Measure of Capital in Böhm-Bawerk's Flow Input Model*

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One of the shortcomings of Böhm-Bawerk's capital theory is his neglect of the compound interest rate in the calculation of the value of capital. However, this paper shows that the introduction of the compound interest rate itself may not invalidate his assertion that the increase of value of capital will lengthen the production period and thus decrease the interest rate, if there exists a sensible measure of the "roundaboutness", as is the case with Böhm-Bawerk uniform flow input model. The fundamental problem with Böhm-Bawerk is that the average production period cannot serve as a sensible measure of capital under a general time distribution with multiple parameters, contrary to his conjecture. An alternative measure of production period under less restrictive conditions is also suggested with a textual evidence.

I. Introduction

This paper intends to examine the consistency of Böhm-Bawerk's interest theory, by investigating the question of under what conditions his assertion that a more "roundabout" production process will bring about the decrease of interest rate holds true. Especially, the problem of finding a sensible measure of "roundaboutness" will be dealt with.

In his book, *The Positive Theory of Capital* (1888), he constructs the tables (Vol. 2, pp. 356-58, Table I-III)[1], in which a more "round-

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[1]Vol. and page numbers in the parenthesis indicate corresponding volume and pages of Böhm-Bawerk (1889), *Capital and Interest,* 3 Vol.s, translated by G. D. Huncke and H. F. Sennholz (1959), which is used as a main text in this paper.

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about" production process is assumed to be superior to a less "roundabout" production process. These tables are based on one commodity flow input model with uniform time distribution where the "roundaboutness" is measured in terms of the production period.

However, his model is plagued with one crucial problem, namely the simple interest calculation which he adopts in measuring the amount of "subsistence fund" by which workers are supported during the production period. This error has been commonly pointed out by many scholars since Wicksell (1893), but no rigorous attempt has been made to examine whether Böhm-Bawerk's insight remains true after this error is corrected.[2]

Thus, in the next section, we will present a correctly refomulated version of Böhm-Bawerk's model, namely uniformly distributed flow input-point output model, to examine the validity of Böhm-Bawerk's assertion that an increase of "capital" will lengthen the production period, thereby decreasing the interest rate. One innocuous difference between Böhm-Bawerk's model and our model is that the absolute period of production is used to express production function instead of Böhm-Bawerk's average period of production, because two measures can be used interchangeably under uniform distribution.

Then, the questions are raised, why Böhm-Bawerk needs the concept of average production period and whether he succeeds to explain the equilibrium value of interest rate, by using this concept. The answer to the first question is that Böhm-Bawerk inadvertently assumes that a general time distribution can be characterized in terms of the average, and production processes with the same average period are equally profitable regardless of the interest rate. However, as is suggested by his follower like Wicksell (1923) and later by his critics like Sraffa (1960), the answer to the second question is in the negative. It is the non-sensible notion of average production period that casts doubts on his entire theory of capital and interest.

The third section deals with one measure of "roundaboutness" suggested by Hicks (1946) and Weizsäcker (1971). Our finding is

[2]Brems (1986, 1988) presented a refomulated Böhm-Bawerkian model. But the optimization problem which Brems solved is not profit maximization with the constraint of capital at stationary state where a new production process begins at every point of time, but present value maximization without capital constraint in which a new production process starts immediately after an old production process ends. In other words, Brem's model is not Böhm-Bawerk's, but Faustmann's [see Manz (1986) on this point].

that this measure is not applicable to the general equilibrium model of Böhm-Bawerk, since it makes sense only in the vicinity of a switching point. In the fourth section, we will attempt to make clear what simplifying assumptions are needed to justify Böhm-Bawerk's reasoning and suggest the alternative measure of "roundaboutness". Our main conclusion is that the assumption of given time distribution of input has to be abandoned to construct the sensible index of measuring "roundaboutness". This implies that, under the assumption of given time distribution of input, any attempt to measure the "roundaboutness" with a single index is doomed to failure, unless we posit special time distributions like uniform or exponential distribution. The last section contains brief concluding remarks.

II. Böhm-Bawerk's Flow Input-Point Output Model

In his flow input-point output model, Böhm-Bawerk assumes that a commodity can be produced by labor only, which is employed evenly during the absolute production period of t. In the same vein, we shall assume that identical firms are producing the same commodity under perfect competition. In addition, to avoid the complicated use of language, we shall assume that an analytical time unit is a year.

To get a stationary equilibrium where the output and consumption are the same over periods, and the interest rates relating to the different periods are also equal, we can construct the staggered structure of production, in which (L/t) amount of labor is newly employed in the newly begun production process at every point of time, in order to produce X amount of output annually.

Since Böhm-Bawerk assumes that the more output is produced, the longer each production process will become, and that the production process reveals constant returns with respect to labor input, L, we can express the annual output of each firm, X, as follows:

$$X = Lf(t)$$

where $f'(t) > 0$, $f''(t) < 0$.

Or writing in terms of per worker, we have:

$$y = (X/L) = f(t) \tag{1}$$

In this world, we don't have to worry about the difference between value product and physical product, because every amount can be measured in terms of physical quantity. Actually, as is pointed out by Wicksell (1893), even Böhm-Bawerk himself has never answered to the question "*why the product of capital, when it becomes due, should be more valuable than the sacrificed capital commodity itself,*"[3] even though he takes the productivity theorists to task for not explaining the *value* production of capital, in his critique of the earlier theories of capital. The economic justification of signs of derivatives of production function may be given by simple parables, like making nets to catch fish, but the positivity of the first derivative, at least, in the case of uniform distribution looks sensible, because a businessman will not choose a production process with a longer production period which costs more, unless it generates a larger output. An optimal condition will be reached, when marginal cost due to the lengthening of production period is equal to marginal revenue, or marginal output in this one commodity world.

Under prefect competition, an individual firm will choose the optimum period of production, and the level of employment, so as to maximize the annual profit with the given amount of capital, $\bar{K}$, regarding the annual wage rate, w, and the annual rate of interest rate, r, as given:

$$\text{Choose } t \text{ and } L \text{ to max } \Pi = L\{f(t) - w\}$$
$$\text{subject to } K = (wL/t)\int_0^t (1/r)(e^{rs} - 1)ds = \bar{K}$$

where s is the number of years which has to elapse before the completion of each production process.

We will shortly explain how to calculate the stationary amount of K in flow input model. In this maximization problem, it can be noted that the amount of labor employed does not affect the profit rate, at the level of firm's decision, in which the wage rate is considered to be given exogenously. Writing the Lagrangean function, we have:

$$L\{f(t) - w\} + \lambda\,[\bar{K} - (wL/t)\int_0^t (1/r)(e^{rs} - 1)ds]$$

The first-order condition for the above constrained maximization problem is:

$$f'(t) = \{f(t) - w\}\{rt\,e^{rt} - (e^{rt} - 1)\} / t(e^{rt} - 1 - rt) \tag{2}$$

[3]Wicksell (1893, pp. 112-13). However, Wicksell's criticism applies only to Böhm-Bawerk's table, since Böhm-Bawerk offered two reasons that a positive interest rate is necessary for the supply of capital, which explain why the physical productivity implies the value of productivity.

At long-run equilibrium, we have the following break-even condition:

$$y = (w/t) \int_0^t e^{rs} ds = (w/t)(1/r)\{e^{rt} - 1\} \tag{3}$$

The equivalent condition is that the profit rate is equal to the interest rate[4]:

$$\frac{L\{f(t) - w\}}{[wL(e^{rt} - 1 - rt)/rt^2]} = r \tag{3$'$}$$

Inserting (3)′ into (2), we have:

$$we^{rt} = f(t) + tf'(t) \tag{2$'$}$$

Now, let us calculate the amount of subsistence fund per worker, which is the accumulated costs a firm has to invest to maintain y amount of annual output per worker:

$$q = K/L = (w/t) \int_0^t (1/r)(e^{rs} - 1) ds = (w/t)(1/r^2)\{e^{rt} - 1 - rt\} \tag{4}$$

If the initial endowments in an economy are given, i.e. $nL = n\bar{L}$ and $nK = n\bar{K}$, where n is the number of identical firms in an economy and a bar ($\bar{\ }$) indicates an exogenous variable, we have the final one equation to determine five unknowns, y, q, t, r, and w, assuming that the solution exists:

$$q = \bar{q} \tag{5}$$

However, the amount of labor supplied, and "capital" which is calculated by wages cost multiplied by compound interest rate cannot be treated on the equal footing, since labor is regarded as the original factor, whereas the "capital" is one of the produced means of production, or intermediate goods. Since "capital" or a "subsistence fund" actually depends on the distributive variables, the same physical input with the same time pattern may have different value. Thus Hirshleifer (1967) rightly asks why the same amount of subsistence fund is maintained throughout many periods at stationary state. For, unless the given amount of subsistence fund happens to be equal to its optimum amount at stationary state, the

[4]If we insert (2) into the definition of Π, we can show that profit rate is equal to interest rate. In addition, we can get the same result as (2)′, by maximizing the interest rate, or internal rate of return, from the break-even condition.

presupposition of stationarity is incompatible with the givenness of subsistence fund.[5] Since this is not a main focus of this paper, we just mention that this problem can be overcome, by introducing the overlapping generation model, as Negishi (1982) demonstrates.

Now that we have the equal number of equations to determine five unknowns, if five equations guarantee the existence of positive solutions, let us try some comparative-static experiments, taking the assumption of given $\bar{q}$. We will examine Böhm-Bawerk's argument which seems similar to results in modern Neo-classical growth model[6]:

> The interest rate in a given economy will rise in inverse ratio to the subsistence fund, in direct ratio to the working population, which that fund must support, and in direct ratio to the degree of productivity that marks continued prolongation of the production period. (Vol. 2, p. 365)

If we take total differentiations of all equations, the signs of effect of increase of $\bar{q}$ on w, r, t, and y can be obtained. First, by differentiating (3) totally, and using (2)′, we can get one interesting relation, which implies that the slope of interest-wage curve is equal to the product of wage rate and the average period of investment where units of input are valued at their accumulated value at the time of completion:

$$\begin{aligned} dw/dr &= -(w/r)\{rte^{rt} - (e^{rt}-1)\}/(e^{rt}-1) \\ &= -w\int_0^t se^{rs}ds \Big/ \int_0^t e^{rs}ds \\ &= -wT(r) < 0 \end{aligned} \tag{3''}$$

where $T(r)$ is a period of investment, defined conformably.

The other derivatives are reported below:

$$dy = f'dt \tag{1'}$$

[5]The same inconsistency may occur, when we restrict our analysis to steady-state in which all quantities grow at the same rate, keeping their proportionalities.

[6]Böhm-Bawerk includes the durability of machine in the meaning of "roundaboutness" in his exposition (Vol. 2, pp. 325-37), as if the adoption of more durable capital had the same effect with the use of more capital with the same durability (see the second appendix of Wicksell (1923) for his discussion of so-called Åkerman problem). However, it is impossible to say that the increase of r will reduce the economic life-span of machine, because there is no one-to-one relationship between interest rate and durability, if the rate of technical depreciation varies over time. More interesting thing is that the same economic life-expectancy of durable machine, or truncation period can be optimal at different interest rates, when several finished goods are used to produce a machine. (See Schefold, 1976)

$$twe^{rt}dr + e^{rt}dw = (2f' + tf'' - wre^{rt})dt \tag{2)''$$

where $(2f' + tf'' - wre^{rt}) < 0$, from the second-order condition of maximization problem.

$$\begin{aligned} tdp = {} & (w/tr^2)(rte^{rt} - e^{rt} + 1)dt + (1/r^2)(e^{rt} - 1 - rt)dw \\ & + (w/r^3)(rte^{rt} + rt - 2e^{rt} + 2)dr \end{aligned} \qquad (4)'$$

$$dq = d\bar{q} \qquad (5)'$$

Substituting (2)'' and (3)'' into (4)', we have:

$$\begin{aligned} tdq = dt[& (w/tr^2)(rte^{rt} - e^{rt} + 1) \\ & - \{(e^{rt} - 1)^2 - (rt)^2e^{rt}\}(2f' + tf'' - wre^{rt})/ \\ & (e^{rt} - 1 - rt)e^{rt}r^2] \end{aligned}$$

The sign of $dt/d\bar{q}$ is positive, because $\{-(rt)^2e^{rt} + (e^{rt} - 1)^2\}$ is positive. The sign of this term is equal to that of the variance of uniform distribution:

$$\begin{aligned} Var = {} & \int_0^t s^2e^{rs}ds / \int_0^t e^{rs}ds - \{T(r)\}^2 \\ = {} & \{(rt)^2e^{rt} - 2rte^{rt} + 2e^{rt} - 2\}/r^2(e^{rt} - 1) \\ & - \{rte^{rt} - (e^{rt} - 1)\}^2/r^2(e^{rt} - 1)^2 \\ = {} & \{(e^{rt} - 1)^2 - (rt)^2e^{rt}\}/r^2(e^{rt} - 1)^2 > 0 \end{aligned}$$

Thus, $dt/d\bar{q} > 0$, $dw/d\bar{q} > 0$, $dy/d\bar{q} > 0$, and $dr/d\bar{q} < 0$, as is predicted by Böhm-Bawerk. However, the derivative of y with respect to $\bar{q}$ is not equal to r, due to "Wicksell effect". Moreover, we cannot tell whether or not this derivative is smaller than r, contrary to Wicksell (1893)'s explanation that this derivative is smaller than r, since the increase of subsistence fund is partly absorbed by the increase of wage.

III. A Criticism of Böhm-Bawerk's Average Period of Production

Böhm-Bawerk himself is using a different formula, to calculate the number of workers employed, with the given amount of subsistence fund:

$$K = \frac{twL}{2}$$

where $(t/2)$ is what he calls the average period of production.

This formula turns out to be a poor approximation, justifiable only when r is small enough to ignore the terms after the fourth in Taylor's expansion. Expanding e^{rt} around zero and eliminating the terms after $(1/3!)(rt)^3$ from the equation (4), we obtain:

$$\begin{aligned} K &= (wL/t)(1/r^2)\{e^{rt} - 1 - rt\} \\ &= (wL/t)(1/r^2)(1/2)(rt)^2 \\ &= \frac{wLt}{2} \end{aligned}$$

A more interesting fact in Böhm-Bawerk's formula is that the subsistence fund can be calculated directly from his concept of average period of production, T_b, which is the average amount of labor input weighted by its time structure:

$$T_b = \int_0^\infty s\,\phi(s)ds / \int_0^\infty \phi(s)ds$$

where $\phi(s)$ is the amount of labor spent at s periods before the end of each production process, and total amount of labor is set to be equal to one, i.e. $\int_0^\infty \phi(s)ds = 1$. Since the labor is used evenly for t periods in Böhm-Bawerk's model, the average period of production will be the following[7]:

$$T_b = \int_0^t s(L/t)ds/L = t/2$$

Now the subsistence fund can be rephrased as the wages fund supporting the workers for the average period of production, i.e. $K = wLt/2$.

Under a uniform distribution, this average period of production, $(t/2)$, and length of production, t, can be used interchangeably to express the production function, since a technical characteristic of input structure can be captured by both parameters. In general, however, it may be impossible to define the production function in terms of the latter, i.e. $Lf(t)$, because the different time structure of input use will produce the different amount of output, and thus different rates of return.

This is why Böhm-Bawerk invents the concept of average period

[7]As Garegnani (1960) shows, the conception of an average production period as a measure of capital intensity cannot be justified, without the following three conditions:

i) Compound interest calculation is left out of consideration.

ii) There are no fixed capital goods.

iii) There is only one primary input which is homogeneous.

of production. As he himself clearly mentions, it is an index designed to measure the roundabout method of production, or the degree of capitalist production, when time distribution of primary inputs is not uniform:

> The length or brevity, the prolongation or abbreviation of the circuitous path is not to be measured in terms of the absolute duration of the process of production beginning and ending with the first jot and the last little of labor expended. ···Only in the case of production methods in which the expenditure of originally productive powers is uniformly distributed over the entire production period, does the absolute duration of the production period afford an appropriate criterion of the degree of capitalist character···For want of a better term, I shall use "average production period" to distinguish it from the absolute production period. (Vol. 2, p. 87)

If output can be expressed as a function of only the average period of production, a production function will be written as follows:

$$X = Lf(T_b) \tag{6}$$

where $f(T_b)$ has the same property as before.

However, the average is not always a sensible measure of capital, because there is no one-to-one relation between q and T_b. For example, a larger value of average might be compatible with the smaller value of q depending on the interest rate, as Samuelson (1966) demonstrated. Thus, as Blyth (1956) points out, the distribution of labor input cannot be characterized by a single parameter, say average, unless the distribution of inputs is a uni-parametric distribution, like a uniform, or an exponential distribution. (see the Appendix for the case of an exponental distribution)

In general, the characteristics of time structure can only be expressed by all parameters of distribution. If the time structure of inputs can be characterized by k parameters of distribution, the production function will have k arguments:

$$X = Lf(m_1, m_2, \cdots, m_k) \tag{6)'}$$

where m_i is i-th parameter of distribution.

In the case of a general distribution, one way of measuring "roundaboutness" is suggested by Weizsäcker (1971). His measure is a new concept of investment period, which is defined as follows:

$$T(r) = \frac{\int_0^\infty e^{rs} s\, \phi\,(s)ds}{\int_0^\infty e^{rs}\, \phi\,(s)ds}$$

This measure is actually the weighted average of production period, because the wage costs incurred for periods t will be compensated at compound rate e^{rt}, at long-run equilibrium. Böhm-Bawerk's average production period, $T_b = t/2$, is a special case which can be obtained, only when the interest rate is zero and time distribution is uniform.

Having defined this measure, Weizsäcker asserts that the absolute value of the slope of factor-price frontier is equal to the product of the wage rate and the investment period, i.e. $dw/dr = -wT(r)$. The proof of this assertion is as follows:

Proof: If we donote $\phi\{s;\, m_1, m_1, \cdots, m_k\}$ as time distribution of input to produce output per worker, y, we can write the production function:

$$y = f(m_1, m_2, \cdots, m_k) \tag{6''}$$

In the profit maximization problem, each parameter of distribution will be adjusted to solve the following problem[8]:

$$\begin{aligned} &\text{Max } \Pi = f(m_1, m_2, \cdots, m_k) - w \int_0^\infty e^{rs} \phi\{s; m_1, m_2, \cdots, m_k\}\, ds \\ &\text{subject to } \int_0^\infty \phi\{s; m_1, m_2, \cdots, m_k\}\, ds = 1 \end{aligned} \tag{7}$$

Writing the Lagrangean function to solve the above constrained maximization problem, we have:

$$\begin{aligned} \mathscr{L}(m_1, m_2, \cdots, m_k, \lambda) = {} & f(m_1, m_2, \cdots, m_k) \\ & - w \int_0^\infty e^{rs} \phi\{s; m_1, m_2, \cdots, m_k\}\, ds \\ & + \lambda \left[1 - \int_0^\infty \phi\{s; m_1, m_2, \cdots, m_k\}\, ds\right] \end{aligned}$$

The necessary condition is that there exists λ such that:

$$\begin{aligned} f_i - wr \int_0^\infty (\partial \phi / \partial m_i) e^{rs} ds = \lambda \int_0^\infty (\partial \phi / \partial m_i) ds \\ \text{for } i = 1, 2, \cdots, k \end{aligned} \tag{8}$$

where f_i is the derivative of $f(\cdot)$ with respect to the i-th parameter. At long-run equilibrium, we have the following break-even condi-

[8]To get a rigorous derivation, we have to solve the problem of maximizing profit with the constraint of the given amount of capital, as we did in the second section. However, to expedite the calculation, we take an innocuous assumption that an individual producer can use as much capital as she wants without increasing the market interest rate.

tion:

$$y = w \int_0^\infty e^{rs} \phi \{ s; m_1, m_2, \cdots, m_k \} ds \tag{9}$$

Adding (8) over $i = 1, 2, \cdots, k,$ and using

$$\Sigma \int_0^\infty (\partial \phi / \partial m_i)(dm_i/dr) ds = 0$$

from the constraint:

$$\begin{aligned} &\Sigma f_i(dm_i/dr) \\ &- wr\Sigma \int_0^\infty (\partial \phi / \partial m_i)(dm_i/dr) e^{rs} ds = 0 \end{aligned} \tag{10}$$

Differentiating (9) totally and inserting (10), we find:

$$\begin{aligned} \Sigma f_i(dm_i/dr) &= (dw/dr) \int_0^\infty e^{rs} \phi (s) ds + w \int_0^\infty s e^{rs} \phi (s) ds \\ &\quad + w \int_0^\infty \Sigma (\partial\phi / \partial m_i)(dm_i/dr) r e^{rs} ds \\ \text{or } 0 &= (dw/dr) \int_0^\infty e^{rs} \phi (s) ds + w \int_0^\infty s e^{rs} \phi (s) ds \end{aligned}$$

Arranging the terms, we have:

$$dw/dr = -wT(r) < 0 \tag{11}$$

From this result, Weizsäcker (1971) asserts that $T(r)$ serves as a useful index of "roundaboutness", since the wage-interest curve of newly adopted technique (B) is always flatter than that of old technique (A) at the given interest rate, r_0, as is easily seen from the following inequality:

$$-w^B \cdot T^B(r_0) > -w^A \cdot T^A(r_0)$$

Dividing each side by wage rate, since the wage-rate at switch point is equal, i.e. $w^A(r_0) = w^B(r_0)$, we obtain:

$$T^B(r_0) < T^A(r_0)$$

However, this measure is not exogenously given, but affected by distributive variables, so that the monotonic increase of $T(r)$ with respect to the increase of r does not prevent the reswitching or "capital" reversal phenomena. As a matter of fact, Weizsäcker's proposition holds true locally, namely, in the context of partial equilibrium in which the interest rate can be considered to be given. In the investigation of market equilibrium which we are concerned about in the Böhm-Bawerkian model, it does not have any meaning.

As long as a production function is written in terms of purely technical parameters, we cannot have a sensible single index of the time structure of primary input,[9] except some special cases I mentioned. In other words, the result of (11) cannot lead us very far since there is no one-to-one relation between "roudaboutness" and the index of $T(r)$, which Weizsäcker considers as an appropriate measure of "capital intensity".

IV. An Alternative Measure: A Suggested Interpretation

Throughout this paper, it is assumed that there is a certain restriction on businessman's choice; they can choose only given numbers of parameters of input time distribution. Under this restriction, it is shown that no single parameter of distribution can be used as an index of "roundaboutness". The task of this section is to examine under what conditions we can construct the measure of "roundaboutness".[10]

First, let us assume that a businessman can choose the rate of work performed t years before completion of production, which will be characterized by time path of labor inputs, or $\phi(s)$, where $\int_0^\infty \phi(s)ds = 1$, as is defined earlier. Then, we can calculate the accumulated value of labor weighted by compound interest rate for its time invested in the production process. If we denote a subsistence fund per worker as q, which is needed for a firm to maintain the y amount of annual output per worker, we have the following formula at stationary equilibrium:

$$q = w \int_0^\infty \phi(s) \{(e^{rs} - 1)/r\} ds$$

In this formula, the integrated part can be called as a measure of "roundaboutness", namely:

$$k(\phi) = \int_0^\infty \phi(s) \{(e^{rs} - 1)/r\} ds \qquad (12)$$

Using this index, we can construct the following "production func-

[9]In the controversies between Böhm-Bawerk (1889) and other contemporary economists, such as J. B. Clark, the focus of criticism was given on the concept of average period of production, against which the following arguments were brought forth: i) Is an average production period finite, even if the production began in the infinite past? ii) Is it possible to formulate an average production period, when produced goods enter the production process of other goods? iii) Is it necessary to consider the time structure of production at a stationary state in which a synchronization of production and consumption occurs and thus a period of production appears to be zero? I think that this line of critique is misguided, because a calculation of average production period can be done, as Weizsäcker (1971) demonstrated.

[10]The discussion in this section depends heavily on the suggestion of Prof. R. Dorfman.

tion", as a maximum output a firm can get from given amount of $k(\phi)$, with the given distributive variables. We can further assume that, whenever more subsistence fund is available, a firm can produce more output, by choosing the most efficient time path of labor inputs, and that this amount will increase at a decreasing rate, due to whatever barriers a firm may face, as they increase the "intensity" of production. Thus we can write the production function as follows:

$$y = g(k) = \sup f(\phi(t)) \text{ subject to the given } k(\phi), w, \text{ and } r \tag{13}$$

where $g(k)$ is twice differentiable, and $g'(k) > 0$, $g''(k) < 0$.

A representative firm is assumed to choose the amount of labor, L, and the degree of "roundaboutness", k, so as to maximize profit, π, with the given wage rate and the given amount of fund, i.e. $\bar{K} = wLk$:

$$\text{Max } \pi = L\{g(k) - w\}$$
$$\text{subject to } wLk = \bar{K}$$

Writing the Lagrangean function, we have:

$$\mathscr{L}(L, k, \lambda) = L\{g(k) - w\} + \lambda(\bar{K} - wLk)$$

The first-order conditions are:

$$\partial\mathscr{L}/\partial L = g(k) - w - \lambda wk = 0 \tag{14}$$

$$\partial\mathscr{L}/\partial k = Lg'(k) - \lambda wL = 0 \tag{15}$$

$$\partial\mathscr{L}/\partial \lambda = \bar{K} - wLk = 0 \tag{16}$$

Eliminating λ from (14) and (15), we get:

$$w = g(k) - g'(k)k \tag{17}$$

The result in (17) looks quite similar to the relation obtained in Böhm-Bawerk's own model.[11]

At long-run equilibrium, the rate of profit equals the interest rate, i.e. $\pi/\bar{K} = r$. From the definition of π and (16), we find that the annual output per worker is totally exhausted by the sum of wage and interest:

$$y = w + r(wk) \tag{18}$$

Once again in this model, if we assume that the amount of subsistence fund and the amount of labor supply is given in an economy, we

[11] See equation (2) in Dorfman (1959a)

can determine the equilibrium set of two distributive variables at which total demand for and supply of both capital and labor in an entire economy are matched through the adjusted factor prices. This assumption boils down to the assumption of given subsistence fund per worker.

$$q = wk = \bar{q} \tag{19}$$

Now that we have five equations, to determine five unknowns, i.e. y, w, r, k, and q, it can be easily shown that the increase of "roundaboutness" will decrease the interest rate and increase the wage rate, as is predicted by Böhm-Bawerk:

$$dk/d\bar{q} = 1 \,/\, \{-k^2 g'' + w\} > 0$$

$$dw/d\bar{q} = -kg'' \,/\, \{-k^2 g'' + w\} > 0$$

$$\text{or } \; 0 < (\bar{q}/w)(dw/d\bar{q}) = 1 \,/\, \{1 - (1/k^2 g'')\} < 1$$

$$dr/d\bar{q} = g''(1 + rk)/w\,\{-k^2 g'' + w\} < 0$$

The reformulated model presented above establishes what Böhm-Bawerk might have in mind when he was defending his position against his critics. In the first essay of *Capital and Interest,* Vol. 3, Böhm-Bawerk restates his thesis of greater productivity of the roundabout ways of capitalist production:

> *a wisely chosen* selection or extension of a roundabout way of production generally results in greater productivity, that is, in more or better goods with the expenditure of the same production factors. (Vol. 3, p. 2, my emphasis)

In the same essay, he elaborates on the meaning of a "wisely chosen" technique:

> we apply production methods that, if wisely chosen, represent the technically best that can be applied with our present fund of subsistence. (Vol. 3, p. 13)

It is now certain that what Böhm-Bawerk had in mind in arguing the superiority of roundabout way of production is the existence of the aggregate production function presented in (13). He thinks that the output maximization with the given subsistence fund and the optimal choice of average production period could be separable. However, this separable two-stage optimization is only permissible, when in maximizing output with given subsistence fund, a businessman choose the time structure of input without considering the profitability condition, contrary to the usual assumption in interest theory.

As Fisher (1907) argues, if a businessman takes the interest calculation into consideration, the choice of production period and time structure cannot be separable.

V. Concluding Remarks

We have shown that, under the assumption of given structure of time distribution, Böhm-Bawerk's explication of interest rate is permissible only in one commodity flow input-point output model with uni-parametric distribution of one primary input. However, if we allow this structure to be chosen freely by a firm, we can construct the measure of "roundaboutness", thereby establishing his insights. Without this assumption of an optimally chosen time path, we cannot extend the basic results into more complicated model with an arbitrarily given input structure and durable capital goods model, which is actually tantamount to flow input-flow output model. These findings are analogous to results agreed by most participants in modern capital controversy in the 1960s, namely the fact that the long-run equilibrium interest rate can be explained within Neo-Classical general equilibrium framework, if we allow the amount and composition of initial endowments, except one input, to change conformably, as is typically argued by Hahn (1982).

One remaining problem, even though it was not a main topic of this paper, is that, in Böhm-Bawerk's reasoning, he thinks that he succeeds in explaining the interest rate at long-run equilibrium, as if the result obtained in one commodity model could be extended into the entire economy where there are multiple goods with heterogeneous methods of production:

> If there is then more capital left over, or if there is an accretion of capital, they will seek out the next most profitable employment··· What effect does this heterogeneity of conditions in the world of actuality have upon the law that we have worked out for the rate of interest? *It does not disturb it in the least. For all the essential relations on which it is founded remain intact*··· It is still true that under this arrangement of a certain stratum of prolongations with a given surplus return become the last one still economically permissible··· And, finally, it is still true that the surplus returns of these "marginal employment" also mark the margins, that is to say the limits, of the interest rate. (Vol. 2, pp. 367–68, original emphasis)

However, as is made clear in the third section, we cannot construct such aggregate production function as $f(T_b)$, even in one commodity flow input-point output model. Furthermore, when there are heterogeneous capital goods whose production structures are different, we will meet the problem of valuation of capital goods, even though we can construct the average period of production. In this case, whether Böhm-Bawerk's insights hold true is doubtful.

These basic shortcomings account for the reason why Böhm-Bawerk's theory could not draw much attention in modern capital theory, even though the originality of his insights about time structure of input cannot be too highly emphasized.

Appendix

Exponential Distribution

Let us consider another simple distribution which can be characterized by a single parameter only. It will be shown that in the case of exponential distribution, the derivative of annual output with respect to subsistence fund, or "marginal product of capital", is positive and equal to interest rate.

Let us assume that the time distribution of labor input follows an exponential distribution with an average $(1/\beta)$, i.e. $\phi(t) = \beta e^{-\beta t}$, where t $(0 \leq t \leq \infty)$ indicates t years before the end of each production process.

First, if we assume that annual output increases at a decreasing rate, as $(1/\beta)$ increases, we can write the production function as follows:

$$y = f(1/\beta) \tag{A1}$$

where $f'(1/\beta) > 0$ and $f''(1/\beta) < 0$.

From the maximization problem, we have the first-order condition:

$$f'(1/\beta) = wr\beta^2/(\beta - r)^2 > 0 \tag{A2}$$

At long-run equilibrium, annual output per worker is equal to annual costs per worker:

$$\begin{aligned} y &= w \int_0^\infty \beta e^{-(\beta - r)s} ds \\ &= w\beta/(\beta - r), \text{ if } \beta > r. \end{aligned} \tag{A3}$$

If $\beta \leq r$, the cost will be infinitely large, so that such time path will become economically irrelevant, or a firm will choose zero length of production, if $f'(\infty) = \infty$.

Next, we can calculate the amount of subsistence fund per worker:

$$\begin{aligned} q &= w \int_0^\infty \beta e^{-\beta s}(1/r)(e^{rs} - 1)ds \\ &= (w/r)\beta \{ \int_0^\infty e^{-(\beta - r)s}ds - \int_0^\infty e^{-\beta s}ds \} \qquad \text{(A4)} \\ &= w/(\beta - r) \end{aligned}$$

If we assume that the amount of subsistence fund and the supply of labor available in an economy is given, i.e. $q = \bar{q}$, we can determine five unknowns, i.e. y, w, r, q, and β.

Differentiating (A1) through (A4), we can obtain the derivatives. Some interesting relations will be reported below:

$$dw/dr = -w/(\beta - r) = -f(1/\beta)/\beta < 0$$

This implies that the slope of wage-interest frontier is negative, as is expected. A more interesting thing is that it is always linear, irrepective of distributive variables. Taking the second derivative, we obtain:

$$d^2w/dr^2 = -\{(\beta - r)(\partial w/\partial r) + w\}/(\beta - r)^2 = 0$$

Moreover, the increase of subsistence fund per worker will lengthen the average production period, $(1/\beta)$, and thus increase the annual output per worker. And, not surprisingly, the derivative of y with respect to $\bar{q}$, or "marginal product of capital", is equal to interest rate:

$$d(1/\beta)/d\bar{q} = w\beta^2/(\beta - r)^2 > 0$$

$$dy/d\bar{q} = f'(1/\beta)w\beta^2/(\beta - r)^2 = r$$

In the case of exponential function, there is a directly linear inverse relation between wage rate and interest rate and the effect of increase of subsistence fund on annual output is unambiguously positive, like the case of a textbook version of aggregate production function model where there is one commodity and the time distribution of input is left out of consideration.

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