An N-Person Bargaining Process with Alternating Demands

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We analyze an n-person bargaining game where players alternatingly demand their shares of a pie, and show that the set of perfect equilibria is a singleton if the common discount factor is below a certain cirtical level, and a continuum otherwise.

I. Introduction

Previously, we (1988) set up an n-person bargaining game in which the bargaining process is recursive. A player's acceptance of a proposal in an n-person subgame leads to an $(n - 1)$-person subgame excluding the player. His rejection leads to another n-person subgame with a permutation of players. There we showed that the game has a unique perfect equilibrium for the case where the players have the same discount factors applied to their shares of the pie. Subsequently, we (1989) have extended this result to a more general case where different players may have different utility functions and discount factors.

In the present paper, we will consider an alternative bargaining process where a player demands his share rather than proposes the responder's share. We will show that the set of perfect equilibria is a singleton if the common discount factor is below a certain critical level, and a continuum otherwise.

For the case where the equilibrium is unique, the players partition a pie in the ratio $(1, \delta, \delta^2, \ldots, \delta^{n-1})$, where δ is the discount factor, in the demand game of the present paper as compared to (1,

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δ, δ,..., δ) in the proposal game of the previous paper.[1] One lesson we learn by comparing the outcomes of the two alternative bargaining processes is that there is a premium for an initiator. In the perfect equilibrium of the proposal game, player 1 remains the only initiator until the end of the game, proposing the other players' shares successively. In the perfect equilibrium of the demand game, an accepting responder becomes the initial demander in the reduced bargaining round. Thus, the order of the players other than the initial player does not influence the outcome in the proposal game, while it does in the demand game.

It is obvious that the two-person case of either game is a Rubinstein game (1982). In a two-person game, demanding one's own share is equivalent to proposing the responder's share.

II. The Model and Result

An n-person bargaining game $G_n(\pi, \tau; P_1,..., P_n)$ with total pie π (≥ 0), starting period τ ($= 0, 1,..., \infty$), and players $P_1,..., P_n$ ($n \geq 1$) is defined recursively as follows:

Players bargain over the partition $(s_1,..., s_n)$ of the pie, where $s_1,..., s_n \geq 0$, $s_1 + ... + s_n \leq \pi$. When all players reach consensus, player i receives s_i. If the consensus is reached in period t ($= \tau, \tau + 1, ..., \infty$), player i's utility is $u_i = \delta^t s_i$ where $0 < \delta < 1$.

In period τ, P_1 demands his share s_1. If P_2 accepts the demand, then the remaining game becomes $G_{n-1}(\pi - s_1, \tau; P_2,..., P_n)$. If P_2 rejects the demand, then the remaining game becomes $G_n(\pi, \tau + 1; P_2, P_3,..., P_n, P_1)$.[2] In a one-person game $G_1(\pi, \tau; P_i)$, P_i can choose any share reaching the self-agreement in any $t = \tau, \tau + 1,..., \infty$. If $\pi = 0$, then the game $G_n(\pi, \tau; P_1,..., P_n)$ is defined as the trivial game where players receive $(0,..., 0)$ in period τ.

For $n \geq 2$, let $\phi_n(x) = x(1 + ... + x^{n-2})$ for $x > 0$, and let δ_n be the unique solution of the equation $\phi_n(x) = 1$. Then $1 = \delta_2 > ... > \delta_n > \delta_{n+1} > ... > 1/2$. One has $\phi_n(\delta) < 1$ if and only if $\delta < \delta_n$.

[1] Jun (1987) obtains (1, δ, δ) as the unique subgame perfect equilibrium of a three-person bargaining game.

[2] In the present model, if an offer is accepted. then no time is needed before the next round of bargaining, whereas if an offer is rejected, one period is lost. This particular formulation was chosen to reflect the cost of changing the proposer. In general, however, whether there is any loss of time after an acceptance is not an essential characteristic of a bargaining game. Any delay after an acceptance has only the effect of shrinking the Pareto frontier.

For notational convenience, let also $\delta_1 = 1$.
Our result is

Theorem
1) If $\delta < \delta_n$, the game $G_n(\pi, 0; P_1, ..., P_n)$ has a unique perfect equilibrium with the outcome $(u_1, ..., u_n) = \pi \sigma_n(1, \delta, ..., \delta^{n-1})$ where $\sigma_n = (1 + \delta + ... \delta^{n-1})^{-1}$. 2) If $\delta \geq \delta_n$, the game $G_n(\pi, 0; P_1, ..., P_n)$, where $\pi > 0$, has a continuum of perfect equilibrium outcomes.

Proof: 1) Assume $\delta < \delta_n$. The theorem is trivially true if $\pi = 0$. In proving the theorem for the case where $\pi > 0$, we may normalize the pie so that $\pi = 1$. We use mathematical induction. First, the theorem is true for a one-person game, for the lone player P_1 will choose the whole pie in period 0. Second, if the theorem is true for an $(n - 1)$-person game, then the theorem is also true for an n-person game as shown below.

By the induction hypothesis, since $\delta_n < \delta_{n-1}$, one may replace the subgame $G_{n-1}(1 - s_i, t; P_{i+1}, ..., P_n, P_1, ..., P_{i-1})$ after the first acceptance (by P_{i+1}) of a demand (s_i by P_i) during any play of the game by the outcome $(u_{i+1}, ..., u_n, u_1, ..., u_{i-1}) = \delta^t(1 - s_i)\sigma_{n-1}(1, \delta, ..., \delta^{n-2})$. (We will use the convention that subscript $i + 1$ means subscript 1 if $i = n$, and subscript $i - 1$ means subscript n if $i = 1$.) The original game reduces to a simpler game where the acceptance of any demand ends the play. We have only to show that this reduced game has a unique perfect equilibrium with the outcome $(u_1, ..., u_n) = \sigma_n(1, \delta, ..., \delta^{n-1})$.

We will first show that there is a perfect equilibrium with the above outcome. Consider a strategy profile, one strategy for each player, such that any demander demands his share σ_n, and any responder accepts any demand less than or equal to σ_n and rejects any demand greater than σ_n. This strategy profile is a perfect equilibrium with the above outcome. (Step 1 below) In this perfect equilibrium, the game ends in period 0 with P_1's share σ_n.

We will now show that the set of perfect equilibrium utilities of P_1 in the game $G_n(1, 0; P_1, ..., P_n)$, denoted by U, is a singleton, viz., $U = \{\sigma_n\}$. Put $\underline{u} = \inf U$ and $\bar{u} = \sup U$. Then $\underline{u} \geq 1 - \sigma_{n-1}^{-1}\delta\bar{u}$ (Step 2 below) and $\bar{u} \leq 1 - \sigma_{n-1}^{-1}\delta\underline{u}$ (Step 3 below). (Notice here that $\delta\sigma_{n-1}^{-1} = \phi_n(\delta) < 1$ because $\delta < \delta_n$.) From these two inequalities, we obtain $\underline{u} \geq \sigma_n$ and $\bar{u} \leq \sigma_n$. Thus $U = \{\sigma_n\}$.

Furthermore, we can show that any perfect equilibrium play of the game $G_n(1, 0; P_1, ..., P_n)$ ends in period 0 with P_2's acceptance of P_1's demand σ_n. Suppose it ends in the starting period t of the subgame $G_n(1, t; P_i, \ldots, P_n, P_1, \ldots, P_{i-1})$. P_i's unique perfect equilibrium utility in this subgame is $\delta^t \sigma_n$, and thus the game ends with P_{i+1}'s acceptance of P_i's demand σ_n. The utilities of the remaining players are $(u_{i+1}, \ldots, u_n, u_1, \ldots, u_{i-1}) = \delta^t(1 - \sigma_n)\sigma_{n-1}(1, \delta, \ldots, \delta^{n-2}) = \delta^t \sigma_n \delta(1, \ldots, \delta^{n-2})$. Therefore P_1's utility is $\delta^s \sigma_n$ for some $s = t, \ldots, t+n-1$. Since P_1's unique perfect equilibrium utility is σ_n, it must be that $t = 0$.

Now we want to show that any perfect equilibrium is a strategy profile described earlier, i.e., one in which any demander demands σ_n, and any responder accepts any demand less than or equal to σ_n and rejects any demand greater than σ_n. We already know that σ_n is the only perfect equilibrium demand, and that it is accepted. This implies that a responder's rejection leads to his share σ_n with the delay of one period. On the other hand, his acceptance of a demand s_i leads to his share $(1 - s_i)\sigma_{n-1}$ in the current period. Since $(1 - \sigma_n)\sigma_{n-1} = \delta\sigma_n$, he accepts any demand less than σ_n, and rejects any demand greater than σ_n.

Step 1 : In a subgame where P_i is the initial demander, his initial demand σ_n is accepted according to such a strategy, and the shares for the remaining players are $(s_{i+1}, \ldots, s_n, s_1, \ldots, s_{i-1}) = (1 - \sigma_n)\sigma_{n-1}(1, \delta, \ldots, \delta^{n-2}) = \sigma_n \delta(1, \delta, \ldots, \delta^{n-2})$. The demander P_i will not gain by demanding less than σ_n, because any demand less than or equal to σ_n will be accepted. The demander will not gain by demanding more than σ_n either, because any demand greater than σ_n will be rejected, and this rejection will lead to P_i's share $\sigma_n \delta^{n-1}$ even lagged one period. If the responder P_{i+1} rejects P_i's demand, P_{i+1} receives σ_n lagged one period. Thus P_{i+1} will not gain by rejecting a demand less than σ_n or accepting a demand greater than σ_n. Since P_{i+1} is indifferent between accepting and rejecting σ_n, accepting the demand is an optimal action.

Step 2 : If P_2 rejects P_1's initial demands s_1, P_2 attains at most utility $\delta \bar{u}$ in the subgame $G_n(1, 1; P_2, P_3, \ldots, P_n, P_1)$. If P_2 accepts s_1, P_2's utility is $(1 - s_1)\sigma_{n-1}$. Let $\underline{s}_1 = 1 - \sigma_{n-1}^{-1}\delta\bar{u}$ so that $(1 - \underline{s}_{-1})\sigma_{n-1} = \delta\bar{u}$. Since P_2 would accept any demand less than $\underline{s}_1$, P_1 can guarantee himself any utility smaller than $\underline{s}_1$, and thus $\underline{u} \geq 1 - \sigma_{n-1}^{-1}\delta\bar{u}$.

Step 3 : Since $\bar{u} = \sup U$, for any $\varepsilon > 0$ there exists some subgame $G_n(1, t; P_i, \ldots, P_n, P_1, \ldots, P_{i-1})$ such that a prefect equilibrium play ends in the starting period t of the subgame with P_{i+1}'s acceptance of P_i's demand s_i yielding P_1's perfect equilibrium utility greater than or equal to $\bar{u} - \varepsilon$.

Suppose $i \neq 1$, i.e., P_1 is not the demander in period t. Since P_i's demand accepted by P_{i+1} is at least as great as $\underline{u}$, P_1's utility after P_{i+1}'s acceptance is at most $\delta^t(1 - \underline{u})\sigma_{n-1}$. Since $\underline{u} \geq 1 - \sigma_{n-1}^{-1}\delta\bar{u}$ from Step 2, one has $(1 - \underline{u})\sigma_{n-1} \leq \delta\bar{u}$, and thus $\delta^t(1 - \underline{u})\sigma_{n-1} \leq \delta^t\delta\bar{u} \leq \delta\bar{u}$, which is impossible if $\varepsilon < (1 - \delta)\bar{u}$.

Assume $\varepsilon < (1 - \delta)\bar{u}$, so that $i = 1$. Let $\bar{s}_1 = 1 - \sigma_{n-1}^{-1}\delta\underline{u}$ so that $(1 - \bar{s}_1)\sigma_{n-1} = \delta\underline{u}$. Since P_2 would have rejected P_1's initial demand if the demanded share s_1 had been greater than $\bar{s}_1$, P_1's utility $\delta^t s_1$ after P_2's acceptance must be at most $\delta^t\bar{s}_1$. Since $\delta^t s_1 \geq \bar{u} - \varepsilon$, one has $\delta^t\bar{s}_1 \geq \delta^t s_1 \geq \bar{u} - \varepsilon$, and thus $1 - \sigma_{n-1}^{-1}\delta\underline{u} \geq \delta^t(1 - \sigma_{n-1}^{-1}\delta\underline{u}) \geq \bar{u} - \varepsilon$. Since $1 - \sigma_{n-1}^{-1}\delta\underline{u} \geq \bar{u} - \varepsilon$, for arbitrarily small ε, one obtains $1 - \sigma_{n-1}^{-1}\delta\underline{u} \geq \bar{u}$.

Q.E.D.

2) Assume $\delta \geq \delta_n$. In proving the theorem, we normalize the pie so that $\pi = 1$. We will show that $(u_1,..., u_n)$ is a perfect equilibrium outcome if $u_1 = x$ and $(u_2,..., u_n) = (1 - x)\sigma_{n-1}(1, \delta,..., \delta^{n-2})$ for some $x \in [\underline{x}, \bar{x}]$ where

$$\underline{x} = \delta^n\{\phi_n(\delta) - 1\}/\{\phi_n(\delta)^2 - \delta^n\}$$

and $\bar{x} = [\delta^n\{\phi_n(\delta) - 1\} + (1 - \delta^n)\phi_n(\delta)^2]/\{\phi_n(\delta)^2 - \delta^n\}$.

Note that $0 \leq \underline{x} < \bar{x} \leq 1$ and $\bar{x} = 1 - \phi_n(\delta)\underline{x}$.

In the proof of 1), we replaced a subgame by its unique perfect equilibrium outcome. The condition $\delta < \delta_n$ would not have been necessary if we had wanted to replace the subgame by the same perfect equilibrium outcome without requiring uniqueness. In other words, the outcome $(u_1,..., u_n) = \sigma_n(1, \delta,..., \delta^{n-1})$ is a perfect equilibrium outcome for the case where $\delta \geq \delta_n$ as well. Thus replace the subgame $G_{n-1}(1 - s_i, t; P_{i+1},..., P_n, P_1,...,P_{i-1})$ after the first acceptance (by P_{i+1}) of a demand (s_i by P_i) during any play of the game by the perfect equilibrium outcome $(u_{i+1},..., u_n, u_1,..., u_{i-1}) = \delta^t(1 - s_i)\sigma_{n-1}(1, \delta,..., \delta^{n-2})$. For the reduced game, consider the following strategy profile: In period $t = 0, 1,...$, the demander demands d_t, where $d_0 = x$ and $d_t = (1 - d_{t-1})/\phi_n(\delta)$ for $t \geq 1$, and the responder accepts a demand if it is less than or equal to d_t and rejects otherwise. Note that $\underline{x} < d_t < \bar{x}$ for any t. This strategy profile is a perfect equilibrium as shown below.

In period t, the demand d_t is accepted according to such a strategy, and the shares of the responder P_{i+1} and the demander in the previous period $t-1$, P_{i-1}, are $\sigma_{n-1}(1-d_t)$ and $\sigma_{n-1}\delta^{n-2}(1-d_t)$. The demander P_i will not gain by demanding less than d_t, because any demand less than or equal to d_t will be accepted. The demander will not gain by demanding more than d_t either, because any demand greater than d_t will be rejected, and this rejection will lead to P_i's share $\sigma_{n-1}\delta^{n-2}(1-d_{t+1})$ lagged one period. Here we have $d_t > \delta\sigma_{n-1}\delta^{n-2}(1-d_{t+1})$ since

$$d_t - \delta\ \sigma_{n-1}\delta^{n-2}(1-d_{t+1}) = [d_t\{\phi_n(\delta)^2 - \delta^n\} - \delta^n\{\phi_n(\delta)-1\}]/\phi_n(\delta)^2 \geq 0.$$

Thus demanding d_t is an optimal action. If the responder P_{i+1} rejects P_i's demand d_t, P_{i+1} receives d_{t+1} lagged one period. One has $\sigma_{n-1}(1-d_t) = (1-d_t)\delta/\phi_n(\delta) = \delta d_{t+1}$. Thus P_{i+1} will not gain by rejecting a demand less than d_t or accepting a demand greater than d_t. Since P_{i+1} is indifferent between accepting and rejecting d_t, accepting the demand is an optimal action.

Q.E.D.

III. Discussions

Shaked (see Sutton 1986) gives an example of a three-person bargaining game of which the perfect equilibrium is not unique. Herrero (1985) analyzes an n-person generalization of Shaked's example. In this game, a player proposes the shares of the other players, who respond sequentially, and if any one player rejects the proposal, the game starts all over with a permutation of players. She claims that (using our notation) if $\delta \geq 1/(n-1)$ any partition is a perfect equilibrium outcome, and that if $\delta < 1/(n-1)$ there exists a unique perfect equilibrium with the outcome $(u_1,\ldots, u_n) = \pi\sigma_n(1, \delta,\ldots, \delta^{n-1})$. Haller (1986) expresses reservations about the validity of Herrero's proof of the latter claim, and shows that for a variation of the model where players respond simultaneously, any partition is a perfect equilibrium outcome for any $\delta < 1$.

For our model, we have established a result similar to Herrero's claim for her model: if $\delta \geq \delta_n$ there exist a continuum of perfect equilibrium outcomes, and if $\delta < \delta_n$ there exists a unique perfect equilibrium with the outcome $(u_1,\ldots, u_n) = \pi\sigma_n(1, \delta,\ldots, \delta^{n-1})$. The similarity is not complete, for in our model it is not true that any partition is a perfect equilibrium outcome when $\delta \geq \delta_n$. This is because in our model the game reduces to a two-person Rubinstein game after the acceptance of $n-2$ players, and thus there exist at least two players whose payoffs are in the ratio 1: δ.

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