Survival, Arbitrage, and Equilibrium with Financial Derivatives in Constrained Asset Markets

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Survival conditions ensure the presence of consumptions that cost less than the total contingent income of agents in general equilibrium models. These conditions are generally fulfilled in competitive equilibrium. This paper shows the existence of equilibrium for incomplete-market economies where individuals' asset holdings are subject to portfolio constraints by introducing a new survival condition. Based on McKenzie's irreducibility assumption, Gottardi and Hens (1996) provide the GEI irreducibility condition for the existence of equilibrium in unconstrained asset markets. The GEI irreducibility condition, however, leaves no room for redundant assets such as financial derivatives simply because they do not contribute to the creation of risk-sharing opportunities in unconstrained asset markets. Thus, such condition is no longer valid in constrained asset markets where redundant assets are empowered to affect the financial ability of agents to possess 'cheaper' consumptions in equilibrium. This paper extends the irreducibility assumption of Gottardi and Hens (1996) to constrained asset markets by considering the capability of financial derivatives to create intertemporal income transfers.

Keywords: Equilibrium, Portfolio constraints, Redundant assets, Incomplete markets, Arbitrage, Irreducibility, Survival conditions

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I. Introduction

Survival conditions ensure the presence of consumptions that cost less than the total (contingent) income of agents in general equilibrium models. These conditions are generally fulfilled in competitive equilibrium and thus, are needed for the existence of equilibrium. Discussions about survival conditions are mainly associated with the Arrow-Debreu general equilibrium model, where a complete system of contingent claims markets is available. Survival conditions are rarely studied in the general equilibrium model with incomplete markets (GEI model) or with constrained incomplete markets (GEIC model). The GEI and GEIC models differ from the Arrow-Debreu model in that, when a full set of contingent claims markets are unavailable at the time of decision making, contingent plans for consumptions and portfolio holdings are to be rescheduled sequentially as the state of nature is resolved over time. Thus, the full-fledged body of survival conditions for the Arrow-Debreu model is not carried over to the GEI or GEIC models simply because such conditions do not consider the risk-sharing role of asset markets over the investment horizon.

This paper shows the existence of equilibrium for the GEIC model where individuals' asset holdings are subject to portfolio constraints by introducing a new survival condition. The survival condition is an extension of McKenzie's (1959) irreducibility assumption to the GEIC economies. Gottardi and Hens (1996) is the first major attempt to embody the idea of McKenzie's irreducibility assumption in the GEI model. The GEI irreducibility condition of Gottardi and Hens (1996) integrates the capacity of existing asset markets to create intertemporal income transfers into the classical irreducibility condition. Given that Gottardi and Hens (1996) is limited to the case with unconstrained asset markets, the GEI irreducibility condition leaves no room for redundant assets such as financial derivatives simply because they do not contribute to the creation of risk-sharing opportunities in unconstrained asset markets. Thus, such condition is no longer valid in constrained asset markets where redundant assets are empowered to affect the financial ability of agents to possess 'cheaper' consumptions in equilibrium. To our best knowledge, this paper is the first attempt to extend McKenzie's irreducibility condition to GEIC economies. As illustrated in Hahn and Won (2008), competitive equilibrium fails to exist in constrained asset markets with redundant assets although strictly positive income is warranted in each contingency of the second period (strong form of survival condition). The strong form of survival condition, which is sufficient for unconstrained asset markets to have equilibrium, does not guarantee the existence of equilibrium in asset markets where portfolio restrictions are too tight to generate positive income transfers to the second period. This paper formulates the GEIC irreducibility condition that accounts for the manner in which financial derivatives are held to meet the intertemporal need for income transfers under portfolio constraints.

The remainder of the paper is organized as follows: Section II provides a description of the economy and discusses assumptions that are required for the existence of competitive equilibrium. The GEIC survival condition is presented in Section III. We exemplify that equilibrium may not exist only because the GEIC economy fails to meet the survival condition. Section IV provides the primary result of the paper by introducing a sequence of GEIC economies that satisfy strong survival conditions for both goods and assets markets. Such economies produce a sequence of competitive equilibria that subsequently converge to a quasiequilibrium of the original GEIC economy. The quasiequilibrium becomes an equilibrium of the economy as it satisfies the survival condition. Concluding remarks are made in Section V.

II. The Model

A two-period economy is considered where asset markets are open in the first period (denoted by 0) and markets for consumption goods are open in the second period (denoted by 1). The uncertainty is described by the set of states of nature in period 1, denoted by $\mathcal{S}=\{1, \ldots, S\}$. In period 0, agents are unaware of which state will occur in period 1. Assets deliver monetary payoffs contingent on the states of the second period.[1] In each state $s\in\mathcal{S}$, L consumption goods exist, the set of which is denoted by $\mathcal{L}=\{1, 2, \ldots, L\}$. Considering that consumption is available only in the second period, the number of commodities equals $\ell := LS$. The Euclidean space $\mathbb{R}^{\ell}$ is taken as the commodity space of the economy.

Let $\mathcal{I}=\{1, 2, \ldots, I\}$ denote the set of agents, $\mathcal{J}=\{1, 2, \ldots, J\}$ the set of

[1] Instead, we can assume that assets pay units of the numeraire good because nominal assets can be converted into real assets and *vice versa*. For details, see Magill and Shafer (1991).

financial assets. Each agent $i \in I$ chooses consumption x_i from the consumption set $X_i \subset \mathbb{R}^{\ell}$, which contains the initial endowment e_i of consumption goods. Agent i's preferences over X_i are represented by a preference relation $\succ_i$ on X_i. Let us define correspondence $P_i : X_i \to 2^{X_i}$ by $P_i(x_i) = \{x_i' \in X_i : x_i' \succ_i x_i\}$, the set of consumption bundles that agent i prefers to x_i. In the real world, asset markets face market frictions such as short-selling constraints, bid-ask spreads, or proportional transaction costs (see Luttmer 1996). We assume that market frictions constrain agent i to choose portfolios in the set Θ_i in $\mathbb{R}^J$, and every agent has an initial endowment $0 \in \Theta_i$ of financial assets.

The sets $X := \Pi_{i \in I} X_i$ and $\Theta := \Pi_{i \in I} \Theta_i$ indicate the set of consumption and portfolio allocations, respectively. Let A_X and A_Θ denote the set of market-clearing consumption and portfolio allocations, respectively, that is,

$$A_X = \{x \in X : \Sigma_{i \in I} x_i = \Sigma_{i \in I} e_i\} \text{ and } A_\Theta = \{\theta \in \Theta : \Sigma_{i \in I} \theta_i = 0\}.$$

For each $i \in I$, let $\hat{X}_i$ denote the projection of A_X on X_i. We set $A = A_X \times A_\Theta$ and $\hat{X} = \Pi_{i \in I} \hat{X}_i$.

Each asset $j \in J$ pays $r_j(s)$ in state s. The payoff of J assets in state s is given by the J-dimensional row vector $r(s) = (r_j(s))_{j \in J}$, whereas that of asset j in S states is given by a S-dimensional column vector $r_j = (r_j(s))_{s \in S}$. The asset (payoff) structure is described by the $S \times J$ matrix $R = [(r(s))_{s \in S}]$, where either $S \geq J$ or $S < J$ may hold. In particular, redundant assets exist when J is greater than the rank of R. Let $p \in \mathbb{R}^{\ell}$ denote the commodity price vector and $q \in \mathbb{R}^J$ denote the asset price vector. For a price pair $(p, q) \in \mathbb{R}^{\ell} \times \mathbb{R}^J$, we introduce the following notation:

$$p \,\square\, (x_i - e_i) := \begin{bmatrix} 0 \\ p \,\square_1 (x_i - e_i) \end{bmatrix}, \quad W(q) := \begin{bmatrix} -q \\ R \end{bmatrix},$$

where $p \,\square_1 (x_i - e_i)$ indicates the S-dimensional column vector $(p(s) \cdot (x_i(s) - e_i(s)))_{s \in S}$. Notably, the zero in $p \,\square\, (x_i - e_i)$ indicates the fact that no consumption arises in the first period. The open budget correspondence $\mathcal{B}_i : \mathbb{R}^{\ell} \times \mathbb{R}^J \to 2^{X_i \times \Theta_i}$ of agent i is defined by

$$\mathcal{B}_i(p, q) := \{(x_i, \theta_i) \in X_i \times \Theta_i : p \,\square\, (x_i - e_i) \ll W(q) \cdot \theta_i\}^2$$

[2] For two vectors v and v' in an Euclidean space, $v \geq v'$ implies that $v - v' \in \mathbb{R}^{\ell}_{+}$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbb{R}^{\ell}_{++}$.

and the budget set is defined by the correspondence $cl\mathcal{B}_i : \mathbb{R}^\ell \times \mathbb{R}^J \to 2^{X_i \times \Theta_i}$, which takes the value $cl\mathcal{B}_i(p, q) := cl[\mathcal{B}_i(p, q)]$ at a price pair (p, q).[3] For a given price pair $(p, q) \in \mathbb{R}^\ell_+ \times \mathbb{R}^J$, agent $i \in I$ chooses a $\succ_i$-maximal element (x_i, θ_i) in $cl\mathcal{B}_i(p, q)$. An element (x_i, θ_i) is $\succ_i$-maximal in $cl\mathcal{B}_i(p, q)$ if x_i satisfies $(P_i(x_i) \times \Theta_i) \cap cl\mathcal{B}_i(p, q) = \varnothing$. Let $\mathcal{E} = \langle (X_i, \succ_i, e_i, \Theta_i)_{i \in I}, R \rangle$ denote the economy described above.

DEFINITION 2.1: A *competitive equilibrium* of economy $\mathcal{E}$ is a profile $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^J \times X \times \Theta$ such that

(i) $(x_i^*, \theta_i^*) \in cl\mathcal{B}_i(p^*, q^*)$, $\forall i \in I$,
(ii) $(P_i(x_i^*) \times \Theta_i) \cap cl\mathcal{B}_i(p^*, q^*) = \varnothing$, $\forall i \in I$,
(iii) $(x^*, \theta^*) \in A$.

We make the following assumptions for every $i \in I$.

(A1) X_i is closed, convex, and bounded from below in $\mathbb{R}^\ell$, and $0 \in X_i$.
(A2) $\succ_i$ is irreflexive, complete, and transitive on X_i.
(A3) $\succ_i$ is continuous and convex on X_i.[4]
(A4) $\succ_i$ is state-wise locally nonsatiated (*i.e.*, $\forall x_i \in \hat{X}_i$, $x_i \in \partial P_i(x_i, s)$, $\forall s \in S$).[5]
(A5) $e_i \in X_i$.
(A6) Θ_i is a closed convex set in $\mathbb{R}^J$ with $0 \in \Theta_i$.

To discuss the effect of redundant assets on risk-sharing in constrained asset markets, let $V = \text{span}\{r(1), r(2), \ldots, r(S)\}$ and $V^\perp = \{\theta \in \mathbb{R}^J : R \cdot \theta = 0\}$. Redundant assets exist if and only if the rank of the payoff matrix R is less than J, *i.e.*, $V^\perp \neq \{0\}$. Portfolios in $V^\perp$ are called a *zero-income portfolio*, which generates zero income transfer in each state of the second period. A portfolio θ in $\mathbb{R}^J$ has the direct sum $\hat{\theta} + \tilde{\theta}$ where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^\perp$. The portfolio $\tilde{\theta}$ does not change the size of income transfers but may account for the feasibility of θ under the portfolio constraints. Let C_i be the recession cone of Θ_i and N_i be the lineality space of Θ_i for each $i \in I$.[6] We set $N = \Sigma_{i \in I}(N_i \cap V^\perp)$ and denote by $N^\perp$ the orthogonal

[3] Let A be a nonempty subset of an Euclidean space. We denote the closure of A by clA, the interior of A by $intA$, and the boundary of A by ∂A.

[4] Recall that $\succ_i$ is continuous if $P_i(x_i)$ and $P^{-1}(x_i)$ are open for every $x_i \in X_i$ and is convex if P_i is convex-valued.

[5] For each $s \in S$, let $P_i(x_i, s) := \{x_i' \in X_i : x_i' \in P_i(x_i), x_i'(-s) = x_i(-s)\}$, where $x(-s) = (x(1), \ldots, x(s-1), x(s+1), \ldots x(S))$.

[6] Let A be a nonempty convex subset of an Euclidean space. The recession

complement of N in $\mathbb{R}^J$.

The asset structure is required to satisfy the following condition:

(A7) For each $i \in I$, there is $v_i^\circ \in C_i$ with $R \cdot v_i^\circ \gg 0$.

Assumption (A7) states that there exists a portfolio in C_i that guarantees strict positive income transfers in each state. In unconstrained asset markets, Gottardi and Hens (1996) assume that there exists $v \in \mathbb{R}^J$ with $R \cdot v > 0$. This condition is slightly weaker than (A7). However, Gottardi and Hens (1996) introduce stronger assumptions on preferences and the initial endowments than (A4) and (A5). More specifically, they assume that preferences are numerically representable and strictly monotone, and e_i is in $\mathbb{R}^\ell_+$ for each i with $\Sigma_{i \in I} e_i \gg 0$. Here, preferences are locally nonsatiated in each state and X_i need not coincide with $\mathbb{R}^\ell_+$. A trade-off exists between conditions on preferences and on the capability of the asset structure to meet the need for survival in the second period. The trade-off taken in this work is more relevant for addressing preferences for which monotonicity is less demanding.

Asset markets admit no arbitrage in equilibrium. The following definition provides a notion of arbitrage for constrained incomplete markets:

DEFINITION 2.2: An asset price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for agent i in economy $\mathcal{E}$ if no $\theta_i \in C_i$ exists which satisfies $q \cdot \theta_i \leq 0$ and $R \cdot \theta_i > 0$. An asset price vector $q \in \mathbb{R}^J$ admits *no constrained arbitrage* for economy $\mathcal{E}$ if it admits no constrained arbitrage for every agent $i \in I$.

Let Q_i denote the set of asset prices that admit no constrained arbitrage for agent i. The set $Q = \cap_{i \in I} Q_i$ denotes the set of asset prices that admit no constrained arbitrage for economy $\mathcal{E}$. As shown in Proposition 3.1 of Hahn and Won (2011), under Assumptions (A4) and (A7), equilibrium asset prices belong to $Q \cap N^\perp$. We define the sets of normalized prices: $\Delta = \Delta_0 \times \Delta_1$ where $\Delta_0 = \{q \in clQ \cap N^\perp : \| q \| = 1\}$ and $\Delta_1 = \Pi_{s \in S} \Delta_s$ with $\Delta_s = \{p(s) \in \mathbb{R}^L : \| p(s) \| = 1\}$.[7]

cone of A is the set $\Gamma(A) = \{v \in E : A + v \subset A\}$ and the lineality space $\mathcal{L}(A)$ is the maximal linear subspace in A. When A is closed, $\Gamma(A)$ is also closed and can be expressed as $\Gamma(A) = \{v \in \mathbb{R}^m : \exists \{x^n\}$ in A and $\{a^n\}$ in $\mathbb{R}$ with $a^n \to 0$ such that $v = \lim_{n \to \infty} a^n x^n\}$.

[7] $\| \cdot \|$ is the Euclidean norm.

(A8) For every $(p, q) \in \Delta$, there is an agent i with $\theta_i \in \Theta_i$ which satisfies $q \cdot \theta_i < 0$ and $p \square_1 e_i + R \cdot \theta_i \geq 0$.

Assumption (A8) guarantees that asset markets enable at least one agent to transfer positive income from the first period to achieve non-negative total income in each state of the second period. This condition is assumed for unconstrained asset markets ($\Theta_i = \mathbb{R}^J$) in Gottardi and Hens (1996). The condition (A8) is always fulfilled in unconstrained asset markets with monotone preferences when $e_i \gg 0$ for some $i \in I$. However, this is not the case with constrained asset markets.

When redundant assets exist in constrained asset markets, the set of market-clearing and budget-feasible portfolio allocations need not be bounded. To manage this problem, the literature imposes additional conditions on portfolio constraints. For example, Siconolfi (1986) requires that $C_i \cap V^{\perp} = \{0\}$, $\forall i \in I$. This condition severely restricts the risk-sharing role of redundant assets. The following condition enables redundant assets to create richer risk-sharing opportunities than that of Siconolfi (1986):

(A9) If $v_i \in C_i \cap V^{\perp}$ for each $i \in I$ and $\Sigma_{i \in I}\, v_i = 0$, then $v_i \in N_i \cap V^{\perp}$ for all $i \in I$.

We note that $v_i \in C_i$ indicates a scale-free feasible portfolio in that any scale of v_i satisfies the portfolio constraint Θ_i. Thus, (A9) requires that if a zero-income portfolio v_i is scale-free feasible for each i and (v_i) is attainable, that is, $\Sigma_{i \in I} v_i = 0$, then both v_i and $-v_i$ be scale-free feasible. As will be shown later, portfolios in $N_i \cap V^{\perp}$ have the null price and thus have no impact on the set of income transfers between periods 0 and 1 in equilibrium. Thus, portfolios in the subspace generated by a set of scale-free feasible zero-income portfolios $\{v_i,\ i \in I\}$ with $\Sigma_{i \in I} v_i = 0$ will have a null value with no risk-sharing role in equilibrium. To compare (A9) with Siconolfi (1986), we take an example where S_1 indicates the price of a stock at maturity, whereas K denotes a strike price of the call and put options with the stock as the underlying asset. The following relation thus holds:

$$\max\{S_1 - K,\ 0\} - \max\{K - S_1,\ 0\} - S_1 + K = 0.$$

The relation is represented by the zero-income portfolio $v = (1, -1, -1, K)$ which represents long call, short put, short stock, and long K units of a risk-free bond. The assumption (A9) covers the case that some agent i

is allowed to take v as well as $-v$ in a scale-free manner, *i.e.*, $v \in N_i$. Let $q=(C_0, P_0, S_0, B_0)$ denote the equilibrium price vector. Either $q \cdot v<0$ or $q \cdot v>0$ would provide an arbitrage opportunity for agent i. Thus, $q \cdot v=0$, that is, the put-call parity holds in equilibrium. The case, however, violates the condition $C_i \cap V^{\perp}=\{0\}$ in Siconolfi (1986), which generally prevents the put-call parity. Assumption (A9) covers numerous interesting classes of portfolio constraints in the literature.[8]

III. Survival Condition for the GEIC Model

The general equilibrium literature provides the conventional wisdom that a competitive equilibrium can exist when every agent is allowed to have feasible consumptions cheaper than available total income in each state of the world. The survival condition for each agent is necessary for competitive equilibrium to exist in general. Total income in each state consists of both the contingent endowments and the income transfers attained by trading marketed assets in the GEI model. Gottardi and Hens (1996) were the first to attempt to adapt the classical irreducibility assumption to the GEI model by considering the capability of marketed assets to create intertemporal income transfers. They show that the irreducibility condition combined with (A8) enables every agent to possess cheaper consumptions in quasiequilibrium of unconstrained asset markets.

Given that Gottardi and Hens (1996) assume $\Theta_i=\mathbb{R}^J$ in their GEI irreducibility condition, such condition is not applicable to the case of asset markets with portfolio constraints that restrict the income-spanning behavior of the asset market structure. We build a survival condition which incorporates the constrained capability of marketed assets to create intertemporal income transfers by extending the GEI irreducibility condition to the GEIC model in the following way:

(A10) Let $\{I_1, I_2\}$ be any nontrivial partition of I and p be any price in Δ_1. Then for every $x \in A_X$ that admits $\theta_i \in \Theta_i$ such that $p \,\Box_1 (x_i - e_i) \leq R \cdot \theta_i$, $\forall i \in I$, there exist $z_i \in \mathbb{R}^{\ell}$ and $v_i \in C_i$ for each $i \in I$ such that

(i) $x_k + z_k \in P_k(x_k)$ for some $k \in I_1$, and $x_i + z_i \in clP_i(x_i)$ for each $i \in I_1 \backslash \{k\}$,
(ii) $p \,\Box_1 z_i = R \cdot v_i$ for each $i \in I$,
(iii) $e_i + z_i \in X_i$, $\forall i \in I_2$, and
(iv) $\Sigma_{i \in I} v_i = 0$.

[8] For review of the literature, see Hahn and Won (2011).

The condition (A10) is an extension of Gottardi and Hens (1996) to constrained asset markets. This extension reflects the need to consider the risk-sharing role of redundant assets. Two major divergences of (A10) from Gottardi and Hens (1996) include:

(D1) For each $i \in I$, the unconstrained portfolio set $\mathbb{R}^J$ is replaced by the portfolio constraint $\Theta_i \subset \mathbb{R}^J$ that satisfies (A9);
(D2) The goods-market redistribution condition $\Sigma_{i \in I} z_i = 0$ of Gottardi and Hens (1996) is replaced by the portfolio redistribution condition that $\upsilon_i \in C_i$ for all $i \in I$ and $\Sigma_{i \in I} \upsilon_i = 0$.

The portfolio redistribution condition of (A10) is more natural in the context of portfolio constraints that enable redundant assets to expand risk-sharing opportunities. Specifically, the goods-market redistribution condition alone does not induce the portfolio redistribution with redundant assets. This condition has distinct implications on the irreducibility conditions of constrained and unconstrained markets. It does not matter in unconstrained asset markets. To see this, consider a case in which the set of individual portfolios, which supports the goods-market redistribution, is aggregated into a nonzero portfolio. Given that the aggregate portfolio is in $V^{\perp}$ (this point is remarked below), it has null market value and, thus, facilitates no income transfers. Eventually, the individual portfolios can be reorganized into a redistribution of portfolios without changing the initial goods-market redistribution in unconstrained asset markets. Consequently, both redistribution conditions play the same role in the irreducibility condition for unconstrained asset markets. However, this case is not true for constrained asset markets. In constrained asset markets, redundant assets can be mispriced such that there exists a portfolio in $V^{\perp}$ that is feasible at the aggregate level and has positive market value. Then the nonzero aggregate portfolio in $V^{\perp}$ induced by the goods-market redistribution may have positive market value. In this case, the initially-taken goods-market redistribution cannot be supported by the portfolio redistribution resulting from the portfolio reorganization. Thus, the arguments based on the goods-market redistribution condition are no longer valid for the GEIC model with redundant assets.

As previously mentioned, the goods-market redistribution does not induce the portfolio redistribution in general except for the two very special cases shown in the following remark.

REMARK 3.1: For convenience, the notation of (A10) is retained in the following discussion.

(1) Suppose that $\Theta_i=\mathbb{R}^J$ for all $i\in I$ and that no redundant assets exists, that is, $V^{\perp}=\{0\}$. The goods-market redistribution induces the portfolio redistribution because the goods market redistribution condition $\Sigma_{i\in I}z_i=0$ implies that $p\,\Box_1\Sigma_{i\in I}z_i=R\cdot\Sigma_{i\in I}v_i=0$ or $\Sigma_{i\in I}v_i\in V^{\perp}$, which gives $\Sigma_{i\in I}v_i=0$. Conversely, $\Sigma_{i\in I}v_i=0$ gives $p\,\Box_1\Sigma_{i\in I}z_i=0$, which yields $\Sigma_{i\in I}z_i=0$ in a one-good economy. Thus, both redistribution conditions coincide in unconstrained asset markets of a one-good economy without redundant assets.

(2) Suppose that $V^{\perp}=N$. In this case, the goods-market redistribution condition represents the portfolio redistribution condition in constrained asset markets. If $\Sigma_{i\in I}z_i=0$, we have $\Sigma_{i\in I}v_i\in N$ or $-\Sigma_{i\in I}v_i\in N$. Then there exists $\eta_i\in N_i\cap V^{\perp}$ such that $\Sigma_{i\in I}(v_i+\eta_i)=0$ and $v_i+\eta_i\in C_i$ for all $i\in I$. We set $v_i'=v_i+\eta_i$ for each $i\in I$. Since $\eta_i\in V^{\perp}$, we have $R\cdot v_i=R\cdot v_i'$. Thus, it follows that $v_i'\in C_i$, $p\,\Box_1 z_i=R\cdot v_i'$ for all $i\in I$, and $\Sigma_{i\in I}v_i'=0$. □

Two examples are given to show the importance of (A10) for the existence of equilibrium. The first example is an economy with unconstrained asset markets that has an equilibrium because it satisfies all the conditions (A1)–(A10). The second example is the same as the first one except that portfolio constraints are imposed on asset holdings. The GEIC economy in the second example satisfies all the assumptions except for (A10). Thus, this economy has no equilibrium only because it violates (A10). The example shows that the irreducibility condition (A10) is indispensable for the existence of equilibrium.

EXAMPLE 3.1: An economy with $I=2$, $S=3$, and $J=2$ is considered where only one good is consumed in each state. The good is also used as a numeraire. The payoff matrix is given by the 3×2 matrix

$$R=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}.$$

Both agents have the same endowment of goods and distinct preferences.

$$u_1(x)=\sqrt{x(1)}+2\sqrt{X(2)}+\sqrt{x(3)},\quad e_1=(1,1,1),\qquad X_1=\mathbb{R}^3_+,$$
$$u_2(x)=2\sqrt{x(1)}+\sqrt{X(2)}+\sqrt{x(3)},\quad e_2=(1,0,0.5),\quad X_2=\mathbb{R}^3_+.$$

We suppose that $\Theta_1 = \Theta_2 = \mathbb{R}^2$, *i.e.*, asset markets are unconstrained. Evidently, the economy satisfies the conditions (A1)–(A9). It is also easy to see that (A10) holds here. The economy has an equilibrium $(q^*, x_1^*, x_2^*, \theta_1^*, \theta_2^*)$ where $(x_1^*, x_2^*) = ((1.1086, 0.9662, 0.9662), (0.8914, 0.0338, 0.0338))$, $(\theta_1^*, \theta_2^*) = ((0.1086, -0.0338), (-0.1086, 0.0338))$, and $q^* = (0.3112, 1)$. □

EXAMPLE 3.2: This example illustrates an economy that has no equilibrium only because it fails to meet the condition (A10). We add a risk-free asset that pays one dollar in each state to the economy in Example 3.1. The payoff matrix is augmented as follows:

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The risk-free asset is redundant. Thus, it is priced by arbitrage in equilibrium of the economy with unconstrained asset markets. We then impose a portfolio constraint on each agent described as follows:

$$\Theta_1 = \{(\theta^1, \theta^2, \theta^3) \in \mathbb{R}^3 : \theta^1 + \theta^2 \geq -1, \theta^3 \geq 0\},$$
$$\Theta_2 = \{(\theta^1, \theta^2, \theta^3) \in \mathbb{R}^3 : \theta^1 + \theta^2 \geq 0, \theta^3 \geq 0\}.$$

The constrained asset markets turn out to violate (A10) and have no equilibrium.

Assumption (A7) holds because we can choose $v_i^\circ = (0, 0, 1)$ to obtain $R \cdot v_i^\circ \gg 0$ for each $i = 1, 2$. Clearly, agent 1 fulfills the requirement of (A8). Since $C_1 = C_2 = \Theta_2$ and $V^\perp = \{v \in \mathbb{R}^3 : v = \lambda(1, 1, -1) \text{ for some } \lambda \in \mathbb{R}\}$, we see $C_i \cap V^\perp = N_i \cap V^\perp = \{0\}$ for each $i = 1, 2$, implying that (A9) holds trivially.

We show that (A10) does not hold in this example. Let $x_i = (x_i(1), x_i(2), x_i(3))$ denote a point in X_i for each i such that $(x_1, x_2) \in A_X$ and for some $\theta_i = (\theta_i^1, \theta_i^2, \theta_i^3) \in \Theta_i$,

$$x_i - e_i \leq R \cdot \theta_i. \tag{1}$$

Let us take $I_1 = \{1\}$ and $I_2 = \{2\}$. Suppose that there exist $z_i = (z_i(1), z_i(2), z_i(3)) \in \mathbb{R}^3$ and $v_i = (v_i^1, v_i^2, v_i^3) \in C_i$ for each $i = 1, 2$ such that

(i) $x_1+z_1 \in P_1(x_1)$,
(ii) $z_i = R \cdot v_i$ for each $i=1, 2$,
(iii) $e_2+z_2 \in X_2$, and
(iv) $v_1+v_2=0$.

Considering $\theta_i \in \Theta_i$ and $v_i \in C_i$ for each $i=1, 2$, (1) and (ii) then yields

$$\theta_1^1+\theta_1^2 \geq -1,\ \theta_2^1+\theta_2^2 \geq 0,\ \theta_i^3 \geq 0,\ v_i^1+v_i^2 \geq 0,\ v_i^3 \geq 0,\ i=1, 2 \tag{2}$$

and

$$\begin{bmatrix} x_1(1)-1 \leq \theta_1^1+\theta_1^3 \\ x_1(2)-1 \leq \theta_1^2+\theta_1^3 \\ x_1(3)-1 \leq \theta_1^2+\theta_1^3 \end{bmatrix}, \begin{bmatrix} x_2(1)-1 \leq \theta_2^1+\theta_2^3 \\ x_2(2) \leq \theta_2^2+\theta_2^3 \\ x_2(3)-0.5 \leq \theta_2^2+\theta_2^3 \end{bmatrix}, \begin{bmatrix} z_i(1) = v_i^1+v_i^3 \\ z_i(2) = v_i^2+v_i^3 \\ z_i(3) = v_i^2+v_i^3 \end{bmatrix}. \tag{3}$$

The last inequality of (2) and the requirement of (iv) for the third asset give $v_1^3=v_2^3=0$. The fourth inequalities of (2) and the condition of (iv) for the first two assets yield

$$v_1^1+v_2^1=0,\ v_1^2+v_2^2=0,\ v_1^1+v_1^2=0,\ v_1^2+v_2^2=0. \tag{4}$$

The inequalities in the third square bracket of (3) and the first two equalities of (4) result in the relation $z_1+z_2=0$. The condition (iii) gives $z_2(1) \geq -1$, $z_2(2) \geq 0$, and $z_2(3) \geq -0.5$. In particular, $z_2(2) \geq 0$ implies that $v_2^2 \geq 0$. Since $v_1^2+v_2^2=0$, it holds that $v_1^2 \leq 0$ and thus, $z_1(2) \leq 0$ and $z_1(3) \leq 0$. The third relation of (4) and the first two inequalities in the third square bracket of (3) yield $z_1(1)+z_1(2)=0$. In summary, z_1 satisfies the following relations:

$$z_1(1)+z_1(2)=0,\ z_1(2) \leq 0, \text{ and } z_1(3) \leq 0. \tag{5}$$

Let us take $x_1=(2, 1, 1)$ and $x_2=(0, 0, 0.5)$. This allocation is in A_X and is supported by $\theta_1=(0, -1, 1) \in \Theta_1$ and $\theta_2=(0, 1, -1) \in \Theta_2$ in the relations of the first two square brackets of (3). Then (i) leads to the following inequality:

$$\sqrt{2}+3 < \sqrt{2+z_1(1)}+2\sqrt{1+z_1(2)}+\sqrt{1+z_1(3)}. \tag{6}$$

The function in the right-hand side of (6) attains the maximum $\sqrt{2}+3$ at $z_1=0$ subject to the constraints (5). This result contradicts the strict inequality of (6). Consequently, (A10) fails in the economy.

We now show that the economy has no equilibrium. We set $\hat{u}_1(a, b, c) =\sqrt{a+c+1}+3\sqrt{b+c+1}$, $\hat{u}_2(a, b, c)=2\sqrt{a+c+1}+\sqrt{b+c}+\sqrt{b+c+0.5}$, and $\mathcal{A}_i(q)=\{(a, b, c)\in\mathbb{R}^3 : q_1a+q_2b+q_3c\leq 0, (a, b, c)\in\Theta_i\}$ for each $i=1$, 2. The function $\hat{u}_i$ is a reduced-form utility function defined over feasible portfolios, whereas $\mathcal{A}_i(q)$ is the budget set for agent i at asset price q. The utility maximization problem for agent $i=1$, 2 is then reduced to the following relations.

$$\max_{(a, b, c)\in\mathcal{A}_i(q)} \hat{u}_i(a, b, c).$$

Suppose that there exists an equilibrium $\{(q_1, q_2, q_3), (a_1, b_1, c_1), (a_2, b_2, c_2)\}$. Since $c_i\geq 0$ for each $i=1, 2$, by the market clearing condition we have $c_1=c_2=0$. The equilibrium profile would satisfy the first order conditions for utility maximization as follows:

$$-\lambda_0 q_1 + \frac{1}{2\sqrt{1+a_1}} + \lambda_1 = 0 \tag{7}$$

$$-\lambda_0 q_2 + \frac{3}{2\sqrt{1+b_1}} + \lambda_1 = 0 \tag{8}$$

$$-\lambda_0 q_3 + \frac{1}{2\sqrt{1+a_1}} + \frac{3}{2\sqrt{1+b_1}} + \lambda_2 = 0 \tag{9}$$

$$-\mu_0 q_1 + \frac{1}{\sqrt{1+a_2}} + \mu_1 = 0 \tag{10}$$

$$-\mu_0 q_2 + \frac{1}{2\sqrt{b_2}} + \frac{1}{2\sqrt{0.5+b_2}} + \mu_1 = 0 \tag{11}$$

$$-\mu_0 q_3 + \frac{1}{\sqrt{1+a_2}} + \frac{1}{2\sqrt{b_2}} + \frac{1}{2\sqrt{0.5+b_2}} + \mu_2 = 0 \tag{12}$$

where λ_k's and μ_k's ($k=0, 1, 2$) indicate the Lagrangian multipliers for the budget constraints and portfolio constraints of agents 1 and 2, respectively.

The following cases are considered to verify whether the first-order conditions hold in equilibrium.

(C1) $\mu_1>0$: The complementary slackness condition implies that $a_2+b_2=0$, with which market clearing yields $a_1+b_1=0$. Recalling that $c_1=0$ and $a_1q_1+b_1q_2=0$, we have $(q_1-q_2)a_1=0$. If $q_1=q_2$, then (7) and (8) give

$$\frac{1}{2\sqrt{1+a_1}}-\frac{3}{2\sqrt{1+b_1}}=\frac{1}{2\sqrt{1+a_1}}-\frac{3}{2\sqrt{1-a_1}}=0 \tag{13}$$

whereas (10) and (11) result in

$$\frac{1}{\sqrt{1+a_2}}-\frac{1}{2\sqrt{b_2}}-\frac{1}{2\sqrt{0.5+b_2}}=\frac{1}{\sqrt{1+a_2}}-\frac{1}{2\sqrt{-a_2}}-\frac{1}{2\sqrt{0.5-a_2}}=0. \tag{14}$$

The two equations produce $a_1=-0.8$ and $a_2=-0.4113$, which contradicts the market clearing condition $a_1+a_2=0$. If $a_1=0$, then $b_1=0$. This implies that $b_2=0$, which contradicts the requirement $b_2>0$ in (11).

(C2) $\mu_1=0$ and $\lambda_1=0$: In this case, (7), (8), (10), and (11) along with $a_1q_1+b_1q_2=0$ bring out the equations

$$\frac{a_1}{2\sqrt{1+a_1}}+\frac{3b_1}{2\sqrt{1+b_1}}=0$$
$$\frac{a_1}{\sqrt{1-a_1}}+\frac{b_1}{2\sqrt{-b_1}}+\frac{b_1}{2\sqrt{0.5-b_1}}=0.$$

The equations have the solution $a_1=0.1086$ and $b_1=-0.0338$. The solution yields $a_2+b_2=-0.0748$, which contradicts the portfolio constraint $a_2+b_2\geq 0$.

(C3) $\mu_1=0$ and $\lambda_1>0$: The restriction $\lambda_1>0$ implies that $a_1+b_1=-1$. This result along with the relations (10), (11) and $a_1q_1+b_1q_2=0$ is summarized as

$$a_1 + b_1 = -1, \quad \eta_0 q_1 = \frac{a_1}{\sqrt{1 - a_1}},$$

$$\mu_0 q_2 = \frac{1}{2\sqrt{-b_1}} + \frac{1}{2\sqrt{0.5 - b_1}}, \quad a_1(\mu_0 q_1) + b_1(\mu_0 q_2) = 0.$$

The four equations produce $a_1 = 0.6845$ and $b_1 = -1.6845$, which contradicts the requirement that $b_1 > -1$ in (9).

Therefore, the economy has no equilibrium. □

IV. Existence of Equilibrium

To establish equilibrium existence for economy $\mathcal{E}$, we introduce a sequence $\{\mathcal{E}^n\}$ of GEIC economies which are identical to $\mathcal{E}$ except that they satisfy strong survival conditions for both goods and assets markets. These economies are built by perturbing the portfolio constraint set Θ_i and the initial endowment e_i for each $i \in \mathcal{I}$. By applying the existence theorem of Hahn and Won (2011), we show that a competitive equilibrium exists for each $\mathcal{E}^n$. We find that the sequence of equilibria for $\{\mathcal{E}^n\}$ has a subsequence convergent to a point that yields a quasiequilibrium of economy $\mathcal{E}$. The irreducibility condition (A10) ensures that the quasi-equilibrium is a competitive equilibrium.

A. Equilibrium of an Economy with Strong Survival Assumptions

Let $\mathcal{E}'$ denote an economy that is identical to $\mathcal{E}$ except that it satisfies the following strong survival conditions instead of Assumptions (A5) and (A6).

(A5′) For each $i \in \mathcal{I}$, $e_i \in int X_i$.
(A6′) For each $i \in \mathcal{I}$, Θ_i is a closed convex set with $0 \in int\,\Theta_i$.

The following existence theorem for $\mathcal{E}'$ can be obtained by applying the existence theorem of Hahn and Won (2011).

THEOREM 4.1: Suppose that Assumptions (A1)–(A4), (A5′), (A6′), (A7), and (A9) hold in economy $\mathcal{E}'$. Then it has a competitive equilibrium.

Proof: Since economy $\mathcal{E}'$ satisfies the conditions for the existence theorem of Hahn and Won (2011), the theorem yields a competitive equilibrium of $\mathcal{E}'$. ■

B. Quasiequilibria of Economy $\mathcal{E}$

We now define the notion of quasiequilibrium for economy $\mathcal{E}$.

DEFINITION 4.1: A quasiequilibrium of economy $\mathcal{E}$ is a profile $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^J \times X \times \Theta$ such that

(i) for every $i \in \mathcal{I}$, $(x_i^*, \theta_i^*) \in cl\mathcal{B}_i(p^*, q^*)$,

(ii) for every $i \in \mathcal{I}$, $(P_i(x^*) \times \Theta_i) \cap \mathcal{B}_i(p^*, q^*) = \varnothing$,

(iii) $\Sigma_{i \in \mathcal{I}}(x_i^* - e_i) = 0$ and $\Sigma_{i \in \mathcal{I}} \theta_i^* = 0$.

The above definition is distinct from that of Gottardi and Hens (1996), which takes a stronger version of quasiequilibrium.[9] For each i, we take a sequence of endowments $\{e_i^n\}$ in $intX_i$ convergent to e_i, and a sequence of portfolio constraints $\{\Theta_i^n\}$ defined by $\Theta_i^n := \Theta_i + B_{1/n}(0)$, where $B_{1/n}(0)$ is a closed ball in $\mathbb{R}^J$ centered at 0 with radius $1/n$. Notably, Θ_i^n is a closed convex set with $0 \in int\Theta_i^n$ and $\Theta_i = \cap_{n=1}^{\infty} \Theta_i^n$. We then consider a sequence of GEIC economies $\{\mathcal{E}^n\}$ where $\mathcal{E}^n = \langle (X_i, P_i, e_i^n, \Theta_i^n)_{i \in \mathcal{I}}, R \rangle$. Observe that each $\mathcal{E}^n$ satisfies Assumptions (A1)–(A4), (A5′), (A6′), (A7), and (A9).

PROPOSITION 4.1: Under (A1)–(A9), economy $\mathcal{E}$ has a quasiequilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta^* \times X \times \Theta$.

Proof: The fact that each $\mathcal{E}^n$ satisfies the conditions of Theorem 4.1 leads to the following result.

Claim 1: Each economy $\mathcal{E}^n$ has a competitive equilibrium $(p^n, q^n, x^n, \hat{\theta}^n) \in \Delta \times X \times \Theta^n$ that

(1) $p^n \square (x_i^n - e_i) = W(q^n) \cdot \hat{\theta}_i^n$ for every i,

(2) $(P_i(x_i^n) \times \Theta_i^n) \cap cl\mathcal{B}_i^n(p^n, q^n) = \varnothing$ for every i,

(3) $\Sigma_{i \in \mathcal{I}}(x_i^n - e_i) = 0$ and $\Sigma_{i \in \mathcal{I}} \hat{\theta}_i^n = 0$.

For each i and each n, $\hat{\theta}_i^n$ is written as $\hat{\theta}_i^n = \theta_i^n + \delta_i^n$ where $\theta_i^n \in \Theta_i$ and $\delta_i^n \in B_{1/n}(0)$. Recalling that each X_i is closed and bounded from below, $\hat{X}_i$ is compact and, thus, $\hat{X}$ is compact. Since $\{(p^n, q^n, x^n)\}$ are in $\Delta \times \hat{X}$, the sequence is bounded. Without loss of generality, we can assume that $\{(p^n, q^n, x^n)\}$ converges to a point $(p^*, q^*, x^*) \in \Delta \times X$. In particular, it holds that $\| p^*(s) \| = 1$, $\forall s \in S$ and $\| q^* \| = 1$, and

[9] Definition 4.1 is also adopted in Hahn and Won (2011) as well as Aouani and Cornet (2011).

$$\sum_{i\in I}(x_i^*-e_i)=0. \tag{4.1}$$

Claim 2: There exists $\theta_i^*\in\Theta_i$ for each $i\in I$ such that $\Sigma_{i\in I}\,\theta_i^*=0$ and $(x_i^*,\ \theta_i^*)\in cl\mathcal{B}_i(p^*,\ q^*)$.

Proof: For each $i\in I$ and each n, we decompose θ_i^n as $\theta_i^n=\phi_i^n+\eta_i^n$ where $\phi_i^n\in(N_i\cap V^\perp)^\perp$ and $\eta_i^n\in N_i\cap V^\perp$, where $(N_i\cap V^\perp)^\perp$ is the orthogonal complement of $N_i\cap V^\perp$ in $\mathbb{R}^J$. Moreover, by Claim 1, $\Sigma_{i\in I}\,\theta_i^n=0$ and thus, $\Sigma_{i\in I}\,\phi_i^n+\Sigma_{i\in I}\,\eta_i^n=0$ for all n. Let $\eta^n:=\Sigma_{i\in I}\,\eta_i^n$. The arguments of Hahn and Won (2011) can be invoked to verify that $\{\phi_i^n\}$ for each $i\in I$ and $\{\eta^n\}$ are bounded and without loss of generality, they may be assumed to converge to points ϕ_i and η, respectively. By the same arguments of Hahn and Won (2011), there exist $\theta_i^*\in\Theta_i$ and $\eta_i\in N_i\cap V^\perp$ for each $i\in I$ such that $\theta_i^*=\phi_i+\eta_i$ and $\Sigma_{i\in I}\,\theta_i^*=0$. On the other hand, by (1) of Claim 1, we have

$$p^n\,\square_1(x_i^n-e_i)=R\cdot\hat{\theta}_i^n=R\cdot(\theta_i^n+\delta_i^n)=R\cdot(\phi_i^n+\eta_i^n+\delta_i^n)=R\cdot(\phi_i^n+\delta_i^n),$$

which implies that

$$p^*\,\square_1(x_i^*-e_i)=\lim_{n\to\infty}R\cdot\hat{\theta}_i^n=\lim_{n\to\infty}R\cdot(\phi_i^n+\delta_i^n)=R\cdot\phi_i=R\cdot(\phi_i+\eta_i)=R\cdot\theta_i^*. \tag{4.2}$$

Recalling that $q^n\in Q\cap N^\perp$ for all n and thus $q^*\in clQ\cap V^\perp$, by (1) of Claim 1, we also see that for all n and $i\in I$,

$$0=q^n\cdot\hat{\theta}_i^n=q^n\cdot(\theta_i^n+\delta_i^n)=q^n\cdot(\phi_i^n+\eta_i^n+\delta_i^n)=q^n\cdot(\phi_i^n+\delta_i^n),$$

which yields the relation

$$0=\lim_{n\to\infty}q^n\cdot\hat{\theta}_i^n=\lim_{n\to\infty}q^n\cdot(\phi_i^n+\delta_i^n)=q^*\cdot\phi_i=q^*\cdot(\phi_i+\eta_i)=q^*\cdot\theta_i^*. \tag{4.3}$$

This result combined with (4.2) gives $p^*\,\square(x_i^*-e_i)=\lim_{n\to\infty}W(q^n)\cdot\hat{\theta}_i^n=W(q^*)\cdot\theta_i^*$. Thus, it follows that $(x_i^*,\ \theta_i^*)\in cl\mathcal{B}_i(p^*,\ q^*)$, $\forall i\in I$. □

Claim 3: For every $i\in I$, it holds that $(P_i(x^*)\times\Theta_i)\cap\mathcal{B}_i(p^*,\ q^*)=\varnothing$.

Proof: Suppose to the contrary that, for some i, there exists $(\hat{x}_i,\ \hat{\theta}_i)\in(P_i(x_i^*)\times\Theta_i)\cap\mathcal{B}_i(p^*,q^*)$. That is, $\hat{x}_i\in P_i(x_i^*)$ and $p^*\,\square(\hat{x}_i-e_i)\ll W(q^*)\cdot\hat{\theta}_i$. Since $\succ_i$ is continuous, there exists a sequence $\{\hat{x}_i^n\}$ converging to $\hat{x}_i$ such that $\hat{x}_i^n\in P_i(x_i^n)$ and $p^n\,\square(\hat{x}_i^n-e_i)\ll W(q^n)\cdot\hat{\theta}_i$ for sufficiently large n. This result contradicts the optimality of $(x_i^n, \hat{\theta}_i^n)$ in $\mathcal{E}^n$. Thus, the claim holds true. □

By Claims 2–3 and (4.1), the profile $(p^*, q^*, x^*, \theta^*)$ satisfies (i), (ii), and (iii) of Definition 4.1 and is thus a quasiequilibrium of economy $\mathcal{E}$. ■

C. Equilibrium and the GEIC Irreducibility Condition

In this subsection, we shall show that, under Assumption (A10), the quasiequilibrium obtained above is a competitive equilibrium. To impose (A10) on the quasiequilibrium $(p^*, q^*, x^*, \theta^*)$, let us take $I_1 := \{i \in I : \mathcal{B}_i(p^*, q^*) \neq \varnothing\}$ in view of (A10). Then, I_1 represents the set of agents, each of whom has a cheaper consumption in the quasiequilibrium. The following lemma shows that I_1 is not empty.

Lemma 4.1: Under Assumptions (A7) and (A8), it holds that $I_1 \neq \varnothing$.

Proof: By Assumption (A8), there exists θ_i for some $i \in I$ such that $q^* \cdot \theta_i < 0$ and $p^* \square_1 e_i + R \cdot \theta_i \geq 0$. Assumption (A7) yields $v_i^\circ \in C_i$ such that $R \cdot v_i^\circ \gg 0$. For sufficiently small $\alpha > 0$, it holds that

$$0 < -q^* \cdot (\theta_i + \alpha v_i^\circ),$$
$$0 \ll p^* \square_1 e_i + R \cdot (\theta_i + \alpha v_i^\circ).$$

Since e_i and 0 belong to X_i, the second inequality implies the existence of $x_i \in X_i$ which satisfies $p^* \square_1 (x_i - e_i) \ll R \cdot (\theta_i + \alpha v_i^\circ)$, that is, $p^* \square (x_i - e_i) \ll W(q^*) \cdot (\theta_i + \alpha v_i^\circ)$. Therefore, $\mathcal{B}_i(p^*, q^*) \neq \varnothing$. ■

Lemma 4.2: For every agent $i \in I_1$, the following hold.

(i) (x_i^*, θ_i^*) is optimal in $cl\mathcal{B}_i(p^*, q^*)$, *i.e.*, $(P_i(x_i^*) \times \Theta_i) \cap cl\mathcal{B}_i(p^*, q^*) = \varnothing$.

(ii) If $(x_i, \theta_i) \in P_i(x^*) \times \Theta_i$ and $p^* \square_1 (x_i - e_i) \leq R \cdot \theta_i$, then $q^* \cdot \theta_i > 0$.

(iii) If $(x_i, \theta_i) \in clP_i(x^*) \times \Theta_i$ and $p^* \square_1 (x_i - e_i) \leq R \cdot \theta_i$, then $q^* \cdot \theta_i \geq 0$.

Proof: Since $P_i(x_i^*)$ and $\mathcal{B}_i(p^*, q^*)$ are open and $\mathcal{B}_i(p^*, q^*) \neq \varnothing$, the standard argument yields (i). The second statement (ii) is straightforward from (i). To prove (iii), suppose otherwise. Then $p^* \square_1 (x_i - e_i) \leq R \cdot \theta_i$ and $q^* \cdot \theta_i < 0$ for some $i \in I_1$. Take a point $(x_i^\circ, \theta_i^\circ) \in \mathcal{B}_i(p^*, q^*) \neq \varnothing$. Then, for $\alpha \in (0, 1]$ sufficiently close to 0, it holds that $(1-\alpha)x_i + \alpha x_i^\circ \in P_i(x_i^*)$ and $p^* \square [(1-\alpha)x_i + \alpha x_i^\circ - e_i] \ll W(q^*) \cdot [(1-\alpha)\theta_i + \alpha \theta_i^\circ]$, which is a contradiction to (i). ■

The following shows that the quasiequilibrium $(p^*, q^*, x^*, \theta^*)$ becomes a competitive equilibrium when irreducibility condition (A10) is satisfied.

Theorem 4.2: Under Assumptions (A1)–(A10), economy $\mathcal{E}$ has a competitive equilibrium.

Proof: Let us take $I_2 := I \setminus I_1$, *i.e.*, $I_2 = \{i \in I : \mathcal{B}_i(p^*, q^*) = \varnothing\}$. To claim that (x_i^*, θ_i^*) is optimal in $cl\mathcal{B}_i(p^*, q^*)$ for every $i \in I$, it suffices to show that $I_2 = \varnothing$ because of (i) of Lemma 4.2. Suppose to the contrary that $I_2 \neq \varnothing$. Then $\{I_1, I_2\}$ is a nontrivial partition. By Assumption (A10), there exist $z_i \in \mathbb{R}^\ell$ and $\theta_i \in C_i$ for each $i \in I$ such that $p^* \square_1 z_i = R \cdot \theta_i$ for each $i \in I$, $x_k^* + z_k \in P_k(x_k^*)$ for some $k \in I_1$ and $x_i^* + z_i \in clP_i(x_i^*)$, $\forall i \in I_1 \setminus \{k\}$, $e_i + z_i \in X_i$, $\forall i \in I_2$, and $\Sigma_{i \in I}\, \theta_i = 0$.

Since $(x_k^* + z_k) \in P_k(x_k^*)$ and $p^* \square_1 (x_k^* + z_k - e_k) \leq R \cdot (\theta_k^* + \theta_k)$, by (ii) of Lemma 4.2, we obtain $q^* \cdot (\theta_k^* + \theta_k) > 0$. Moreover, since $x_i^* + z_i \in clP_i(x_i^*)$ and $p^* \square_1 (x_i^* + z_i - e_i) \leq R \cdot (\theta_i^* + \theta_i)$ for every $i \in I_1 \setminus \{k\}$, by (iii) of Lemma 4.2, we have $q^* \cdot (\theta_i^* + \theta_i) \geq 0$. Summing over all agents of I_1 leads to

$$q^* \cdot \sum_{i \in I_1} (\theta_i^* + \theta_i) > 0. \tag{4.4}$$

Notice that $\theta_i \in C_i$ implies $\theta_i^* + \theta_i \in \Theta_i$ for every $i \in I$. On the other hand, (iv) of (A10) leads to $q^* \cdot \Sigma_{i \in I}\, \theta_i = 0$ and thus $q^* \cdot \Sigma_{i \in I} (\theta_i^* + \theta_i) = 0$. In view of (4.4), this yields

$$q^* \cdot \sum_{i \in I_2} (\theta_i^* + \theta_i) < 0.$$

Therefore, there exists $i_0 \in I_2$ such that $q^* \cdot (\theta_{i_0}^* + \theta_{i_0}) < 0$. Recalling from (4.3) that $q^* \cdot \theta_{i_0}^* = 0$, we have $q^* \cdot \theta_{i_0}^* < 0$. By (iii) of Assumption (A10), $x_{i_0} := e_{i_0} + z_{i_0} \in X_{i_0}$. Observe that $p^* \square_1 (x_{i_0} - e_{i_0}) = p^* \square_1 z_{i_0} = R \cdot \theta_{i_0}$. By Assumption (A7), there exists $v_{i_0}^\circ \in C_{i_0}$ with $R \cdot v_{i_0}^\circ \gg 0$. For sufficiently small $\lambda > 0$, we see that

$$0 < -q^* \cdot (\theta_{i_0} + \lambda v_{i_0}^\circ),$$
$$p^* \square_1 (x_{i_0} - e_{i_0}) \ll R \cdot (\theta_{i_0} + \lambda v_{i_0}^\circ).$$

As a consequence, $(x_{i_0}, \theta_{i_0} + \lambda v_{i_0}^\circ)$ is in $\mathcal{B}_i(p^*, q^*)$. This result implies $i_0 \in I_1$, which is impossible. Then it follows that $I_2 = \varnothing$, *i.e.*, $I_1 = I$. By (i) of Lemma 4.2, (x_i^*, θ_i^*) is optimal in $cl\mathcal{B}_i(p^*, q^*)$ for every $i \in I$. Therefore $(p^*,$

q^*, x^*, θ^*) is a competitive equilibrium.[10] ■

V. Conclusion

This paper established the existence of equilibrium in the GEIC model based on the new irreducibility condition (A10). The condition (A10) is an extension of Gottardi and Hens (1996) to constrained asset markets. This extension is particularly important because of the need to consider the risk-sharing role of redundant assets. Redundant assets affect the existence of equilibrium in the GEIC model in two ways. First, such assets expand risk-sharing opportunities in quasiequilibrium under portfolio constraints.[11] Indeed, the assumption (A9) specifies the manner in which portfolio constraints limit the risk-sharing capability of redundant assets. Second, redundant assets enrich intertemporal income transfers which affect the financial ability of agents to possess cheaper consumptions in equilibrium. This fact is properly embedded in the irreducibility condition (A10) for constrained asset markets. A challenging task is to extend the condition (A10) to multi-period GEIC models. Multi-period GEIC models are distinct from the current two-period model in that they involve long-lived assets the future prices of which affect intertemporal income transfers. This fact, we believe, need to be incorporated into formulating a multi-period version of the GEIC irreducibility condition.

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