Monotonicity and Independence Axioms for Quasi-linear Social Choice Problems*

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We consider *quasi-linear social choice problems*. A society must choose one among a finite number of costless public decisions; money is available to perform side payments; each agent has quasi-linear preferences. We are interested in determining what public decision should be chosen and what side payments among agents should be performed. By formulating monotonicity and independence axioms relating various changes in the set of public decision, we characterize egalitarianism.

I. Introduction

We consider the following class of *quasi-linear social choice problems*. A society must choose one among a finite number of costless public decisions; money is available to perform side payments; each agent has quasi-linear preferences (separably additive with respect to the public decisions and money, and linear with respect to money). The question is to determine what public decision should be chosen and what side payments among agents should be performed.

An axiomatic analysis of quasi-linear social choice problems has been initiated recently by Moulin (1985a). In Moulin (1985a), he proposed various axioms concerning certain changes in the utility profiles and established characterizations of utilitarianism and egalitarianism. Following this pioneering work, Moulin (1985b) and Chun (1986) proposed axioms relating solutions for societies of different sizes and gave axiomatic characterizations of utilitarianism and egalitarianism. In Moulin (1986), he introduced binary quasi-linear social choice problems where a society is faced with a

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choice between two public decisions and characterized utilitarianism and egalitarianism.

Here we formulate various axioms relating various changes in the set of public decisions and characterize egalitarianism. The axioms can be divided into two subgroups: *Monotonicity Axioms* specify how the final utilities of agents change when the set of public decisions is subject to some transformations, while *Independence Axioms* require the invariance of the final utilities in similar circumstances. Although this work was originally done by Thomson and Myerson(1980) for bargaining theory, all of their axioms are reformulated to be suitable for quasi-linear social choice problems and several new axioms are introduced.

The paper is organized as follows. Section II contains some preliminaries and introduces the concept of a soultion and the basic axioms. Section III presents various axioms of our interests. In Section IV, the logical implications between these axioms are established and a characterization of all solutions satisfying all these axioms is provided.

II. Preliminaries

Let $\mathcal{A}$ be the (infinite) universe of "potential" *public decisions* and Σ be the class of all subsets of $\mathcal{A}$. Given $A \in \Sigma$, *society* $N \equiv \{1, \cdots, n\}$ is supposed to choose an *outcome* (a, t) where $a \in A$ and $t \equiv (t_i)_{i \in N}$ is a vector of balanced monetary transfers across agents:

$$\sum_{i \in N} t_i = 0.$$

Every agent $i \in N$ has "quasi-linear preferences" over the set $A \times I\!R$, so that his utility for outcome (a, t) is $u_i(a) + t_i$.

Definition

Given a society $N = \{1, \cdots, n\}$ and a finite set A of public decisions, a *solution* is a function S: $I\!R^{\,\,n} \rightarrow I\!R^n$, which associates to every utility profile $u(A) \equiv (u(a))_{a \in A}$ a vector $S(u(A)) \equiv (S_1(u(A)), \cdots, S_n(u(A)))$ of utility levels. $S(u(A))$ is called the *solution outcome*.

In addition to satisfy certain monotonicity and independence axioms, we will always require a solution to satisfy the following axioms.

Pareto Optimality (PO): For all $A \in \Sigma$,

$$\sum_{i \in N} S_i(u(A)) = \max_{a \in A} \sum_{i \in N} u_i(a).$$

Trivial Independence (TI): For all $A \in \Sigma$,

$$S((u(A))_{a \in A}, S(u(A))) = S(u(A)).$$

PO requires that society picks a decision which maximizes the sum of individual utilities. This maximal sum is called the *social optimum*. *TI* requires that, if a society chooses a solution outcome and later the society finds an additional public decision which gives the same utility to all agents as the solution outcome, then the society does not change its decision. As we will show in the remark at the end of Section IV, since *TI* is implied by most of our monotonicity and independence axioms, it is very weak.

Given $x \in IR$, we define $\max x(A) \equiv \max_{a \in A} x(a)$ and for any coalition $T \subseteq N$, we define $S_T = \sum_{i \in T} S_i$, $u_T = \sum_{i \in T} u_i$, and so on. Thus Pareto optimality is rewritten as: for all $A \in \Sigma$, $S_N(u(A)) = \max u_N(A)$. Vector inequalities are as follows: $\geqq$, $\geq$, $>$.

III. The Axioms

Most of the axioms introduced here were already discussed in bargaining theory by Thomson and Myerson(1980), but all of them are reformulated to be suitable for quasi-linear social choice problems.

Strong Monotonicity (SM): For all $A, B \in \Sigma$, if $A \subseteq B$, then either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$.

This axiom was introduced in bargaining theory by Luce and Raiffa(1957). Our formulation requires that an expansion in the set of public decisions affects all agents in the same way: all strictly gain or remain at the original solution outcome.

Individual Monotonicity (IM): For all $A, B \in \Sigma$ and for all $i \in N$, if $A \subseteq B$ and $\max u_i(B) = \max u_i(A)$, then either

(a) $\max u_N(B) > \max u_N(A)$ and $S_j(u(B)) > S_j(u(A))$ for all $j = i$

or

(b) $\max u_N(B) = \max u_N(A)$ and $S(u(B)) = S(u(A))$.

This axiom was introduced in bargaining theory by Kalai and Smorodinsky (1975), and variants appeared in Kalai (1977), Rosenthal(1976), and Thomson and Myerson(1980). Our formulation can be regarded as a strengthening of these axioms, since we require the invariance of a solution outcome under the condition that social optimum remains the same.

Independence of Irrelevant Alternatives(IIA): For all $A, B \in \Sigma$, if $B \subseteq A$ and $\max u_N(B) = \max u_N(A)$, then $S(u(B)) = S(u(A))$.

Weak Independence of Irrelevant Alternatives (WIIA): For all $A, B \in \Sigma$, if $B \subseteq A$, $a, \in B$ implies $u(a) \leqq S(u(A))$ and there exists $a^* \in B$ such that $u(a^*) = S(u(A))$, then $S(u(B)) = S(u(A))$.

Monotonicity with respect to Cutting Alternatives (MC): For all $A, B \in \Sigma$, if $B \subseteq A$ and $a \in B$ implies $u(a) \leqq S(u(A))$, then either $S(u(B)) = S(u(A))$ or $S(u(B)) < S(u(A))$.

IIA was introduced in bargaining theory by Nash(1950) and *WIIA* by Thomson and Myerson(1980). Our formulation of *IIA* implies that the existence of socially unoptimal public decisions does not affect the solution outcome. *WIIA* is a considerable weakening of *IIA*, since its application is very limited. *MC* is a strengthening of *WIIA*, in that it is applicable to certain problems with different social optima. However, it is not logically related to *IIA*.

Independence of Undominating Alternatives(IUA): For all $A, B \in \Sigma$, if $A \subseteq B$ and $a \in B \backslash A$ implies $u_N(a) \leq \max u_N(A)$, then $S(u(B)) = S(u(A))$.

Weak Independence of Undominating Alternatives (WIUA): For all $A, B \in \Sigma$, if $A \subseteq B$, $a \in A$ implies $u(a) \leqq S(u(A))$ and $\max u_N(B) = \max u_N(A)$, then $S(u(B)) = S(u(A))$.

Monotonicity with respect to Adding Alternatives (MA): For all $A, B \in \Sigma$, if $A \subseteq B$, $a \in A$ implies $u(a) \leqq S(u(A))$ and there does not exist $a^* \in B$ such that $u(a^*) > S(u(A))$, then either $S(u(B)) = S(u(\mathrm{A}))$ or $S(u(B)) > S(u(A))$.

IUA was introduced in bargaining theory by Thomson(1981) and *WIUA* by Thomson and Myerson(1980). Our formulation of *IIA* implies that the addition of socially unoptimal public decisions does not affect the solution outcome. *WIUA* is a considerable weakening of *IUA*, since its application is very limited. *MA* is a strengthening of *WIUA*, in that it is applicable to certain problems with different social optima. However, it is not logically related to *IUA*.

Twisting (Tw): For all $A, B \in \Sigma$ and for all $i \in N$, if

(i) $a \in B \setminus A$ implies $u_j(a) \leq S_j(u(A))$ for all $j \neq i$ and
(ii) $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$,

then either

(a) $S_i(u(B)) \geq S_i(u(A))$ or
(b) $S_j(u(B)) < S_j(u(A))$ for all $j \neq i$.

Cutting (Cu): For all A, $B \in \Sigma$ and for all $i \in N$, if $B \subseteq A$ and $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) \geq S_i(u(A))$ or
(b) $S_j(u(B)) < S_j(u(A))$ for all $j \neq i$.

Adding (Ad): For all A, $B \in \Sigma$ and for all $i \in N$, if $A \subseteq B$ and $a \in B \setminus A$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) \leq S_i(u(A))$ or
(b) $S_j(u(B)) > S_j(u(A))$ for all $j \neq i$.

These axioms were introduced in bargaining theory by Thomson and Myerson (1980), and variants appeared in Peters (1986). They require that a certain change in the set of public decisions which is in favor of an agent results in either the gain of that agent or the loss of all the other agents.

The next three axioms are weakenings of these axioms, in that their applicability is limited to certain problems with the same social optimum.

Weak Twisting(W.Tw): For all A, $B \in \Sigma$ and for all $i \in N$, if

(i) $\max u_N(B) = \max u_N(A)$,
(ii) $a \in B \setminus A$ implies $u_j(a) \leq S_j(u(A))$ for all $j \neq i$ and
(iii) $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$,

then either

(a) $S_i(u(B)) \geq S_i(u(A))$ or
(b) $S_j(u(B)) < S_j(u(A))$ for all $j \neq i$.

Weak Cutting (W. Cu): For all A, $B \in \Sigma$ and for all $i \in N$, if $B \subseteq A$, $\max u_N(B) = \max u_N(A)$ and $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) \geq S_i(u(A))$ or
(b) $S_j(u(B)) < S_j(u(A))$ for all $j \neq i$.

Weak Adding (W.Ad): For all A, $B \in \Sigma$, and for all $i \in N$, if $A \subseteq B$.

max $u_N(B)$ = max $u_N(A)$ and $a \in B \setminus A$ implies $u_i(a) \leq S_i\,(u(A))$, then either

(a) $S_i(u(B)) \leq S_i(u(A))$ or
(b) $S_j(u(B)) > S_j(u(A))$ for all $j \neq i$.

Dominance (D): For all A, $B \in \Sigma$, either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$ or $S(u(B)) < S(u(A))$.

This last axiom was introduced in bargaining theory by Thomson and Myerson (1980). It requires that for any pairs of problems A and B, independently of their relation, all agents gain or lose together. It can be interpreted as a requirement of strong solidarity among agents.

In addition to those axioms demanding changes of a solution outcome in intuitive directions, we also discuss a parallel set of axioms demanding changes in counterintuitive directions. These axioms are useful in explicating the relation between the previous ones.

Perverse Individual Monotonicity(IM)*: For all A, $B \in \Sigma$ and for all $i \in N$, if $A \subseteq B$ and max $u_i(B)$ = max $u_i(A)$, then either

(a) max $u_N(B)$ > max $u_N(A)$ and $S_i(u(B)) > S_i(u(A))$ or
(b) max $u_N(B)$ = max $u_N(A)$ and $S(u(B)) = S(u(A))$.

Perverse Twisting (Tw)*: For all A, $B \in \Sigma$ and for all $i \in N$, if

(i) $a \in A \setminus B$ implies $u_j(a) \leq S_i(u(A))$ for all $j \neq i$ and
(ii) $a \in B \setminus A$ implies $u_i(a) \leq S_i(u(A))$,

then either

(a) $S_i(u(B)) > S_i(u(A))$ or
(b) $S_j(u(A)) \leq S_j(u(A))$ for all $j \neq i$.

Perverse Cutting (Cu)*: For all A, $B \in \Sigma$ and for all $i \in N$, if $B \subseteq A$ and $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) < S_i(u(A))$ or
(b) $S_j(u(B)) \geq S_j(u(A))$ for all $j \neq i$.

Perverse Adding (Ad)*: For all A, $B \in \Sigma$ and for all $i \in N$, if $A \subseteq B$ and $a \in B \setminus A$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) > S_i(u(A))$ or
(b) $S_j(u)B)) \leq S_j(u(A))$ for all $j \neq i$.

Perverse Weak Twisting (W.Tw)*: For all A, $B \in \Sigma$ and for all $i \in$

N, if

(i) $\max\, u_N(B) = \max\, u_N(A)$,
(ii) $a \in A \setminus B$ implies $u_j(a) \leq S_j(u(A))$ for all $j \neq i$ and
(iii) $a \in B \setminus A$ implies $u_i(a) \leq S_i(u(A))$,

then either

(a) $S_i(u(B)) > S_i(u(A))$ or
(b) $S_j(u(B)) \leq S_j(u(A))$ for all $j \neq i$.

Perverse Weak Cutting ($W.Cu^*$): For all $A,\ B \in \Sigma$ and for all $i \in N$, if $B \subseteq A$, $\max\, u_N(B) = \max\, u_N(A)$ and $a \in A \setminus B$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) < S_i(u(A))$ or
(b) $S_j(u(B)) \geq S_j(u(A))$ all $j \neq i$.

Perverse Weak Adding ($W.Ad^*$): For all $A,\ B \in \Sigma$ and for all $i \in N$, if $A \subseteq B$, $\max\, u_N(B) = \max\, u_N(A)$ and $a \in B \setminus A$ implies $u_i(a) \leq S_i(u(A))$, then either

(a) $S_i(u(B)) > S_i(u(A))$ or
(b) $S_j(u(B)) \leq S_j(u(A))$ for all $j \neq i$.

IV. The Logical Implications

Now we investigate the logical implications between these axioms.

Lemma 1
$SM + PO \rightarrow IIA$.

Proof: Let A and B satisfy: $B \subseteq A$, $\max\, u_N(A) = \max\, u_N(B)$. By SM. either $S(u(A)) = S(u(B))$ or $S(u(A)) > S(u(B))$. If $S(u(A)) > S(u(B))$, then $S_N(u(A)) > S_N(u(B))$. Now PO applied to both sides implies that $\max\, u_N(a) > \max\, u_N(B)$, a contradiction.

Lemma 1′
$SM + PO \rightarrow IUA$.

Proof: Let A and B satisfy: $A \subseteq B$, $a \in B \setminus A$ implies $u_N(a) \leq \max\, u_N(A)$, which implies that $\max\, u_N(B) = \max\, u_N(A)$. By SM, either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$. If $S(u(B)) > S(u(A))$, then $S_N(u(B)) > S_N(u(A))$. Now PO applied to both sides implies that $\max\, u_N(B) > \max\, u_N(A)$, a contradiction.

Lemma 2
$SM + PO \rightarrow IM$

Lemma 2′
$SM + PO \rightarrow IM^*$

Proof: Let A and B satisfy the hypotheses of IM (respectively IM^*). Then, by SM, either (i) $S(u(B)) = S(u(A))$ or (ii) $S(u(B)) > S(u(A))$. If max $u_N(B) >$ max $u_N(A)$, then (i) is incompatible with PO, so that (ii) holds. If may $u_N(B) =$ max $u_N(A)$, then (ii) is incompatible with PO, so that (i) holds.

Lemma 3
$IIA \rightarrow WIIA$.

Lemma 3′
$IUA \rightarrow WIUA$.

Lemma 4
$Tw \rightarrow W.Tw$.

Lemma 4′
$Tw^* \rightarrow W.Tw^*$.

Lemma 5
$Cu \rightarrow W.Cu$.

Lemma 5′
$Cu^* \rightarrow W.Cu^*$

Lemma 6
$Ad \rightarrow W.Ad$.

Lemma 6′
$Ad^* \rightarrow W.Ad^*$.

Proof: $WIIA$ is obtained from IIA by a strengthening of the hypothesis, whence the first implication. The other seven are obtained in a similar fashion.

Lemma 7
$IIA \rightarrow IUA$.

Proof: Since the conclusions of both axioms are the same, it is enough to show that the hypotheses of IUA implies those of IIA. Now let A and B satisfy the hypotheses of IUA. Since $A \subseteq B$, we have max $u_N(A) \leq$ max $u_N(B)$. Since $a \in B \backslash A$ implies $u_N(a) <$ max

$u_N(A)$, in fact we have $\max u_N(A) = \max u_N(B)$.

Lemma 7′
$IUA \rightarrow IIA$.

Proof: It is enough to show that the hypotheses of IIA imply those of IUA. Now let A and B satisfy the hypotheses of IIA. Since $B \subseteq A$ and $\max u_N(A) = \max u_N(B)$, there does not exist $a \in A \setminus B$ such that $u_N(a) > \max u_N(B)$.

Lemma 8
$IM \rightarrow IIA$.

Proof: Let A and B satisfy: $B \subseteq A$ and $\max u_N(B) = \max u_N(A)$. We will reconstruct A from B by successive additions. First, let B_1 be obtained from B by adding all public decisions of $A \setminus B$ whose first coordinate is smaller than $\max u_1(B)$, i.e.,: $a \in B_1 \leftrightarrow a \in A$ and $u_1(a) \leq \max u_1(B)$). Since $\max u_1(B_1) = \max u_1(A)$, IM applied to B_1 and A yields

$$S(u(B_1)) = S(u(A)).$$

This addition process is then iterated. At step k, we define B_k by $[a \in B_k \leftrightarrow a \in A$ and $u_l(a) \leq \max u_l(B)$ for some $l = 1, \cdots, k]$. By applying IM to B_k and B_{k-1}, we have

$$S(u(B_k)) = S(u(B_{k-1})).$$

When $k = n$, we eventually have:

$$S(u(B)) = S(u(B_n)) = S(u(B_{n-1})) = \cdots = S(u(B_1)) = S(u(A)),$$

as desired.

Lemma 8′
$IM^* \rightarrow IUA$.

Proof: Let A and B be satisfy the hypotheses of IUA. Then we can reconstruct B from A by successive additions, whose procedure is described in the proof of Lemma 8, so that we have $S(u(B)) = S(u(A))$.

Lemma 9
$IM \rightarrow Tw$.

Proof: Suppose, to the contrary, that IM holds, but not Tw. Then there exist A, B and i satisfying the hypotheses of Tw, but with

(a) $S_i(u(B)) < S_i(u(A))$ and (b) $\exists\ j \neq i \mid S_j(u(B)) \geq S_j(u(A))$.

Let A', B', C_1, C_2 and C' be defined by

$A' \equiv A \cup \{a^*\}$ where a^* is a public decision such that $u(a^*) = S(u(A))$,
$B' \equiv B \cup \{b^*\}$ where b^* us a public decision such that $u(b^*) = S(u(B))$,
$C_1 \equiv A \cup B \cup \{a^*\}$,
$C_2 \equiv A \cup B \cup \{b^*\}$ and
$C' \equiv A' \cup B' = C_1 \cup C_2$.

By IM applied to A and A', and B and B', we have

$$S(u(A')) = S(u(A)) \text{ and } S(u(B')) = S(u(B)).$$

Two situations are possible:

(i) If $\max u_N(B) \geq \max u_N(A)$, then applying IM to C_1 and C', C' and C_2, and C_2 and B' yields

$$S(u(B')) = S(u(C_2)) = S(u(C')) = S(u(C_1)).$$

Since $u_j \leq S_j(u(A))$ for all $j \neq i$ and for all $a \in B \backslash A$ from the hypotheses of Tw, we have $\max u_j(C_1) = \max u_j(A')$ for all $j \neq i$. Therefore, by applying IM to A' and C_1, we have

$$S_i(u(C_1)) \geq S_i(u(A')),$$

so that we have

$$S_i(u(B)) \geq S_i(u(A)),$$

which is incompatible with (a).

(ii) If $\max u_N(B) < \max u_N(A)$, the applying IM to A' and C_1, C_1 and C', and C' and C_2 yields

$$S(u(C_2)) = S(u(C')) = S(u(C_1)) = S(u(A')).$$

Since $u_i \leq S_i(u(A))$ for all $a \in A \backslash B$ from the hypotheses of Tw, we have $\max u_i(C_2) = \max u_i(B')$. Therefore, by applying IM to B' and C_2, we have

$$S_j(u(C_2)) > S_j(u(B')) \text{ for all } j \neq i.$$

so that we have

$$S_j(u(A)) > S_j(u(B)) \text{ for all } j \neq i,$$

which is incompatible with (b).

Lemma 9′
$IM^* \rightarrow Tw^*$.

Proof: Suppose, to the contrary, that IM^* holds, but not Tw^*. Then there exist A, B and i satisfying the hypotheses of Tw^*, but with

(a) $S_i(u(B)) \leq S_i(u(A))$ and (b) $\exists\ j \neq i \mid S_j(u(B)) > S_j(u(A))$.

Let A', B', C_1, C_2 and C' be defined by

$A' \equiv A \cup \{a^*\}$ where a^* is a public decision such that $u(a^*) = S(u(A))$,
$B' \equiv B \cup \{b^*\}$ where b^* is a public decision such that $u(b^*) = S(u(B))$,
$C_1 \equiv A \cup B \cup \{a^*\}$,
$C_2 \equiv A \cup B \cup \{b^*\}$ and
$C' \equiv A' \cup B' = C_1 \cup C_2$.

By IM^* applied to A and A', and B and B', we have

$S(u(A')) = S(u(A))$ and $S(u(B')) = (S(u(B))$.

Two situations are possible:

(i) If $\max u_N(B) > \max u_N(A)$, then applying IM^* to C_1 and C', C' and C_2, and C_2 and B' yields

$S(u(B')) = S(u(C_2)) = S(u(C')) = S(u(C_l))$.

Since $u_i \leq S_i(u(A))$ for all $a \in B \backslash A$ from the hypotheses of Tw^*, we have $\max u_i(C_1) = \max u_j(A')$. Therefore, by applying IM to A' and C_1, we have

$S_i(u(C_1)) > S_i(u(A'))$,

so that we have

$S_i(u(B)) > S_i(u(A))$,

which is incompatible with (a).

(ii) If $\max u_N(B) \leq \max u_N(A)$, then applying IM to A' and C_1, C_1 and C', and C' and C_2 yields

$S(u(C_2)) = S(u(C')) = S(u(C_1)) = S(u(A'))$.

Since $u_j \leq S_j(u(A))$ for all $j \neq i$ and for all $a \in A \backslash B$ from the hypotheses of Tw, we have $\max u_j(C_2) = \max u_j(B')$ for all $j \neq i$. Therefore, by applying IM to B' and C_2, we have

$S_j(u(C_2)) \geq S_j(u(B'))$ for all $j \neq i$,

so that we have

$S_j(u(A)) \geq S_j(u(B))$ for all $j \neq i$,

which is incompatible with (b).

Lemma 10
$IIA \rightarrow W.Tw^*$.

Proof: Let A and B satisfy the hypotheses of $W.Tw$. By applying IIA to $A \cup B$ and A, we have $S(u(A)) = S(u(A \cup B))$. Again, by applying IIA to $A \cup B$ and B, we have $S(u(B)) = S(u(A \cup B))$. Therefore, we have $S(u(B)) = S(u(A))$, as desired.

Lemma 10′
$IUA \rightarrow W.Tw^*$.

Proof: Let A and B satisfy the hypotheses of $W.Tw^*$. By applying IUA to A and $A \cup B$, we have $S(u(A \cup B)) = S(u(A))$. Again, by applying IUA to B and $A \cup B$, we have $S(u(A \cup B)) = S(u(B))$. Therefore, we have $S(u(B)) = S(u(A))$, as desired.

Lemma 11
$Tw \rightarrow Cu$.

Lemma 11′
$W.Tw \rightarrow W.Cu$.

Lemma 12
$Tw^* \rightarrow Ad^*$.

Lemma 12′
$W.Tw^* \rightarrow W.Ad^*$.

Proof: Cu is obtained from Tw by a strengthening of the hypotheses, whence the first implication. The other three are obtained in a similar fashion.

Lemma 13
$Cu \rightarrow Ad$.

Lemma 13′
$W.Cu \rightarrow W.Ad$.

Proof: Suppose, to the contrary, that Cu(respectively $W.Cu$) holds but not $Ad(W.Ad)$. Then there exist A, B and i satisfying the hypotheses of $Ad(W.Ad)$ but with

(a_1) $S_i(u(B)) > S_i(u(A))$ and (b_1) $\exists\ j \neq i \mid S_j(u(B)) \leq S_j(u(A))$.

Note that $S_i(u(B)) > S_i(u(A))$ and $S_i(u(A)) \geq u_i(a)$ for all a imply $S_i(u(B)) \geq u_i(a)$ for all $a \in B \backslash A$. Therefore, by $Cu(W.Cu)$, either

(a_2) $S_i(u(A)) \geq S_i(u(B))$ or

(b_2) $S_j(u(A)) < S_j(u(B))$ for all $j \neq i$.

Two cases may occur:

(i) (a_1) + (b_1) + (a_2) → From (a_1) and (a_2), we have a contradiction.
(ii) (a_1) + (b_1) + (b_2) → From (b_1) and (b_2), we have a contradiction.

Therefore, $Cu(W.Cu)$ cannot hold without $Ad(W.Ad)$ holding.

Lemma 14
$Ad^* \to Cu^*$.

Lemma 14′
$W.Ad^* \to W.Cu^*$.

Proof: Suppose, to the contrary, that Ad^*(respectively $W.Ad^*$) holds but not $Cu^*(W.Cu^*)$. Then there exist A, B and i satisfying the hypotheses of $Cu^*(W.Cu^*)$ but with

(a_1) $S_i(u(B)) \geq S_i(u(A))$ and (b_1) $\exists\ j \neq i \mid S_j(u(B)) < S_j(u(A))$.

Not that $S_i(u(B)) \geq S_i(u(A))$ and $S_i(u(A)) \geq u_i(a)$ for all $a \in A\ \ B$ imply $S_i(u(B)) \geq u_i(a)$ for all $a \in A \backslash B$. Therefore, by $Ad^*(W.Ad^*)$, either

(a_2) $S_i(u(A)) > S_i(u(B))$ or

(b_2) $S_j(u(A)) \leq S_j(u(B))$ for all $j \neq i$.

Two cases may occur:

(i) (a_1) + (b_1) + (a_2) → From (a_1) and (a_2), we have a contradiction.
(ii) (a_1) + (b_1) + (b_2) → From (b_1) and (b_2), we have a contradiction.

Therefore, $Ad^*(W.Ad^*)$ cannot hold without $Cu^*(W.Cu^*)$ holding.

Lemma 15
$Cu + Cu^* + PO \to MC$.

Lemma 15′
$W.Cu + W.Cu^* + PO \to WIIA$.

Proof: Let A and B satisfy the hypotheses of MC.

(i) If $\max u_N(B) = \max u_N(A)$, then given A and its solution $S(u(A))$ let B_1 be obtained from A by eliminating all public decisions of $A \setminus B$ whose first coordinate is smaller than $S_1(u(A))$, i.e.,: $a \in B_1 \leftrightarrow a \in B$ or $[a \in A \setminus B$ and $u_1(a) > S_1(u(A))]$. Since $B_1 \subseteq A$ and $a \in A \setminus B_1$ implies $u_1(a) \leq S_1(u(A))$, Cu and Cu^* apply. Four cases may occur:

1) $Cu(a) + Cu^*(a) \rightarrow S_1(u(B_1)) \geq S_1(u(A)) > S_1(u(B_1))$, a contradiction.
2) $Cu(a) + Cu^*(b) \rightarrow S(u(B_1)) \geq S(u(A))$, which implies $S(u(B_1)) = S(u(A))$ since PO implies $S_N(u(B_1)) = S_N(u(A))$.
3) $Cu(b) + Cu^*(a) \rightarrow S(u(A)) > S(u(B_1))$, a contradiction to $\max u_N(B_1) = \max u_N(A)$.
4) $Cu(b) + Cu^*(b) \rightarrow S_j(u(A)) > S_j(u(B_1)) \geq S_j(u(A))$ for all $j \neq 1$, a contradiction.

This truncation procedure is then iterated. At step k, we define B_k by: $a \in B_k \leftrightarrow a \in B$ or $[a \in A \setminus B$ and $u_l(a) > S_l(u(A))$ for all $l = 1, \cdots, k]$. Applying Cu and Cu^* yields $S(u(B_k)) = S(u(B_{k-1}))$. When $k = n$, we eventually have:

$$S(u(B)) = S(u(B_n)) = S(u(B_{n-1})) = \cdots = S(u(B_1)) = S(u(A)),$$

which is a conclusion of MC.

(ii) If $\max u_N(B) < \max u_N(A)$, then let B' be defined by $B' \equiv B \cup \{a^*\}$ where a^* is a public decision such that $u_N(a^*) = \max u_N(A)$. Since $\max u_N(B') = \max u_N(A)$, applying the same argument as in (i), we have $S(u(B')) = S(u(A))$. Since $u_N(a^*) = \max u_N(A) = S_N(u(A))$, there exists at least one agent, say l, such that $u_l(a^*) \leq S_l(u(A))$, so that Cu and Cu^* apply to B and B'. Four cases may occur:

1) $Cu(a) + Cu^*(a) \rightarrow S_l(u(B)) \geq S_l(u(B')) > S_l(u(B))$, a contradiction.
2) $Cu(a) + Cu^*(b) \rightarrow S(u(B)) \geq S(u(B'))$, which violates PO since $\max u_N(B) < \max u_N(B') = \max u_N(A)$.
3) $Cu(b) + Cu^*(a) \rightarrow S(u(B')) > S(u(B)) \rightarrow S(u(A)) > S(u(B))$.
4) $Cu(b) + Cu^*(b) \rightarrow S_j(u(B')) > S_j(u(B)) \geq S_j(u(B'))$ for all $j \neq l$, and contradiction.

Therefore, we have either $S(u(B)) = S(u(A))$ or $S(u(B)) < S(u(A))$, as desired. Finally, we note that the proof for Lemma 15′ is identical to (i).

Lemma 16
$Ad + Ad^* + PO \rightarrow MA$.

Lemma 16′
$W.Ad + W.Ad^* + PPO \rightarrow WIUA$.

Proof: Let A and B satisfy the hypotheses of MA. We will reconstruct B from A by successive additions.

(i) If $\max u_N(B) = \max u_N(A)$, then let B_1 be defined by: $a \in B_1 \leftrightarrow a \in B$ and $u_1(A) \leq S_1(u(A))$. Since $A \subseteq B_1$ and $a \in B_1 \setminus A$ implies $u_1(a) \leq S_1(u(A))$, Ad and Ad^* apply. Four cases may occur:

1) $Ad(a) + Ad^*(a) \rightarrow S_1(u(A)) \geq S_1(u(B_1)) > S_1(u(A))$, a contradiction.
2) $Ad(a) + Ad^*(b) \rightarrow S(u(A)) \geq S(u(B_1))$, which implies $S(u(B_1)) = S(u(A))$ since PO implies $S_N(u(B_1)) = S_N(u(A))$.
3) $Ad(b) + Ad^*(a) \rightarrow S(u(B_1)) > S(u(A))$, a contradiction to $\max u_N(B_1) = \max u_N(A)$.
4) $Ad(b) + Ad^*(b) \rightarrow S_j(u(B_1)) > S_j(u(A)) \geq S_j(u(B_1))$ for all $j \neq 1$, a contradiction.

This addition procedure is then iterated. At step k, we define B_k by $[a \in B_k \leftrightarrow a \in B$ and $u_l(a) \leq S_l(u(A))$ for some $l = 1, \cdots, k]$. Applying Ad and Ad^* yields $S(u(B_k)) = S(u(B_{k-1}))$. When $k = n$, we eventually have:

$$S(u(B)) = S(u(B_n)) = S(u(B_{n-1}) = \cdots = S(u(B_1)) = S(u(A)),$$

which is a conclusion of MA.

(ii) If $\max u_N(B) > \max u_N(A)$, then let A' be defined by $A' \equiv A \cup \{a^*\}$ where a^* is a public decision such that $u_N(a^*) = \max u_N(B)$. Since $\max u_N(A')) = \max u_N(B)$, applying the same argument as in (i), we have $S(u(A')) = S(u(B))$. From the hypothesis of MA, there exists at least one agent, say l, such that $u_l(a^*) \leq S_l(u(A))$, so that Ad and Ad^* apply to A and A'. Four cases may occur:

1) $Ad(a) + Ad^*(a) \rightarrow S_l(u(A)) \geq S_l(u(A')) > S_l(u(A))$, a contradiction.
2) $Ad(a) + Ad^*(b) \rightarrow S(u(A)) \geq S(u(A'))$, which violates PO since $\max u_N(B) = \max u_N(A') > \max u_N(A)$.

3) $Ad(b) + Ad^*(a) \rightarrow S(u(A')) > S(u(A')) \rightarrow S(u(B)) > S(u(A))$.
4) $Ad(b) + Ad^*(b) \rightarrow S_j(u(A')) > S_j(u(A'))$ for all $j \neq l$, a contradiction.

Therefore, we have either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$, as desired. Finally, we note that the proof for *Lemma* 16′ is identical to (i).

Lemma 17
$MC + PO \rightarrow WIIA$.

Proof: Let A and B satisfy the hypotheses of $WIIA$. By MC, we have either (i) $S(u(B)) = S(u(A))$ or (ii) $S(u(B)) < S(u(A))$. Since $\max u_N(B) = \max u_N(A)$, (ii) is incompatible with PO, so that (i) holds.

Lemma 17′
$MA + PO \rightarrow WIUA$.

Proof: Let A and B satisfy the hypotheses of $WIUA$. Since $\max u_N(B) = \max u_N(A)$, there does not exist $a \in B$ such that $u(a) > S(u(A))$. Therefore, by MA, we have either (i) $S(u(B)) = S(u(A))$ or (ii) $S(u(B)) > S(u(A))$. Since $\max u_N(B) = \max u_N(A)$, (ii) is incompatible with PO, so that (i) holds.

Lemma 18
$MC + WIUA + TI + PO \rightarrow D$.

Proof: Suppose, to the contrary, that MC, $WIUA$, TI and PO hold, but not D. Then there exist A and B such that $S(u(B)) \neq S(u(A))$ and neither $S(u(B)) > S(u(A))$ nor $S(u(B)) < S(u(A))$. Let A_1 and B_1 be defined by

$A_1 \equiv A \cup \{a^*\}$ where a^* is a public decision such that $u(a^*) = S(u(A))$ and

$B_1 \equiv B \cup \{b^*\}$ where b^* is a public decision such that $u(b^*) = S(u(B))$.

By TI, we have

$$S(u(A_1)) = S(u(A)) \text{ and } S(u(B_1)) = S(u(B)).$$

Next, we define A_2 and B_2 by

$$A_2 \equiv \{a \in A_1 \mid u(a) \leq S(u(A))\} \text{ and}$$
$$B_2 \equiv \{a \in B_1 \mid u(a) \leq S(u(B))\}.$$

By *MC* and *PO*, we have

$$S(u(A_2)) = S(u(A_1)) \text{ and } S(u(B_2)) = S(u(B_1)).$$

Now we suppose, without loss of generality, that max $u_N(A) \geq$ max $u_N(B)$ and let

$$A_3 \equiv A_2 \cup \{a'\}$$

where a' is a public decision such that

$$u(a') \leq S(u(A)) \text{ and } u_N(a') = \max u_N(B).$$

By *WIUA* applied to A_2 and A_3, we have

$$S(u(A_3)) = S(u(A_2)).$$

Now we define A_4 by

$$A_4 \equiv \{a \in A_3 \mid u(a) \leq u(a')\}.$$

MC applied to A_3 and A_4 yields either

$$S(u(A_4)) = S(u(A_3)) \text{ or } S(u(A_4)) < S(u(A_3)).$$

Since max $u_N(A_4) =$ max $u_N(B_2)$, invoking *WIUA* to $A_4 \cup B_2$ and A_4, and $A_4 \cup B_2$ and B_2 yields

$$S(u(A_4 \cup B_2)) = S(u(A_4)) \text{ and } S(u(A_4 \cup B_2)) = S(u(B_2)).$$

Therefore, we have either

$$S(u(B)) = S(u(A)) \text{ or } S(u(B)) < S(u(A)),$$

which is in contradiction with the hypothesis.

Lemma 18′

$$MA + WIIA + TI + PO \rightarrow D.$$

Proof: Suppose, to the contrary, that *MA, WIIA, TI* and *PO* hold, but not *D*. Then there exist *A* and *B* such that $S(u(B)) \neq S(u(A))$ and neither $S(u(B)) > S(u(A))$ nor $S(u(B)) < S(u(A))$. Let A_1 and B_1 be defined by

$A_1 \equiv A \cup \{a^*\}$ where a^* is a public decision such that $u(a^*) = S(u(A))$ and

$B_1 \equiv B \cup \{b^*\}$ where b^* is a public decision such that $u(b^*) = S(u(B))$.

By *TI*, we have

$S(u(A_1)) = S(u(A))$ and $S(u(B_1)) = S(u(B))$.

Next, we define A_2 and B_2 by

$A_2 \equiv \{a \in A_1 \mid u(a) \leqq S(u(A))\}$ and

$B_2 \equiv \{a \in B_1 \mid u(a) \leqq S(u(B))\}$.

By *WIIA*, we have

$S(u(A_2)) = S(u(A_1))$ and $S(u(B_2)) = S(u(B_1))$.

Now we suppose, without loss of generality, that max $u_N(A) \geq$ max $u_N(B)$. Since $A_2 \subseteq A_2 \cup B_2$ and $B_2 \subseteq A_2 \cup B_2$, invoking *MA* to $A_2 \cup B_2$ and A_2, and $A_2 \cup B_2$ and B_2, together with *PO*, yields

$S(u(A_2 \cup B_2)) = S(u(A_2))$ and

$[S(u(A_2 \cup B_2)) = S(u(B_2))$ or $S(u(A_2 \cup B_2)) > S(u(B_2))]$.

Therefore, we have either

$S(u(B)) = S(u(A))$ or $S(u(B)) < S(u(A))$,

which is in contradiction with the hypothesis.

Lemma 19

$D + PO \rightarrow SM$.

Proof: Let A and B be such that $A \subseteq B$. By D, either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$ or $S(u(B)) < S(u(A))$. If $S(u(B)) < S(u(A))$, then $S_N(u(B)) < S_N(u(A))$. Now *PO* applied to both sides implies that max $u_N(B) <$ max $u_N(A)$, a contradiction to $A \subseteq B$. Therefore, we have either $S(u(B)) = S(u(A))$ or $S(u(B)) > S(u(A))$.

All the logical implications are summarized in Figure 1.

Remark: The weakness of *TI* can be shown in the following lemmas.

Lemma 20

$W.Ad + PO \rightarrow TI$ (Also, $Ad + PO \rightarrow TI$).

Lemma 20′

$W.Ad^* + PO \rightarrow TI$ (Also, $Ad^* + PO \rightarrow TI$).

Proof: We need to show only that $W.Ad$(respectively $W.Ad^*$) and *PO* imply *TI*. Let B be defined by

$B \equiv A \cup \{a^*\}$

where a^* is a public decision such that $u(a^*) = S(u(A))$.

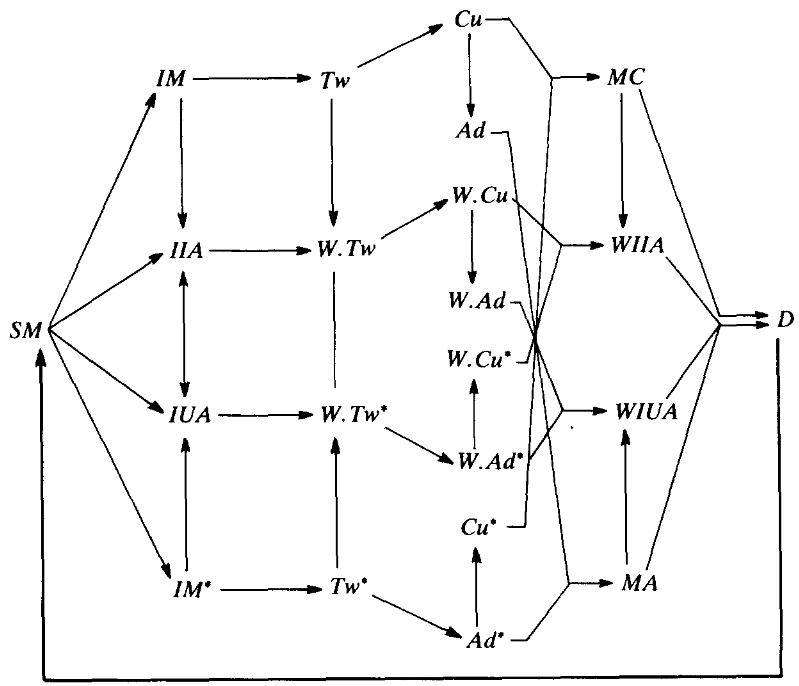

FIGURE 1

Since $u(a^*) \leqq S(u(A))$, we apply $W.Ad$ n times to A and B. The only way to satisfy all n conclusions of $W.Ad(W.Ad^*)$ is either

$$S(u(B)) \leqq S(u(A)) \text{ or } S(u(B)) > S(u(A)).$$

Now by applying PO to both sides, we have

$$S(u(B)) = S(u(A)),$$

as desired.

We conclude with a characterization of all the solutions satisfying D and PO.

Theorem

A solution S satisfies D and PO if and only if there exists a continuous function g: $R \to \mathbb{R}^n$ such that, for all $i = 1, \cdots, n$, g_i is strictly increasing and that, for all $A \in \Sigma$,

$$S(u(A)) = g(\alpha) \text{ where } g_N(\alpha) = \max u_N(A).$$

Proof[1]: Since the proof for the sufficiency part of the Theorem is straightforward, we prove only the necessity part of the Theorem. First, we note that given two sets of public decisions A, $B \in \Sigma$, if $\max u_N(A) = \max u_N(B)$, then D and PO together imply that $S(u(A)) = S(u(B))$. Now let S defined on Σ satisfying D and PO be given. For each $\alpha \in \mathbb{R}$, let $A_\alpha \in \Sigma$ be a set of public decisions such that $\max u_N(A_\alpha) = \alpha$ and let $g(\alpha) \equiv S(u(A))$. Given $\alpha, \beta \in \mathbb{R}$, if $\alpha > \beta$, then D and PO imply that $g(\alpha) > g(\beta)$. Therefore, $g(\cdot)$ must be strictly increasing in all components.

We now show that $g(\cdot)$ is continuous. Suppose that, for some $\alpha \in \mathbb{R}$ and some $i \in N$, $g_i(\alpha) > \sup_{\beta < \alpha} g_i(\beta)$. Then consider $A_\gamma \in \Sigma$ such that $\max u_N(A_\gamma) = \frac{1}{2}\{g_N(\alpha) + \sup_{\beta < \alpha} g_N(\beta)\}$. Since $g(\cdot)$ is strictly increasing in all components, we have $\sup_{\beta < \alpha} g_N(\beta) < \max u_N(A) < g_N(\alpha)$, which implies $g_N(\beta) < \max u_N(A_\gamma)$ for all $\beta < \alpha$ and $\max u_N(A_\gamma) < g_N(\beta)$ for all $\beta \geq \alpha$. So there does not exist an $\alpha \in \mathbb{R}$ such that $g_N(\alpha) = \max u_N(A_\gamma)$, a contradiction. A similar argument proves $g_i(\alpha) = \inf_{\beta < \alpha} g_i(\beta)$.

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[1]This proof is closely related to that of Thomson and Myerson (1981, p. 74).