Wealth and Optimal Consumption of Retired Consumer: Two-Phase Life Cycle Model

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In testing the life cycle theory of saving, the question whether the bequest motive is a significant determinant of saving has become a controversial issue. Someone argue that if the wealth of the retired consumers declines at a slow rate, it is an indication of existence of a significant bequest motive or it is because of uncertainty of expected life span. In this paper, optimal consumption path is calculated, and some simulated results are presented on the time path of wealth for retired consumers. It is shown that if wealth is measured in nominal values, the wealth curve can take increasing and decreasing shapes depending upon the expected life span, the rate of interest, and the growth rate of consumption, irrespective of significance of the bequest motive. Thus, the rate of decumulation of nominal wealth alone cannot be used as evidence for significance or insignificance of the bequest motive.

I. Introduction

The life cycle theory of saving maintains that a consumer accumulates savings during the work life, and the accumulated savings will be used to support the retirement life. Thus the wealth should decline with age to zero at the time of death if the date of death is certain and if there is no bequest motive. In effect, the major motive for saving is to finance the retirement life, and the bequest motive should be a minor factor in saving decision (Modigliani and Brumberg 1954; Ando and Modigliani 1963).

Controversy occurs on the empirical significance of the bequest

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motive in the determination of personal saving. There are two empirical approaches. In the first approach, if the bequest motive is not significant, the savings of the retired consumer should be close to zero at the time of death except for the precautionary savings balances. In the second approach, if the bequest motive is not significant, the inherited wealth should be very small in the share of intergenerational transfer of total accumulated wealth.

As for the empirical results of the first approach, some studies have found rapidly declining wealth suggesting an insignificant bequest motive (King and Dicks-Mireaux 1982; Hamermesh 1984; Hurd 1987). However, some other studies have found increasing wealth or slow rates of decumulation of wealth for the retired consumers (Mirer 1979; Davies 1981; Blinder-Gordon-Wise 1983). Those who have found increasing wealth, or slow rates of decumulation of wealth argue that such phenomena must be explained in terms of a significant bequest motive and/or uncertainty on the date of death.[1]

As for the second approach, some argue that the inherited wealth (intergenerational transfer of wealth) ranged from 45-80% of total accumulated wealth, and that such large shares of intergenerational transfer of wealth must imply existence of a significant bequest motive (Kotlikoff and Summers 1981; Kotlikoff-Spivak-Summers 1982; Kotlikoff 1988). Modigliani (1986, 1988) argues that such large shares are resulted from different definitions of inherited wealth. For instance, in Kotlikoff and Summers, educational expenditures are included in the inherited wealth, and the income from inherited wealth is regarded as a bequest. Eliminating such amounts, Modigliani estimates that the share of the inherited wealth is about 20% of the total wealth in the U.S.

The above controversies suggest existence of the following problems: First, there is the need for the clarification of the definitions of a bequest motive and saving. For instance, should educational expenditures for children be regarded as saving, and the motive to educate children be regarded as a bequest motive? The answer may depend upon the intended use of the theory (Shin 1978). If the theory of saving is to explain the determination of national income in the framework of the Keynesian model, saving should be defined

[1]If the retirement life span is expected to increase, a retired consumer must further reduce annual consumption, if the initial wealth is given. Other factors that may contribute to the slow rate of wealth decumulation may include uncertainty on the future cost of living and uncertainty on health condition and medical cost.

as a leakage from the expenditure stream, and thus educational expenditures should not be regarded as saving since it is a current expenditure and not a leakage. Thus, the motive to educate children should not be regarded as a bequest motive.

On the other hand, if the theory intends to explain different types of expenditures, it is clear that expenditures on education and durable goods are different from expenditures on food, clothing, and services. Such classification may be useful for industrial demand analysis, and in some cases to explain productivity growth in the economy. In such cases, educational expenditures and child rearing costs may be regarded as saving and investment expenditures. Since the life cycle model is intended mainly for the Keynesian model of income determination, expenditures on education and durable goods should be regarded as consumption expenditures rather than saving. For instance, when a retired consumer purchases durable goods, they may be bequeathed later to an heir, but such expenditures should be regarded as consumption expenditures rather than saving.

Second, there is the question of scope of a bequest. In most studies, no explicit distinction was made between personal and public bequests, and planned and unplanned bequests. A personal bequest is the bequest to his personal heirs, relatives, and friends, and a public bequest is the bequest to public or private organizations such as church, school, and other social and political organizations. Also, Kessler and Masson (1989) argue that saving motives are not necessarily limited to life cycle saving, and personal and public bequest saving. They argue that additional motives are power, entrepreneurship, and social prestige. A question is whether all of the above types of savings or bequests should be regarded as savings from the bequest motive.

The problem is that the bequest motive is invisible, and it can be inferred only from the effects or results. Furthermore, an effect or a result can be either an intentional or unintentional one, or a planned or unplanned one. If it is possible to make such a distinction, only planned private bequest should be regarded as savings from the bequest motive. Public bequest should be excluded since most public bequests are made from the wealth already accumulated, and the cause of wealth accumulation was not the bequest motive, but the bequest was an effect of wealth accumulation. Thus, most public bequest and wealth left over after death without an heir specified should not be included in the savings resulted from the bequest motive. In effect, only the private planned bequest should be counted as the savings resulted from the bequest motive.

A third problem is concerned with the methodology of testing the significance of the bequest motive. A possible method may be to use multiple regression analysis in which dependent variable is the consumption expenditures of retired consumers and independent variables should include all relevant variables such as wealth, age, expected life span, health conditions, education, taste, marital status, amount and types of wealth, and detailed attributes of the heir (number of heirs, dependent and independent; age and income), etc. In such a model, if the heir variables are significant, we could infer that the bequest motive is significant in consumption and saving behavior.

However, the conventional methodology is to observe the time path of wealth of retired consumers using cross section and longitudinal panel data using only very few controlled variables. As stated before, in such studies, a slow rate of wealth decumulation is taken as an evidence for the existence of a significant bequest motive. However, as will be shown in this paper, when wealth is observed in nominal values, the wealth curve can take various shapes such as slowly decreasing and increasing time paths. Thus, Hurd (1987) observed changes in wealth in real values rather than in nominal values, and found no significant differences in the consumption patterns between the retired consumers with or without children, and concluded that there exists no significant bequest motive.

Using Canadian data, Burbidge and Robb (1985) found a concave wealth curve for blue collar workers. The peak wealth was around age 60. However, for white collar workers, the wealth curve reached a plateau at age 60, but started to rise from age 65 on. From the above results, one cannot conclude that blue collars workers have no bequest motive, while white collar workers have a bequest motive. They argue that these results are due to early retirement for blue collar workers and late retirement for white collar workers. They cite the following factors for the differences in the shape of the wealth curve: health, expected life span, expectations on intergenerational transfers, and differences in retirement income.

This paper is mainly concerned with theoretical reasons for existence of various shapes of wealth time path. There are two major objectives in this paper. First, it is to present the life cycle model in terms of a two-phase life cycle theory. The two-phase life cycle model is needed to show that consumers face different utility functions, choice problems and constraints during the work life and the retirement life. In terms of the two-phase life cycle model, it is simple and easy to show the determination of optimal consumption level for retired consumers.

The second objective is to show some simulated time paths of the wealth of the consumer who follows the optimal level of consumption. It is shown that for the utility maximizing retired consumer, the annual real consumption expenditures should be constant during the retirement life, when the initial wealth and the rate of interest are given, and that the wealth curve can take·various shapes of time path under such conditions.

In section II, the Modigliani-Brumberg life cycle theory is represented in terms of a two-phase life cycle theory. In section III, some simulation results are presented, and a summary and conclusions are provided in the final section IV. Some mathematical proofs are provided in the appendix notes.

II. Two-Phase Life Cycle Model

In the Modigliani-Brumberg life cycle model of consumption and saving (1954), the consumer is to maximize total utility over his life time. In our model, the life cycle is divided into two phases: In phase 1, the objective is to maximize his aggregate utility during his work life by allocating his income on consumption and saving, and his time on work and leisure. In phase 2, the objective is to maximize his aggregate utility during his retirement life span by appropriately allocating his given amount of wealth on annual consumption and the final bequest. In this phase 2, the retired consumer has no choice problem between work and leisure since his entire time consists of leisure by definition of retirement.

This scenario of the two-phase life cycle model is depicted in Figure 1. In phase 1, the Y_0Y_1 is the income path over his work life, and C_0C_1 is the consumption path over his work life. At age A_1, the worker retires with accumulated wealth W_0 and phase 2 begins, and ends at age A_2. The wealth decreases to zero at the time of death along W_0A_2. The wealth may monotonically fall to zero at A_2, or it may reach a peak at W_1 and then decline to zero at A_2, when bequest motive is absent. The consumption path C_1C_2 or C_1C_3 is an example of many possible consumption paths of the retired consumer over the retirement life. A_2 is the expected age of death. But, it is possible that the consumer's retirement life can suddenly end at

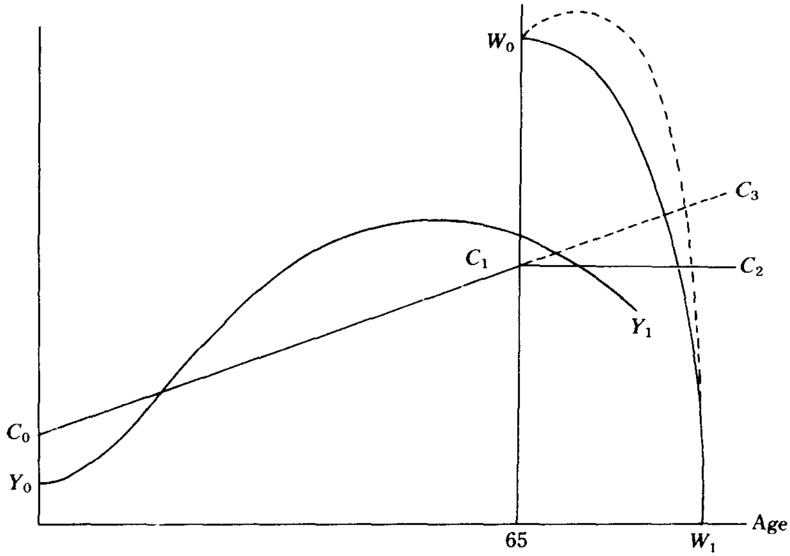

FIGURE 1
TWO-PHASE LIFE CYCLE MODEL

somewhere between A_1A_2, and the wealth remaining at the time of sudden death is regarded as the bequest.

In Figure 1, the retirement consumption path starts at C_1, which is equal to the peak consumption level at the end of his work life. However, the retirement consumption need not start at C_1. The initial retirement consumption can be higher or lower than C_1. The initial amount of consumption would depend upon various factors. Indeed, the determination of the initial optimal consumption level and its subsequent time path is the major objective of this paper.

In the conventional model, wealth is supposed to decrease monotonically along W_0A_2 path, if the bequest motive is zero. However, in this paper, it will be shown that even in the absence of a bequest motive, initial wealth W_0 can first increase to the peak W_1, and then fall to zero along $W_0W_1A_2$ path. Thus, it is possible that empirical observation reveals the upward sloping phase of W_0W_1, or the downward sloping phase of W_1W_2, depending upon the observation period.

The above two-phase cycle model can be represented in terms of equations. The objective functions and the constraints for consumers in each phase are restated below in terms of equations:

First, in phase 1, the objective of a consumer is to maximize

$$u = \sum_{t=0}^{T-1} \frac{1}{(1+\alpha)^t} u_t(c_t, L_t) + \frac{1}{(1+\alpha)^T} u_T(c_T, L_T, W_T) \tag{1}$$

$$\text{subject to} \quad c_t = c_t(y_t, W_t, e_1) \tag{2}$$

$$s_t = y_t - c_t \tag{3}$$

$$y_t = y_t(w_t, L_t, W_t, i, e_2) \tag{4}$$

$$W_t = W_t(s_1, s_2, s_3, \ldots, W_h, i, e_4) \tag{5}$$

where u_t = aggregate utility from consumptions during the work life at time t

$1/(1+\alpha)^t$ = time preference factor
c = annual real consumption
w = real wage rate
W = real wealth
W_h = inherited real wealth
y = annual real income
s = annual real saving
L = labor supply
i = real interest rate
e = other omitted variables
t = year
T = work life, i.e., the final year of work life

All variables are measured in real values. The utility function is assumed to be time separable and concave.

For a working consumer, the objective is to maximize his aggregate utility which is a function of annual consumption, labor supply and the accumulated wealth in the final year T (1). Equation (2) is a consumption function which depends upon income and wealth. Equation (3) is the definition of saving. Equation (4) states that income depends upon wage rate, labor supply, wealth, and interest rate. In the final equation (5), the wealth depends upon annual savings, inherited wealth, and interest rate. e_4 is the error term that includes all other omitted variables including a luck for lottery fortune.

There are two implications of the model of phase 1. First, the consumer decides the optimal annual consumption and labor supply subject to budget constraint. Second, the consumer decides the time to retire. In the above model, a consumer will decide to retire when he has accumulated a sufficient amount of wealth that can support the retirement life and possibly meet the planned amount of bequest. There are also some involuntary reasons for early retirement. They include poor health conditions, legal age of retirement, and unemployment. However, the question when to retire, and other consumer problems in phase 1 are not the main concern of this paper.[2]

Next, in phase 2, which is the main subject of this paper, the retired consumer is to maximize his aggregate utility which is a function of annual consumptions and the amount of planned bequest at the end of retirement life. The choice of leisure and work time is no longer a determinant of his utility since his entire retirement life consists of leisure, as stated before. The problem the retired consumer faces at the beginning of retirement life is the determination of annual optimal consumption expenditures. If his annual consumptions are too high in the early retirement years, there is the risk of low or zero consumption in the later years, and the risk of zero bequest. If the annual consumption expenditures are too small, he will leave an unplanned bequest at the time of death, and thus his aggregate utility will not be maximized.

To show the determination of the optimal consumption path, we assume that the objective of the retired consumer is to maximize his utility from his annual consumption expenditures over his retirement life, and the final amount of bequest at the time of death subject to the wealth constraint:

$$\max \quad U = \sum_{t=0}^{N-1} \frac{1}{(1+\beta)^t} U_t(c_t) + \frac{1}{(1+\beta)^N} U_N(c_N, b_N) \tag{6}$$

$$\text{subject to} \quad W_0 = \sum_{t=0}^{N} [c_t/(1+i)^t] + b_N/(1+i)^N \tag{7}$$

where $U =$ total utility over the retirement life span
$1/(1+\beta)^n =$ time preference factor
$c_t =$ annual real consumption in year t
$W_0 =$ initial present value of wealth at the time of retirement
$i =$ real interest rate, i.e., the real discount rate for future consumption to obtain the present value
$b_N =$ the planned real amount of bequest at the time of death

[2]For the above type of models and solutions, see McCafferty (1990). Burtless and Munnell (1990) has the following utility function: $U = U(C, R)$, where $C =$ identical annual real consumption, $R =$ age of retirement. The income constraint is given by $Y = RE$, where $E =$ identical annual income, $Y =$ lifetime income. The optimal age of retirement is determined where the indifference curve is tangent to the income constraint.

$N =$ retirement life, i.e., the final year of retirement life.

All variables are expressed in terms of real values. The utility function is assumed to be time separable and concave.

Equation (6) states that the consumer is to maximize his aggregate utility which is a function of annual consumptions and the amount of bequest at the time of death. Equation (7) is the income constraint that the present value of the initial wealth must equal the sum of present values of future consumptions and the present value of the future bequest.

To solve it by the method of Lagrange multipliers, the augmented Lagrange function is:

$$V = \sum_{t=0}^{N-1} \frac{1}{(1+\beta)^t} U_t(c_t) + \frac{1}{(1+\beta)^N} U_N(c_N, b_N) - \lambda [W_0 - c_0 - \frac{c_1}{1+i} - \frac{c_2}{(1+i)^2} - \dots - \frac{c_N}{(1+i)^N} - \frac{b_N}{(1+i)^N}] \tag{8}$$

The first order conditions are:

$$\begin{aligned}
\frac{\delta V}{\delta c_0} &= \frac{\delta U}{\delta c_0} - \lambda = 0 \\
\frac{\delta V}{\delta c_1} &= \frac{1}{1+\beta} \frac{\delta U}{\delta c_1} - \frac{\lambda}{(1+i)} = 0 \\
\frac{\delta V}{\delta c_2} &= \frac{1}{(1+\beta)^2} \frac{\delta U}{\delta c_2} - \frac{\lambda}{(1+i)^2} = 0 \\
&\dots\dots \\
\frac{\delta V}{\delta c_N} &= \frac{1}{(1+\beta)^N} \frac{\delta U}{\delta c_N} - \frac{\lambda}{(1+i)^N} = 0 \\
\frac{\delta V}{\delta b_N} &= \frac{1}{(1+\beta)^N} \frac{\delta U}{\delta b_N} - \frac{\lambda}{(1+i)^N} = 0 \\
\frac{\delta V}{\delta \lambda} &= W_0 - c_0 - \frac{c_1}{1+i} - \frac{c_2}{(1+i)^2} - \dots \\
&\quad - \frac{c_N}{(1+i)^N} - \frac{b_N}{(1+i)^N} = 0
\end{aligned} \tag{9}$$

where λ = the Lagrange multiplier.

Dividing the first equation by the successive equations, and rearranging the terms, the above system of equations can be expressed by the following equilibrium condition of utility maximization:

$$\frac{\delta U}{\delta c_0} = \frac{\delta U}{\delta c_1}\frac{1+\beta}{1+i} = \frac{\delta U}{\delta c_2}\frac{(1+\beta)^2}{(1+i)^2} = \cdots = \frac{\delta U}{\delta c_N}\frac{(1+\beta)^N}{(1+i)^N} = \frac{\delta U}{\delta b_N}\frac{(1+\beta)^N}{(1+i)^N} \tag{10}$$

In the above equations, $1/(1+\beta)^t$ and $1/(1+i)^t$ are the time preference and the real interest rate factors respectively. Since the social time preference can be represented in terms of the interest rate, it can be reasonably assumed that $i = \beta$. Then, Equation (10), except for the last term, states that the marginal utilities derived *from annual real consumptions should be the same every year.*

A question is, under what conditions, will the marginal utilities from the real consumption expenditures be the same each year? The answer is, under the law of diminishing marginal utility or the law of diminishing marginal rate of substitution, the marginal utilities should be the same only when the total real consumption expenditures are the same every year! For instance, if the total real consumption expenditure in year 1 is greater than in year 2, the marginal utility in year 1 should be smaller than in year 2. If the total real consumption expenditure in year 1 is smaller than in year 2, the marginal utility in year 1 should be greater than in year 2. The equilibrium is reached only when the total real consumption expenditures are the same each year.[3]

[3]This result is of course an elementary law of equimarginal utilities on goods and services applied to the case of intertemporal annual consumption expenditures. The law of equimarginal utilities on goods and services is a first order condition for the utility maximization when consumption expenditures are made on different goods and services:

$$MU_1/P_1 = MU_2/P_2 = MU_3/P_3 = \cdots = MU_N/P_N \tag{N1}$$

where MU_i = marginal utility of good i and P_i = price of good i. Equation (N1) states that the marginal utility of good 1 per dollar should be equal to the marginal utility of good 2 per dollar, etc. When it is applied to the intertemporal consumption expenditures, we have

$$MU_1 = MU_2 = MU_3 = \cdots = MU_N \tag{N2}$$

where MU_t = marginal utility from real consumption expenditures in year t. The implicit assumption is that annual real consumption expenditures are made every year on the same goods and services:

Equation (11) states that annual real consumption expenditures should be the same every year. However, it does not imply that the total amount of bequest should be the same as the annual real consumption expenditures. The bequest and the real consumption expenditures are two different types of "goods and services". In such a case, the marginal utilities will not necessarily be the same when real consumption expenditures and the real amount of bequest are equal.

In effect, Equation (10) suggests that the economic problem of finding an optimal consumption path to maximize the aggregate utility of the consumer is reduced to a mathematical problem of finding the initial optimal consumption expenditures that meet the given conditions:

Find the optimal consumption time path:

$$C_0, C_1, C_2, \dots, C_N \tag{11}$$

subject to
$$W_0 = C_0 + \frac{c_1}{(1+i)} + \frac{c_1}{(1+i)^2} + \cdots + \frac{c_N}{(1+i)^N} + \frac{b_N}{(1+i)^N} \tag{12}$$

Equation (12) may be expressed in terms of the current values:[4]

$$W_0 = C_0 + \frac{C_1}{(1+g)} + \frac{C_2}{(1+g)^2} + \cdots + \frac{C_N}{(1+g)^N} + \frac{B_N}{(1+g)^N} \tag{13}$$

$$\delta U/\delta c_0 = \delta U/\delta c_1 = \delta U/\delta c_2 = \delta U/\delta c_3 = \cdots = \delta U/\delta c_N = \delta U/\delta b_N \tag{N3}$$

Equation (N3) states that marginal utilities should be the same in terms of real values.

[4]Equation (12) can be expressed in terms of current values by multiplying each term with $(1+p)^t/(1+p)^t$:

$$W_0 = C_0 + \frac{c_1(1+p)}{(1+p)(1+i)} + \frac{c_2(1+p)^2}{(1+p)^2(1+i)^2} + \cdots + \frac{c_N(1+p)^N}{(1+p)^N(1+i)^N} + \frac{B_N(1+p)^N}{(1+p)^N(1+i)^N}$$

Since real consumption is constant every year, $c_t(1+p)^t = C_0(1+p)$, and $(1+p)(1+i) = (1+g)$, where $g =$ nominal interest rate, or the nominal discount rate, which is equal to the growth rate of nominal wealth.

where g = nominal interest rate, or the nominal growth rate of wealth, $(1+p)(1+i)=(1+g)$

B = the amount of bequest in current values

$B_N = b_N(1+p)^N$

C_t = consumption in current values

C_0 is the consumption in present value, so it is the same as the real consumption. Since the annual consumption in the current value is $c_t(1+p)^t$, and the annual real consumption should be constant, the current consumption can be expressed as $C_t = C_0(1+p)^t$. Thus, Equation (13) can be rewritten as

$$W_0 = C_0[1 + \frac{1+p}{1+g} + \frac{(1+p)^2}{(1+g)^2} + \cdots + \frac{(1+p)^N}{(1+g)^N}] + \frac{B_N}{(1+g)^N} \tag{14}$$

Letting $k=(1+p)/(1+g)$, the geometric series of the expression in the brackets can be simplified, and Equation (14) be can rewritten as[5]

$$W_0 = C_0 \frac{1-k^{N+1}}{1-k} + \frac{B_N}{(1+g)^N} \tag{15}$$

$$\begin{aligned} \text{Thus, } C_0 &= W_0 \frac{1-k}{1-k^{N+1}} - \frac{B_N}{(1+g)^N}\frac{1-k}{1-k^{n+1}} \\ &= [W_0 - \frac{B_N}{(1+g)^N}]\frac{1-k}{1-k^{N+1}} \end{aligned} \tag{16}$$

In the above model, the only unknown variable for the consumer is the initial real consumption level C_0. The subsequent consumption expenditures are determined once the initial real consumption expenditure C_0 is found since inflation rate is assumed to be constant. The amount of intended bequest B_N is predetermined. It should be either positive or zero.[6] All other variables such as W_0, g, p, and N are given parameters for the retired consumer.

[5]Equation (14) is solved using the present value of annuity formula: Letting $k=(1+p)/(1+g)$, the geometric series in the brackets is equal to

$$(1+k+k^2+k^3+\cdots+k^n) = (1-k^{n+1})/(1-k)$$

Another derivation is given in Appendix A.

[6]In reality, it can be even negative, that is, some people die leaving debt.

The meaning of the above equations can be restated in the following equations:

Find the optimal consumption time path:

$$C_0, C_1, C_2, \ldots, C_N \tag{17}$$

subject to $B_t = (W_t - C_t)$ (18)

$$B_N = (W_N - C_N) \tag{19}$$

$$C_t = C_0(1 + p)^t \tag{20}$$

$$W_t = B_{t-1}(1 + g) = (W_{t-1} - C_{t-1})(1 + g) \tag{21}$$

$$W_t, C_t > 0 \text{ if } t \leq N \tag{22}$$

The retired consumer starts with the initial wealth W_0. In (18), annual consumption C_t is subtracted from the wealth at the beginning of each year, and the balance is an unplanned bequest B_t, if the death occurs before the planned date. If the death occurs at the end of the retirement life, i.e., in year N, the actual bequest is equal to the planned bequest B_N (Equation 19). Annual consumption expenditures in current values are expected to increase at the inflation rate p (Equation 20). The current value of wealth is equal to the previous year's wealth balance plus current year's growth in wealth (Equation 21), where $g =$ the growth rate of wealth, i.e., interest rate $i = g$. Finally, wealth and annual consumptions should be positive during the retirement life (Equation 22).

III. Optimal Consumption and Wealth

The question is to find the initial optimal real consumption and the subsequent annual consumption path for the retired consumer. The solution depends upon assumptions concerning the amount of planned bequest. The solutions are derived from Equation (16), and they are summarized below for three different amounts of the planned bequest (see Appendix A and B):

Case 1. The planned bequest is zero, i.e., $W_N - C_N = B_N = 0$

$$C_0 = \begin{cases} W_0(1 - k)/(1 - k^{N+1}), & \text{if } k \neq 1.0 \quad (23) \\ W_0/(N + 1), & \text{if } k = 1.0 \quad (24) \end{cases}$$

where $k = (1 + p)/(1 + i) = (1 + p)/(1 + g)$, so $k = 1.0$ if $\mathrm{p} = g$.

Case 2. The planned bequest is positive, i.e., $W_N - C_N = B_N > 0$

$$C_0 = \begin{cases} [W_0 - B_N(1+g)^{-N}](1-k)/(1-k^{N+1}), & \text{if } k \neq 1.0 \quad (25) \\ [W_0 - B_N(1+g)^{-N}]/(N+1), & \text{if } k = 1.0 \quad (26) \end{cases}$$

Case 3. The planned bequest is a positive number including the final year's interest, i.e., $(W_N - C_N)/(1+g) = B_N > 0$,

$$C'_0 = \begin{cases} [W_0 - B_N(1+g)^{-N-1}](1-k)/(1-k^{N+1}), & \text{if } k \neq 1.0 \, (27) \\ [W_0 - B_N(1+g)^{-N-1}]/(N+1), & \text{if } k = 1.0 \, (28) \end{cases}$$

To try some simulations, we assume that initial wealth holding at the beginning of retirement at age 65 is equal to $W_0 = \$100{,}000$.[7] The growth rate of asset, i.e., the rate of return on wealth is $g = 0.10$. Further assume that the cost of living rises annually at the inflation rate $p = 0.05$. The objective is to find the optimal initial consumption and the time paths of wealth and consumption. As we have seen before, the equilibrium condition (10) requires that the initial real consumption must be maintained constant over his entire retirement life. The retirement life span is assumed to be $N = 20$. That is, we assume that he retires at age 65 and dies at age 85 (21st year). Further assume that consumption expenditures are made at the beginning of the year, and the balance of the wealth is reinvested each year at the rate of return g. The wealth is divisible and liquid (like a savings account), and transaction cost of wealth is zero.[8]

Substituting the initial conditions in equation (23), we obtain the initial optimal consumption expenditure \$7,289.85.[9] In Table 1, the

[7]We assume that age 65 is the retirement age 1. When life span (N) is, for instance, 20 years, we assume that he dies anytime during the 21st year, spending the full amount of the annual consumption expenditures.

[8]A house is usually illiquid and indivisible. A reverse mortgage plan is available under which the mortgage bank provides monthly payments to the home owner such that home equity value is reduced to zero at the time of death. However, the plan is not popular among the retired consumers. Diamond and Hausman (1984) examined the National Longitudinal Survey of Mature Men (1966), and reported that in 1966, 7.6% of men aged 45-59 had negative wealth, and 12.1% had net wealth below \$1,000. A large fraction of the population held wealth primarily in home ownership. Excluding the home equity, 30% of the population had no wealth, 39% had wealth below \$490, and 50% had wealth below \$1,500.

If the transaction cost is not zero, it can be deducted from the growth rate of wealth, or from the annual consumption expenditures.

[9]
$$k = (1+p)/(1+g) = (1+0.05)/(1+1.10) = 0.954545$$

$$C_0 = 100{,}000(1-0.954545)/(1-0.954545^{20+1}) = 7289.85$$

TABLE 1
WEALTH AND CONSUMPTION ($p = 0.05$, $g = 0.10$)

Age	Wealth at beginning	Consump.	Balance	Wealth at end with interest	Real consump.	Rate of change in wealth	Real wealth	Rate of change in real wealth
65	100,000.00	7,289.85	97,710.15	101,981.17	7,289.85		100,000.00	
66	101,981.17	7,654.34	94,326.82	103,759.50	7.289.85	0.0198	97,124.92	—0.0288
67	103,759.50	8,037.06	95,722.45	105,294.69	7,289.85	0.0174	94,112.93	—0.0310
68	105,294.69	8,438.91	96,855.78	106,541.35	7.289.85	0.0148	90,957.51	—0.0335
69	106,541.35	8,860.86	97,680.50	107,448.55	7,289.85	0.0118	87,651.84	—0.0363
70	107,448.55	9,303.90	98,144.64	107,959.11	7.289.85	0.0085	84,188.75	—0.0395
71	107,959.11	9,769.10	98,190.01	108,009.01	7,289.85	0.0048	80,560.75	—0.0431
72	108,009.01	10,257.55	97,751.46	107,526.61	7.289.85	0.0005	76,759.99	—0.0472
73	107,526.61	10,770.43	96,756.18	106,431.80	7,289.85	—0.0045	72,778.24	—0.0519
74	106,431.80	11,308.95	95,122.85	104,635.13	7.289.85	—0.0102	68,606.89	—0.0573
75	104,635.13	11,874.40	92,760.74	102,036.81	7,289.85	—0.0169	64,236.90	—0.0637
76	102,036.81	12,468.12	89,568.69	98,525.56	7.289.85	—0.0248	59,658.81	—0.0713
77	98,525.56	13,091.52	85,434.04	93,977.44	7,289.85	—0.0344	54,862.72	—0.0804
78	93,977.44	13,746.10	80,231.34	88,254.48	7.289.85	—0.0462	49,838.24	—0.0916

TABLE 1
(CONTINUED)

Age	Wealth at beginning	Consump.	Balance	Wealth at end with interest	Real consump.	Rate of change in wealth	Real wealth	Rate of change in real wealth
79	88,254.48	14,433.40	73,821.07	81,203.18	7,289.85	−0.0609	44,574.51	−0.1056
80	81,203.18	15,155.07	66,048.11	72,652.92	7.289.85	−0.0799	39,060.12	−0.1237
81	72,652.92	15,912.83	56,740.09	62,414.10	7,289.85	−0.1053	33,283.14	−0.1479
82	62,414.10	16,708.47	45,705.63	50,276.19	7.289.85	−0.1409	27,231.06	−0.1818
83	50,276.19	17,543.89	32,732.30	36,005.53	7,289.85	−0.1945	20,890.80	−0.2328
84	36,005.53	18,421.09	17,584.44	19,342.88	7.289.85	−0.2838	14,248.61	−0.3179
85	19,342.88	19,342.14	0.74	0.82	7.289.85	−0.4628	7,290.13	−0.4884
Total	1,864,276.02	260,387.99			153,086.85	−1.3874 0.0111[1] −0.1127[2]	1,267,916.84	−2.2737
Average	88,775.05	12,399.43			7,289.85	−0.0694[3]	100,000.00	−0.1137
Average 65–75								−0.0432

Note : Initial wealth $W_0 = \$100,000$. Initial consumption $C_0 = \$7,289.85$

1. average annual rate of increase in nominal wealth during ages 66–72.
2. average annual rate of decrease in nominal wealth during ages 73–85.
3. average annual rate of change in nominal wealth during ages 66–85.

time paths of nominal and real consumption, nominal and real wealth, and their rates of change are presented. It should be noted that the nominal wealth increases from $100,000 to $108,009.01 at age 71 and then monotonically declines to zero at the final age of 85. The time paths of wealth and consumption are shown in Figure 1. The nominal wealth rises up to age 71, and then falls, and in the final year, the remaining wealth balance is equal to the amount of final consumption at age 85.[10]

In Figure 1, what is to be emphasized is that even when the date of death is certain and the planned bequest is zero, the nominal wealth level first increases, and reaches a peak, and then falls. The nominal wealth increases at the average annual rate 1.11% during ages 66–72, and decreases at 11.27% during ages 73–85. The overall average annual rate of nominal wealth decumulation is 11.27%. The real wealth, on the other hand, falls monotonically at the average annual rate of 11.37%. However, the annual rate of real wealth decumulation is 4.32% during the first 10 years, i.e., ages 66–75.[11]

The initial increases in nominal wealth in the early years are due to the fact that the retirement life span is so long that the wealth has to be stretched to cover the cost of living over the long retirement years. In the early years, the growth rate of wealth is higher than the growth rate of consumption, but consumption level increases at a compound rate, but wealth increases at a simple rate because annual consumption is subtracted from the wealth at the beginning of each year. As a result, at a certain point of time, annual consumption expenditure becomes to exceed the annual increase in wealth, and so the nominal wealth starts to fall.[12]

The effects of increasing and decreasing retirement life span are shown in Figures 3 and 4. Figure 3 is based on the following assumptions: $p = 0.10$, $g = 0.05$, and the age of death is increased

[10]The above simulation results are quite realistic. According to the US Census Bureau report (1991), the median net worth in 1988 increased with age, but began to decline among households over 70 years old. See Appendix C.

[11]Hurd (1987) estimated that for couples with initial positive real wealth, the wealth decreased by 2.9% when housing wealth is included, and 16.9% when housing wealth is not included during 1969–79. For singles, the initial wealth decreased by 25.2% when housing is included, and by 39.8% when housing is not included.

[12]The age at which the nominal wealth reaches a peak is given by

$$t_{\max} = [\ln(\ln G/\ln P)]/\ln(P/G) + N + 1$$

where $t_{\max}$ = retirement age at which the nominal wealth reaches a peak, $G = 1 + g$, $P = 1 + p$, N = retirement life span. See Appendix B for the derivation.

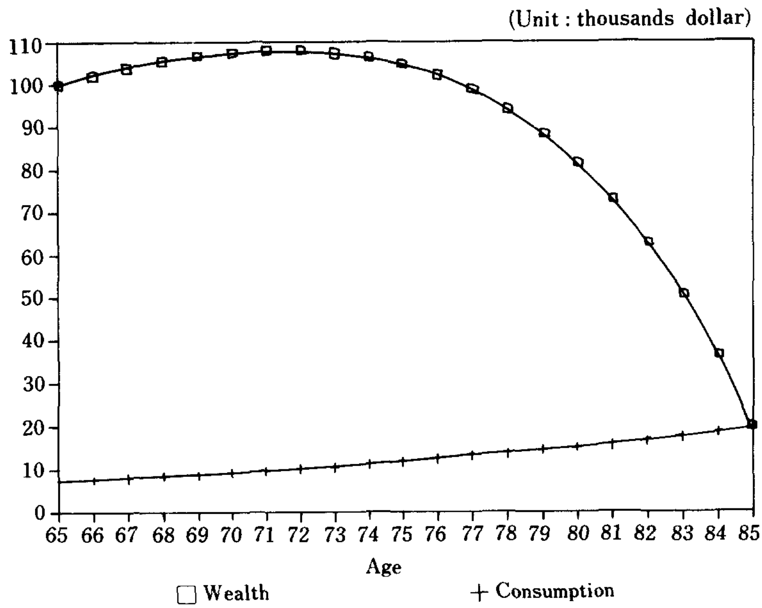

FIGURE 2
WEALTH, CONSUMPTION, AND AGE

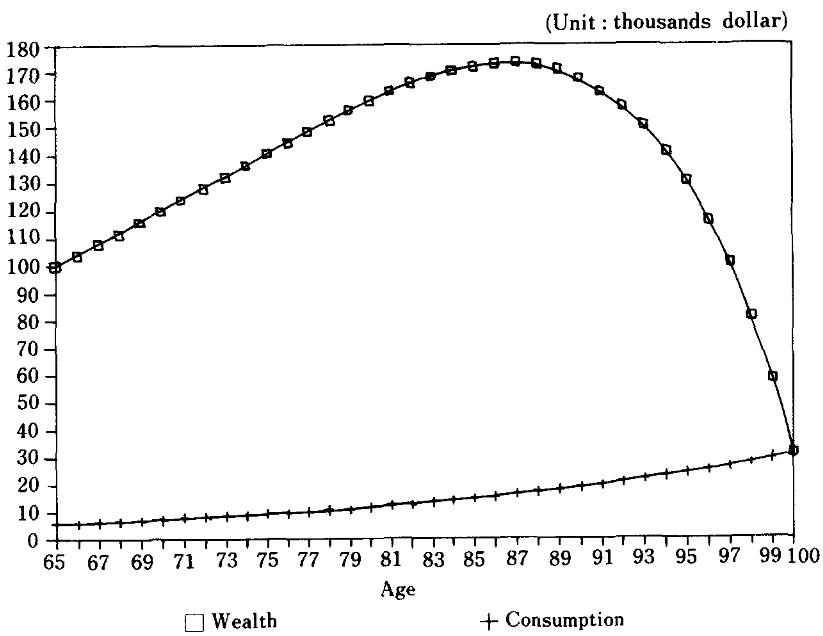

FIGURE 3
WEALTH, CONSUMPTION, AND AGE

to 100 years. The initial optimal consumption is found to be \$5,593.44. The peak of wealth is reached at age 86.61. Figure 4 is based on the following assumptions: $p = 0.05$, $g = 0.10$, and the age of death is reduced to 75 years. The initial optimal consumption is found to be \$11,348.41.[13] In Figure 4, the wealth decreases monotonically without a peak (See Appendix B and Table B1).[14]

Figure 5 is the case of unplanned bequest due to an unanticipated early sudden death at age 85, though the retired consumer planned to live until age 100. The other assumptions are: $g = 0.05$, and $p = 0.10$. So the initial optimal consumption expenditure is \$5,593.45. In Figure 5, the wealth increases monotonically until the time the consumer dies at age 85.[15]

The general shape of the wealth curve is the same even when the planned bequest is positive. If the planned bequest is \$50,000 at the beginning of age 85 without including that year's interest, the initial optimal consumption is \$6,748.06. If the amount of bequest \$50,000 includes interest income in the final year, the initial consumption is \$6,797.31.[16]

13

$$k = (1 + 0.05)/(1 + 1.10) = 0.954545$$

$$C_0 = 100{,}000(1 - 0.95454545)/(1 - 0.954545^{35+1}) = 5{,}593.44$$

$$C_0 = 100{,}000(1 - 0.95454545)/(1 - 0.954545^{10+1}) = 11{,}348.41$$

14 The first order condition for existence of the peak of the nominal wealth is that the retirement life span must be sufficiently large such that

$$N > [\ln(\ln P/\ln G)]/\ln(P/G) - 1$$

If $p = 0.05$, and $g = 0.1$, the retirement life span should be greater than 13.39 years:

$$N > [\ln(\ln 1.05/\ln 1.10)/\ln(1.05/1.1) - 1]$$

$$= 14.39 - 1 = 13.39$$

See Appendix B.

15

$$C_0 = W_0(1 - k)/(1 - k^{N+1})$$

$$= 100{,}000(1 - 1.05/1.10)/[1 - (1.05/1.10)^{35+1}] = 5{,}553.45$$

16 When bequest includes interest:

$$C_0 = [100{,}000 - 50{,}000(1 + 0.10)^{-20}]\ (1 - 0.954545)/(1 - 0.954545^{21})$$

$$= 6{,}748.06$$

When bequest does not include interest:

$$C'_0 = [100{,}000 - 50{,}000(1 + 0.10)^{-21}]\ (1 - 0.954545)/(1 - 0.954545^{21})$$

$$= 6.797.31$$

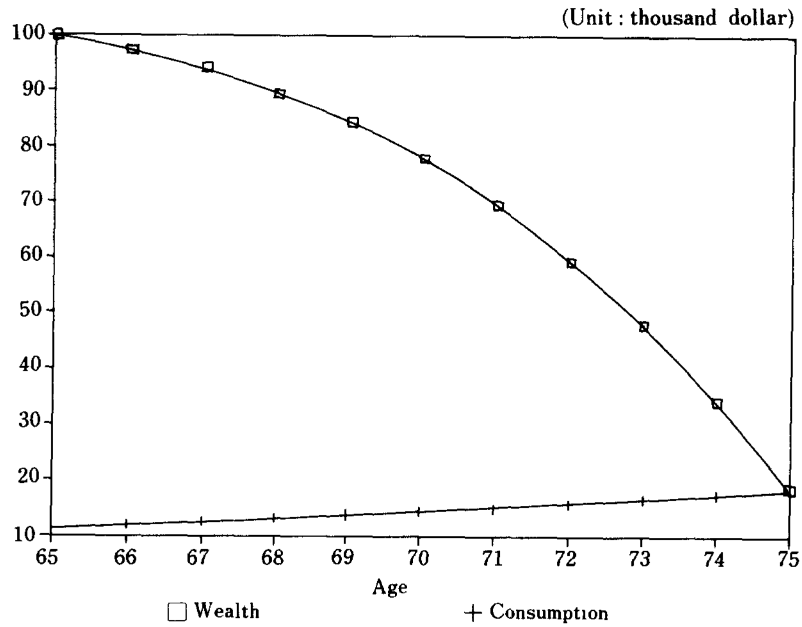

FIGURE 4
WEALTH, CONSUMPTION, AND AGE

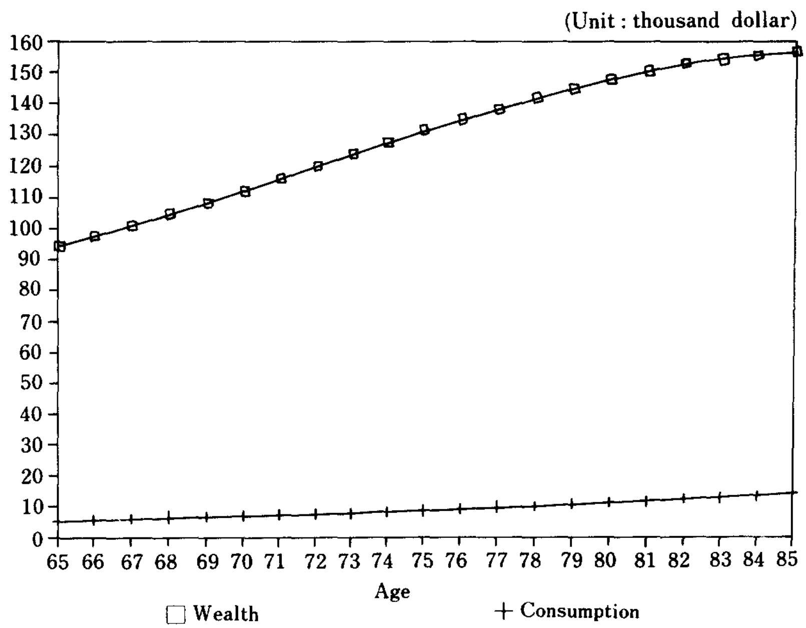

FIGURE 5
WEALTH, CONSUMPTION, AND AGE

IV. Summary and Conclusions

In the two-phase life cycle model, a consumer has quite different sets of determinants of utility, and constraints in the two phases of a life cycle. In phase 1, his utility depends upon consumption, and leisure as well as the accumulated wealth. In phase 1, some consumers may make retirement planning and bequest planning by trying to increase the target wealth as much as possible. However, the target wealth may not necessarily be attainable by the time of retirement because of illness, unemployment, marriage, divorce, children to rear, low wage rate, and other causes.

When the consumer moves into phase 2, i.e., the stage of retirement, he may have to set up an entirely new consumption and bequest planning over the retirement life. His utility now depends upon annual consumption expenditures and a planned bequest. His annual consumption plan is determined by the given wealth, life span, interest rate, inflation rate, and the amount of planned bequest. It is quite possible that his desire to bequeath may be quite strong, but he cannot plan any bequest if the accumulated wealth is very small and he can barely cover the subsistence of living. It is also possible that he may have no desire to bequeath, and yet he may have to leave a large bequest if his wealth is very large and he is unable to spend it due to poor health or changes in taste and in the philosophy of life. In any case, we have shown that to maximize his aggregate utility over the retirement life, the annual real consumption expenditures must remain constant, and the wealth path can take various shapes depending upon the initial wealth, interest rate, inflation rate, and the life span, irrespective of the bequest motive.

A controversial empirical issue in the life cycle model is whether the bequest motive is significant in determining consumption and saving. In the strict life cycle theory, bequest does not affect his utility, and the bequest motive drops out of the utility function. In the modified life cycle theory, bequest is a significant determinant of his utility, and the bequest motive is a significant motive for saving.

In our two-phase life cycle model, bequest is not explicitly included as an independent variable in the utility function of phase 1, though it may be included implicitly in the wealth variable. The consumer tries to maximize his utility by increasing his terminal wealth as much as possible to support his retirement life and possibly to meet his planned bequest. In the utility function of phase 2, the bequest motive or the planned bequest may become more important and may enter his utility function as an explicit determinant. The retired consumer must rebuild his consumption plan based on the given wealth and economic conditions such as inflation, interest rate, life span, and the economic conditions of heirs and personal relationships with them.

In the previous studies of significance of the bequest motive, they examined mainly the time path of wealth of retired consumers. Those who have found increasing wealth and slow rates of decumulation of wealth argued that such phenomena must be explained in terms of a significant bequest motive and/or uncertain date of death. In this paper, we have shown that the nominal wealth of the retired consumer can increase in the early years of retirement, and then falls in the later years even under the conditions of certainty on the date of death and no bequest motive.

In the early years of retirement, if the consumption expenditure is less than the return from the wealth, the nominal wealth increases. However, the annual consumption expenditures increase at a compound rate, while the wealth increases at a simple rate due to the fact that consumption expenditures are subtracted from the wealth every year. As a result, the annual level of consumption expenditures becomes to exceed the annual increase in wealth, and ultimately wealth starts to decline. The shape of the wealth function depends upon mainly the initial wealth, life span, interest rate, inflation rate, and the initial level of consumption, and the amount of planned bequest is a secondary factor. Thus, by the rate of capital decumulation alone, or by the shape of wealth curve alone, we will be unable to tell the significance of the bequest motive. A better method may be to test real consumption functions for retired consumers and working consumers using multivariate analysis.[17]

[17]The two-phase life cycle model assumes that the two groups of consumers have different utility functions and constraints. To support such an approach, empirical real consumption functions should be different for the two groups.

Appendix A

Derivation of Equations (23)-(28)

Case 1. Bequest $B_N = 0$,

$$\text{Let } p = (1 + p), \text{ and } G = (1 + g)$$

$$\text{Then } \quad W_t = (W_{t-1} - C_{t-1})G$$

$$C_t = C_0 P^t$$

$$\text{Thus,} \quad W_t = W_{t-1}G - C_0P^{t-1}G \tag{A1}$$

$$W_1 = W_0G - C_0P^0G$$

$$W_2 = W_0G^2 - C_0P^0G^2 - C_0P^1G$$

$$W_3 = W_0G^3 - C_0P^0G^3 - C_0P^1G^2 - C_0P^2G$$

$$W_4 = W_0G^4 - C_0P^0G^4 - C_0P^1G^3$$
$$- G_0P^2G^2 - C_0P^3G$$
$$\cdots$$

$$W_N = W_0G^N - C_0P^0G^N - C_0P^1G^{N-1}$$
$$- C_0P^2G^{N-2} - \cdots - C_0P^{N-1}G$$

If the planned bequest is zero in the final year N, $W_N - C_N = B_N = 0$.
Thus,

$$C_0P^N = W_0G^N - C_0[P^0G^N + P^1G^{N-1} + P^2G^{N-2} + \cdots + P^{N-1}G]$$

or,

$$C_0 = W_0G^N/[P^N + (P^0G^N + P^1G^{N-1} + P^2G^{N-2} + \cdots + P^{N-1}G)]$$
$$= W_0/[1 + (P/G) + (P/G)^2 + \cdots + (P/G)^N]$$

Since $1 + x + x^2 + x^3 + \cdots + x^n = (1 - x^{n+1})/(1 - x)$, assume $k = (1 + p)/(1 + g)$
Then, we have

$$C_0 = \begin{cases} W_0(1-k)/(1-k^{N+1}), & \text{if } k \neq 1.0 \quad \text{(A2)} \\ W_0/(N+1), & \text{if } k = 1.0 \quad \text{(A3)} \end{cases}$$

Case 2. Bequest (excluding final year's interest) $B_N > 0$

If the planned bequest at the time of certain death is positive, i.e., $W_N - C_N = B_N > 0$, then Equation (A1) can be rewritten as

$$\begin{aligned} W_0G^N - C_0[P^0G^N - P^1G^{N-1} - P^2G^{N-2} \\ - \cdots - P^{N-1}G] - C_0P^N = B_N \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned} C_0 &= (W_0G^N - B_N)/[P^N + (P^0G^N + P^1G^{N-1} + P^2G^{N-2} \\ &\quad + \cdots + P^{N-1}G)] \\ &= (W_0 - B_N/G^N)/[1 + (P/G) + (P/G)^2 + \cdots \\ &\quad + (P/G)^N] \end{aligned}$$

$$C_0 = \begin{cases} (W_0 - B_NG^{-N})(1-k)/(1-k^{N+1}), & \text{if } k \neq 1.0 \quad \text{(A5)} \\ (W_0 - B_NG^{-N})/(N+1), & \text{if } k = 1.0 \quad \text{(A6)} \end{cases}$$

Note : In the above case, if the bequest is 50,000, and interest rate is 0.10, then the beneficiary will get 50,000 plus accrued interest earnings, 5,000.

Case 3. Bequest (including final year's interest) $B_N > 0$

If the planned bequest is to include the final year's interest, i.e., $(W_N - C_N)/(1+g) = B_N$, then Equation (A4) can be rewritten as

$$\begin{aligned} W_0G^N - C_0[P^0G^N - P^1G^{N-1} - P^2G^{N-2} \\ - \cdots - P^{N-1}G] - C_0P^N = B_NG \end{aligned} \quad \text{(A7)}$$

And Equations (A5) and (A6) can be rewritten as

$$\begin{aligned} C'_0 &= (W_0G^N - B_NG)/[P^N + (P^0G^N + P^1G^{N-1} + P^2G^{N-2} \\ &\quad + \cdots + P^{N-1}G)] \\ &= (W_0 - B_NG/G^N)/[1 + (P/G) + (P/G)^2 \\ &\quad + \cdots + (P/G)^N] \end{aligned}$$

$$C'_0 = \begin{cases} (W_0 - B_NG^{-N-1})(1-k)/(1-k^{N+1}), & \text{if } k \neq 1.0 \quad \text{(A8)} \\ (W_0 - B_NG^{-N-1})/(N+1), & \text{if } k = 1.0 \quad \text{(A9)} \end{cases}$$

Note : In the above case, if the bequest is 50,000 including interest, the retired consumer should leave 45,454.54, which will generate interest income 4,545.46 by the end of the year, and the total bequest will add up to 50,000.

Appendix B

Existence of Maximum Wealth Age

This note shows the first order conditions under which the wealth has a maximum value, and the age at which the wealth is a maximum.

Case 1. $p \neq g$

$$\begin{aligned}
\text{Given}\quad W_t &= W_0G^t - C_0[P^0G^t + P^1G^{t-1} + P^2G^{t-2} \\
&\quad + \cdots + P^{t-1}G)] \qquad \text{(B1)}\\
&= (W_0G^t - C_0G^t[1 + (P/G) + (P/G)^2 \\
&\quad + \cdots + (P/G)^{t-1}] \\
&= G^t\,\{W_0 - C_0[(1-k^t)/(1-k)]\} \\
&\quad \text{where } k = (1+p)/(1+g) \\
&= G^t\,\{W_0 - W_0[(1-k)/(1-k^{N+1})] \\
&\quad \times [(1-k^t)/(1-k)]\} \\
&= W_0G^t[1-(1-k^t)/(1-k^{N+1})] \\
&= W_0G^t[(k^t - k^{N+1})/(1-k^{N+1})] \qquad \text{(B2)}
\end{aligned}$$

The maximum of W_t occurs at

$$\begin{aligned}
0 = dW_t/dt = W_0t(\ln\ G)G^t[(k^t - k^{N+1})/(1-k^{N+1}) \\
+ G^t(\ln\ k)(k^t/(1-k^{N+1})] \qquad \text{(B3)}
\end{aligned}$$

$$(\ln\ G)(k^t - k^{N+1}) + (\ln\ k)k^t = 0$$

$$(\ln\ G + \ln\ k)k^t = (\ln\ G)k^{N+1} \qquad \text{(B4)}$$

$$k^t = (\ln\ G/\ln\ P)k^{N+1}$$

$$\begin{aligned}
\text{Thus, } t_{\max} &= [\ln(\ln\ G/\ln\ P)]/\ln\ k + N + 1 \\
&= [\ln(\ln\ G/\ln\ P)]/\ln(P/G) + N + 1
\end{aligned} \qquad \text{(B5)}$$

The first order condition for the existence of a maximum is $t_{\max} >$

0. Thus,

$$[\ln(\ln G/\ln P)]/\ln(P/G) + N + 1 > 0 \tag{B8}$$

or,

$$N > [\ln(\ln P/\ln G)]/\ln(P/G) - 1 \tag{B9}$$

Case 2. $p = g$

$$\begin{aligned} W_t &= W_0G^t - C_0G^t[1 + (P/G) + \cdots + (P/G)^{t-1}] \\ &= W_0G^t - [W_0/(N+1)]G^t \\ &= W_0G^t[1 - t/(N+1)] \end{aligned} \tag{B10}$$

To find the maximum wealth age, the first order condition is

$$\begin{aligned} 0 = dW_t/dt &= W_0(\ln G)G^t[1 - t/(N+1)] \\ &+ W_0G^t[-1/(N+1)] \\ &= \ln G - (t \ln G)/(N+1) - 1/(N+1) \end{aligned} \tag{B11}$$

Thus,

$$\begin{aligned} t_{\max} &= \frac{[\ln G - 1/(N+1)]}{(\ln G)/(N+1)} \\ &= N + 1 - 1/(\ln G) \end{aligned} \tag{B12}$$

The existence condition is $t_{\max} > 0$. So

$$N + 1 - 1/(\ln G) > 0$$

Thus, $$N > 1/(\ln G) - 1 \tag{B13}$$

Case 3. When G and P values are interchanged.

$$\text{Since } \frac{\ln(\ln G/\ln P)}{\ln(P/G)} = \frac{-\ln(\ln G/\ln P)}{-\ln(P/G)} = \frac{\ln(\ln P/\ln G)}{\ln(G/P)} \tag{B14}$$

Equation (B5) can be rewritten as

$$t_{\max} = [\ln(\ln P/\ln G)]/(\ln G/P) + N + 1 \tag{B15}$$

Equations (B5) and (B15) suggest that $t_{\max}$ remains the same when P and G values are interchanged.

In the final note, we have assumed that when the retirement life span is N, the retirement ends in the $(N+1)$th year. If we assume that retirement life ends in the Nth year, in the above formulas, the constant N should be replaced with $(N-1)$. For instance, when N is replaced with $(N-1)$, Equations (A2) and (A3) will be changed to

$$C_0 = \begin{cases} W_0(1-k)/(1-k^N), & \text{if } k \neq 1.0 \quad (\text{A2})' \\ W_0/N, & \text{if } k = 1.0 \quad (\text{A3})' \end{cases}$$

Numerical Examples :

Let $p = 0.05$, $g = 0.10$, then $G = (1 + 0.10) = 1.1$, and $P = (1 + 0.05) = 1.05$; and $N = 20$ years. Then $\ln G/\ln P = 1.95347$, $k = (1 + 0.05)/(1 + 1.10) = 0.954545$, and $\ln k = \ln 0.954545 = -0.04652$. Substituting these numbers in (B5), we get

$$\begin{aligned} t_{\max} &= \ln 1.95347/\ln 0.954545 + 20 + 1 \\ &= 0.669623/(-0.04652) + 20 + 1 \\ &= -14.39429 + 21 = 6.6057 \end{aligned}$$

When p and g values are interchanged such that $p = 0.10$, and $g = 0.05$, $\ln P/\ln G = \ln 1.1/\ln 1.05 = 0.0953/0.04879 = 1.95347$, and $k = (1+g)/(1+p) = 1.05/1.10 = 0.954545$, and $\ln(0.9545) = -0.04652$.

$$\begin{aligned} t_{\max} &= \ln 1.95347/\ln 0.954545 + 20 + 1 \\ &= 0.669623/(-0.04652) + 20 + 1 \\ &= -14.39429 + 21 = 6.6057 \end{aligned}$$

However, when p and g values are interchanged, the initial optimal consumption changes: Assuming $p = 0.05$, $g = 0.10$, $N = 20$, and bequest $= 0$, from Equation (A2), we get

$$\begin{aligned} C_0 &= W_0(1-k)/(1-k^{N+1}) \\ &= 100{,}000(1 - 1.05/1.10)/[1 - (1.05/1.10)^{20+1}] \\ &= 7{,}289.86 \end{aligned}$$

If $p = 0.10$, $g = 0.05$, and $N = 20$, the initial optimal consumption is

$$\begin{aligned} C_0 &= 100{,}000(1 - 1.10/1.05)/[1 - (1.10/1.05)^{20+1}] \\ &= 2{,}875.09 \end{aligned}$$

TABLE B1
AGE AT WHICH THE NOMINAL WEALTH IS A MAXIMUM

Retirement life span (N)	10	15	20	25	30	35
Maximum wealth age (after retirement)	−3.39	1.61	6.61	11.61	16.61	21.61
Maximum wealth age (total personal age)	ne	66.61	71.61	76.61	81.61	86.61
Expected age of death	75	80	85	90	95	100
Initial consumption	11,348.42	8,658.96	7,289.86	6,478.15	5,952.87	5,593.45

Note : $p = 0.05$, and $g = 0.10$
n = a maximum does not exist during the retirement life span.

Since the age at which the nominal wealth reaches a peak depends upon the retirement life span as well as p and g values, we have tried various values of the retirement life span N, and have obtained the following results. As stated above, the maximum wealth ages are the same either for the case when $p = 0.05$, and $g = 0.10$, or for the case $p = 0.10$, and $g = 0.05$. However, the initial optimal consumption changes, if p and g values are reversed:

The above table shows that, for a retired consumer with retirement life span 20 years, for instance, if the inflation rate is 5%, and the interest rate on wealth is 10%, the nominal maximum wealth is reached at age 71.61 years (i.e., 6.61 years after the retirement), and the initial optimal consumption expenditure should be $7,289.86. If the expected life span is 100 years, then the nominal wealth reaches a peak at age 86.61 years (i.e., 21.61 years after retirement), and the initial optimal consumption expenditure should be $5,593.45. (A computer software is available from the authors. It calculates the initial optimal consumption, time paths of annual consumption and wealth, and the maximum age at which the wealth reaches a peak.)

Appendix C

1988 Census Bureau Survey

According to the 1988 Census Bureau report, "the median net worth was $35,752 in 1988 compared with $37,012 in 1984. The net worth was $62,390 for the typical married white couple, and

$17,640 for the typical married black couple. The wealthiest 3% of all households hold 27% of the approximately $8.4 trillion of net worth. Those whose heads are between 35 and 44 years old in 1988 had median net worth of $40,264, while those between 55 and 64 years old had median net worth of $83,750. Median net worth begins to decline among households over 70." "Net worth included savings, securities, real estate, autos, mortgage, but excluded jewelry, pension plans, insurance policies, furniture, art, and antiques. About 43% of all net worth is tied up in equity in homes" (reported in the *Wall Street Journal,* January 11, 1991, p. A3).

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