Improved Beta Approximation to the Critical Point of the Durbin-Watson Test Statistic*

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The Durbin-Watson test in linear regression models is widely known to have problems in its application. The problem of the calculation burden in the exact test and that of the inconclusive region in the bounds test have motivated several approximations to the exact critical point. This paper improves upon the beta approximation by considering that the range of the Durbin-Watson statistic obtainable *a priori* is reducible to a subinterval of [0, 4], which Durbin and Watson (1951) assumed when devising their beta approximation. The proposed approximation is found to outperform the beta, $a + bd_u$, and Jeong's approximations. Its computational burden is virtually the same as that of Beta, heavier than those of $a + bd_u$ and Jeong's. However, the additional burden of computing beta critical points is removed if one uses either our tables or built-in inverse beta function.

I. Introduction

The Durbin-Watson test in linear regression models is widely known to have problems in its application. First, since the distribution of the Durbin-Watson test statistic, d, depends on the X matrix, tabulation of exact critical points is impossible and thus a rather serious burden is encountered in the calculation of the exact critical point. Second, although the critical points of the lower and upper bounds of d are available in tabulated forms, a rather wide inconclusive region appears in a bounds test, due to ignoring some information about X.

The problem of the calculation burden in the exact test and that

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of the inconclusive region in the bounds test have motivated several approximations to the exact critical point. Durbin and Watson (1971) compared various approximations and recommended their $a + bd_u$ and beta approximations as the "optimum" choice between the calculation burden and accuracy. Jeong (1985) has improved the $a + bd_u$ approximation by considering the lower as well as upper bound of d.

In this paper I improve upon the beta approximation by considering that the range of d obtainable *a priori* for various classes of X matrices is reducible to a subinterval of $[0, 4]$, which Durbin and Watson (1951) assumed when devising their beta approximation.

The proposed approximation, called $d_\alpha(R)$, is found to perform better than the beta, $a + bd_u$, and Jeong's approximations. Its computational burden is virtually the same as that of beta, heavier than those of $a + bd_u$ and Jeong's. However the additional burden of computing beta critical points is removed if one uses either our tables or built-in inverse beta function.

II. A New Approximation

We develop a new approximation for the following three classes of X matrices which appear frequently in regression analysis.

CLASS 1 : X matrices corresponding to regressions with the constant term, $l = (1, 1,...,1)'$.

CLASS 2 : X matrices corresponding to regressions through the origin.

CLASS 3 : X matrices corresponding to regressions with common regressors, X_c, or order $n \times c$, such as seasonal dummy variables and/or linear trend variable.

Table 1 tabulates the lower and upper bounds of d and the absolute range of d for the above three classes of X matrices. If l is an element of the subspace spanned by the columns of X_c,[1] the following relations hold:

1) With respect to the lower and upper bounds of d,

$$d_M < d_L \leq d_L^c \leq d \leq d_U^c \leq d_U.^2$$

[1]It is true for instance when X_c contains seasonal dummy variables in it.

[2]Every inequality holds in a stochastic sense, i.e., holds with probability one.

TABLE 1

LOWER AND UPPER BOUNDS AND ABSOLUTE RANGE OF d

	Notation	Lower and Upper Bounds
CLASS 1		$d_L \leq d \leq d_U$ $d_L = \dfrac{\sum_{i=1}^{n-k} \lambda_{i+1} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$ $\quad d_U = \dfrac{\sum_{i=1}^{n-k} \lambda_{i+k} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$
CLASS 2	subscript "*M*"	$d_M \leq d \leq d_U$ $d_M = \dfrac{\sum_{i=1}^{n-k} \lambda_{i+1} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$ $\quad d_U = \dfrac{\sum_{i=1}^{n-k} \lambda_{i+k} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$
CLASS 3	superscript "*c*"	$d^c_L \leq d \leq d^c_U$ $d^c_L = \dfrac{\sum_{i=1}^{n-k} \theta_i^{(c)} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$ $\quad d^c_U = \dfrac{\sum_{i=1}^{n-k} \theta_{i+k-c}^{(c)} \zeta_i^2}{\sum_{i=1}^{n-k} \zeta_i^2}$

	Absolute Range	Reference on Theory	Reference on Critical Points
CLASS 1	$I = [\lambda_2, \lambda_n]$	Durbin and Watson (1950)	Savin and White (1977)
CLASS 2	$I_M = [0, \lambda_n]$	Kramer (1971)	Farebrother (1980)
CLASS 3	$I^c = [\theta_1^{(c)}, \theta_{n-c}^{(c)}]$	King (1981)	King (1981; 1983)

Note: 1. X is of order $n \times k$.

2. Every inequality holds in a stochastic sense.
3. $\zeta_i \sim$ i.i.d. $N(0, 1)$ under the null hypothesis of no serial correlation $i = 1, \cdots, n-k$.
4. $\lambda_i = 2 - 2\cos(i-1)\pi/n$ $(i = 1, \cdots, n)$.
5. $\theta_1^{(c)}, \theta_2^{(c)}, \ldots, \theta_{n-c}^{(c)}$: eigenvalues of $M_C A M_C$ other than C zeros arranged in increasing order where $M_C = I - X_C(X_C' X_C)^{-1} X_C'$ and

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \cdots & 0 \\ -1 & 2 & -1 & 0 \cdots & 0 \\ 0 & -1 & 2 & -1 \cdots & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & 2 & -1 \\ 0 & 0 & \cdot & \cdot\ -1 & 1 \end{bmatrix}$$

2) With respect to the absolute range of d,

$$I^c \subset I \subset I_M \subset [0, 4]$$

It follows from this that the more columns of X we consider *a priori*, the narrower the "range of ignorance" becomes both stochastically and absolutely, and, conversely, the less information about X we consider, the wider the range of ignorance becomes.

Durbin and Watson's beta approximation always regards the absolute range of d as $[0, 4]$ and approximates $d/4$, the linear transformation of d the absolute range of which is reduced to $[0, 1]$, to beta distribution.

The beta approximation leaves room for improvement in that the absolute range of d is reducible to a proper subinterval of $[0, 4]$ *a priori* for each class of X matrices, since approximation on a subinterval is reasonably expected to be more accurate than that on a wider interval. The new proposed approximation builds an extension of beta distribution not on the interval of $[0, 4]$, but on a subinterval of $[0, 4]$.

Denote the absolute range of d, in general, as $I = [r_s, r_g]$, which is a subinterval of $[0, 4]$, and define x as $x \equiv (d - r_s)/(r_g - r_s)$. The range of x is reduced to $[0, 1]$ which is just the range of a beta variate. The approximate critical point by the new method is

$$d_\alpha(R) = r_s + (r_g - r_s) b_\alpha(p, q)$$

where $b_\alpha(p, q)$ is the critical point at level α of the beta distribution with parameters p and q. The parameters p and q are obtained by equating the 1st and 2nd moments of x respectively with those of a beta variate:[3]

$$p + q = \frac{\{\mathrm{E}(d) - r_s\}\{r_g - \mathrm{E}(d)\}}{\mathrm{Var}(d)} - 1,$$

$$p = (p + q)\frac{\mathrm{E}(d) - r_s}{r_g - r_s}.$$

We will denote $d_\alpha(R)$ applied concretely to Classes 1, 2, and 3 as $d_\alpha(R)$, $d_{M\alpha}(R)$, and $d^c_\alpha(R)$ respectively.[4]

[3]Moments of d are available in Durbin and Watson (1950, pp. 418-21), and those of a beta variate in both Theil and Nagar (1961, p. 800) and Henshaw (1966, p. 650).

[4]Here $d_\alpha(R)$ means both the method itself in a wider sense and the method applied to Class 1 in a narrower sense. We use it interchangeably. It is hoped not to cause any confusion.

Since the values of $[r_s, r_g]$ for each class of X matrices of appropriate orders are already known (see Appendix), the calculation burden of $d_\alpha(R)$ is virtually the same as that of the beta approximation. But $d_\alpha(R)$ is expected to give a more accurate result than the beta approximation.

III. Comparison of Various Approximations

Table 2 classifies the approximations considered in this paper on two criteria: i) the information used and ii) the distribution approximated.

TABLE 2
CLASSIFICATION OF VARIOUS APPROXIMATIONS

Distribution Information	Lower & Upper Bounds Distribution of d	Beta Distribution
1st and 2nd Moments of d	$d_\alpha(U), d_\alpha(J)$	$d_\alpha(B), d_\alpha(R)$
1st, 2nd, 3rd and 4th Moments of d	not available	$d_\alpha(H)$

Note : For consistency in notation with that of Jeong (1985), denote $a+bd_u$, Jeong's, beta, and Henshaw's as $d_\alpha(U)$, $d_\alpha(J)$, $d_\alpha(B)$, and $d_\alpha(H)$, respectively. For details of these approximations, see Durbin and Watson (1971), Henshaw (1966) and Jeong (1985).

To compare the accuracy of various approximations including their extended forms,[5] we make three classes of data sets from four published data sets. They are listed below.

CLASS 1 : *Regressions with Constant Term*

Code(n, k)		Descriptions
$A(20, 3)$	$A=(l:X_A)$	Spirits Data in Durbin and Watson (1951)
$B(16, 4)$	$B=(l:X_B)$	Pears Data excluding linear trend variable in Henshaw (1966)
$C(21, 3)$	$C=(l:X_C)$	Consumption Data in Klein (1950), 1921-41
$D(22, 3)$	$D=(l:X_D)$	Investment Data in Klein (1950), 1920-41

[5]For the extended forms of $d_\alpha(U)$, $d_\alpha(L)$, and $d_\alpha(J)$, see Jeong (1985). We denote these extended forms by attaching subscript "M" or superscript "c" according to the Class concerned.

CLASS 2 : *Regressions through the Origin*

Code(n, k)	Descriptions
$A_M(19, 2)$	First difference of X_A
$B_M(15, 3)$	First difference of X_B
$C_M(20, 2)$	First difference of X_C
$D_M(21, 2)$	First difference of X_D

CLASS 3 : *Regressions with Common Regressors*[6]

Code(n, k)		Descriptions
$A^c(20, 6)$	$A^c = (X_c : X_A)$	$X_c : 20 \times 4$ Matrix of four seasonal dummy variables
$B^c(16, 7)$	$B^c = (X_c : X_B)$	$X_c : 16 \times 4$ Matrix of four seasonal dummy variables
$C^c(20, 6)$	$C^c = (X_c : X_C^0)$	$X_c : 20 \times 4$ Matrix of four seasonal dummy variables
$D^c(20, 6)$	$D^c = (X_c : X_D^0)$	$X_c : 20 \times 4$ Matrix of four seasonal dummy variables

Table 3, 4, and 5 summarize the comparisons of various approximations corrsponding to Classes 1, 2, and 3 respectively.[7] For each approximation method we calculated the mean absolute error of the critical points from d_α and that of the actual sizes of test from the assumed nominal significance level, 5%.

From these tables we can derive the following implications:

1) Henshaw's approximation, which uses the biggest amount of information among the approximations considered, is best. But its calculation burden is serious.
2) Among the approximations using the 1st and 2nd moments of d (1st row of Table 2), the approximations based on the beta distribution (right side of Table 2) appear to outperform those

[6] $X_C{}^0$ and $X_D{}^0$ are X_C, X_D truncated at the 20th observation respectively.

[7] Comparisons for Class 3 when X_C consists of i) seasonal dummy variables and linear trend variable, ii) constant term and linear trend variable are also available in Tables A1 and A2 in the Appendix.

TABLE 3
REGRESSIONS WITH CONSTANT TERM (Class 1)

($\alpha = 5\%$)

Data (n, k)	Critical Points						Actual Sizes of Test (%)				
	d_α	$d_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$	$d_\alpha(U)$	$d_\alpha(H)$	$d_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$	$d_\alpha(U)$	$d_\alpha(H)$
$A(20, 3)$	1.482	1.484	1.484	1.486	1.485	1.482	5.044	5.044	5.094	5.069	4.995
$B(16, 4)$	1.273	1.270	1.270	1.271	1.251	1.272	4.937	4.937	4.961	4.493	4.985
$C(21, 3)$	1.501	1.501	1.501	1.502	1.501	1.501	4.994	4.994	5.020	4.994	4.994
$D(22, 3)$	1.491	1.492	1.492	1.493	1.491	1.491	5.025	5.025	5.051	4.999	4.999
Mean Absolute Error		0.001	0.001	0.002	0.006	0.000	0.034	0.034	0.051	0.146	0.007

TABLE 4
REGRESSIONS THROUGH THE ORIGIN (Class 2)

($\alpha = 5\%$)

Data (n, k)	Critical Points						Actual Sizes of Test (%)				
	d_α	$d_{M\alpha}(R)$	$d_\alpha(B)$	$d_{M\alpha}(J)$	$d_\alpha(U)$	$d_\alpha(H)$	$d_{M\alpha}(R)$	$d_\alpha(B)$	$d_{M\alpha}(J)$	$d_\alpha(U)$	$d_\alpha(H)$
$A_M(19, 2)$	1.287	1.282	1.283	1.306	1.277	1.285	4.880	4.905	5.551	4.756	4.956
$B_M(15, 3)$	0.938	0.946	0.948	0.978	0.910	0.937	5.206	5.257	6.056	4.353	4.984
$C_M(20, 2)$	1.269	1.272	1.273	1.292	1.266	1.269	5.088	5.114	5.628	4.935	5.011
$D_M(21, 2)$	1.323	1.325	1.326	1.345	1.322	1.324	5.044	5.070	5.586	4.967	5.018
Mean Absolute Error		0.004	0.005	0.026	0.010	0.001	0.114	0.134	0.695	0.247	0.022

TABLE 5
REGRESSIONS WITH COMMON REGRESSORS (Class 3)
(X_c = SEASONAL DUMMIES)

(α = 5%)

Data (n, k)	Critical Points							Actual sizes of Test (%)					
	d_α	$d_\alpha(R)$ / $d^c_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ / $d^c_\alpha(J)$	$d_\alpha(U)$ / $d_\alpha(U)$	$d_\alpha(L)$ / $d^c_\alpha(L)$	$d_\alpha(H)$	$d_\alpha(R)$ / $d^c_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ / $d^c_\alpha(J)$	$d_\alpha(U)$ / $d^c_\alpha(U)$	$d_\alpha(L)$ / $d^c_\alpha(L)$	$d_\alpha(H)$
		1.321		0.934	0.534	1.509		5.125		0.506	0.005	10.979	
$A^c(20, 6)$	1.316	1.319	1.321	1.322	1.319	1.358	1.316	5.079	5.125	5.149	5.079	6.043	5.009
		1.113		0.563	−0.221	1.315		5.094		0.025	0.000	11.444	
$B^c(16, 7)$	1.109	1.110	1.113	1.115	1.063	1.194	1.109	5.022	5.094	5.142	3.979	7.280	4.998
		1.337		0.951	0.562	1.523		5.037		0.462	0.007	10.932	
$C^c(20, 6)$	1.335	1.335	1.338	1.339	1.336	1.375	1.335	4.990	5.061	5.085	5.103	5.995	4.990
		1.347		0.957	0.574	1.533		5.065		0.463	0.008	10.951	
$D^c(20, 6)$	1.344	1.345	1.347	1.348	1.346	1.385	1.344	5.018	5.065	5.089	5.041	6.022	4.994
Mean Absolute		0.003		0.450	0.914	0.194		0.080		4.636	4.995	6.076	
Error		0.001	0.004	0.005	0.013	0.052	0.000	0.032	0.086	0.116	0.288	1.335	0.007

based on the lower and upper bounds distributions of d (left side of Table 2) on the whole. The former are more robust against changes of X matrices than the latter since the beta distribution allows a wider range of flexibility according to the changes of two parameters than the lower and upper bounds distributions of d, which are fixed if only n and k are given.

3) The performance of $d_\alpha(J)$, in general, turns out to be better than that of $d_\alpha(U)$, but does not always dominate that of $d_\alpha(U)$. This is because $d_\alpha(J)$ has not only the merit that it considers the lower bound in addition to the upper bound of d, but also the demerit that it lowers the quality (dimension) of information since the weight used in averaging $d_\alpha(U)$ and $d_\alpha(L)$ consists of 1st moment only. Furthermore, the calculation burden of $d_\alpha(J)$ is not always the same as that of $d_\alpha(U)$. Only for Class 1 is it so, due to the fact that $d_L + d_U = 4$ holds.
4) Overall, the performance of our $d_\alpha(R)$ (in the wider sense) is better than that of $d_\alpha(B)$. In Table 3, both $d_\alpha(R)$ and $d_\alpha(B)$ approximate the true critical point d_α quite well. Since the approximation of $d_\alpha(B)$ is already good enough (note that the mean absolute error for the critical points is only 0.001), our $d_\alpha(R)$ cannot improve on $d_\alpha(B)$. But in Tables 4 and 5, $d_\alpha(R)$ outperforms $d_\alpha(B)$. Particularly, in Table 5, $d_\alpha^c(R)$ performs much better than $d_\alpha(B)$: the mean absolute error is reduced to a quarter for critical points, and to a third for actual sizes of test. This improvement is due to the fact that $d_\alpha(R)$ considers additional *a priori* information about the range of d. The calculation burden of $d_\alpha(R)$ is virtually the same as that of $d_\alpha(B)$.
5) For Class 3 both the approximations with no superscript (indicated in the upper-half of the rows in Table 5) and those with superscript "c" (lower-half) are available.[8] But the latter performs better than the former on the whole since it utilizes more information about X.[9] This fact also supports our argument that approximation on a subinterval is more accurate than that on a wider interval, another evidence in favor of $d_\alpha(R)$ (in the wider sense) over $d_\alpha(B)$.

[8] Refer to footnote 1.

[9] Jeong (1985) expected that the performance of $d_\alpha^c(J)$ would be better than that of $d_\alpha(J)$.

IV. Concluding Remarks

We suggest the following when the Durbin-Watson bounds test is inconclusive:

1) To get a very accurate approximation, use Henshaw's.
2) To get a rather accurate approximation with relatively moderate calculation burden, use $d_\alpha(R)$.

We tabulated the 5% and 1% (lower) critical points of the beta distribution with parameters p and q of the following combinations:

$$\{(p, q) \mid p, q = 3.0, 3.5, 4.0, \cdots, 13.5, 14.0\}$$

Using these tables we eliminate the additional calculation burden in getting $b_\alpha(p, q)$. So far as we rely on these tables the computational burden of our new approximation becomes the same as those of $a + bd_u$ and Jeong's as well as that of Beta. If either p or q is not a multiple of 0.5, we can use interpolation to get the critical point of the corresponding beta distribution almost accurately to the third decimal point. When using interpolation, we have two alternatives. One is to interpolate along p first and then along q (Method 1). The other is to interpolate in reverse order (Method 2). But as can be seen from Table 6, the order of interpolation does not make any significant difference.

The following relation holds between the upper critical points and lower critical points of a beta distribution:

$$b_{1-\alpha}(p, q) = 1 - b_\alpha(q, p)$$

This allows our tables to be used also for testing negative serial correlation.

While Durbin and Watson (1971) recommended their $a + bd_u$ and beta approximations when the bounds test is inconclusive, we recommend our new approximation, which gives more reliable results than theirs. The computational burden is virtually the same as that of Beta, heavier than those of $a + bd_u$ and Jeong's. However, the additional burden of obtaining beta critical points is removed if one uses our tables or built-in inverse beta probability distribution function routine such MDBETI (written in FORTRAN).

TABLE 6
COMPARISON OF THE TRUE AND INTERPOLATED VALUES OF $b_\alpha(p, q)$ AND $d_\alpha(R)$
WHEN $\alpha = 1\%$

Data (n, k)	Method	$b_\alpha(p, q)$	$d_\alpha(R)$	$d_\alpha(R)$ rounded off at the 3rd decimal point
$A(20, 3)$	true	0.3000095	1.2100375	1.210
	by method 1	0.3000261	1.2101030	〃
	by method 2	0.3000260	1.2101027	〃
$B(16, 4)$	true	0.2430896	0.9918835	0.992
	by method 1	0.2431163	0.9919883	〃
	by method 2	0.2431164	0.9919887	〃
$C(21, 3)$	true	0.3066110	1.2349531	1.235
	by method 1	0.3065858	1.2348534	〃
	by method 2	0.3065771	1.2348192	〃
$D(22, 3)$	true	0.3096152	1.2460761	1.246
	by method 1	0.3096354	1.2461561	〃
	by method 2	0.3096353	1.2461560	〃

Note : We choose $\alpha = 1\%$ instead of $\alpha = 5\%$ since the tail probabilities are generally supposed to be more sensitive.

Appendix

TABLE A1
REGRESSION WITH COMMON REGRESSORS (Class 3)
(X_c = SEASONAL DUMMIES + LINEAR TREND)

(α = 5%)

Data (n, k)	Critical Points							Actual Sizes of Test (%)					
	d_α	$d_\alpha(R)$ $d^c_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ $d^c_\alpha(J)$	$d_\alpha(U)$ $d_\alpha(U)$	$d_\alpha(L)$ $d^c_\alpha(L)$	$d_\alpha(H)$	$d_\alpha(R)$ $d^c_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ $d^c_\alpha(J)$	$d_\alpha(U)$ $d^c_\alpha(U)$	$d_\alpha(L)$ $d^c_\alpha(L)$	$d_\alpha(H)$
$A^{c2}(20, 7)$	1.447	1.454 1.452	1.454	1.059 1.453	0.604 1.449	1.933 1.491	1.447	5.161 5.116	5.161	0.531 5.139	0.004 5.048	25.974 6.063	5.003
$B^{c2}(16, 8)$	1.270	1.284 1.282	1.284	0.747 1.280	−0.138 1.221	1.902 1.366	1.274	5.315 5.269	5.315	0.082 5.223	0.000 3.988	31.970 7.422	5.088
$C^{c2}(20, 7)$	1.425	1.431 1.430	1.432	1.060 1.431	0.583 1.424	1.907 1.466	1.425	5.145 5.122	5.168	0.595 5.145	0.004 4.984	26.044 6.005	5.007
$D^{c2}(20, 7)$	1.510	1.512 1.511	1.513	1.097 1.511	0.689 1.509	1.980 1.550	1.510	5.057 5.033	5.080	0.364 5.033	0.002 4.986	26.091 6.014	5.010
Mean Absolute Error		0.007 0.006	0.008	0.422 0.006	0.981 0.013	0.517 0.055	0.001	0.169 0.135	0.181	4.607 0.135	4.997 0.272	22.520 1.376	0.027

TABLE A2
REGRESSION WITH COMMON REGRESSORS (Class 3)
(X_c = CONSTANT + LINEAR TREND)

(α = 5%)

Data (n, k)	Critical Points							Actual sizes of Test (%)					
	d_α	$d_\alpha(R)$ $d^\kappa_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ $d^\kappa_\alpha(J)$	$d_\alpha(U)$ $d_\alpha(U)$	$d_\alpha(L)$ $d^\kappa_\alpha(L)$	$d_\alpha(H)$	$d_\alpha(R)$ $d^\kappa_\alpha(R)$	$d_\alpha(B)$	$d_\alpha(J)$ $d^\kappa_\alpha(J)$	$d_\alpha(U)$ $d^\kappa_\alpha(U)$	$d_\alpha(L)$ $d^\kappa_\alpha(L)$	$d_\alpha(H)$
		1.596		1.617	1.598	1.844		4.970		5.503	5,019	14.293	
$A^{c3}(20, 4)$	1.597	1.598	1.597	1.600	1.598	1.617	1.597	5.019	4.994	5.068	5.019	5.503	4.994
		1.402		1.512	1.385	1.733		4.927		8.060	4.538	17.957	
$B^{c3}(16, 5)$	1.405	1.405	1.403	1.406	1.385	1.428	1.403	4.998	4.951	5.022	4.538	5.570	4.951
		1.580		1.606	1.580	1.826		5.032		5.705	5.032	14.323	
$C^{c3}(20, 4)$	1.579	1.581	1.580	1.583	1.580	1.599	1.579	5.057	5.032	5.106	5.032	5.518	5.007
		1.648		1.657	1.650	1.890		4.946		5.177	4.997	14.333	
$D^{c3}(20, 4)$	1.650	1.649	1.648	1.651	1.650	1.669	1.650	4.971	4.946	5.022	4.997	5.497	4.997
Mean Absolute		0.002		0.040	0.005	0.265		0.047		1.111	0.129	10.226	
Error		0.001	0.001	0.002	0.005	0.020	0.000	0.027	0.035	0.054	0.129	0.522	0.015

TABLE A3

λ_2 AND λ_n

n	λ_2	λ_n	n	λ_2	λ_n
15	0.044	3.956	28	0.013	3.987
16	0.038	3.962	29	0.012	3.988
17	0.034	3.966	30	0.011	3.989
18	0.030	3.970	31	0.010	3.990
19	0.027	3.973	32	0.010	3.990
20	0.025	3.975	33	0.009	3.991
21	0.022	3.978	34	0.009	3.991
22	0.020	3.980	35	0.008	3.992
23	0.019	3.981	36	0.008	3.992
24	0.017	3.983	37	0.007	3.993
25	0.016	3.984	38	0.007	3.003
26	0.015	3.985	39	0.006	3.994
27	0.014	3.986	40	0.006	3.994

TABLE A4

$\theta_1^{(c)}$ AND $\theta_{n-c}^{(c)}$ WHEN X_c = SEASONAL DUMMY VARIABLES

n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$	n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$
16	0.094	3.848	32	0.024	3.962
20	0.060	3.902	36	0.019	3.970
24	0.042	3.932	40	0.015	3.975
28	0.031	3.950			

TABLE A5

$\theta_1^{(c)}$ AND $\theta_{n-c}^{(c)}$ WHEN X_c = SEASONAL DUMMY VARIABLES + LINEAR TREND VARIABLE

n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$	n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$
16	0.152	3.848	32	0.038	3.962
20	0.098	3.902	36	0.030	3.970
24	0.068	3.932	40	0.025	3.975
28	0.050	3.950			

TABLE A6

$\theta_1^{(c)}$ AND $\theta_{n-c}^{(c)}$ WHEN X_c = CONSTANT TERM + LINEAR TREND VARIABLE

n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$	n	$\theta_1^{(c)}$	$\theta_{n-c}^{(c)}$
15	0.173	3.956	28	0.050	3.987
16	0.152	3.962	29	0.047	3.988
17	0.135	3.966	30	0.044	3.989
18	0.121	3.970	31	0.041	3.990
19	0.108	3.973	32	0.038	3.990
20	0.098	3.975	33	0.036	3.991
21	0.089	3.978	34	0.034	3.991
22	0.081	3.980	35	0.032	3.992
23	0.074	3.981	36	0.030	3.992
24	0.068	3.983	37	0.029	3.993
25	0.063	3.984	38	0.027	3.993
26	0.058	3.985	39	0.026	3.994
27	0.054	3.986	40	0.025	3.994

TABLE A7

CRITICAL POINTS OF BETA DISTRIBUTION AT $\alpha = 5\%$

p \ q	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
3.0	0.1893	0.1693	0.1532	0.1399	0.1288	0.1193	0.1111	0.1040	0.0977	0.0922	0.0873
3.5	0.2318	0.2089	0.1902	0.1746	0.1614	0.1501	0.1403	0.1317	0.1241	0.1173	0.1113
4.0	0.2713	0.2461	0.2253	0.2078	0.1929	0.1800	0.1688	0.1588	0.1500	0.1422	0.1351
4.5	0.3078	0.2808	0.2583	0.2393	0.2229	0.2087	0.1962	0.1851	0.1752	0.1664	0.1584
5.0	0.3413	0.3130	0.2892	0.2689	0.2514	0.2360	0.2224	0.2104	0.1996	0.1898	0.1810
5.5	0.3720	0.3428	0.3181	0.2968	0.2782	0.2619	0.2475	0.2345	0.2229	0.2124	0.2029
6.0	0.4003	0.3704	0.3449	0.3229	0.3035	0.2865	0.2713	0.2576	0.2453	0.2341	0.2240
6.5	0.4263	0.3960	0.3700	0.3473	0.3274	0.3097	0.2938	0.2796	0.2667	0.2549	0.2442
7.0	0.4504	0.4198	0.3934	0.3703	0.3498	0.3316	0.3152	0.3005	0.2870	0.2748	0.2636
7.5	0.4725	0.4419	0.4152	0.3918	0.3709	0.3523	0.3355	0.3203	0.3065	0.2938	0.2822
8.0	0.4931	0.4624	0.4356	0.4120	0.3909	0.3719	0.3548	0.3392	0.3250	0.3120	0.3000
8.5	0.5122	0.4816	0.4547	0.4309	0.4096	0.3905	0.3731	0.3572	0.3427	0.3294	0.3170
9.0	0.5299	0.4995	0.4727	0.4488	0.4274	0.4080	0.3904	0.3743	0.3596	0.3460	0.3334
9.5	0.5464	0.5162	0.4895	0.4656	0.4441	0.4247	0.4069	0.3906	0.3757	0.3618	0.3490
10.0	0.5619	0.5319	0.5054	0.4815	0.4600	0.4404	0.4226	0.4061	0.3910	0.3770	0.3640
10.5	0.5763	0.5467	0.5203	0.4965	0.4750	0.4554	0.4375	0.4209	0.4057	0.3915	0.3784
11.0	0.5899	0.5606	0.5343	0.5107	0.4892	0.4696	0.4517	0.4351	0.4197	0.4054	0.3922
11.5	0.6026	0.5736	0.5476	0.5241	0.5028	0.4832	0.4652	0.4485	0.4331	0.4188	0.4054
12.0	0.6146	0.5860	0.5602	0.5369	0.5156	0.4961	0.4781	0.4614	0.4460	0.4315	0.4181
12.5	0.6259	0.5976	0.5721	0.5490	0.5278	0.5084	0.4904	0.4737	0.4582	0.4438	0.4303
13.0	0.6366	0.6086	0.5834	0.5605	0.5395	0.5201	0.5022	0.4855	0.4700	0.4555	0.4420
13.5	0.6466	0.6191	0.5942	0.5714	0.5505	0.5313	0.5134	0.4968	0.4813	0.4668	0.4532
14.0	0.6562	0.6290	0.6044	0.5818	0.5611	0.5420	0.5242	0.5077	0.4992	0.4777	0.4641

TABLE A7
(CONTINUED)
CRITICAL POINTS OF BETA DISTRIBUTION AT $\alpha = 5\%$

p \ q	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0
3.0	0.0828	0.0788	0.0752	0.0719	0.0688	0.0660	0.0635	0.0611	0.0589	0.0568	0.0557	0.0531
3.5	0.1058	0.1009	0.0964	0.0922	0.0885	0.0850	0.0818	0.0788	0.0760	0.0734	0.0710	0.0688
4.0	0.1287	0.1229	0.1175	0.1127	0.1082	0.1040	0.1002	0.0967	0.0933	0.0903	0.0874	0.0846
4.5	0.1511	0.1445	0.1385	0.1329	0.1277	0.1230	0.1186	0.1145	0.1107	0.1071	0.1037	0.1006
5.0	0.1730	0.1657	0.1589	0.1527	0.1470	0.1417	0.1367	0.1321	0.1278	0.1238	0.1200	0.1164
5.5	0.1942	0.1862	0.1789	0.1721	0.1658	0.1599	0.1545	0.1494	0.1447	0.1402	0.1360	0.1321
6.0	0.2146	0.2061	0.1982	0.1909	0.1841	0.1778	0.1719	0.1664	0.1612	0.1563	0.1518	0.1475
6.5	0.2343	0.2253	0.2169	0.2091	0.2018	0.1951	0.1888	0.1829	0.1773	0.1721	0.1672	0.1626
7.0	0.2533	0.2437	0.2349	0.2267	0.2190	0.2119	0.2052	0.1990	0.1931	0.1875	0.1823	0.1773
7.5	0.2714	0.2615	0.2523	0.2437	0.2357	0.2282	0.2212	0.2146	0.2084	0.2025	0.1970	0.1917
8.0	0.2889	0.2786	0.2690	0.2601	0.2518	0.2440	0.2366	0.2297	0.2232	0.2171	0.2113	0.2057
8.5	0.3056	0.2950	0.2852	0.2759	0.2673	0.2592	0.2516	0.2444	0.2376	0.2313	0.2252	0.2194
9.0	0.3217	0.3108	0.3007	0.2912	0.2823	0.2739	0.2661	0.2587	0.2516	0.2450	0.2387	0.2327
9.5	0.3371	0.3260	0.3156	0.3059	0.2968	0.2882	0.2801	0.2724	0.2652	0.2584	0.2518	0.2457
10.0	0.3519	0.3406	0.3300	0.3201	0.3108	0.3020	0.2937	0.2858	0.2784	0.2713	0.2646	0.2582
10.5	0.3661	0.3546	0.3439	0.3338	0.3242	0.3152	0.3068	0.2987	0.2911	0.2839	0.2770	0.2705
11.0	0.3797	0.3681	0.3572	0.3469	0.3373	0.3281	0.3195	0.3113	0.3035	0.2961	0.2891	0.2824
11.5	0.3928	0.3811	0.3700	0.3596	0.3498	0.3405	0.3317	0.3234	0.3155	0.3079	0.3008	0.2939
12.0	0.4054	0.3936	0.3824	0.3719	0.3620	0.3525	0.3436	0.3351	0.3271	0.3194	0.3121	0.3051
12.5	0.4176	0.4056	0.3944	0.3837	0.3737	0.3642	0.3551	0.3465	0.3384	0.3306	0.3231	0.3160
13.0	0.4292	0.4172	0.4059	0.3952	0.3850	0.3754	0.3663	0.3576	0.3493	0.3414	0.3338	0.3266
13.5	0.4404	0.4284	0.4170	0.4062	0.3960	0.3863	0.3770	0.3683	0.3599	0.3519	0.3443	0.3370
14.0	0.4512	0.4391	0.4277	0.4168	0.4066	0.3968	0.3875	0.3786	0.3702	0.3621	0.3544	0.3470

TABLE A8

CRITICAL POINTS OF BETA DISTRIBUTION AT $\alpha = 1\%$

p \ q	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
3.0	0.1056	0.0940	0.0847	0.0771	0.0708	0.0654	0.0608	0.0568	0.0533	0.0503	0.0475
3.5	0.1396	0.1251	0.1134	0.1038	0.0956	0.0887	0.0827	0.0775	0.0729	0.0688	0.0652
4.0	0.1731	0.1561	0.1423	0.1307	0.1210	0.1126	0.1053	0.0989	0.0932	0.0882	0.0837
4.5	0.2054	0.1864	0.1707	0.1575	0.1462	0.1365	0.1280	0.1205	0.1138	0.1079	0.1025
5.0	0.2365	0.2155	0.1982	0.1835	0.1710	0.1600	0.1504	0.1420	0.1344	0.1276	0.1215
5.5	0.2656	0.2433	0.2247	0.2088	0.1951	0.1831	0.1725	0.1631	0.1547	0.1471	0.1403
6.0	0.2932	0.2698	0.2500	0.2331	0.2183	0.2054	0.1940	0.1838	0.1746	0.1663	0.1588
6.5	0.3192	0.2949	0.2742	0.2563	0.2407	0.2270	0.2148	0.2039	0.1940	0.1851	0.1770
7.0	0.3437	0.3186	0.2971	0.2785	0.2622	0.2478	0.2349	0.2233	0.2129	0.2034	0.1947
7.5	0.3667	0.3410	0.3189	0.2997	0.2828	0.2677	0.2543	0.2421	0.2311	0.2211	0.2120
8.0	0.3883	0.3621	0.3396	0.3198	0.3024	0.2869	0.2729	0.2603	0.2488	0.2383	0.2287
8.5	0.4086	0.3821	0.3592	0.3390	0.3212	0.3052	0.2908	0.2777	0.2658	0.2550	0.2450
9.0	0.4277	0.4010	0.3778	0.3573	0.3391	0.3227	0.3080	0.2945	0.2823	0.2710	0.2607
9.5	0.4457	0.4189	0.3955	0.3747	0.3562	0.3396	0.3245	0.3107	0.2981	0.2866	0.2759
10.0	0.4627	0.4358	0.4122	0.3913	0.3726	0.3556	0.3403	0.3263	0.3134	0.3016	0.2906
10.5	0.4787	0.4518	0.4282	0.4071	0.3882	0.3711	0.3555	0.3412	0.3281	0.3160	0.3048
11.0	0.4938	0.4670	0.4433	0.4222	0.4031	0.3858	0.3701	0.3556	0.3423	0.3300	0.3186
11.5	0.5082	0.4814	0.4578	0.4365	0.4174	0.4000	0.3840	0.3694	0.3559	0.3435	0.3319
12.0	0.5217	0.4951	0.4715	0.4502	0.4310	0.4135	0.3975	0.3827	0.3691	0.3564	0.3447
12.5	0.5346	0.5082	0.4846	0.4633	0.4441	0.4265	0.4104	0.3955	0.3818	0.3690	0.3571
13.0	0.5468	0.5206	0.4971	0.4759	0.4566	0.4390	0.4228	0.4097	0.3940	0.3811	0.3691
13.5	0.5585	0.5324	0.5090	0.4879	0.4686	0.4510	0.4347	0.4197	0.4058	0.3928	0.3806
14.0	0.5695	0.5436	0.5204	0.4993	0.4801	0.4625	0.4462	0.4311	0.4171	0.4041	0.3918

TABLE A8
(CONTINUED)
CRITICAL POINTS OF BETA DISTRIBUTION AT $\alpha = 1\%$

p \ q	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0
3.0	0.0450	0.0428	0.0408	0.0390	0.0373	0.0358	0.0344	0.0331	0.0318	0.0307	0.0297	0.0287
3.5	0.0619	0.0589	0.0563	0.0538	0.0516	0.0495	0.0476	0.0458	0.0442	0.0427	0.0412	0.0399
4.0	0.0796	0.0759	0.0725	0.0695	0.0666	0.0640	0.0616	0.0595	0.0573	0.0554	0.0536	0.0519
4.5	0.0977	0.0933	0.0893	0.0856	0.0822	0.0791	0.0762	0.0735	0.0710	0.0686	0.0664	0.0644
5.0	0.1159	0.1108	0.1062	0.1019	0.0980	0.0944	0.0910	0.0878	0.0849	0.0822	0.0796	0.0772
5.5	0.1340	0.1283	0.1231	0.1183	0.1139	0.1097	0.1059	0.1023	0.0990	0.0959	0.0929	0.0902
6.0	0.1520	0.1457	0.1399	0.1346	0.1296	0.1251	0.1208	0.1168	0.1131	0.1096	0.1063	0.1032
6.5	0.1696	0.1627	0.1565	0.1507	0.1453	0.1402	0.1356	0.1312	0.1271	0.1233	0.1196	0.1162
7.0	0.1868	0.1795	0.1727	0.1665	0.1606	0.1552	0.1502	0.1454	0.1410	0.1368	0.1329	0.1292
7.5	0.2036	0.1958	0.1886	0.1820	0.1758	0.1700	0.1646	0.1595	0.1547	0.1502	0.1460	0.1420
8.0	0.2199	0.2117	0.2041	0.1971	0.1905	0.1844	0.1787	0.1733	0.1682	0.1634	0.1589	0.1546
8.5	0.2357	0.2272	0.2193	0.2119	0.2050	0.1985	0.1925	0.1868	0.1814	0.1764	0.1716	0.1671
9.0	0.2511	0.2422	0.2340	0.2263	0.2191	0.2124	0.2060	0.2001	0.1944	0.1891	0.1841	0.1793
9.5	0.2660	0.2569	0.2483	0.2403	0.2328	0.2258	0.2192	0.2130	0.2071	0.2016	0.1963	0.1914
10.0	0.2805	0.2710	0.2622	0.2540	0.2462	0.2390	0.2321	0.2257	0.2196	0.2138	0.2083	0.2031
10.5	0.2944	0.2847	0.2757	0.2672	0.2592	0.2518	0.2447	0.2380	0.2317	0.2257	0.2201	0.2147
11.0	0.3080	0.2980	0.2888	0.2801	0.2719	0.2642	0.2569	0.2501	0.2436	0.2374	0.2316	0.2260
11.5	0.3210	0.3109	0.3015	0.2926	0.2842	0.2763	0.2689	0.2618	0.2552	0.2488	0.2428	0.2370
12.0	0.3337	0.3234	0.3138	0.3047	0.2962	0.2881	0.2805	0.2733	0.2664	0.2599	0.2538	0.2479
12.5	0.3459	0.3355	0.3257	0.3165	0.3078	0.2996	0.2918	0.2845	0.2775	0.2708	0.2645	0.2584
13.0	0.3578	0.3472	0.3373	0.3280	0.3191	0.3108	0.3028	0.2953	0.2882	0.2814	0.2749	0.2688
13.5	0.3693	0.3586	0.3485	0.3391	0.3301	0.3216	0.3136	0.3059	0.2987	0.2918	0.2852	0.2789
14.0	0.3804	0.3696	0.3594	0.3498	0.3408	0.3322	0.3240	0.3163	0.3089	0.3018	0.2951	0.2887

References

Dhrymes, P. J. *Introductory Econometrics*. New York: Springer-Verlag, 1978.

Durbin, J., and Watson, G. S. "Testing for Serial Correlation in Least Squares Regression I." *Biometrika* 37 (1950): 409-28.

__________. "Testing for Serial Correlation in Least Squares Regression II." *Biometrika* 38 (1951): 159-78.

__________. "Testing for Serial Correlation in Least Squares Regression III." *Biometrika* 58 (1971): 1-19.

Farebrother, R. W. "The Durbin-Watson Test for Serial Correlation When There Is No Intercept in the Regression." *Econometrica* 48 (1980): 1533-63.

Henshaw, R. C. "Testing Single-Equation Least Squares Regression Models for Autocorrelated Disturbances." *Econometrica* 34 (1966): 646-60.

Jeong, Ki-Jun. "A New Approximation of the Critical Point of the Durbin-Watson Test for Serial Correlation." *Econometrica* 53 (1985): 477-82.

King, M. L. "The Durbin-Watson Test for Serial Correlation: Bounds for Regressions with Trend and/or Seasonal Dummy Variables." *Econometrica* 49 (1981): 1571-81.

__________. "The Durbin-Watson Test for Serial Correlation-Bounds for Regressions Using Monthly Data." *Journal of Econometrics* 21 (1983): 357-66.

Klein, L. R. *Economic Fluctuations in the United States, 1921-1941*. New York: Wiley, 1950.

Koerts, J., and Abrahamse, A. P. J. *On the Theory and Application of the General Linear Model*. Rotterdam: Rotterdam University Press, 1969.

Kramer, G. "On the Durbin-Watson Bounds Test in the Case of Regression through the Origin." *Jahrbücher für Nationalökonomie und Statistik* 185 (1971): 345-58.

Savin, N. E., and White K. J. "The Durbin-Watson Test for Serial Correlation with Extreme Sample Sizes or Many Regressors." *Econometrica* 45 (1977): 1989-96.

Theil, H., and Nagar A. L. "Testing the Independence of Regression Disturbances." *Journal of the American Statistical Society* 56 (1961): 793-806.