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—Research Paper—

A Study for Multistage, Multiproduct Production Planning With Concave Costs*

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Abstract

This paper is concerned with the multistage, multiproduct production planning model with concave production, inventory and conversion costs. This model does not allow backlogging. This model considers that products require the same raw material at the first stage of production and that the flow is possible among different products. In order to obtain optimal solution, 0-1 integer formulation, Lagrangian relaxation, and branch-and-bound methodology are suggested. In this approach the special structure of the problem is exploited effectively.

1. Introduction

This paper is concerned with the multistage production planning with conversion process at every stage.

Zangwill observed that the Wagner and Whitin model [9], could be represented as a single source network model [10]. Zangwill [12] developed properties of the extreme flow of a single source concave network model, which states that given a single source concave network, each node can have at most one positive input in an extreme flow.

Using the above result, he developed a dynamic recursive equation for solving a multistage, single product production planning problem with concave costs [11].

Motivated by the Zangwill's model, we consider a multistage, multiproduct production planning model with concave costs.

In order to represent our multiproduct production planning problem as a single source concave network problem, we assume that process is capable of conversion among different items. According to the above assumption this production system consists of the similar products that may differ only in colour, shape, package, etc., which require the same material.

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This model is represented by Figure 1. The material at the source is shipped from the node to a destination node for each item along the arcs of the network.

In this single source concave network model, the act of determining the optimal production scheduling for the production system is equivalent to that of finding the corresponding network optimal flow.

Zangwill [12] has shown that an optimal flow is in a class of flows known as extreme flow. Since feasible region has a finite number of extreme flows, a search of all extreme flows will lead to the optimal flow, if it exists.

However, compared with the convex problem in which case a local minimum is also a global minimum, the concave cost problem may have many local minima. The problem of searching all the local minima is in a class of problems known as NP- complete except for the case of special structure of the problem [6].

There have been many papers dealing with the minimization problem in concave cost network [1, 3, 10]. However, all the algorithms developed until now for solving the minimum concave cost flow problem only lead to good local minimum on the problem with special structure. In order to get a global minimum by the above algorithms, exhaustive search is inevitable.

Fortunately, Zangwill [12] developed a dynamic approach to get a exact optimal solution for the single product, multistage production system. However, Zangwill's approach could not be applied for the multiproduct, multistage production system. In order to obtain an exact optimal solution for the above production system, alternative methodology should be considered.

In this paper, we develop 0-1 integer formulation and use Lagrangian relaxation approach, and branch-and-bound method.

2. Basic Assumptions

The basic assumptions for the model are as follows:

(a) Demands occur periodically for the finished product and demands of each item per period are known.

(b) Backlogging is not permitted and demands should be met

(c) A unit of production of any stage is used as an input unit of production at the next stage, and a unit of conversion at any product is corresponded with a unit of next production.

(d) There is no constraint on production capacity at any stage.

(e) Without loss of generality, it is assumed that there are no time lag involved in the transmission of goods from one stage to the next. Also it is assumed that conversion process is instantaneous.

(f) Production cost, holding cost, and conversion cost are all concave. The assumption of concave coincides with "economy of scale."

3. 0-1 Integer Formulation

Since our network model is single source network with concave costs and has the special structure without permitting backlogging, the amounts of flow between two nodes are limitted to several cases. Hence, we can have a transformed network model with so many arcs corresponded to a number of cases.

It follows that in our transformed network model several arcs exist among nodes. However, in a feasible flow only one way of arcs is selected without regard to the case.

Therefore, we can formulate a 0-1 integer programming by considering which way of arcs is selected.

The formulation of original problem is as follows:

$$\text{Minimize} \sum_{k=1}^{m} \sum_{i\in A_k} \sum_{j=1}^{n_i} C_{ij}\, x_{ij}$$

subject to

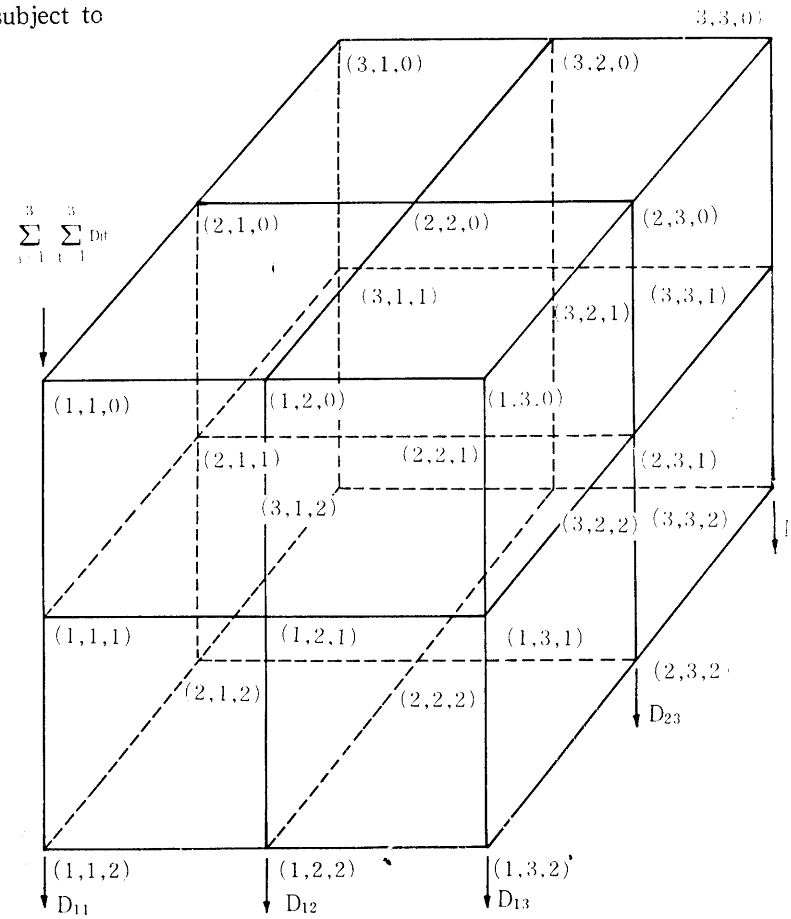

Fig. 1. Multistage, Multiproduct Production Plannng Model.

$$\sum_{i\in A_k}\sum_{j=1}^{n_i} a_{ij}x_{ij}=\sum_{l\in B_k}\sum_{j=1}^{n_l} a_{lj}x_{lj}+D_k,\ k=0,\cdots\cdots,\ m \tag{1}$$

$$\sum_{j\in A_k}\sum_{j=1}^{n_i} x_{ij}=1,\ k=1,\cdots\cdots,\ m \tag{2}$$

$$x_{ij}=0 \text{ or } 1$$

where,

k : node number and node 0 is source node.

m : the numbe of nodes except source node.

A_k : input arc set shipped into k-node in original network.

B_k : output arc set flowed out of k-node in original network.

C_{ij} : cost determined by flow amount a_{ij}

x_{ij} : the variable corresponding to j-th flow amount among amounts that can be flowed along the original arc i.

D_k : demand at k-node. At any node k except destination, D_k is zero and $D_0=-\sum D_k$

n_i : the number of flow amount that can be shipped along original arc i.

In above formulation, it is noticeable that input flow into any node does not exist in a feasible flow. Hence, the variable corresponding to zero input flow is added,

Constraints (1) represent input/output capacity balance equation for each node,

Constraints (2) represent equation from characterization of the extreme point of single source network and the independence of case.

4. Lagrangian Relaxation and Subgradient Algorithm

In the original problem, constraints (1) make the problem difficult to solve. Hence, we relax constraints (1) using Lagrangian relaxation.

Let u be the Lagrangian vector and u_i the Lagrangian multiplier corresponding to the i-th constraint in (1).

Then, the relaxed Lagrangian problem becomes,

$$\text{Minimize } \sum_{i=1}^{m}\sum_{j=1}^{n_i}(C_{ij}+\alpha_{ij}a_{ij})x_{ij}-\sum_{i=0}^{m}u_iD_i$$

subject to

$$\sum_{j=1}^{n_i} x_{ij}=1,\ i=1,\ \cdots\cdots,\ m$$

$$x_{ij}=0 \text{ or } 1$$

In the above relaxed problem, the decision variable x_{ij} is the variable corresponding to j-th flow amount among amounts that can be shipped into i-node. α_{ij} is the constant determined by input/ouput balance equation (1), whose value is the difference of two Lagrangian multipliers corresponding to variable x_{ij}.

The relaxed problem is decomposed into m independent subproblems, and each subproblem has a simple multiple choice constraint. So, the solution of the relaxed problem can be easily obtained in one variable which has a smallest objective functien coefficient in the subproblem.

Let $\bar{x}$ be the solution obtained from the above relaxed problem and $L(u:\bar{x})$ be the objective function value when u is given. Then $L(u:\bar{x})$ becomes lower bound of the original problem.

To obtain a good Lagrangian multiplier vector, subgradient algorithm is used.

Starting with an initial Lagrangian multiplier vector, u^0, the subgradient optimization algorithm obtains a sequence of vectors $u^0, u^1, u^2, \cdots,$. This sequence converges asymptotically to an optimal solution, However, there is no guarantee that $L(u^p:\bar{x})$ is monotone increasing with p. Since $L(u:\bar{x})$ is only used as a lower bound for the candidate problem in the branch-and-bound approach, no attempt is made to achieve actual convergence. Our aim is to find a Lagrangian multiplier vector u, which gives as high a value of $L(u:\bar{x})$ as possible.

Given u^j, the next Lagrangian vector can be obtained from

$$u^{j+1}=u^j+t_jV(u^j),\quad j=0,1,\cdots, \tag{3}$$

$$\text{if } u^0\cdot u^{j+1}<0, \text{ then } u^{j+1}=0$$

Here, t_j is positive scalar, called step size, $V(u^j)$ is subgradient vector given u^j.

In [5], it is suggested to take

$$t_j=\lambda_j\frac{\bar{L}-L(u^j)}{||V(u^j)||} \tag{4}$$

where $\bar{L}\leqq\max\{L(u)\}$, $0<\lambda_j\leqq 2$

$||V(u^j)||$ denotes the Euclidean norm.

In this paper, $\bar{L}$ is taken to be the objective value of the current incumbent.

For the value λ, a good rule is to start with $\lambda_0=2$. If no improvement of $L(u^j)$ occurs in the several steps, then λ is halved. [7]

5. Branch-and-bound Method

The solution obtained by applying the subgradient algorithm is not necessarily optimal. So, branch-and-bound method is used to obtain exact optimal solution.

In the branch-and-bound process of applying Langrangian relaxation on a candidate problem, it may happen that the optimal solution of relaxed problem satisfies all the constraints in the candidate problem and has an objective value equal to the lower bound corresponding to this candidate problem. If it happens, this solution is an optimal solution of a candidate problem.

If a candidate problem is not fathomed, we only have a lower bound for the minimum objective value in it. Then, it should be possible to apply the branching strategy on any candidate problem and to generate two candidate subproblems.

Since our problem is a pure zero-one integer problem, by adding one more constraint, "$x_j=0$" or "$x_j=1$" respectively, to the constraints on the candidate problem, two candidate subproblem are generated.

The branching variable should be selected so as to make the lower bound on the candidate problem as high as possible. Therefore, as a branching variable there is selected the variable, not fixed, with maximum objective coefficient.

By searching procedure, a branching tree is made. The branching tree is composed of nodes and branches. A node represents a candidate problem, a branch represents a branching variable and joins two nodes which differ in the state of only one variable. Now, we define the level of a node as the number of variables fixed at 1. In our relaxed problem the level does not exceed m, because the problem is decomposed into m subproblem each subproblem needs only one variable to be fixed at 1. The search terminates when the level is negative.

The best Lagrangian vector first obtained at any level is stored for later use as an initial Lagrangian vector. If branching step is occurred, the Lagrangian vector obtained from previous level is used as an initial Lagrangian vector. If pruning step is occurred, the Lagrangian vector stored previously at the same level is used.

6. Concluding Remarks

In this paper, concave cost network model is transformed into 0-1 integer programming. To solve the problem effectively, Lagrangian relaxation in the context with branch-and-bound method is used.

Usefulness of this method in our problem is easy to solve the problem and reduction of computational time and iteration number as shown in table 1.

Table 1. Computational Results for Test Problems

Problem size	network size		No. of enumeration solution	No. of iteration	CPU time(msec)
	No. of node	No. of destination			
11×11	6	2	48	34	297
15×16	8	2	288	58	668
15×29	8	4	16128	239	3607
17×27	9	3	9072	244	3976
23×40	12	3	571536	1542	42085
23×54	12	4	22127616	2283	76990

This model can be used in several applications. First, it can deal with production planning, in which case material flow is possible among different productions, and substitution of different items is possible.

Secondly, the model can be applied to the problem to obtain where the items are fabricated. In this case product index indicates the location to be fabricated and conversion amount means the transportation amount and conversion cost denotes the transportation cost.

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