Characterizing the Proportional Reasoning of Middle School Students

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Abstract

The purpose of this study was to identify characteristics of middle school students' proportional reasoning. The participants were 70 7th grade, 68 8th grade, and 69 9th grade students in Seoul. This study was conducted using written test to analyze students' proportional reasoning strategies and interview to investigate their thoughts.

In the results, there was no statistically significant difference between the grades. Also the students often chose solving strategies depending on the context of the problems, and they showed a lack of multiplicative understanding. In conclusion, we identified a pseudo-formal stage, which jumps from the additive strategy to the formal strategy with procedural understanding and not conceptual understanding. Therefore the multiplicative strategy could serve the role of a bridge from an additive strategy to a formal strategy on proportional reasoning.

Key words: ratio, proportion, proportional reasoning

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Ⅰ. Introduction

The concepts of ratio and proportion are applied in mathematics and science as well as in everyday life. In the realm of mathematics, the notion of proportionality is related to rational numbers, slope, velocity, probability and similarity, while in science, it has more to do with core concepts of density, force, and acceleration. Although the concepts of ratio and proportion are familiar to people in diverse ways, students often have difficulty understanding the relationship existing between two groups of quantities.

Several studies showed that despite the importance of understanding proportion in middle school, only a few students are skillful at it (Nunez, Schliemann & Carraher, 1993; Tourniaire & Pulos, 1985). Moreover, even math teachers in middle schools are not proficient in concepts of ratio and proportion (Behr, Harel, Post, & Lesh, 1992). This lack of understanding does not necessarily mean that students cannot make up a proportional relationship or solve problems. Understanding proportionality extends beyond the application of mathematical formula such as $a/b=c/d$ to solve the problems. For further understanding, proportionality requires proportional reasoning.

Inhelder and Piaget (1958) believed that understanding of proportionality can be acquired in the formal stages of cognitive development and regarded proportional reasoning as one index of cognitive development. Lesh, Post and Behr (1988) claimed that proportional reasoning is the capstone of elementary mathematics and the cornerstone of high school mathematics. According to NCTM Curriculum and Evaluation Standards (1989), 'the ability to reason proportionally develops in students throughout grades 5-8. It is of such great importance that it merits whatever time and effort expended to assure its careful development.'(p.82). Despite the importance of proportional reasoning in the middle school mathematics curriculum, most of the research on proportional reasoning (Kaput & West, 1994; Noelting, 1980; Lamon, 1993) is focused on the elementary school curriculum and elementary students' understanding. The reason can be the concept of ratio is first introduced in elementary school.

In summary, Inhelder & Piaget (1958) suggested that proportional reasoning does not appear until the formal stages of cognitive development. Lesh et al. (1988) asserted the importance of proportional reasoning in elementary and middle school mathematics education. Thus it is necessary and useful to analyze the characteristics of proportional reasoning amongst middle school students who are in the process of progressing from the concrete stage to the formal stage.

Therefore our research questions are to investigate students' problem solving strategies used in proportional situations and to identify characteristics of middle school students' proportional reasoning. To answer these questions, we gave written tests to middle school students so that we could analyze their proportional reasoning strategies, and we also interviewed them to investigate their thoughts.

Ⅱ. Background

A. Proportional reasoning

The study of proportional reasoning was initially introduced by Piaget and his colleagues, and their definitions slightly differ with other scholars. Inhelder and Piaget (1958) didn't use the word "proportional reasoning," but their initial study introduced characteristics of students' proportional thinking stage by stage. They suggested that proportional reasoning included a secondary relationship, that is, the relation between two proportions. However, the study was challenged in terms of its adequacy, because the problem situations utilized in their study contained a number of situations that required the concept of balance and scientific knowledge as well as proportional reasoning. They utilized problem situations, such as balance beam problem, hauling weight on an inclined plane or the projection of shadow. Their study was more appropriate for using a proportion-linked expression $a \times b = c \times d$ than $a/b = c/d$. Tourniaire and Pulos (1985) also criticized their study, claiming that strategies used by students was different depending on the context, numbers, and their understanding level of proportionality.

Karplus, Pulos and Stage (1983) defined reasoning that involves a linear relationship($y=ax$, $a \neq 0$) between two variables as proportional reasoning. Lesh et al. (1988) defines proportional reasoning as a form of mathematical reasoning that involves a sense of co-variation and a sense of multiple comparisons, as well as the ability to mentally store and process several pieces of information. In addition, proportional reasoning is concerned largely with inference and prediction, and includes both qualitative and quantitative methods. While the proportional reasoning of Inhelder and Piaget (1958) is a global cognitive index, the definitions given by Karplus et al. (1983) and Lesh et al. (1988) are a part of mathematical reasoning, recognizing the expressions "$f(x)=kx(k \neq 0)$ and $a/b=c/d$" in particular.

As Lamon (1989) indicated, many researchers applied the proportional reasoning to achieve correct answers without recognizing the structural similarities on both sides of the proportional equation. Actually, all students that solved the proportional problem didn't use the proportional reasoning. Cramer, Post, Currier (1993) indicated that the algorithms $ad = bc$ from the relation of the equivalent relation for two ratios could be procedural knowledge and the ability of proportional reasoning could not have been developed although the students well solved the proportional problems. Lamon (1989, 2007) explained that proportional reasoning would occur when a person, is interacting with a situation simultaneously involving covariance and invariance. In this, covariation referred to the simultaneous change in two variables between which exists some binding relationship and invariance referred to the constancy of the existing relationship between two variables under one or more transformations. The idea of Lamon (1989, 2007) for proportional reasoning seemed to be equivalent to the idea of Lesh et al. (1988).

In our study, we accepted Lesh et al. (1988)'s definition. As

Inhelder and Piaget's definition represents an element of general cognitive ability, and the one of Karplus et al. (1983) indicates a limited mathematical concept, it is difficult to accept their definitions as it is. More concretely, we are studying proportional reasoning as mathematical thinking including covariance and multiplicative comparison of two quantities where there exist linear functions with y intercept 0, as well as a sense of ratio useful to form conjecture about real phenomenon. Although this definition reflects Piaget's point of view in that it includes a sense of ratio, it is more closely related to mathematical reasoning in a sense that the definition focuses more on recognition of invariance and covariance of the ratio-equivalent relationship.

B. Proportional reasoning problem types and problem solving strategies

The research on proportional reasoning is divided principally into two problem types: numerical comparison problems and missing value problems (Cramer & Post, 1993; Kaput & West, 1994; Noelting, 1980). In a numerical comparison problem, four values are given (a, b, c, and d) and the goal is to determine an order relation between ratio a/b and c/d. Missing value problems provide three out of the four values in the proportion a/b=c/d, from which one should find the missing value. In the past, numerous studies on defining proportional reasoning dealt with students' strategies on missing value problems. This is because the missing value problems are represented by the form, a/b=c/d. However, this is but one part of the proportional reasoning types and not the whole.

Lamon (1993) understood that the missing value problems and numerical comparison problems include various contexts, she identified four semantic problem types according to problem situations (well-chunked measures, associated sets, part-part-whole, and stretchers/shrinkers). The well-chunked measures type involves the comparison of two extensive measures, resulting in an intensive measure or rate. In the part-part-whole type, the extensive measure of a single subset of a whole is given in terms of the cardinalities of two or more subsets. Associated sets involve two sets which may have no commonly known connection or an ill-defined connection until some explicit statement indicates that rate pairs should be formed. Stretchers/shrinkers refer to a problem that involves growth or shrinkage according to a fixed ratio (Lamon, 1993).

Several studies have examined different strategies employed in solving numerical comparisons and missing value problems. For instance, Kaput and West (1994) divided students' problem solving strategies into four types: the building‐up process, abbreviated building-up process, unit factor approach, and the formal equation-based approach. Consider the following example:

> *Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for three. Later, they decided to go back to the store and buy enough balloons for everyone in the class. How much would they pay for 24 balloons? (Lamon, 1993, p.53)*

A typical response to this problem by means of the building‐up process is shown in Table 1.

Table 1. The example of building-up process

The number of balloons	3	6	9	12
Price	2	4	6	8

In this pattern, we know that the price of the 24 balloons is 16 dollars. The abbreviated building-up process is to omit the procedure of repeated addition, and to directly go to multiplication process of "$24 \div 3 = 8$ and $2 \times 8 = 16$". The unit factor approach means that by recognizing the price per one balloon is 2/3 dollars, we get the answer by using "$24 \times 2/3 = 16$ dollars". The formal equation-based approach is a problem solving process by means of the expression "$a/b = c/d$".

Carpenter et al. (1999) and Steinthorsdottir (2003) analyzed problem-solving strategies according to students' proportional reasoning development, but Singh (2000) reported from the analysis result of two grade six students' proportional reasoning schemes that the more important thing in proportional reasoning was how to understand the proportional problem situations than which problem strategies was used. So in our study, these strategies were the only basis for the analysis of our students' problem solving strategies and the purpose of our study was to investigate how the middle school students understand the proportional situations including the invariance and covariance quantities.

Ⅲ. Method

This research used mixed methods (Creswell, 2003) to investigate middle school students' proportional reasoning. It was conducted in two phases; in the first phase, we analyzed the data that were scores and strategies for students' proportional reasoning and in second phase, we analyzed the data that were the transcripts for the students' understanding about the ratio concepts and the invariant quantities.

A. Participants

The participants for this study were 70 7th grade, 68 8th grade, and 69 9th grade students in Seoul, Korea. Most of the participants belonged to families of lower middle or middle socio-economic status, and generally had relatively low average scholastic ability. Each student was given 45 minutes to complete a written test consisting of eleven items. After the written test, a semi-structured interview was conducted with some students in order to find out the students' understanding of invariance and covariance. Fourteen students selected based on their written tests were interviewed individually regarding each problem on the test. The interviews were audio taped and transcribed for analysis.

B. The instruments

The written test was composed of four types of problems, following Lamon (1993)'s semantic types: Associated sets, part-part-whole, well-chunked measures, and stretchers/shrinkers. Within each type, there were missing-value problems and numerical comparison problems. The problems were designed to facilitate the evaluation of students' concepts of ratios and proportions, and students' understanding of proportional relationships. We revised and used some problems from other studies conducted previously (Gravemeijer, Keijzer, Galen, & F.H.J. van, 2005; Hines & McMahon, 2005; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997; Lamon, 1993, 1999). The contents of the problems are represented in Table 2 and the Cronbach alpha for the written test was 0.753. Appendix shows the problems in the written test.

Table 2. Semantic types and the contents of problems

Semantic types	Number of problems	Types of problem	The contents of problem
Associated Sets	1-1	Numerical comparison	Comparing taste of orange juice
	1-2	Numerical comparison	Comparing taste of orange juice
	1-3	Missing value	Making orange juice
Part-Part-Whole	2-1	Missing value	Finding the number of cookies
	2-2	Missing value	Attribution of the payment
	2-3	Numerical comparison	Finding the best shooter
Well-Chunked Measures	3-1	Missing value	Finding the distance in the same velocity
	3-2	Missing value	Finding the distance in the same time
	3-3	Numerical comparison	Comparing cars' velocity
Stretchers and Shrinkers	4-1	Missing value	Finding one side of a similar figure
	4-2	Numerical comparison	Finding similar figures to the square

Also, the interviews were conducted to investigate not only how they solved the problems, but also how they thought about ratios and proportions in using those methods. The questions were designed to probe regions difficult to investigate on the written test. The principle interview questions asked students to explain their solving methods, give reasons for their choices, the invariant factor in each situation and express the relation using x, y in each missing value problems.

C. Data analysis

The students' answers were analyzed in two ways. First, we gave a score of 1 for a correct answer and 0 for a wrong answer. Second, we categorize responses according to students' problem strategies. The strategies were diverse according to semantic types of problem and were slightly different according to missing value of problem or numerical comparison problems.

Table 3. The types of students' problem solving strategies

The Type	Students' problem solving strategies	The examples
St0	No response	There is no explanation.
St1	Only guessing	I think … or I guess … or It seems
St2	The difference between two quantities	Since 4-3=1, 5-2=3 then 4-3 < 5-2
St3	Repeated addition	3persons, 2spoons; 6persons, 4spoons; 9persons, 6spoons; 12persons, 8spoons
St4	The unit as 1	For 1 hour, it is possible to go 3km
St5	The unit as arbitrary number	Since 45km= 30km +15km, 90+(30+15)=90+45=135
St6	The multiplication or division	3×5=15
St7	The equivalent fraction or ratio	40/60, 25/60, 30/60, 36/60
St8	Ratio as operand	24000×2/12
St9	The formal proportion formula	A : B = C : D
St10	A mathematical formula	(distance) = (time) ×(speed)

Therefore, in our study, we used grounded theory methods (Strauss & Corbin, 1990) to students' strategies from diverse students' response through discussions with three in-service teachers, a foreign mathematics education professor, and a foreign education technology professor, as well as three co-authors. Table 3 represents the types of students' problem solving strategies developed for analysis.

The students interviewed were selected based upon their problem strategies and their representations. We divided them into 3 groups; from St 0 to St3, identified as Type1 if students used St2 or St3 more than four times, from St4 to St8, identified as Type2 if students used St4 or St5 or St6 or St7 more than four times, and identified as Type 3 if students used St9 or St10 more than 3 times. Even those students using St9 or St10 used St4 or St7 for easy problems, and St 9 cannot be used for value comparison problems, so the count was reduced by one when determining groups. For each type, we investigated their understanding ratio concepts, invariant quantities, and representation of the functional relationships.

Ⅳ. Result

A. Quantitative Results

Analysis of variance (ANOVA) and chi-square analysis were used as appropriate to analyze the written test. Table 4 shows the results of ANOVA on the total score in grades.

Table 4. The results of ANOVA on the total score

Grade Problem types		7	8	9	F	p
Missing value	Mean	2.9439	3.3824	3.7164	3.379	.036*
	SD	(1.5593)	(1.7019)	(1.9680)		
Numerical comparison	Mean	2.7286	3.0147	2.8955	1.048	.353
	SD	(1.1283)	(1.0719)	(1.2926)		
Total	Mean	5.6714	6.3971	6.6119	2.532	.082
	SD	(2.3077)	(2.4564)	(2.9230)		

* (Maximum Score: MV=6, NC=5, Total=11)

Although the mean of total score increased with grade level, there was no statistically significant difference in the total scores. The results showed that proportional reasoning was not affected by grades. However, there was statistically significant difference in missing value problem among the grades. The reason is that the only one problem 4-1(ANOVA, F=7.534, p<.01) had significant differences between grade 9 and the others. The problem 4-1 is about the figure's similarity. The research results that students do not apply the proportionality concept very well for problems in a geometrical context such as 4-1 can be found in many previous researches (Lamon, 1993; Nunez et al., 1993; Singh, 2000). According to the classification types of Lamon (1993), a similarity problem, corresponding to an enlargement/reduction problem, is an area identified as a problem type for which students not having learned a formal proportional expression cannot understand the multiplicative relationship very well.

In our study, the fact that only 9th grade students could solve this problem better compared to students from other grades seems to be due to the influence of the curriculum. As a matter of fact, this study was conducted two months after the ratio of similitude was taught in 8th grade, and the ratio of similitude in the curriculum includes a lot of practice for utilizing the proportional expression in the form of a : b = c : d. Especially for missing value problems in other contexts, there were no statistically significant difference, which suggests that the reason 9th grade students could solve problem 4-1 better was because the study was conducted after they learned the ratio of similitude.

We also analyzed the strategies of students for each problem through chi-square analysis. Statistically significant differences were found only in 3-1 (chi square value= 28.783, p<.05), 4-1 (chi square value=28.919, p<.01), and 4-2 (cui square value=34.415, p<.01). Table 5-7 show the percentages of students' strategies for problem 3-1, 4-1 and 4-2, respectively.

Table 5. The percentages of students' strategies used for problem 3-1 (%)

Strategies	Grade 7	Grade 8	Grade 9
0	48.6	64.7	59.7
1	1.4	0.0	0.0
2	0.0	1.5	0.0
4	12.9	10.3	3.0
5	25.7	7.3	23.9
6	7.1	1.5	1.5
9	4.3	10.3	4.5
10	0.0	4.4	7.5

The percent of correct answers in the problem 3-1 was 57.1%, but the percent of St0 is more than 48% (Table 5). The reason which the rate of no response was higher could be students' difficulty to write explanation. Grade 7 students used more multiplicative strategy (St6) than the other grades, but no one used mathematical formula (St10). Though this main strategy of this problem was using unit as arbitrary number (St5) that students divided given conditions into arbitrary parts easily to use, grade 8 students used it less of ten and used a proportion formula (St9) more often than the other grades. Grade 9 students used mathematical formula (St10) more often as compared with the other grades.

Table 6. The percentages of students' strategies used for problem 4-1 (%)

Strategies	Grade 7	Grade 8	Grade 9
0	65.7	72.1	56.7
1	0.0	1.5	0.0
2	17.1	7.4	1.5
6	4.3	1.5	3.0
7	0.0	0.0	1.5
8	0.0	1.5	0.0
9	12.9	16.2	37.3

The percentage of total correct answers of problem 4-1 was 31.7% and the percentage of no response was higher than in problem 3-1 except in grade 9, while the strategy of the proportion algorithm (St9) was higher in grade 9 than other grades. This may be because similarity had been taught in the second semester of grade 8, so Grade 9 students might have remembered the proportion algorithm as tools to solve the problem. Thus the percentage correct answers for problem 4-1 in grade 9 was higher than in grade 7 and grade 8. On the other hand, the multiplicative strategies (St6) were fewer than in other missing value problems.

Table 7. The percentages of students' strategies used for problem 4-2 (%)

Strategies	Grade 7	Grade 8	Grade 9
0	30.0	38.2	37.3
1	11.4	11.8	35.8
2	52.9	48.5	22.4
6	5.7	1.5	0.0
7	0.0	0.0	1.5
9	0.0	0.0	3.0

Problem 4-2 was similar to the problems in Bright, Joyner, Wallis (2003). But our students' scores was significantly less than the students in Bright, Joyner, Wallis (2003). Problem 4-2 was the only numerical comparison problem that revealed a statistically significant difference in strategies. As the percent of total correct answers in this problem was 3.1%, it was difficult for all grades to solve. Grade 7 and 8 students used the difference between quantities (St2) more frequently than those in grade 9 and their strategies resulted in wrong answers (Table 7). On the other hand, some of grade 9 used the formal proportion formula (St9) while no one in the other grades.

From this result, students' strategies were dependent on the contextual problems. For example, in problem 1-1, 1-2, and 2-3, the frequent strategy was St7, in 1-3 and 2-1, the frequent strategy was St6, and in 3-1, the frequent strategy was St5 except St0. But in problem 4-1 and 4-2, the frequent strategy was St2. The rate of correct answers for 4-1, 4-2 was below than any other problems. Thus, students were inclined to use an additional strategy in an unfamiliar context without using a multiplicative strategy.

B. The Interview Results

We interviewed 14 students. The students were selected according to the problem-solving strategies and their representations on the written test rather than school achievements. The total scores of Type 1 students are from 5 to 7, the scores of Type 2 are from 7 to 9, and the scores of Type 3 are from 8 to 9. After interviewing 14 students, we found only one student who showed understanding of the ratio concept, recognize invariant quantities, and represent y=kx ($k \neq 0$) in the proportional situations. Pseudonyms were used for the students below; Type 1: 7th grade students were labeled S71A or S71B, Type 2: 8th grade students were labeled S82A or S82B, and Type 3: 8th grade students were labeled S83A or S83B.

1. The characteristics of students who used the difference between quantities

S71A's protocol illustrated how to solve the difference of two quantities.

> *Interviewer: The question is which orange juice is the strongest out of four types. Has your mind changed?*
>
> *S71A: It has not changed. Comparing the number of cups and spoons, the orange juice is the strongest when the difference is the smallest. So since the difference is 1 in orange juice A, I think orange juice A is the strongest.*

S71A got the answer right for 1-1, but wrong for 1-2 as the same strategy was used to choose the weakest orange juice. In this process, the student could not recognize the fact that the unit of spoons and cups were not the same and that it is meaningless to subtract them from each other.

Another characteristic of this type is a lack of recognizing the quantities of invariance in this situation. S71B's protocol shows the recognition of invariant quantities in orange juice situations.

> *Interviewer: In this problem 1-3, how would you say the taste of orange juice differs for 3 people and for 15 people?*
> *S71B… Not the same.*
> *Interviewer: Not the same? Which one is stronger?*
> *S71B: The orange juice for 3 people.*
> *Interviewer: Why is that?*
> *S71B: There are fewer people.*

Since the students would assess the taste of orange juice using the difference in the number of spoons and cups, it is natural that students could not recognize the fact that the taste should be the same despite the number of people being different. The students who used the difference of two quantities kept with the same strategy during the representation of mathematical relationship. Without recognizing the multiplicative relation and only focusing on each number of quantities in each situation, it was difficult to get invariant quantities in proportional situations.

2. The characteristics of students who used multiplicative strategies

The problem-solving strategies of this type of student involved using multiplicative strategies, for example, equivalent fraction, scalar ratio, and functional ratio of two quantities. S72A's protocol illustrated how to implement the multiplicative strategies of two quantities.

> *Interviewer: In problem 3-1, could you explain how to solve the problem?*

S72A: Since he went 30km for 1 hour 30minutes, that means it takes 90 minutes and since it was 45km to Blue City, it is 1.5 multiplied by 30km, so it takes the same multiple of 90 minutes.
Interviewer: Good. Would it be possible to suggest another method?
S72A: There are two distances, 30km and 45km. Since it takes 90 minutes to cover 30km, I divide 30km by 6, yielding 5km. I also divide the time it takes by 6. For 5km units, 45km is 9 times of 5km, and then I multiply 9 to the time it takes for 5km.

S72A recognized the proportional relation between distance and time. He explained the first method by indicative if 1.5 multiplied by 30 then it becomes 45, it is possible to get the time if 1.5 is multiplied by 90. When the interviewer asked for another method, he used the unit as an arbitrary number, for example, it takes 15min for 5km, and since 45km is 9 times to 5km, he could multiply 9 by 15min. He solved most of the problems in the written test with multiplication or unit factor strategy.

Most of the students in this type recognized the taste of orange juice for 3 people and for 15 people being the same because they determined the taste of orange juice with fractions.

Interviewer: How would you determine the taste of orange juice for 3 people and for 15 people?
S82A: I think it would be the same.
Interviewer: Why do you think it would be the same?
S82A: Is the consistency the same?

Comparing to Type 1, S82A recognized the invariant quantity in orange juice problem. This is because the strategy using equivalent factions helped determining the taste of the orange juice. Students could compare multiplicatively, recognize the invariant quantity in each situation and represent ratio with a concrete number.

Interviewer: Are there invariant factors in problem 4-1?
S92A: Invariant factors?
Interviewer: Yes, in this problem.
S92A: There is some ratio.
Interviewer: What is the ratio?
S92A: In this problem, I make out that there is a ratio of 7:6 and in between A and B there is a ratio of 5:3.

S92A could recognize 7:6 as the ratio between width and length as well as 5:3 with the ratio between A and B. Students who were able to compare multiplicatively could recognize and represent the invariant factors as fractions or ratios depending on the contextual situations. However, they did not use proportional relationships, thus rendering it difficult to represent mathematical relationships in a proportional situation-involving x and y.

3. The characteristics of students who used formal strategies

The problem solving strategies of this type of student involved using the proportion algorithm and formal mathematical relations. They did not apply formal strategies to the entire problem but only to the missing value problems. Particularly, students in this type used the proportion algorithm to solve the problem 4-1 or used the formula (distance) = (time) × (speed) to solve problems 3-1 and 3-2. In the numerical comparison problems, they used equivalent fractions or ratios to make comparisons between the two groups of quantities.

Interviewer: In this problem the questions is which is the strongest orange juice out of A, B, C, and D. Do you remember how to solve this problem?
S83B: With fractions.
Interviewer: Why did you solve the problem that way?
S83B: It just happened.
Interviewer: Is there another method?

S83B: Proportional relation?
Interviewer: How can it be solved with proportional relation?
S83B: Well, like this … I think it is possible … No, it cannot be done … it is only possible with fractions.

S83B used fractions to solve problem 1-1 but she didn't know why. In addition, when the interviewer asked for another method, she came up with the proportional relation but did not know how to apply it.

Interviewer: How can we find the time relation to travel 2 hours 15 minutes in problem 3-1?
S93A: It gets changed to 90 minutes. 30 to 45 equal 90 to x. Since 30x is 4050, if I divide into 30 then x is 135 minutes. 135 can be divided into 60 then 120 and 15, so therefore that makes 2 hours 15minutes.
Interviewer: Is there another method?
S93A: (Smiling) I don't know.

S93A used the rule of three to solve problem 3-1 but she was not able to find another method. S93B's protocol illustrates a solution with a formal mathematical formula.

Interviewer: Would you explain how to solve problem 3-1?
S93B: I used a formula.
Interviewer: What is the formula? Tell me in words.
S93B: The problem requests the time and time is distance over speed. Since there is no speed, I found the speed.
Interviewer: Ah! Did you find the speed initially?
S93B: And then made distance over speed
Interviewer: How can you find the speed?
S93B: Speed is distance of time and it takes 1 hour 30 minutes, so I found it with 30 over 1.5.

Interviewer: Is there another method?
S93B: I don't know.

The common factors in S93A and S93B are that students used the formal strategies and did not have a flexible method. In this type, despite the application of algorithm or formal mathematical formula correctly, students showed an incomplete ratio concept and a lack in recognizing invariant quantities.

Interviewer: How would you determine the taste of orange juice for 3 people and for 15 people?
S93A: It would be the same.
Interviewer: What were the invariant factors in the situation for 3 people and for 15 people?
S93A: … Well… spoons?
Interviewer: For 3 people, we have 2 spoons of orange flavoring and 3 cups of water and for 15 people we have 10 spoons of orange flavoring and 15 cups of water, O.K?
S93A: What about the number of people and cups?
Interviewer: Are the numbers invariant? Do the numbers remain unchanged?
S93A: Well…
Interviewer: I think there are invariant factors.
S93A: If we divide it into 5 … I don't know.

S93A could solve the numbers of spoons for 15 people and for 20 people but she could not recognize invariant quantities in proportional situations. She also could not understand the interviewer's question so she changed her answer for every question. In this protocol, she focused only on the number of each quantity and not the relation of quantities.

The total scores of Type 3 students was similar or higher to Type 2 students, but the understanding of proportionality in Type 3 were not flexible than Type 2. Thus, we might conclude that multiplicative strategies are necessary for understanding invariant quantity in proportional situations. In addition, this type of student might show not a relative understanding but rather a procedural understanding of proportion. Particularly, when we analyse the interviews done with them, we found few multiplicative strategies. In our study, we identified a stage that seems to jump from A-B=C-D to A:B=C:D without a multiplicative stage as a pseudo formal stage.

Piaget et al. (1977) declared pre-proportional reasoning although students achieved correct answers without recognizing the structural similarities on both sides of the proportion equation. In this view, the pseudo-formal stage is similar to pre-proportional reasoning. However, we thought that the reason for the pseudo-formal stage was owing to mathematics curriculum. Students had learned the proportion algorithm on the course in grade 6 without a connection to other mathematical concepts. In the middle school mathematics curriculum, although direct relationship, slope, probability and similarity are all closely connected to ratio and proportion, students had learned the contents independently and used proportion algorithm with a problem-solving tool. In addition, students did not have a chance to learn the multiplicative strategy in the mathematics curriculum. Therefore, they had had to jump from the additive strategy to the formal strategy.

Ⅴ. Conclusions

The purposes of our study are to investigate the students' problem solving strategies on proportional situations and to identify characteristics of middle school students' proportional reasoning. In the results of the written test, there was no statistically significant difference between the grades in middle school. The distribution of strategies used in each type of problem showed that students might choose strategies depending on the context of the problems and few students used multiplicative strategies.

Results of the interview showed that students' problem solving strategies were closely related to their understanding of invariant quantities. The students using the difference strategy had poor ratio concepts and a lack of understanding of ratio-equivalent relationships. The students using the multiplicative strategy had a little ratio concept and recognized the ratio-equivalent relationships. However, some students using the formal strategy showed the pseudo-formal stage, which jumps from the additive strategy to the formal strategy with procedural understanding, and applied an algorithm or mathematical formula with tools to solve the problems.

Lamon (1993) showed that 6th grade students' conceptual competencies were far greater than their symbolic competence through pre-instructional interviews. Lamon (1993) suggested that the use of algorithm was no guarantee that proportional reasoning was occurring. Conversely, proportional reasoning might occur without the use of the formal mathematical symbolism. In comparing our study to Lamon's research, our participants were middle school students, but they still had difficulty in using symbolic competence and applying proportional reasoning. Although some students could apply the proportion formula to problems, they had the potential of omitting multiplicative strategy in the development process.

Therefore, we concluded that the multiplicative strategy can serve the role of a bridge from an additive strategy to a formal strategy on proportional reasoning. Although in the middle school mathematics curriculum ratio and proportion are closer to other mathematical concepts, each concept is learned separately without a connection. This study contributes to that research base by exemplifying have to implement one of the basic mathematics education curriculums – namely teaching ratio and proportion and mathematics connection.

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Appendix A

1. Min-Su and Ju-Hyun are going to test different orange juice tastes.

A	B	C	D
Orange taste: 2 spoons Water : 3 cups	Orange taste: 1 spoons Water : 4 cups	Orange taste: 4 spoons Water : 8 cups	Orange taste: 3 spoons Water : 5 cups

1-1. Which has the strongest orange taste? Explain the reason.

1-2. Which has the weakest orange taste? Explain the reason.

1-3. If you make orange juice according to formula A, it is enough for 3 people. Which juice would be enough for 15 people? How many spoons of orange flavor would be needed?

2-1. Each cookie bag contains 7 cookies. Three of the seven cookies in the bag have star shapes and the rest have heart shapes. Sung-Ju brought a total of 28 cookies. How many of those 28 cookies have star shapes? How many are heart-shaped cookies?

2-2. Jun-Su helps to clean rooms twice a week, his brother cleans 4 times in a week, and his sister cleans 4 times in a week. His mother gives 24,000 won as payment for cleaning the rooms. How much does each contribute to get their fair share?

2-3. A group of students were practicing basketball shooting. The following table shows the results:

Player Name	Shots Taken	Successes
Kim	6	3
Na	12	4
Lee	24	7

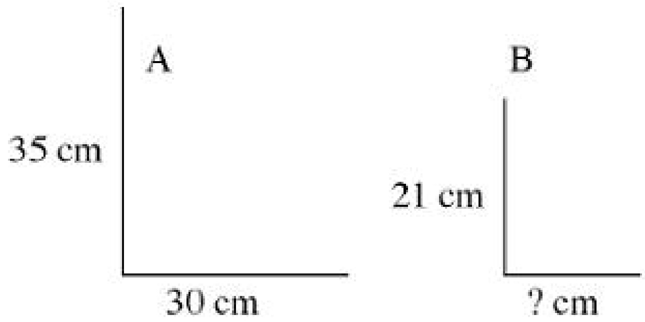
3. Coming from his house, Ji-Wu plans to go to Blue City through Green City by bicycle. The distance from his house to Green City is 30 km and the traveling time would be 1 hour and 30 minutes. The distance from Green City to Blue City is 45km. Solve the following questions:

3-1. How much time does it take to travel from Green City to Blue City?

3-2. The traveling time from JiWu's house to Green City and the time it takes to go from Green City to Blue City are just about the same. How long is the distance to go from Green city to Blue city in 30 minutes?

3-3. Which vehicle has a faster average speed, a car that travels 126 km in 90 minutes or a car that travels 135 km in 105 minutes?

4-1. The two sides of Figure A are 35 cm high and 30 cm long. Figure B has the same shape but it is smaller. If one side of Figure B is 21 cm high, how long is the other side?



4-2. There are two rectangles: 35 feet vs. 39 feet and 22 feet and 25 feet. Which is "more" square?

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