin{bmatrix}1 & & \\ & J & \\ & & 1\end{bmatrix},\ A=\begin{bmatrix}1 & & \\ & -J & \\ & & -1\end{bmatrix},\ B=\begin{bmatrix}-1 & & \\ & J & \\ & & -1\end{bmatrix},\ C=\begin{bmatrix}-1 & & \\ & -J & \\ & & 1\end{bmatrix}$$

where J is the identity matrix of size $n-2$. Then $\Phi=\{I,A,B,C\}$ is a subgroup of $SO(n)$ which is isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. Let $e_i=[0,\cdots,1,\cdots,0]^t$ $(1\leq i\leq n)$ be the standard basis for $\boldsymbol{R}^n$. Let

$$a=\frac{1}{2}e_1\ ,\quad b=\frac{1}{2}\sum_{i=2}^{n}e_i\ ,\quad c=\frac{1}{2}\sum_{i=1}^{n}e_i$$

Let π be the subgroup of $E^+(n)$ generated by

$$\alpha=(a,\ A),\ \beta=(b,\ B),\ \gamma=(c,\ C),\ t_i=(e_i,\ I)\quad(1\leq i\leq n).$$

Notice that π can be described by the following presentation:

generators:

$$\alpha,\beta,\gamma,t_i\ (1\leq i\leq n)$$

relations:

$$\alpha t_1\alpha^{-1}=t_1,\quad \alpha t_i\alpha^{-1}=t_i^{-1}\ (1<i<n),\quad \alpha t_n\alpha^{-1}=t_n^{-1},\quad \alpha^2=t_1,$$
$$\beta t_1\beta^{-1}=t_1^{-1},\quad \beta t_i\beta^{-1}=t_i\ (1<i<n),\quad \beta t_n\beta^{-1}=t_n^{-1},\quad \beta^2=t_2\cdots t_{n-1},$$
$$\gamma t_1\gamma^{-1}=t_1^{-1},\quad \gamma t_i\gamma^{-1}=t_i^{-1}\ (1<i<n),\quad \gamma t_n\gamma^{-1}=t_n,\quad \gamma^2=t_n.$$

Clearly π has a normal subgroup $\boldsymbol{Z}^n$ generated by $t_1,\ t_2,\cdots,\ t_n$ with quotient $\Phi=\pi/\boldsymbol{Z}^n\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. Therefore, there is a short exact sequence

$$1 \to \boldsymbol{Z}^n \to \pi \to \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \to 1.$$

An element of $\pi/\boldsymbol{Z}^n$ is of the form $t\cdot\alpha$, $t\cdot\beta$ or $t\cdot\gamma$ where

$$t = \sum_{i=1}^{n} l_i t_i \in \boldsymbol{Z}^n$$

(we use the additive notation for $\boldsymbol{Z}^n$). Then

$$(t\cdot\alpha)^2 = (2l_1+1)t_1, \quad (t\cdot\beta)^2 = \sum_{i=2}^{n-1}(2l_i+1)t_i, \; (t\cdot\gamma)^2 = (2l_n+1)t_n.$$

These are all non-zero elements of $\boldsymbol{Z}^n$. This implies that π is torsion-free, that is, π does not have elements of finite order.

The action of the holonomy group $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ on the linear space $\boldsymbol{R}^n$ has no fixed points other than the origin. This implies that rank of $H_1(\pi; \boldsymbol{Z})$ is zero. As an alternate argument, we could consider the commutator subgroup $[\pi, \pi]$. From the above presentation, we can obtain easily

$$[\alpha, t_i] = t_i^{-2} \quad (1 < i \leq n), \qquad [\beta, t_1] = t_1^{-1} \text{ etc.}$$

This implies that $[\pi, \pi]$ contains $(2\boldsymbol{Z})^n$. This shows that $\pi/[\pi, \pi]$ is finite. Thus, $H_1(\pi; \boldsymbol{Z})$ = (abelianization of π) has free rank 0 so that $b_1 = 0$. Since $\Phi \subset SO(n)$, the manifold $\boldsymbol{R}^n/\pi$ is orientable.

Case $n=2k$ $(k>2)$. Let

$$I = \begin{bmatrix} K & & \\ & J & \\ & & K \end{bmatrix}, A = \begin{bmatrix} K & & \\ & -J & \\ & & -K \end{bmatrix}, B = \begin{bmatrix} -K & & \\ & J & \\ & & -K \end{bmatrix}, C = \begin{bmatrix} -K & & \\ & -J & \\ & & K \end{bmatrix}$$

where K and J are the identity matrices of size 2 and $2k-4$ respectively. Note that since $k>2$, we have $2k-4>0$. In fact, the construction fails for $k=2$. As before,

$\Phi = \{I, A, B, C\} \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \subset SO(n)$. Let

$$a = \frac{1}{2}\sum_{i=1}^{2} e_i, \quad b = \frac{1}{2}\sum_{i=3}^{n} e_i, \qquad c = \frac{1}{2}\sum_{i=1}^{n} e_i.$$

Let π be the subgroup of $E^+(n)$ generated by

$$\alpha = (a, A), \; \beta = (b, B), \; \gamma = (c, C), \; t_i = (e_i, I) \; (1 \leq i \leq n).$$

Notice that π has a presentation as follows:

generators:

$$\alpha, \beta, \gamma, t_i \; (1 \leq i \leq n)$$

relations:

$$\alpha t_i \alpha^{-1} = t_i \; (i \leq 2), \quad \alpha t_j \alpha^{-1} = t_j^{-1} \; (2 < j < n-1), \quad \alpha t_l \alpha^{-1} = t_l^{-1} \; (l \geq n-1),$$

$$\beta t_i \beta^{-1} = t_i^{-1} \; (i \leq 2), \quad \beta t_j \beta^{-1} = t_j \; (2 < j < n-1), \quad \beta t_l \beta^{-1} = t_l^{-1} \; (l \geq n-1),$$

$$\gamma t_i \gamma^{-1} = t_i^{-1} \; (i \geq 2), \quad \gamma t_j \gamma^{-1} = t_j^{-1} \; (2 < j < n-1), \quad \gamma t_l \gamma^{-1} = t_l \; (l \geq n-1),$$

$$\alpha^2 = t_1 t_2, \qquad \beta^2 = t_3 \cdots t_{n-2}, \qquad \gamma^2 = t_{n-1} t_n.$$

Again π has a normal subgroup $\boldsymbol{Z}^n$ generated by $t_1, t_2, \cdots, t_n$ with quotient $\Phi = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. There is an exact sequence

$$1 \to \boldsymbol{Z}^n \to \pi \to \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \to 1.$$

For $t = \sum l_i t_i$, we have

$$(t \cdot \alpha)^2 = \sum_{i \leq 2} (2l_i + 1) t_i, \quad (t \cdot \beta)^2 = \sum_{2 < i < n-1} (2l_i + 1) t_i, \quad (t \cdot \gamma)^2 = \sum_{i \geq n-1} (2l_i + 1) t_i$$

all of which are non-zero elements of $\boldsymbol{Z}^n$. This shows that π is torsion-free. Further, $[\alpha, t_i] = t_i^{-2}$ for $i > 2$ and $[\gamma, t_i] = t_i^{-1}$ for $i \leq 2$. This implies that $\pi / [\pi, \pi]$ is finite. Thus, π is a Bieberbach group with $b_1 = 0$. Since $\Phi \subset SO(n)$, the manifold $\boldsymbol{R}^n / \pi$ is orientable. This completes the proof of theorem.

References

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1차 베티수 영인 평탄한 다양체

김 연 식

〈요 약〉

평탄한 다양체는 compact, connected, Riemannian manifold를 뜻한다. 평탄한 다양체의 구성에 대한 연구는 1957년의 Calabi의 논문과 같은 시기에 발표된 Auslander와 Kuranishi의 논문에서 볼 수 있다.

Betti number 0인 flat manifold에 대한 최근의 연구는 1985년의 A. Szezepanski, 1986년의 H. Hiller와 C.H. Sah의 논문에서 볼 수 있으며, 이들 논문에서는 각각 holonomy group $(\boldsymbol{Z}_2)^{2n+1}$이고 Betti number 0인 $(2n+1)$차원 flat manifold의 구성과 holonomy group $(\boldsymbol{Z}_2)^{2n+1}$이고 Betti number 0인 $2n$차원 flat manifold의 구성을 다루고 있다.

이 논문에서는 holonomy group $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$이고 Betti number 0인 orientable flat manifold를 구성하고자 한 것이다. 주정리는 1, 2, 4차원을 제외한 holonomy group $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$이고 Betti number 0인 orientable flat manifold가 존재한다는 것이고 이것을 증명한 것이다.