On the Length of the Decision Period under Uncertainty: The Case of the Price-setting Commercial Banking Firm

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The purpose of this essay is to determine how often a price-setting commercial banking firm facing uncertainty should optimally change its loan and deposit rates. Defining the decision period as the span of time over which the banking firm sets and does not intend to alter its interest rates, it examines the relationship between the length of the decision period and the optimizing behavior of the bankig firm under uncertainty.

I. Introduction

Period analysis is a tool of economic analysis in which synchronous and sequential studies are made of events in a continuous economic process divided into a sequence of time segments, or periods. However, the consideration hitherto given in economics to periods has not been extensive. The few instances include Marshall's three time periods and Foley's postulate that, considering the properties of a theoretical model, the length of the period is not an important parameter.[1] Most analyses in economics have, therefore, assumed the length of the period as exogenously given. Nevertheless, it is an undeniable fact that economic agents show different patterns of optimizing behavior for different periodizations. Furthermore, the significance of periodization increases when the closely related issue of uncertainty is involved.

Focusing on the much ignored matter of periodization, this paper examines the relationship between the length of the decision period and the optimizing behavior of economic agents under uncertainty. Specifically, we will consider the optimizing behavior of a

[1] Foley(1975).

price-setting commercial banking firm. If the decision period is defined as the span of time over which the banking firm sets and does not intend to alter interest rates on loans and deposits, then the problem is how often it should optimally change interest rates.

II. The Model

A. Uncertainty and Banking

Edgeworth may be cited as the first economist to perceive that uncertainty is the basis of commercial banking and to thereupon attempt to elucidate this matter by means of an inventory theoretic approach.[2] The behavior of Edgeworth's commercial banking firm can be summarized as follows:

The expected-profit-maximizing banker holds a certain amount of deposits at the beginning of a period. He allocates part of this sum to loans and securities, and retains the remainder as reserve in cash. The banker now faces a kind of uncertainty since the deposits he initially held may be withdrawn anytime. There will be no problem if the withdrawal is of a negligible size which can be adequately covered by his cash reserve. However, if the withdrawal demand exceeds the reserve, the banker must borrow against his securities as collateral. If this is not enough to meet the withdrawal demand, the banker will be forced to resort to the central bank's discount window for additional funds against his loans as collateral. Both of these types of borrowing entail transaction costs, which are larger in the latter case due to the penal nature of the higher interest rates on central-bank loans.

Since cash bears no explicit earnings, the banker tries to minimize his cash holdings. However, because of the transaction costs of borrowing to supplement a deficient reserve, he cannot reduce his cash holdings excessively. A balance must be struck between liquidity and profitability. Therefore, if the banker knows the probability distribution of the withdrawal demand, a random variable, he will determine the optimal amounts of loans and securities (and, accordingly, cash) in his attempt to maximize expected profit.

Such an attempt by Edgeworth to study uncertainty by grafting probability theory onto the theory of the banking firm was undoubtedly an unparalleled contribution to the 19th-century money

[2]Edgeworth(1888).

TABLE 1
BALANCE SHEET OF A COMMERCIAL BANKING FIRM

Assets	Liabilities
Loans Cash	Deposits

and banking theory. However, notwithstanding the achievement of having formalized the problem, Edgeworth's analysis has its limitation in that it lacks a quantitative approach using calculus.

B. The Model

We will set up an optimization model based on Edgeworth's framework, incorporating the concept of the decision period. For simplicity, assume that there are no securities, so that the bank handles only loans and deposits.[3]

Table 1 shows the balance sheet of such a hypothetical commercial banking firm. The bank receives depostis, loaning a part of the sum and holding the remainder in cash.

We assume that the bank is a price-setter in both the loan and deposit markets, and that its deposit and loan functions are as follows:

A) The Deposit Function

Deposits received by the bank during a certain period are assumed to be

$$d = \beta_1 + \beta_2 i - \theta \quad (\beta_1,\ \beta_2 > 0), \tag{1}$$

where i is the interest rate on deposits. $\beta_2 > 0$ since higher interest rates attract more deposits, and $\beta_1 > 0$ since the bank still receives deposits even when $i = 0$, i.e., even when all deposits are demand deposits that do not pay interest.[4] θ is a random variable showing the amount of unforeseen deposit withdrawals, assumed to have probability density function

$$f(\theta) = \frac{1}{2Z} \quad (-Z \leq \theta \leq Z,\ Z > 0). \tag{2}$$

[3]This assumption does not affect the essential features of the analysis. See Appendix A for an analysis with securities.

[4]For example, checking accounts.

Z determines the upper and lower bounds of θ, and its magnitude is assumed to be dependent on the length of the decision period. This property of the variable Z will be discussed further.

B) The Loan Function

The bank's loan function, free from defaults on the part of borrowers, is assumed to be

$$l = \alpha_1 - \alpha_2 r \ \ (\alpha_1, \ \alpha_2 > 0), \tag{3}$$

where r is the interest rate on loans. The expression shows that the demand for loans falls as the interest rate on loans rises and that there always exists a demand for loans when r = 0.

C) Cash Reserve

Since balance sheets must always balance, from Table 1 we have

$$c = d - l \tag{4}$$

Since $E(\theta) = 0$ from (2), the expectation of (1) is

$$\begin{aligned} E(d) &= \beta_1 + \beta_2 i - E(\theta) \\ &= \beta_1 + \beta_2 i, \end{aligned} \tag{1$'$}$$

from which follows

$$d = E(d) - \theta. \tag{5}$$

Substituting (5) into (4), we obtain

$$c = E(d) - \theta - l. \tag{4$'$}$$

D) The Length of the Decision Period and the Probability Distribution of Cash Withdrawls

Let T denote the length of the banking firm's decision period. This means that the bank sets and does not alter the interest rates i and r for a period of length T, after which it newly determines i' and r' for the subsequent period of length T'. A large T indicates that the bank does not alter i and r for a considerable length of time (i.e., T), despite continually changing circumstances. The range of unforeseen withdrawals (or, for a negative θ, deposits) widens in this case.

On the other hand, a small T means that the bank responds to external changes by continually adjusting i and r and thereby reduces the domain θ (i.e., Z). Thus Z is an increasing function of T.

Specifically, we assume that

$$Z = zT \ (z > 0), \tag{6}$$

where z is a parameter reflecting the extent to which changes in the length of the decision period affect the degree of uncertainty.

E) The Cost Function

As is previously noted, reducing T constricts the domain of θ, thereby reducing uncertainty. However, reducing T makes it necessary for the banker to adjust i and r more frequently, which entails various costs.

First, there is the search cost involved in determining new values of i and r. The parameters α_1, α_2, β_1, and β_2 must be frequently re-estimated in order to find the new optimal values i' and r'. In addition, there is also the cost of informing customers of these new rates, which may be called the advertising cost. Finally, there is the intangible cost of negative effects on customers' confidence caused by frequent alterations of interest rates, which is a kind of psychological cost. Since banking operations should essentially pursue stability, frequent adjustments of interest rates might cause customers to feel insecure about the bank. If this prompts a customer to take his money to another bank, the psychological effects constitute a real cost to the banker.

Therefore, in general, the cost C of reducing T can be said to be inversely related to T. We assume that

$$C = \frac{a}{T} \ (a > 0). \tag{7}$$

F) Penalties

We discussed earlier the case where the banking firm, faced with insufficient cash to cover withdrawals, borrows against its securities or loans as collateral. However, since we assume that securities do not exist, we will discuss only the case of borrowing from the central bank against loans as collateral. Assume that a heavy penalty is imposed on such borrowings at the rate of n for each dollar. Then the amount of the penalty is

$$n(\theta - \mathrm{E}(c)) = n(\theta - \mathrm{E}(d) + l). \tag{8}$$

C. The Objective Function

Let us now see how the expected-profit-maximizing banking firm behaves under the given assumptions. The bank first estimates its loan, deposit, and cost functions. With this information it then determines the optimal values i_0^* , r_0^* , and T_0^* , which maximize expected profit. After maintaining i_0^* and r_0^* for a period of T_0^* , the banking firm then determines the new values i_1^* , r_1^* , and T_1^* by the same process. It should be noted here that the banking firm does not determine an optimal value of T for altering i and r at regular intervals.[5] Rather, it is concerned only with the duration of the immediately following values of i and r [6]. Thus it is possible for the initial value T_0^* and the subsequent value T_1^* to differ from each other. In other words, the banking firm can alter its interest rates at irregular intervals.[7]

Another point deserves mention in the determination of T. It can be seen from (6) that T influences Z, which determines the domain of θ. A larger T results in an increase in Z, since both i and r remain unaltered for a longer span of time. It then becomes possible that the bank may run short of its cash reserve since θ also might take a large value. The banking firm must then borrow from the central bank at the cost of n for each dollar. On the other hand, a smaller value of T lessens the probability of θ taking a value so large as to result in a cash-reserve shortage. If T is sufficiently small as to make Z smaller than its cash reserve, the banking firm will not face a cash shortage for any θ. In other words, the banking firm need not consider penalties in this case.

We can then think of two kinds of objective functions for the banking firm.

A) The State of Pseudo Certainty

Consider first the case where T is so small as to preclude the possibility of Z taking values larger than the bank's cash reserve. Although uncertainty still exists since θ does vary within its do-

[5]For example, the bank is not concerned with whether or not it should alter its interest rates every 3 months or 6 months.

[6]For example, the current rates of $i^*_0 = 5\%$ and $r^*_0 = 6\%$ will be maintained for 18 months, after which i^*_1, r^*_1, and T^*_1 will be newly detemined.

[7]For example, even if the first period was 18 months, it is possible for the new rates i^*_1 and r^*_1 to be maintained for only 6 months.

main, in effect there is little difference from a state of certainty in the sense that such variations in θ are almost meaningless. Accordingly, this may be called the state of pseudo certainty.

From (4)', $c \geq 0$ is equivalent to

$$c = \mathrm{E}(d) - \theta - l \geq 0,$$

or

$$-z \leq \theta \leq \mathrm{E}(d) - l, \tag{9}$$

which means $c \geq 0$ is always true if $\theta \leq \mathrm{E}(d) - l$. Therefore pseudo certainty is always guaranteed when

$$z \leq \mathrm{E}(d) - l. \tag{10}$$

Substituting (6) into (10) yields

$$zT \leq \mathrm{E}(d) - l,$$

or

$$T \leq \frac{1}{z}(\mathrm{E}(d) - l). \tag{11}$$

This means penalties can be disregarded regardless of θ if T is set at a value smaller than $\frac{1}{z}(\mathrm{E}(d) - l)$.

The expected profit $\mathrm{E}(\pi)_1$ will then be

$$\mathrm{E}(\pi)_1 = rl - i\mathrm{E}(d) - \frac{a}{T}. \tag{12)'$$

B) The State of Pure Uncertainty

Now we will consider the case where T is larger than $\frac{1}{z}(\mathrm{E}(d) - l)$. The expected-profit function then consists of two parts:

$$\begin{aligned}\mathrm{E}(\pi)_2 = &\int_{-z}^{\mathrm{E}(d)-l} (rl - id - \frac{a}{T})\frac{1}{2z}\,\mathrm{d}\theta \\ &+ \int_{\mathrm{E}(d)-l}^{z} \{(rl - id - \frac{a}{T}) - n(\theta - \mathrm{E}(d) + l)\}\frac{1}{2z}\mathrm{d}\theta\,.\end{aligned} \tag{13}$$

It can be seen from (13) that the banking firm never pays penalties when θ is smaller than $\mathrm{E}(d) - l$ since $c \geq 0$, and that it pays a penalty of n for each dollar of the excess amount when θ is larger than $\mathrm{E}(d) - l$.

Rearranging (13), we obtain

$$\mathrm{E}(\pi)_2 = \int_{-z}^{z} (rl - id - \frac{a}{T})\frac{1}{2z}\,\mathrm{d}\theta - \int_{\mathrm{E}(d)-l}^{z} n(\theta - \mathrm{E}(d) + l)\frac{1}{2z}\,\mathrm{d}\theta \qquad (13)'$$

$$= rl - i\mathrm{E}(d) - \frac{a}{T} - \int_{\mathrm{E}(d)-l}^{z} n(\theta - \mathrm{E}(d) + l)\frac{1}{2z}\,\mathrm{d}\theta .$$

Substituing $Z = zT$ into (13)′ finally gives us

$$\mathrm{E}(\pi)_2 = rl - i\mathrm{E}(d) - \frac{a}{T} - \frac{n}{2}[\frac{1}{2}zT - (\mathrm{E}(d) - l) + \frac{1}{2zT}(\mathrm{E}(d) - l)^2]. \qquad (14)$$

The banking firm's maximization problem can now be summarized as follows:

$$\max_{i,r,T} \mathrm{E}(\pi) = \begin{cases} \mathrm{E}(\pi)_1, \text{ if } 0 < T \leq \frac{1}{z}(\mathrm{E}(d) - l). \\ \mathrm{E}(\pi)_2, \text{ if } T > \frac{1}{z}(\mathrm{E}(d) - l). \end{cases} \qquad (15)$$

Several assumptions may be further adopted at this stage before solving the maximization problem. We assume that deposits, cash reserve, and loans are always nonnegative. Mathematically,

$$\begin{aligned} \mathrm{E}(d) &= \beta_1 + \beta_2 i \geq 0 \\ l &= \alpha_1 - \alpha_2 r \geq 0 \\ \mathrm{E}(c) &= \mathrm{E}(d) - l \geq 0. \end{aligned} \qquad (16)$$

III. The Maximization Problem

A. Profit Maximization under Pseudo Certainty

Differentiate $\mathrm{E}(\pi)_1$ with respect to i, r, and T to obtain the first-order conditions of profit maximization:

$$\frac{\partial \mathrm{E}(\pi)_1}{\partial i} = -\beta_1 - 2\beta_2 i \qquad (17)$$

$$\frac{\partial \mathrm{E}(\pi)_1}{\partial r} = \alpha_1 - 2\alpha_2 r \qquad (18)$$

$$\frac{\partial \mathrm{E}(\pi)_1}{\partial T} = \frac{a}{T^2} \qquad (19)$$

Taking (17) first, $\partial E(\pi)_1 / \partial i < 0$ since both β_1 and β_2 are positive and i is nonnegative. This means that $E(\pi)_1$ increases as i falls. But since i is non-negative by assumption,

$$i^* = 0 \tag{20}$$

This tells us that in equilibrium under pseudo certainty, every bank handles only demand deposits.[8]

Next, setting (18) equal to 0 yields

$$r^* = \frac{\alpha_1}{2\alpha_2} \ (> 0). \tag{21}$$

Also, we can see from (19) that $\partial E(\pi)_1 / \partial T > 0$, which means that $E(\pi)_1$ increases with T for constant values of i^* and r^*. This happens because an increase in T lowers costs while leaving revenue unaffected. Since the function $E(\pi)_1$ is defined over the interval $0 < T \leq \frac{1}{z}\ (E(d) - l)$,

$$T^* = \frac{1}{z}\ (E(d)^* - l^*). \tag{22}$$

Let us now consider $E(\pi)_2$. The banking firm's true maximum expected profit will be the larger of the maxima of $E(\pi)_1$ and $E(\pi)_2$.

B. Profit Maximization under Pure Uncertainty

The first-order conditions for maximizing $E(\pi)_2$ are;

$$\begin{aligned} \frac{\partial E(\pi)_2}{\partial i} &= -\beta_1 - 2\beta_2 i - \frac{n}{2}[-\beta_2 + \frac{(E(d) - l)}{zT}\ \beta_2] \\ &= -\beta_1 - 2\beta_2 i + \frac{n\beta_2}{2zT}[zT - (E(d) - l)] = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial E(\pi)_2}{\partial r} &= \alpha_1 - 2\alpha_2 r - \frac{n}{2}[-\alpha_2 + \frac{(E(d) - l)}{zT}\ \alpha_2] \\ &= \alpha_1 - 2\alpha_2 r + \frac{n\alpha_2}{2zT}\ [zT - E(d) - l\ )] = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial E(\pi)_2}{\partial T} &= \frac{a}{T^2} - \frac{n}{2}[\frac{z}{2} - \frac{(E(d) - l)^2}{2zT^2}] \\ &= \frac{1}{T^2}\{a - \frac{n}{4}[zT^2 - \frac{1}{z}\ (E(d) - l)^2]\} = 0 \end{aligned} \tag{25}$$

[8] Deposits with $i = 0$ are regarded as identical to demand deposits. Of course, it is perfectly possible for demand deposits to have $i > 0$ in reality.

First calculate $\left.\frac{\partial \mathrm{E}(\pi)_2}{\partial T}\right|_{T=(\mathrm{E}(d)-l)/z}$

from (25): $\left.\frac{\partial \mathrm{E}(\pi)_2}{\partial T}\right|_{T=(\mathrm{E}(d)-l)/z}$

$$T=\frac{1}{T^2}[a-\frac{n}{3}(zT^2-zT^2)] \tag{26}$$

$$=\frac{a}{T^2}>0.$$

It can be seen that at the borderline between pure uncertainty and pseudo certainty, expected profit can be increased by increasing T. But $\mathrm{E}(\pi)_1$ is maximized at the borderline $T=\frac{1}{z}(\mathrm{E}(d)-l)$. Therefore the maximum of $\mathrm{E}(\pi)_2$ is larger than that of $\mathrm{E}(\pi)_1$. Accordingly, we will focus only on $\mathrm{E}(\pi)_2$ for the remaining discussion.

Adding up (23) and (24) after multiplying through by $-\alpha_2$ and β_2 respectively,

$$\alpha_2\beta_1+2\alpha_2\beta_2 i+\beta_2\alpha_1-2\beta_2\alpha_2 r=0,$$

which yields

$$r^*=i^*+\frac{\alpha_2\beta_1+\beta_2\alpha_1}{2\alpha_2\beta_2}\ (>i^*). \tag{27}$$

Thus in equilibrium r^* is always larger than i^*, and the magnitudes of changes resulting from variations in the exogenous variables, a, n, and z are the same for i^* and r^*.

Multiplying (25) through by T^2,

$$a-\frac{n}{4}(zT^2-\frac{1}{z}\ (\mathrm{E}(d)-l)^2)=0,$$

from which we obtain

$$zT^2=\frac{4a}{n}+\frac{1}{z}(\mathrm{E}(d)-l)^2$$

Therefore,

$$T^*=\sqrt{\frac{4a}{nz}+\frac{1}{z^2}(\mathrm{E}(d)-l)^2}. \tag{28}$$

Since $\frac{4a}{nz}>0$, T^* is larger than $\frac{1}{z}$ $(\mathrm{E}(d)-l)$, which confirms our previous finding that the maximum of $\mathrm{E}(\pi)_2$ is larger than that of

$\mathrm{E}(\pi)_1$.

C. A Comparative Static Analysis

Subtracting (3) from (1)′, we obtain

$$\mathrm{E}(d) - l = \beta_1 + \beta_2 i - \alpha_1 + \alpha_2 r. \tag{29}$$

Substituting (27) into (29) yields

$$\mathrm{E}(d) - l = \beta_1 + \beta_2 i - \alpha_1 + \alpha_2 (i + \frac{\alpha_2 \beta_1 + \beta_2 \alpha_1}{2 \alpha_2 \beta_2})$$

$$= (\alpha_2 + \beta_2) i + k \tag{30}$$

$$(k = \beta_1 - \alpha_1 \frac{\alpha_2 \beta_1 + \beta_2 \alpha_1}{2 \beta_2})$$

Substituting into (23) and (28) and after some manipulations,

$$n \beta_2 [zT - (\alpha_2 + \beta_2) i - k] = 2zT(\beta_1 + 2\beta_2 i) \tag{31}$$

$$z^2 T^2 = \frac{4za}{n} + [(\alpha_2 + \beta_2) i + k]^2, \tag{32}$$

from which we obtain $T^* = T^*(a,n,z)$ and $i^* = i(a,n,z)$. Also, from (27), we obtain r^*. The remaining discussion focuses only on the influences that the exogenous variables a, n and z exert on the equilibrium decision period T^*.

A) T^* and a

Differentiating (32) with respect to a and rearranging in terms of $\frac{\partial i^*}{\partial a}$,[9]

$$\frac{\partial i^*}{\partial a} = \frac{z^2 T \, \partial T^* / \partial a - 2z/n}{(\mathrm{E}(d) - l)(\alpha_2 + \beta_2)} \tag{33}$$

Substituting (33) into (31) differentiated with respect to a and rearranging in terms of $\frac{\partial T^*}{\partial a}$ yields

$$\frac{T^*}{a} \{ n \beta_2 (\alpha_2 + \beta_2) \overbrace{[zT - (\mathrm{E}(d) - l)]}^{(+)} + 2(\mathrm{E}(d) - l)$$

$$(\alpha_2 + \beta_2)(\beta_1 + 2\beta_2 i) + 4\beta_2 z^2 T^2 \} \tag{34}$$

$$= 2\beta_2 (\alpha_2 + \beta_2) + \frac{8\beta_2}{n} zT.$$

[9]See Appendix B for details of calculations below.

Therefore

$$\frac{\partial T^*}{\partial a} > 0. \tag{34)'}$$

A large value of a indicates that it is costly to reduce T. Therefore, the content of (34)′ is that the bank will refrain from reducing T^* excessively when faced with the increasing cost of doing so.

B) T^* and n

Differentiating (32) with respect to n yields

$$\frac{\partial i^*}{\partial n} = \frac{2nz^2\,T\partial T^*/\partial n + [z^2\,T^2 - (\mathrm{E}(d) - l)^2]}{2n(\mathrm{E}(d) - l)(\alpha_2 + \beta_2)} \tag{35}$$

Substituting (35) into (31) differentiated with respect to n and rearranging in terms of $\frac{\partial T^*}{\partial n}$,

$$\begin{aligned}\frac{\partial T^*}{\partial n}\{2n^2\beta_2(\alpha_2 + \beta_2)z\underbrace{[zT - (\mathrm{E}(d) - l)]}_{(+)} + 4nz(\beta_1 + 2\beta_2 i)\\ (\alpha_2 + \beta_2)(\mathrm{E}(d) - l)\} + 8n\beta_2 z^3T^2\}\\ = \underbrace{-n\beta_2(\alpha_2 + \beta_2)[zT - (\mathrm{E}(d) - l)]^2}_{(-)}\\ - \underbrace{4\beta_2 zT[Z^2T^2 - (\mathrm{E}(d) - l)^2]}_{(-)}.\end{aligned} \tag{36}$$

Therefore

$$\frac{\partial T^*}{\partial n} < 0 \tag{36)'}$$

Higher values of n reflect heavier penalties on cash shortages. The bank, accordingly, will adjust T^* to a lower value to avoid the possibility of taking penalties, since shortening T^* reduces uncertainty and consequently the domain of θ. Thus we can conclude that T^* and n move in opposite directions.

C) T^* and z

Differentiating (32) with respect to z and rearranging in terms of $\frac{\partial i^*}{\partial z}$ yields

$$\frac{\partial i^*}{\partial z} = \frac{z^2 T\, \partial T^* / \partial z + (zT^2 - 2a/n)}{(\mathrm{E}(d) - l)(\alpha_2 + \beta_2)} \tag{37}$$

Substituting into (31) differentiated with respect to z and rearranging in terms of $\frac{\partial T^*}{\partial z}$,

$$\frac{\partial T^*}{\partial z}\{n\beta_2(\alpha_2 + \beta_2)z\underbrace{[zT - (\mathrm{E}(d) - l)]}_{(+)} + 2z(\beta_1 + 2\beta_2 i)$$

$$(\mathrm{E}(d) - l)(\alpha_2 + \beta_2) + 4\beta^2 z^3 T^2\}$$

$$= \underbrace{-n\beta_2(\alpha_2 + \beta_2)\frac{1}{2z}[zT - (\mathrm{E}(d) - l)]^2}_{(-)} \tag{38}$$

$$\underbrace{-\ 2T(\beta_1 + 2\beta_2 i)(\mathrm{E}(d) - l)(\alpha_2 + \beta_2)}_{(-)}$$

$$\underbrace{-\ 4\beta_2 z T(zT^2 - \frac{2a}{n})}_{(-)}$$

Therefore

$$\frac{\partial T^*}{\partial z} < 0 \tag{38$'$}$$

An increase in z means that z reacts more sensitively to a change in T^*. That is, uncertainty will be greatly reduced as T^* is shortened, since T^* and uncertainty will become more closely related to each other. (38)$'$ tells us that T^* will become correspondingly smaller the greater the reduction of uncertainty accompanying a shortening of T^*.

IV. Conclusion

So far we have examined how a price-setting commercial banking firm under uncertainty determines its decision period. The results can be summed up as follows:

1) The maximum of $\mathrm{E}(\pi)_2$ is larger than that of $\mathrm{E}(\pi)_1$. This shows that the banking firm's objective of maximizing expected profit is better attained when it risks a certain degree of uncertainty than

when it reduces T excessively in order to reach pseudo certainty.

2) r^* is always larger than i^*. Also, $\frac{\partial r^*}{\partial a} = \frac{\partial i^*}{\partial a}$. Thus not only the directions but also the magnitudes of changes in r^* and i^* resulting from disturbances in exogenous variables are the same.

3) T^* is determined at smaller value the less the costs of reducing T^*, the heavier the penalties for not reducing T^*, and the more the reduction of uncertainty accompanying a shortening of T^*. This is because reducing T^* ultimately constricts the domain of the random variable θ.

Appendix A

A General Model of the Determination of the Length of the Decision Period

For convenience, it was assumed that securities do not exist. Let us now build a model which allows for the existence of securities.[10]

1) Assume the deposit function to take the general form as follows:

$$d = D(i) - \theta \quad (D'(i) > 0). \tag{A-1}$$

2) The probability density function of θ, f(θ ; t), has the following characteristics:

$$\begin{cases} -zT \leq \theta \leq zT \\ \mathrm{E}(\theta) = 0 \\ \dfrac{d\,(\mathrm{Var}(\theta))}{\mathrm{d}T} > 0 \end{cases} \tag{A-2}$$

The content of (A-2) is that the probablity of being more widely distributed around the mean increases with T.

3) Assume the loan function to be

$$l = l(r) \quad (l'(r) \leq 0). \tag{A-3}$$

4) Securities can be obtained in any amount in the market. The rate

[10]We only present a general model and do not go into specific analysis since it is similar to that of the main text.

of return on securities g is an exogenous variable determined in the securities market.

5) Since
$$\begin{aligned} d &\equiv c + l + s, \\ c &\equiv d - l - s \\ &= \mathrm{E}(d) - l - s - \theta, \end{aligned} \tag{A-4}$$

where s denotes the amount of securities held by the bank. Further, assuming that the expectation of the amount of cash the bank wants to hold is positive,

$$\mathrm{E}(c) = \mathrm{E}(d) - l - s \ (\geq 0). \tag{A-5}$$

6) The cost function of keeping the decision period fixed at T is assumed to be

$$C = C(T) \quad (C'(T) < 0,\ C''(T) > 0) \tag{A-6}$$

which means that the cost of shortening T increases progressively.

7) Borrowings necessitated by unexpectedly large withdrawal demands are made against securities as collateral. The cost of such borrowings is assumed to be m for each dollar. When withdrawal demands cannot be met with such borrowings, it is asumed that the bank will borrow the deficient amount from the central bank at the cost of $n(>m)$ for each dollar against its loans as collateral.

Let us now see what values of θ necessitate borrowings. First, if $c \geq 0$, then no borrowings are necessary. Thus any withdrawal demand can be met by the bank's cash reserve if

$$c = \mathrm{E}(d) - l - s - \theta \geq 0,$$

or

$$-zT \leq \theta \leq \mathrm{E}(d) - l - s \ (= \mathrm{E}(c)). \tag{A-7}$$

Now posit the case where a large withdrawal demand can be met only by borrowing against securities as collateral.
Then

$$-s \leq c < 0 \longleftrightarrow -s < \mathrm{E}(d) - l - s - \theta < 0.$$

That is, the cash shortage can be met with borrowings against securities if

$$(\mathrm{E}(c) =)\mathrm{E}(d) - l - s < \theta \leq \mathrm{E}(d) - l \tag{A-8}$$

In the case where borrowings against securities are inadequate to meet a very large θ,

$$c = \mathrm{E}(d) - l - s - \theta < -s,$$

i.e.,

$$\mathrm{E}(d) - l < \theta \leq zT \tag{A-9}$$

Let us now find the specific objective function. First, if $zT \leq \mathrm{E}(c)$, i.e., if $0 < T \leq \frac{1}{z}(\mathrm{E}(d) - l - s)$, then borrowings do not need to be considered since the domain of θ will be as given by (A-7). The bank's expected profit here is

$$\mathrm{E}(\pi)_1 = rl + gs - i\mathrm{E}(d) - C(T). \tag{A-10}$$

Next, if $\mathrm{E}(d) - l - s < zT < \mathrm{E}(d) - l$, i.e., if $\frac{1}{z}\mathrm{E}(c) < T \leq \frac{1}{z}(\mathrm{E}(d) - l)$, the bank may find itself in need of emergency borrowings, which can be met with borrowings against securities. The expected profit in this case is

$$\begin{aligned} \mathrm{E}(\pi)_2 = {} & \mathrm{E}(\pi)_1 \\ & - m \int_{\mathrm{E}(d)-l-s}^{zT} (\theta - \mathrm{E}(d) + l + s)\ \mathrm{f}(\theta;\ T)\ \mathrm{d}\theta. \end{aligned} \tag{A-11}$$

Lastly, borrowings from the central bank are taken into consideration when $zT > \mathrm{E}(d) - l$, i.e., when $T > \frac{1}{z}(\mathrm{E}(d) - l)$. The expected profit then is

$$\begin{aligned} \mathrm{E}(\pi)_3 = {} & \mathrm{E}(\pi)_1 - m \int_{\mathrm{E}(d)-l-s}^{\mathrm{E}(d)-l} (\theta - \mathrm{E}(d) + l + s)\ \mathrm{f}(\theta;\ T)\mathrm{d}\theta \\ & - \int_{\mathrm{E}(d)-l}^{zT} [ms + n\ (\theta - \mathrm{E}(d) + l)]\ \mathrm{f}(\theta;\ T)\mathrm{d}\theta. \end{aligned} \tag{A-12}$$

The bank's maximization problem can be summed up as follows:

$$\max_{l,s,T} \mathrm{E}(\pi) = \begin{cases} \mathrm{E}(\pi)_1, \text{ if } 0 < T \leq \dfrac{1}{z}(\mathrm{E}(d) - l - s). \\ \mathrm{E}(\pi)_2, \text{ if } \dfrac{1}{z}(\mathrm{E}(d) - l - s) < T < \dfrac{1}{z}(\mathrm{E}(d) - l). \\ \mathrm{E}(\pi)_3, \text{ if } \dfrac{1}{z}(\mathrm{E}(d) - l) < T. \end{cases} \tag{A-13}$$

Appendix B

Derivation of the Comparative Static Analysis of III-C

The first-order conditions for maximization are:

$$n\beta_2[zT-(\alpha_2+\beta_2)\,i+k]=2zT(\beta_1+2\beta_2 i) \tag{31}$$

$$z^2T^2=\frac{4za}{n}+[(\alpha_2+\beta_2)i+k]^2 \tag{32}$$

1) *Derivation of* $\frac{\partial T^*}{\partial a}>0$.

Differentiation of (32) with respect to a yields

$$2z^2T^*\frac{\partial T}{\partial a}=\frac{4z}{n}+2(\mathrm{E}(d)-l)(\alpha_2+\beta_2)\frac{\partial i^*}{\partial a'} \tag{B-1}$$

which is (33) when rearranged in terms of $\frac{\partial i^*}{\partial a}$. Next, differentiating (31) with respect to a, we obtain

$$\begin{aligned} &n\beta_2[z\frac{T^*}{\partial a}-(\alpha_2+\beta_2)\frac{\partial i^*}{\partial a}] \\ &=2z(\beta_1+2\beta_2 i)\frac{\partial T^*}{\partial a}+4\beta_2 zT\frac{\partial i^*}{\partial a} \end{aligned} \tag{B-2}$$

Substituting (33) into (B-2) yields

$$\begin{aligned} &n\beta_2[z\frac{\partial T^*}{\partial a}-\frac{1}{\mathrm{E}(d)-l}(z^2T\frac{\partial T^*}{\partial a}-\frac{2z}{n})] \\ &=2z(\beta_1+2\beta_2 i)\frac{\partial T^*}{\partial a}+\frac{4\beta_2 zT}{(\mathrm{E}(d)-l)(\alpha_2+\beta_2)} \\ &\qquad(z^2T\frac{\partial T}{\partial a}-\frac{2z}{n}), \end{aligned} \tag{B-3}$$

which is (34) when rearranged in terms of $\frac{\partial T^*}{\partial a}$. Recalling that $\mathrm{E}(c)=\mathrm{E}(d)-l\geq 0$ and $T\geq\frac{1}{z}(\mathrm{E}(d)-l)$, we obtain inequality (34)′, i.e., $\frac{\partial T^*}{\partial a}>0$.

2) *Derivation of* $\frac{\partial T^*}{\partial n}<0$

Multiplying (32) through by n,

$$nz^2T^2=4za+n[(\alpha_2+\beta_2)i+k]^2 \tag{32$'$}$$

Differentiating (32) with respect to n yields (35). Next, differentiating (31) with respect to n,

$$\begin{aligned} &\beta_2[zT-(\mathrm{E}(d)-l)]+n\beta_2[z\frac{\partial T^*}{\partial n}-(\alpha_2+\beta_2)\frac{\partial i^*}{\partial n}] \\ &=2z(\beta_1+2\beta_2 i)\frac{\partial T^*}{\partial n}+4zT\ \beta_2\frac{\partial i^*}{\partial n} \end{aligned} \tag{B-4}$$

Substituting (35) into (B-4) yields

$$
\begin{aligned}
&\beta_2[zT-(\mathrm{E}(d)-l)]+n\beta_2 z\frac{\partial T^*}{\partial n}-\frac{n\beta_2}{(\mathrm{E}(d)-l)}\\
&\times\{2nz^2T\frac{\partial T^*}{\partial n}+[z^2T^2-(\mathrm{E}(d)-l)^2]\}\\
&=2z(\beta_1+2\beta_2 i)\frac{\partial T^*}{\partial n}+\frac{4\beta_2 zT}{2n(\mathrm{E}(d)-l)(\alpha_2+\beta_2)}\\
&\{2nz^2T\frac{\partial T^*}{\partial n}+[z^2T^2\ (E(d)-l)^2]\}
\end{aligned}
\tag{B-5}
$$

Rearranging (B-5) in terms of $\frac{\partial T^*}{\partial n}$, we obtain

$$
\begin{aligned}
&\frac{\partial T^*}{\partial n}\{2n^2\beta_2(\alpha_2+\beta_2)z[zT-(\mathrm{E}(d)-l)]\\
&+4nz(\beta_1+2\beta_2 i)(\alpha_2+\beta_2)(\mathrm{E}(d)-l)+8n\beta_2 z^3T^2\}\\
&=-n\beta_2(\alpha_2+\beta_2)[h(T)]-4\beta_2 zT[z^2\ T^2-(\mathrm{E}(d)-l)^2]\\
&\big(h(T)=z^2T^2-(\mathrm{E}(d)-l)^2-2(\mathrm{E}(d)-l)[zT-(\mathrm{E}(d)-l)]\big).
\end{aligned}
\tag{B-6}
$$

But since

$$
\begin{aligned}
h(T)&=[zT+(\mathrm{E}(d)-l)][zT-(\mathrm{E}(d)-l)]\\
&\quad-2(\mathrm{E}(d)-l)[zT-(\mathrm{E}(d)-l)]\\
&=[zT-(\mathrm{E}(d)-l)][zT+(\mathrm{E}(d)-l)-2(\mathrm{E}(d-l)]\\
&=[zT-(\mathrm{E}(d)-l)]^2>0,
\end{aligned}
\tag{B-7}
$$

the right-hand side of (B-6) is negative which proves inequality (36)′, i.e., $\frac{\partial T^*}{\partial n}<0$.

3) *Derivation of* $\frac{\partial T^*}{\partial z}<0$.

Differentiating (32) with respect to z, we obtain

$$
\begin{aligned}
&2zT^2+2z^2T\frac{\partial T^*}{\partial z}\\
&=2(\mathrm{E}(d)-l)(\alpha_2+\beta_2 i)\frac{\partial i^*}{\partial z}+\frac{4a}{n},
\end{aligned}
\tag{B-8}
$$

which is (37) when rearranged in terms of $\frac{\partial i^*}{\partial z}$. Next, differentiating (31) with respect to z,

$$
n\beta_2[T+z\frac{\partial T^*}{\partial z}-(\alpha_2+\beta_2)\frac{\partial i^*}{\partial z}]
$$

$$= 2[T(\beta_1 + 2\beta_2 i) + z(\beta_1 + 2\beta_2 i)\frac{\partial T^*}{\partial z} + zT^2\beta_2 \frac{\partial i^*}{\partial z}] \tag{B-9}$$

Substituting (37) into (B-9) yields

$$n\beta_2 T + n\beta_2\{z\frac{\partial T^*}{\partial z} - \frac{1}{E(d)-l}[z^2T\frac{\partial T^*}{\partial z} + (zT^2 - \frac{2a}{n})]\}$$
$$= 2T(\beta_1 + 2\beta_2 i) + 2z(\beta_1 + 2\beta_2 i)\frac{\partial T^*}{\partial z} + \frac{4\beta_2 zT}{(E(d)-l)(\alpha_1+\beta_2)}(z^2T\frac{\partial T^*}{\partial z} + (2T^2 - \frac{2a}{n})). \tag{B-10}$$

Rearranging (B-10) in terms of $\frac{\partial T^*}{\partial z}$,

$$\frac{\partial T^*}{\partial z}\{n\beta_2(\alpha_2+\beta_2)z[zT - (E(d)-l)] + 2z(\beta_1 + 2\beta_2 i)(E(d)-l)(\alpha_2+\beta_2) + 4\beta_2 z^3T^2\}$$
$$= -n\beta_2(\alpha_2+\beta_2)[g(T)] - 2T(\beta_1+2\beta_2 i)(E(d)-l)(\alpha_2+\beta_2) - 4\beta_2 zT(zT^2 - \frac{2a}{n}) \tag{B-11}$$
$$\left(g(T) = zT^2 - \frac{2a}{n} - (E(d)-l)T\right)$$

Next, manipulate (32) into

$$\frac{2a}{n} = \frac{1}{2z}[z^2T^2 - (E(d)-l)^2].$$

Substituting into $g(T)$,

$$\begin{aligned} g(T) &= zT^2 - \frac{1}{2}zT^2 + \frac{1}{2z}(E(d)-l)^2 - (E(d)-l)T \\ &= \frac{1}{2}zT^2 - (E(d)-l)\,T + \frac{1}{2z}(E(d)-l)^2 \\ &= \frac{1}{2z}z^2T^2 - 2(E(d)-l)zT + (E(d)-l)^2 \\ &= \frac{1}{2z}[zT - (E(d)-l)]^2 > 0. \end{aligned} \tag{B-12}$$

Therefore, the right-hand side of (B-11) is negative, which proves inequality (38)′, i.e., $\frac{\partial T^*}{\partial z} < 0$.

References

Edgeworth, F. "The Mathematical Theory of Banking." *Journal of Royal Statistical Society* (March 1888).

Foley, D. "On Two Specifications of Asset Equilibrium in Macroeconomic Models." *Journal of Political Economy* 83 (April 1975).