Conjectures in Cournot Duopoly under Cost Uncertainty

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This paper presents a Cournot duopoly model based on a condition when firms are facing cost uncertainty under risk neutrality and risk aversion. Each firm conjectures about the rival's output level, and its cost function is assumed to be unknown to its rival. The Cournot model shows that the expected utility maximizing firms, under risk aversion, show different behaviors from the expected profit maximizing firms. This implies that each firm can increase or decrease its output, which depends on the interaction between both firms under cost uncertainty, assuming that both firms are risk-averse.

Keywords: Cournot equilibrium, Cost uncertainty, Risk aversion

JEL Classification: D81, L13

I. Introduction

Economists have extensively analyzed the effects of uncertainty on the optimizing behavior of a single agent. These economists include Sandmo (1970, 1971), Leland (1972), Cheng *et al.* (1987), Feder (1977), Meyer and Ormiston (1985), and Kim *et al.* (2005). Sandmo (1971) has specifically conducted a systematic study of the theory of the firm under price uncertainty and risk aversion. He has argued that the uncertainty effect on the optimal output level of risk-averse firm is negative. Dardanoni (1988) has introduced a unified framework for the analysis

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of two-argument utility function with a random budget constraint and presented the effect of uncertainty on an agent under plausible normality conditions. His analysis showed that the optimal level of the choice variable increases for mean-preserving increases in risk in the distribution of the random parameter if the absolute risk aversion increases. His results are applied in deriving the rules for efficient taxation under uncertainty.

Previous studies on the duopoly model under uncertainty have assumed that each firm is risk-neutral and can share or exchange its information on market uncertainty with its rival. Examples of such studies include those of Bresnahan (1981), Clarke (1983a, 1983b), Gal-Or (1986), Kirby (1988), Li (1985), Novshek and Sonnenschein (1982), Sakai (1990, 1991), Sakai and Yamato (1989), Shapiro (1986), and Vives (1984). They have investigated how market uncertainty with either unknown market demand or unknown constant marginal cost affects firms' behavior.

This paper introduces cost uncertainty in a simple Cournot duopoly model and extends previous analyses in two directions. First, we assume that there is no information sharing or exchange between firms. Each firm does not know the other firm's cost function at the time the output is produced. In a Cournot duopoly game, Firm i's optimal output choice depends on its own cost function and Firm j's output choice, which also depends on its cost function and Firm i's output choice (this type of relationship goes on). This implies that, in equilibrium, each firm's optimal output depends on the other firm's cost function as well as its own cost function. Therefore, with uncertainty, each firm has to use its beliefs on the other firm's cost function, and these beliefs can be summarized in its own subjective probability distribution. Second, we assume that firms are risk-averse. We show that under risk neutrality, the well-known Cournot equilibrium varies based on the assumption on the objective functions of the firms. We also demonstrate that the expected utility maximizing firms under risk aversion show different behaviors from the expected profit maximizing firms. We adopt the Sandmo's approach (1971) to compare the uncertainty case with the certainty one. The equilibrium output under cost function uncertainty is lesser or greater than one under certainty, in which it depends on each firm's conjecture about the other firm's output level.

This paper is organized into sections. Section 2 presents our basic duopoly model and analyzes the expected profit maximizing behaviors in it. Section 3 investigates the expected utility maximizing behaviors and compares the results with the certainty case. Finally, Section 4 discusses the concluding remarks.

II. Basic Model

We consider a simple Cournot duopoly model in which two firms producing identical products face a market demand. The inverse demand function is derived through $P=a-q_i-q_j$, where q_i and q_j denote the amount of output produced by Firms i and j, respectively, the demand intercept is a, and the slope of market demand is one for simplicity. We assume that technology exhibits constant returns to scale, so that Firm i has constant unit cost c_i with $c_i>0$. We briefly repeat the well-known results under certainty in which the cost functions are exactly known to each other, for comparison with later results under cost uncertainty.

The profit of Firm i is described as

$$\pi_i = (a - q_i - q_j - c_i)q_i \text{, for } i, j=1, 2 \quad i\neq j. \tag{1}$$

The first-order condition for profit maximization through (1) by Firm i is

$$\partial \pi_i / \partial q_i = a - 2q_i - q_j - c_i = 0. \tag{2}$$

The second-order condition is always satisfied because

$$\partial^2 \pi_i / \partial q_i^2 = -2 < 0. \tag{3}$$

From Equation (2), the optimal reaction function for Firm i is derived through

$$q_i = \frac{1}{2}(a - c_i - q_j). \tag{4}$$

Therefore, the unique Cournot equilibrium for Firm i, q_i^c is derived through:

$$q_i^c = \frac{1}{3}(a - 2c_i + c_j)\ , \text{ for } i, j = 1,\ 2 \quad i \neq j. \tag{5}$$

We then introduce the uncertainty through randomness in the cost function. Each firm knows exactly its own constant marginal cost but has partial information only on the other firm's marginal cost. That is, each firm has its subjective probability distribution on the other firm's constant marginal cost.

We consider the symmetric uncertainty in which both firms have uncertain information on the other firm's marginal cost. Before proceeding, we define the random variables to distinguish them from the non-random variables in the following way:

$$q_j|\Omega_i,\ c_j|\Omega_i, \text{ and } \pi_i|\Omega_j, \text{ for } i, j = 1,\ 2\ i \neq j,$$

where Ω_i is the set of information available to Firm i.

III. Expected Profit Maximizing Behavior (Symmetric Uncertainty)

Each firm is risk-neutral and chooses its output to maximize its expected profit based on its private information. Firm i's expected profit given its private information, Ω_i, is derived through:

$$\begin{aligned} E(\pi_i \big| \Omega_i) &= E\{(a - q_i - q_j - c_i \big| \Omega_i) q_i\} \\ &= \{a - q_i - c_i - E(q_j \big| \Omega_i)\} q_i \text{ for } i, j = 1,\ 2\ i \neq j, \end{aligned} \tag{6}$$

where E is the expectations operator. The first-order condition of (6) is

$$\partial E(\pi_i \big| \Omega_i) / \partial q_i = a - 2q_i - c_i - E(q_j \big| \Omega_i) = 0. \tag{7}$$

The second-order condition is always satisfied because

$$\partial^2 E(\pi_i \big| \Omega_i) / \partial q_i^2 = -2 < 0. \tag{8}$$

The optimal reaction functions from the first-order conditions for both firms are derived through:

$$q_1 = \frac{1}{2}\left\{a - c_1 - E(q_2 \mid \Omega_1)\right\} \tag{9}$$

and

$$q_2 = \frac{1}{2}\left\{a - c_2 - E(q_1 \mid \Omega_2)\right\}. \tag{10}$$

The unique Cournot equilibrium is derived from solving Equations (9) and (10) simultaneously. However, the equilibrium can vary, depending on each firm's expectation, $E(q_i \mid \Omega_j)$. Firm 1 can only expect Firm 2's reaction with its private information through

$$\begin{aligned} E(q_2 \mid \Omega_1) &= E\left\{\frac{1}{2}\left[a - c_2 - E(q_1 \mid \Omega_2)\right] \mid \Omega_1\right\} \\ &= \frac{1}{2}\left\{a - E(c_2 \mid \Omega_1) - E\left[E(q_1 \mid \Omega_2)\right] \mid \Omega_1\right\}. \end{aligned} \tag{11}$$

If we assume that both firms have their private information, Firm 1's expectation on Firm 2's reaction, $E(q_2 \mid \Omega_1)$, is located between the upper bound $[E(q_2 \mid \Omega_1)]^u$ and the lower bound $[E(q_2 \mid \Omega_1)]^\ell$. Therefore, Firm 1's optimal output, q_1^*, is

$$q_1^{\min} \le q_1^* \le q_1^{\max}.$$

Similarly, Firm 2's expectation on Firm 1's reaction, $E(q_1 \mid \Omega_2)$, is located between $[E(q_1 \mid \Omega_2)]^u$, and $[E(q_1 \mid \Omega_2)]^\ell$, and its optimal output, q_2^*, is

$$q_2^{\min} \le q_2^* \le q_2^{\max}.$$

Therefore, the possible Cournot equilibrium for this uncertainty game is shown in Figure 1. The point A represents an equilibrium point with $(q_1^c,\ q_2^c)$ being the pair of equilibrium output strategies if both firms have perfect information on the constant marginal cost of their rivals. The equilibrium is shown in the box of dashed lines if both firms face cost uncertainty. There are multiple equilibrium output strategies for

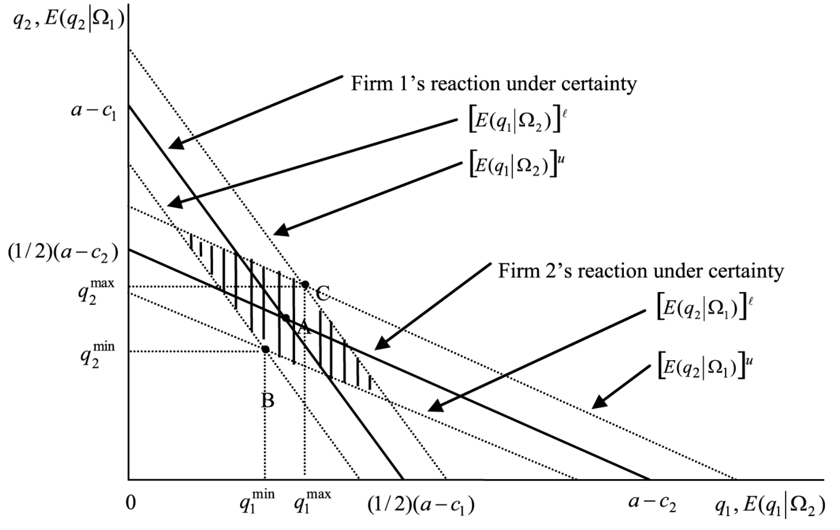

FIGURE 1
COURNOT DUOPOLY EQUILIBRIA UNDER SYMMETRIC UNCERTAINTY

two firms in this case.

Let us then look at how the Cournot equilibrium can be derived. Consider Equations (9) and (10), which have four unknowns, q_1, q_2, $E(q_2|\Omega_1)$, and $E(q_1|\Omega_2)$. We can solve the problem through an iterative substitution method.

$$q_i = \left(\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\cdots\right)a-\left\{\frac{1}{2}c_i+\frac{1}{2^3}E\left[E(c_i|\Omega_j)\ |\Omega_i+\cdots\right]\right\}$$
$$+\left\{\frac{1}{2^2}E(c_j|\Omega_i)+\frac{1}{2^4}E\left\{E\left[E(c_j|\Omega_i)\ |\Omega_j\right]\ |\Omega_i\right\}+\cdots\right\}. \tag{12}$$

We assume that the information set of each firm is exclusive to each other to simplify the expression:

***Assumption* 1:** Let Firm j's expectation on c_i be written as, $E(c_i|\Omega_j)=c_i+b_i$, where b_i is assumed to be a small (relative to c_i) non-systematic bias. This implies that Firm i's expectation on Firm j's expectational bias, b_i, is zero: $E(b_i|\Omega_i)=0$. Then

$$E[E(c_i|\Omega_j)|\Omega_i] = E[(c_i + b_i)|\Omega_i] = c_i + E(b_i|\Omega_i) = c_i$$
$$E\{\cdots E[E(c_i|\Omega_j)|\Omega_i]\cdots|\Omega_i\} = E[E(c_i|\Omega_j)|\Omega_i] = c_i$$
$$E\{\cdots E[E(c_i|\Omega_j)|\Omega_i]\cdots|\Omega_j\} = E(c_i|\Omega_j) = c_i + b_i.$$

The equilibrium output, using Assumption 1 and Equation (12), is derived through

$$q_i^* = \frac{1}{3}\left[a - 2c_i + (c_j + b_j)\right]. \tag{13}$$

Let the maximum and minimum values for the bias, b_j be denoted by b_j^u and b_j^ℓ, respectively, so the equilibrium output for firm i, q_i^*, is resolved as

$$q_i^{\min} \leq q_i^* \leq q_i^{\max},$$

where $q_i^{\min} = 1/3[a - 2c_i + (c_j + b_j^\ell)] = q_i^c + (1/3)b_j^\ell$, $q_i^{\max} = q_i^c + (1/3)b_j^u$, and q_i^c means the Cournot equilibrium output of Firm i without cost uncertainty. This implies that if Firm i's expectation on c_j is greater (smaller) than real c_j, then Firm i's equilibrium output, q_i^*, is greater (smaller) than the certainty equilibrium output, q_i^c. Firm j's expectation on Firm i's reaction is also bounded as

$$\left[E(q_i|\Omega_j)\right]^\ell \leq E(q_i|\Omega_j) \leq \left[E(q_i|\Omega_j)\right]^u, \tag{14}$$

where $\left[E(q_i|\Omega_j)\right]^\ell = \frac{1}{2}\left[a - (c_i + b_i^\ell) - q_j\right]$ and $\left[E(q_i|\Omega_j)\right]^u = \frac{1}{2}\left[a - (c_i + b_i^u) - q_j\right]$.

IV. Expected Utility Maximizing Behavior (Symmetric Uncertainty)

We briefly introduce the Sandmo's approach before proceeding. Sandmo (1971) has studied the theory of the competitive firm under price uncertainty and risk aversion. The expected utility of profits can be written as

$$E\left[u(\pi)\right] = E\left[u(px - C(x) - B)\right], \tag{15}$$

where p is the price of output and a random variable with the expected value $E(p)=\mu$, the variable cost $C(x)$ and the fixed cost B.

The first-order condition of Equation (15) is

$$E\left[u'(\pi)(p-C'(x))\right]=0. \tag{16}$$

$E(p)=\mu$ is the price under certainty to compare with the level of output under certainty. Equation (16) can be written as

$$E\left[u'(\pi)p\right]=E\left[u'(\pi)C'(x)\right]. \tag{17}$$

Subtracting $E[u'(\pi)\mu]$ from each side of Equation (17), we derive

$$E\left[u'(\pi)(p-\mu)\right]=E\left[u'(\pi)(C'(x)-\mu)\right]. \tag{18}$$

We derive that $\pi=E[\pi]+(p-\mu)x$ because $E[\pi]=\mu x-C(x)-B$. Clearly,

$$u'(\pi)\leq u'(E\left[\pi\right]) \quad \text{if} \quad p\geq\mu. \tag{19}$$

It immediately follows that

$$u'(\pi)(p-\mu)\leq u'(E\left[\pi\right])(p-\mu). \tag{20}$$

This inequality holds for all p. The inequality sign of Equation (19) is reversed for $p\leq\mu$. However, multiplying both sides of the inequality by $(p-\mu)$ will still hold the inequality of (20). Taking expectations on both sides of (20) and noting that $E[u'(\pi)]$ is a given number, we obtain

$$E\left[u'(\pi)(p-\mu)\right]\leq u'(E\left[\pi\right])E(p-\mu).$$

However, the right-hand side is equal to zero by definition; hence, the left-hand side is negative. We then know that the right-hand side of (18) is also negative. Meanwhile, Equation (18) can be written as

$$E\left[u'(\pi)\right](C'(x)-\mu)\leq 0,$$

and this implies

$$C'(x) \le \mu \tag{21}$$

because marginal utility is always positive. Therefore, output under price uncertainty is smaller than the certainty output.

We now assume that both firms are risk-averse ($u'>0$ and $u''<0$). The objective function of the expected utility maximizing Firm i is

$$E\left[u_i(\pi_i \,|\, \Omega_i)\right] = E\left\{u_i\left[(a - q_i - q_j - c_i \,|\, \Omega_i)\, q_i\right]\right\}. \tag{22}$$

The first- and second-order conditions of (22) can be written as

$$E\left[u_i'(\pi_i \,\big|\, \Omega_i)\left(a - 2q_i - c_i - (q_j \,|\, \Omega_i)\right)\right] = 0 \tag{23}$$

and

$$E\{u_i''(\pi_i \,|\, \Omega_i)\left[a - 2q_i - c_i - (q_j \,|\, \Omega_i)\right]^2 - 2u_i'(\pi_i \,|\, \Omega_i)\} < 0 \tag{24}$$

because the second-order condition is always satisfied for the risk-averse firms, $u_i'(\cdot)>0$ and $u_i''<0$.

$E[u_i'(\pi_i | \Omega_i)(a-2q_i-c_i)]=E[u_i'(\pi_i | \Omega_i)(q_j | \Omega_i)]$ is derived from the first-order condition (23).

Subtracting $E[u_i'(\pi_i | \Omega_i)E(q_j | \Omega_i)]$ from both sides, we then derive

$$E\left[u_i'(\pi_i \,\big|\, \Omega_i)[a - 2q_i - c_i - E(q_j \,|\, \Omega_i)]\right] = E\{u_i'(\pi_i \,\big|\, \Omega_i)[q_j \,\big|\Omega_i - E(q_j \,|\, \Omega_i)]\}. \tag{25}$$

We know that:

$$\pi_i \,\big|\, \Omega_i = (a - q_i - c_i - q_j \,|\, \Omega_i) q_i \quad \text{and} \quad E(\pi_i \,\big|\, \Omega_i) = \left[a - q_i - c_i - E(q_j \,|\, \Omega_i)\right] q_i .$$

Therefore,

$$\pi_i \,\big|\, \Omega_i - E(\pi_i \,\big|\, \Omega_i) = -\left[q_j \,\big|\, \Omega_i - E(q_j \,|\, \Omega_i)\right] q_i .$$

Clearly, it follows that

$$u_i'(\pi_i \big| \Omega_i) \geq u_i'\left[E(\pi_i \,|\, \Omega_i)\right], \text{ if } q_j \big| \Omega_i \geq E(q_j \,|\, \Omega_i).$$

The following inequality now holds for all $q_j \,|\, \Omega_i$,

$$u_i'(\pi_i \big| \Omega_i)\left[q_j \big| \Omega_i - E(q_j \,|\, \Omega_i)\right] \geq u_i'\left[E(\pi_i \,|\, \Omega_i)\right]\left[q_j \big| \Omega_i - E(q_j \,|\, \Omega_i)\right].$$

Taking expectations on both sides, we derive

$$\begin{aligned} E\{u_i'(\pi_i \big| \Omega_i)[q_j \big| \Omega_i - E(q_j \big| \Omega_i)]\} &\geq E\{u_i'[E(\pi_i \,|\, \Omega_i)][q_j \big| \Omega_i - E(q_j \big| \Omega_i)]\} \\ &= u_i'[E(\pi_i \,|\, \Omega_i)][E(q_j \big| \Omega_i) - E(q_j \,|\, \Omega_i)] = 0. \end{aligned} \tag{26}$$

From Equations (25) and (26), we derive

$$\begin{aligned} &E\{u_i'(\pi_i \big| \Omega_i)[a - 2q_i - c_i - E(q_j \big| \Omega_i)]\} \\ &\quad = [a - 2q_i - c_i - E(q_j \big| \Omega_i)]E[u_i'(\pi_i \big| \Omega_i)] \geq 0. \end{aligned} \tag{27}$$

This implies that the following inequalities must be satisfied for optimal decision:

$$q_1 \leq \frac{1}{2}\left[a - c_1 - E(q_2 \big| \Omega_1)\right] \tag{28}$$

and

$$q_2 \leq \frac{1}{2}\left[a - c_2 - E(q_1 \big| \Omega_2)\right]. \tag{29}$$

The present model shows that each firm does in fact react to the other in a way that depends on conjectures based on its information. Let r_{ij} be Firm i's response to the change in Firm j's output level. r_{ij} is equal to $-(1/2)$ under certainty but r_{ij} is equal to and less than $-(1/2)$ under cost uncertainty. This means that if Firm j reduces its output by

1 unit, Firm i increases its output by equal to and less than 1/2 unit. This makes Firm i's reaction function to move inward by k_i (i=1, 2) under cost uncertainty (compared with certainty situation). Let k_i be the amount of output, q_i which is reduced under cost uncertainty by Firm i.

We follow the Sandmo's approach (1971) and the certainty equilibrium output, q_1^c, can be compared with the uncertainty equilibrium output, q_1^*, when $b_i=b_j=0$. Equations (28) and (29), if $b_i=b_j=0$, become

$$q_1 = \frac{1}{2}(a - c_1 - q_2) - k_1 \tag{30}$$

$$q_2 = \frac{1}{2}(a - c_2 - q_1) - k_2. \tag{31}$$

Equations (30) and (31) show that reaction curves of both firms under cost uncertainty are inside their certainty reaction curves.

Solving (30) and (31), the equilibrium outputs under cost uncertainty are

$$q_1^* = q_1^c + \frac{2}{3}k_2 - \frac{4}{3}k_1 \tag{32}$$

$$q_2^* = q_2^c + \frac{2}{3}k_1 - \frac{4}{3}k_2, \tag{33}$$

where q_1^c is the Cournot equilibrium output under certainty.

Let k_j/k_i be the relative size of the amount of output from both firms (reduced under cost uncertainty). The values of k's measure the difference of the firms' reaction between uncertainty and certainty. Uncertainty in the cost function is generated by random elements, such as uncertainty delivery and timing of inputs, uncertain prices of input factors, and uncertain technological relationship between input and output. Therefore, these risk factors affect the ratio.

We derive three possible cases, when compared with the results under certainty situation (Figure 2).

The following theorem is derived after solving (32) and (33).

Theorem

Supposing that both firms are risk-averse, there exist three Cournot

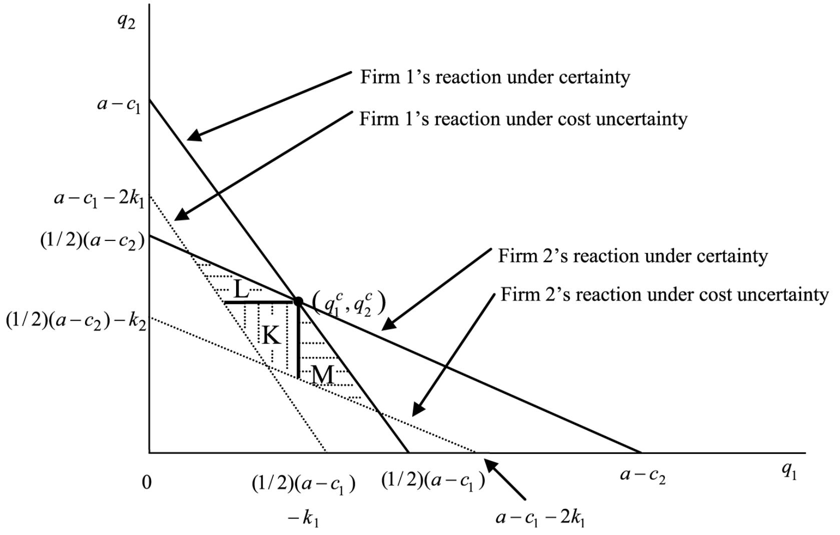

FIGURE 2
COURNOT DUOPOLY EQUILIBRIA UNDER RISK AVERSION

equilibrium cases under cost uncertainty:

Case 1: If $\frac{1}{2} \le \frac{k_2}{k_1} \le 2$, then $q_1^* \le q_1^c$ and $q_2^* \le q_2^c$

(representing Area K in Figure 2).

Case 2: If $0 \le \frac{k_2}{k_1} \le \frac{1}{2}$, then $q_1^* < q_1^c$ and $q_2^* > q_2^c$

(representing Area L in Figure 2).

Case 3: If $\frac{k_2}{k_1} > 2$, then $q_1^* > q_1^c$ and $q_2^* < q_2^c$

(representing Area M in Figure 2).

The equilibrium output level under cost uncertainty depends on the relative size of k_1 and k_2, that is, the interactive behavior of both firms. The followings are observed if the results are compared with the results under certainty: (i) both firms reduce their optimal level of output in Case 1; (ii) Firm 2 increases its optimal output whereas Firm 1 reduces its optimal output in Case 2; and (iii) Firm 2 decreases its optimal output whereas Firm 1 increases its optimal output in Case 3.[1]

V. Concluding Remarks

This paper presents a Cournot duopoly model based on a situation when firms are facing cost uncertainty under risk neutrality and risk aversion, and compares the results with certainty case. The Cournot duopoly model shows that the expected utility maximizing firms show different behaviors from the expected profit maximizing firms under cost uncertainty. A comparison with the results under certainty shows that each risk-averse firm can increase or decrease its output, which depends on the interaction between both firms under cost uncertainty.

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References

Bresnahan, T. F. "Duopoly Models with Consistent Conjectures." *American Economic Review* 71 (No. 5 1981): 934-45.

Cheng, H. C., Magill, M., and Shafer, W. "Some Results on Comparative Statics under Uncertainty." *International Economic Review* 28 (No. 2 1987): 493-507.

Clarke, R. N. "Collusion and the Incentives for Information Sharing." *Bell Journal of Economics* 14 (No. 2 1983a): 383-94.

_______. "Duopolist Don't Wish to Share Information." *Economics Letters* 11 (Nos. 1-2 1983b): 33-6.

Dardanoni, V. "Optimal Choices under Uncertainty: the Case of Two-Argument Utility Functions." *Economic Journal* 98 (No. 391 1988):

[1] We need a strong assumption (for example, that a risk-neutral Firm i's expectation on a risk-neutral Firm j's reaction, $E(q_j|\Omega_i)$ is the same as a risk-averse Firm i's expectation on a risk-averse Firm j's reaction), so we can compare a risk-neutral firm's behavior with a risk-averse firm's behavior under symmetric uncertainty. That is, b_j under risk neutrality is the same as that under risk aversion. The result in this case is similar to that of comparing a risk-averse firm under certainty with a risk-averse firm under uncertainty. Note that we cannot compare a risk-neutral firm's behavior with a risk-averse firm's behavior under symmetric uncertainty if a risk-neutral Firm i's expectation on a risk-neutral Firm j's reaction, $E(q_j|\Omega_i)$, is different from a risk-averse Firm i's expectation on a risk-averse Firm j's reaction.

429-50.
Feder, G. "The Impact of Uncertainty in a Class of Objective Functions." *Journal of Economic Theory* 16 (No. 2 1977): 504-12.
Gal-Or, E. "Information Transmission-Cournot and Bertrand Equilibria." *Review of Economic Studies* 53 (No. 1 1986): 85-92.
Kim, Soojong, Kim, Iltae, and Ryu, Suyeol. "Left-side Relatively Strong Increases in Risk and Their Comparative Statics." *Seoul Journal of Economics* 18 (No. 1 2005): 45-57.
Kirby, A. "Trade Associations as Information Exchange Mechanism." *The RAND Journal of Economics* 19 (No. 1 1988): 138-46.
Leland, H. E. "The Theory of the Firm Facing Uncertain Demand." *American Economic Review* 62 (No. 3 1972): 278-91.
Li, L. "Cournot Oligopoly with Information Sharing." The RAND *Journal of Economics* 16 (No. 4 1985): 521-36.
Meyer, J., and Ormiston, M. "Strong Increase in Risk and Their Comparative Statics." *International Economic Review* 26 (No. 2 1985): 425-37.
Novshek, W., and Sonnenschein, H. "Fulfilled Expectations Cournot Duopoly with Information Acquisition and Release." *Bell Journal of Economics* 13 (No. 1 1982): 214-18.
Sakai, Y. "Information Sharing in Oligopoly: Overview and Evaluation. Part I. Alternative Models with a Common Risk." *Keio Economic Studies* 27 (No. 2 1990): 17-41.
_______. "Information Sharing in Oligopoly: Overview and Evaluation. Part II. Private Risks and Oligopoly Models." *Keio Economic Studies* 28 (No. 1 1991): 51-71.
Sakai, Y., and Yamato, T. "Oligopoly, Information and Welfare." *Journal of Economics* 49 (No. 1 1989): 3-24.
Sandmo, A. "The Effect of Uncertainty on Saving Decisions." *Review of Economic Studies* 37 (No. 3 1970): 353-60.
_______. "On the Theory of the Competitive Firm under Price Uncertainty." *American Economic Review* 61 (No. 1 1971): 65-73.
Shapiro, C. "Exchange of Cost Information in Oligopoly." *Review of Economic Studies* 53 (No. 3 1986): 433-46.
Vives, X. "Duopoly Information Equilibrium: Cournot and Bertrand." *Journal of Economic Theory* 34 (No. 1 1984): 71-94.