Incomplete Markets with Endogenous Portfolio Constraints and Redundant Assets

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This paper shows that a competitive equilibrium exists in an exchange economy with incomplete financial markets where redundant assets are traded and the asset trading of each agent is subject to endogenous portfolio constraints. The set of budget-feasible portfolios need not be bounded in the presence of redundant assets. To address this problem, we impose the positive semi-independence condition on individual portfolio constraints.

Keywords: Incomplete markets, Competitive equilibrium, Endogenous portfolio constraints, Redundant assets, Constrained arbitrage

JEL Classification: C62, D51

I. Introduction

When financial markets are unconstrained, redundant (financial) assets do not play a role in risk-sharing and thus they are useless. Therefore, without loss of generality, we can assume that there is no redundant asset. In reality, however, redundant assets such as futures and options exist because financial markets are subject to portfolio constraints. In financial markets, agents usually face portfolio constraints when they trade financial assets. Portfolio constraints capture market frictions such

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as short-selling constraints, credit limits, bid-ask spreads, margin requirements, and proportional transaction costs. It is noted that many of portfolio constraints (*e.g.*, margin requirements) depend on asset prices. It is important to investigate how redundant assets and endogenous portfolio constraints affect equilibrium asset prices in financial markets.

The purpose of this paper is to show that there exists a competitive equilibrium in an exchange economy with incomplete financial markets, where agents are subject to portfolio constraints depending on asset prices. Aouani and Cornet (2011) and Hahn and Won (2014) among others demonstrate the existence of a competitive general equilibrium in an exchange economy with incomplete markets, where each agent's asset trading is subject to exogenous portfolio constraints, which do not depend on endogenous variables. However, when agents participate in financial markets, they are often faced with endogenous portfolio constraints such as margin requirements, which depend on asset prices.

Several recent papers have studied this problem, including Carosi *et al.* (2009) and Cea-Echenique and Torres-Martinez (2014), among others. Carosi *et al.* (2009) describe portfolio constraints by restriction functions, which depend on first-period consumption and commodity prices, as well as financial asset prices. They assume that portfolio restriction functions are continuously differentible in order to characterize the generic regularity of equilibrium. Thus such approach cannot cover cases in which portfolio constraints are represented by convex cones (*e.g.*, margin requirements). Moreover, by assuming that the payoff matrix has a column full rank, they exclude redundant assets such as financial derivatives, whose raison d'être is portfolio constraints.

Cea-Echenique and Torres-Martinez (2014) employ endogenous trading constraints represented by correspondences that depend on both commodity and asset prices. Restrictions on consumption and portfolio choices are incorporated into a single trading constraint set. Trading constraints are so general and can therefore cover collateralized borrowing constraints and income-based portfolio constraints. In particular, attainable allocations are price-dependent. However, they impose a restrictive assumption that the set of price-dependent attainable allocations is bounded. This assumption may not be fulfilled in constrained incomplete markets with redundant assets, in which asset demand correspondences are unbounded. Therefore, they *de facto* exclude financial derivatives from incomplete financial markets.

The rest of the paper is organized as follows: in Section II, we present the model of an exchange economy with incomplete markets where each agent is faced with endogenous portfolio constraints. In Section III, we define constrained arbitrage and provide additional assumptions for endogenous portfolio constraints. Section IV contains examples of endogenous portfolio constraints. In Section V, we show that a competitive equilibrium exists in the economy and present a numerical example of a competitive equilibrium. Section VI contains the concluding remarks.

II. The Model

The paper considers an exchange economy with financial asset markets, extending over two periods. There are I agents and J financial assets. The uncertainty of the second period is described by a finite set $\mathbf{S}:=\{1, \ldots, S\}$ of states of nature. In the first period, no agent knows which state will be realized in the second period. The payoffs of asset $j\in\boldsymbol{J}:=\{1, 2, \ldots, J\}$ are realized depending on the state in the second period. There are L commodities in each state $s=\mathbf{S}_0:=\mathbf{S}\cup\{0\}$ where the first period is regarded as state $s=0$. Therefore, the commodity space is equal to $\mathbb{R}^{\ell}$ where $\ell:=L(S+1)$.

In the first period, agent $i\in\boldsymbol{I}:=\{1, 2, \ldots, I\}$ makes consumption $x_i(0)$ and invests portfolio θ_i with his endowments. In the second period, agent i makes consumptions $(x_i(s))_{s\in\mathbf{S}}$ with his endowments and payoffs of his portfolio. Hence, agent i chooses consumption bundle $x_i:=(x_i(0), x_i(1), \ldots, x_i(S))$ in his consumption set $X_i\subset\mathbb{R}^{\ell}$, which contains his initial endowment e_i of commodities. Preferences over X_i are represented by a preference relation $\succ_i$ on X_i, which is irreflexive, complete, and transitive. The preference relation $\succ_i$ defines the preference correspondence P_i: $X_i\rightarrow 2^{X_i}$ by $P_i(x_i):=\{x_i'\in X_i : x_i'\succ_i x_i\}$, which is the set of consumption bundles that agent i prefers to x_i. Agent i is subject to portfolio constraints, as represented by correspondence $\Theta_i:\mathbb{R}^J\rightarrow 2^{\mathbb{R}^J}$ of asset price $q\in\mathbb{R}^J$. To finance his consumption in the second period, agent i chooses portfolio $\theta_i\in\Theta_i(q)$ in the first period.

The payoff of asset j in state $s\in\mathbf{S}$ is denoted by $r_j(s)$, and the payoff vector of asset j over S states by an S dimensional column vector $r_j=(r_j(s))_{s\in\mathbf{S}}$. Payoff vector in state s is denoted by a J dimensional row vector $r(s)=(r_j(s))_{j\in\boldsymbol{J}}$. We denote the asset payoffs by an $(S\times J)$ payoff matrix $R=[(r_j)_{j\in\boldsymbol{J}}]$. An asset is called redundant if its payoffs can be replicated by those of the other assets. We allow redundant assets, *i.e.*, $V^{\perp}\neq\{0\}$ where $V^{\perp}=\{\theta\in\mathbb{R}^J: R\cdot\theta=0\}$. We note that redundant assets do not play a risk-sharing role without portfolio constraints because their payoffs can be replicated by those of the other assets. In contrast, redundant assets participate in risk-sharing under portfolio constraints, which may prevent the replication of redundant assets. We represent this economy by $\boldsymbol{E}=\langle (X_i, \succ_i, e_i, \Theta_i)_{i\in \boldsymbol{I}}; R\rangle$.

In the first period, agent i is subject to budget constraint $p(0)\cdot x_i(0)+q\cdot\theta_i\leq p(0)\cdot e_i(0)$, where $(p(0), q)\in\mathbb{R}^L\times\mathbb{R}^J$ is a vector of commodity and asset prices in the first period. In the second period, he is subject to budget constraint $p(s)\cdot x_i(s)\leq p(s)\cdot e_i(s)+r(s)\cdot\theta_i$, $\forall s\in\boldsymbol{S}$, where $p(s)\in\mathbb{R}^L$ is a vector of commodity prices at state $s\in\boldsymbol{S}$. Therefore, given price vector $(p, q)\in\mathbb{R}^\ell\times\mathbb{R}^J$, agent i maximizes his preference $\succ_i$ by choosing a pair (x_i, θ_i) of consumption and portfolio in his budget set:[1]

$$\bar{B}_i(p,\ q):=\left\{(x_i,\ \theta_i)\in X_i\times\Theta_i(q): p\ \square\,(x_i-e_i)\leq W(q)\cdot\theta_i\right\},^{2} \tag{1}$$

where

$$p\ \square\,(x_i-e_i)=\begin{bmatrix} p(0)\cdot(x_i(0)-e_i(0)) \\ p(1)\cdot(x_i(1)-e_i(1)) \\ \vdots \\ p(S)\cdot(x_i(S)-e_i(S)) \end{bmatrix},\qquad W(q)=\begin{bmatrix} -q \\ R \end{bmatrix}. \tag{2}$$

A pair $(x_i, \theta_i)\in\bar{B}_i(p, q)$ is optimal for agent i if $[P_i(x_i)\times\Theta_i(q)]\cap\bar{B}_i(p, q)=\varnothing$.

***Definition* 2.1:** A *competitive equilibrium* of economy $\boldsymbol{E}$ is a profile $(p^*, q^*, x^*, \theta^*)\in\mathbb{R}^\ell\times\mathbb{R}^J\times(\mathbb{R}^\ell)^I\times(\mathbb{R}^J)^I$, such that

(i) $(x_i^*, \theta_i^*)\in\bar{B}_i(p^*, q^*)$, $\forall i\in\boldsymbol{I}$,

(ii) $[P_i(x_i^*)\times\Theta_i(q^*)]\cap\bar{B}_i(p^*, q^*)=\varnothing$, $\forall i\in\boldsymbol{I}$,

(iii) $\sum_{i\in\boldsymbol{I}}(x_i^*-e_i)=0$ and $\sum_{i\in\boldsymbol{I}}\theta_i^*=0$.

We now provide the list of basic assumptions for every agent $i\in\boldsymbol{I}$, which are necessary for our main results.

[1] Let v and v' be vectors in a Euclidean space. Then $v\geq v'$ implies that $v-v'\in\mathbb{R}^\ell_+$; $v>v'$ implies that $v\geq v'$ and $v\neq v'$; and $v\gg v'$ implies that $v-v'\in\mathbb{R}^\ell_{++}$.

[2] Note that $\bar{B}_i$ is a correspondence from $\mathbb{R}^\ell\times\mathbb{R}^J$ to $\mathbb{R}^\ell\times\mathbb{R}^J$.

(A1) X_i is closed, convex, and bounded from below in $\mathbb{R}^{\ell}$.
(A2) $\succ_i$ is irreflexive, complete, and transitive on X_i.
(A3) $\succ_i$ is continuous and convex on X_i.[3]
(A4) $\succ_i$ is state-wise locally nonsatiated.[4]
(A5) $e_i \in \text{int}(X_i)$.[5]
(A6) Θ_i: $\mathbb{R}^J \rightarrow 2^{\mathbb{R}^J}$ is a lower hemicontinuous correspondence with convex values, has a closed graph,[6] and satisfies $\Theta_i(\lambda q) = \Theta_i(q)$ for every $q \in \mathbb{R}^J \backslash \{0\}$ and $\lambda > 0$.[7]

Note that Assumptions (A1)-(A5) are standard assumptions. Assumption (A6) states that the portfolio constraint of agent *i* is represented by a convex-valued correspondence that has a closed graph. Moreover, Assumption A6 requires that portfolio constraints 'nicely' depend on asset prices. This assumption can cover market frictions such as short-selling constraints, bid-ask spreads, margin requirements, and proportional transaction costs.[8] Moreover, Assumption (A6) states that portfolio choice sets depend solely on the relative price of assets.

III. Constrained Arbitrage and Additional Assumptions

When no portfolio constraints are present in incomplete markets, no arbitrage opportunity is admitted and therefore the law of one price holds in equilibrium. However, the law of one price does not hold in incomplete markets with portfolio constraints, and it is not appropriate to apply the notion of arbitrage used for unconstrained incomplete markets to constrained incomplete markets. The notion of constrained arbitrage is employed in Jouini and Kallal (1999) and Luttmer (1996), which study

[3] The preference relation $\succ_i$ is continuous if $P_i(x_i)$ and $P_i^{-1}(x_i) := \{x_i' \in X_i : x_i \succ_i x_i'\}$ are open for every $x_i \in X_i$, and is convex if $P_i(x_i)$ is convex for every $x_i \in X_i$.

[4] For each $x_i \in X_i$ and for each $s \in \mathbf{S}$ there exists $x_i'(s) \in X_i(s)$ such that $(x_i'(s), x_i(-s)) \succ_i x_i$, where

$$x_i(-s) = (x_i(0), \ldots, x_i(s-1), x_i(s+1), \ldots, x_i(S)).$$

[5] Let *A* be a non-empty subset of a Euclidean space. The closure of *A* is denoted by *cl*(*A*) and the interior of *A* is denoted by int(*A*).

[6] Let *X* and *Y* be subsets of Euclidean space. A correspondence $\varphi: X \rightarrow 2^Y$ is lower hemicontinuous if $\{x \in X: \varphi(x) \cap V \neq \emptyset\}$ is open for every open set $V \subset Y$ and has a closed graph if $G_\varphi := \{(x, y) \in X \times Y: y \in \varphi(x)\}$ is closed.

[7] The homogeneity of degree zero for constrained choice sets can be also found in Page and Wooders (1999) and Carosi *et al.* (2009).

[8] See Heath and Jarrow (1987), Luttmer (1996), and Elsinger and Summer (2001).

incomplete markets with *exogenous* portfolio constraints. To introduce an appropriate notion of arbitrage for incomplete markets with *endogenous* portfolio constraints, let $C_i(q)$ denote the recession cone $\Gamma(\Theta_i(q))$ of $\Theta_i(q)$.[9]

***Definition* 3.1:** Asset price $q \in \mathbb{R}^J$ is said to admit a *constrained arbitrage* for agent i if there is a portfolio $\theta_i \in C_i(q)$, such that $W(q) \cdot \theta_i > 0$. Asset price $q \in \mathbb{R}^J$ is said to admit *no constrained arbitrage* for economy $\boldsymbol{E}$ if it admits no constrained arbitrage for every agent $i \in \boldsymbol{I}$.

No constrained arbitrage is equivalent to no arbitrage in unconstrained incomplete markets. Let Q_i denote the set of asset prices that admit no constrained arbitrage for agent i. Then, Q: $\cap_{i\in \boldsymbol{I}} Q_i$ is the set of asset prices that admit no constrained arbitrage for $\boldsymbol{E}$. Let $N_i(q)$ be the lineality space of $\Theta_i(q)$.[10] We define $N_0(q) = \sum_{i \in \boldsymbol{I}} (N_i(q) \cap V^{\perp})$ and denote by $N_0(q)^{\perp}$ its orthogonal complement in $\mathbb{R}^J$. Let us define $Q^* := \{q \in Q\text{: } q \in N_0(q)^{\perp}\}$. The following results show what is appropriate for equilibrium asset prices.

***Proposition* 3.1:** It holds that Q and Q^* are non-empty.

Proof: To show $Q \neq \varnothing$, suppose otherwise, that is, $Q = \varnothing$. Consider $q = \lambda \cdot R$ with $\lambda \in \mathbb{R}^S_{++}$. Since $Q = \varnothing$, we see that $q \notin Q$. Then there is some agent i with $\theta_i \in C_i(q)$ satisfying $W(q) \cdot \theta_i > 0$, which makes $q \cdot \theta_i = \lambda \cdot R \cdot \theta_i \geq 0$ necessary. If $q \cdot \theta_i > 0$, then $q \in Q$, which is a contradiction. If $q \cdot \theta_i = \lambda \cdot R \cdot \theta_i = 0$, then $R \cdot \theta_i = 0$. This implies that $q \in Q$, which is a contradiction. Hence, Q is non-empty.

To show $Q^* \neq \varnothing$, suppose otherwise, that is, $Q^* = \varnothing$. Take any $q \in Q$, and we have $q \notin N_0(q)^{\perp}$. Then there exists $v \in N_0(q)$ such that $q \cdot v < 0$ without loss of generality. Since there exists $v_i \in N_i(q) \cap V^{\perp}$, $\forall i \in \boldsymbol{I}$ such that $v = \sum_{i \in \boldsymbol{I}} v_i$, it follow that $q \cdot v_i < 0$ for some i. Noting that $v_i \in C_i(q)$ and $R \cdot v_i = 0$, we see that v_i is a constrained arbitrage opportunity at q. Therefore, $q \notin Q_i$ and $q \notin Q$, which is a contradiction. Hence, Q^* is nonempty. ■

[9] Let A be a non-empty convex subset of Euclidean space X. The recession cone of A is the set $\Gamma(A) = \{v \in E\text{: } A + v \subset A\}$. When A is closed, $\Gamma(A)$ is also closed and can be expressed as

$$\Gamma(A) = \left\{v \in X : \exists\{x^n\} \text{ in } A \text{ and } \{a^n\} \text{ in } \mathbb{R}_+ \text{ with } a^n \to 0 \text{ such that } a^n x^n \to v\right\}.$$

[10] The lineality space $\mathcal{L}(A)$ is the maximal subspace in A, that is, $\mathcal{L}(A) = \Gamma(A) \cap [-\Gamma(A)]$.

***Proposition* 3.2:** Under Assumption (A4), an equilibrium asset price q belongs to Q^*.

Proof: Let (p, q, x, θ) be an equilibrium of $\boldsymbol{E}$. Suppose that $q \in Q$. Then there is some $i \in \boldsymbol{I}$ with $v_i \in C_i(q)$ satisfying $W(q) \cdot v_i > 0$. This implies that $\theta_i + v_i \in \Theta_i(q)$ and $W(q) \cdot \theta_i < W(q) \cdot (\theta_i + v_i)$. Due to Assumption (A4), there exists a consumption bundle $x_i' \in X_i$, such that $x_i' \succ_i x_i$ and $(x_i', \theta_i + v_i) \in \bar{B}_i(p, q)$, which contradicts the optimality of (x_i, θ_i) in $\bar{B}_i(p, q)$. Hence, $q \in Q$.

We now show that $q \in N_0(q)^{\perp}$, that is, $q \cdot v = 0$ for all $v \in N_0(q)$. Suppose otherwise. Then there exists $v \in N_0(q)$ such that $q \cdot v < 0$ without loss of generality. Since there exists $v_i \in N_i(q) \cap V^{\perp}$, $\forall i \in \boldsymbol{I}$ such that $v = \sum_{i \in \boldsymbol{I}} v_i$, it follow that $q \cdot v_i < 0$ for some i. Noting that $v_i \in C_i(q)$ and $R \cdot v_i = 0$, we see that v_i is a constrained arbitrage opportunity at q. That is, $q \notin Q_i$, and therefore $q \notin Q$, which is a contradiction. Hence, $q \in N_0(q)^{\perp}$. ■

From Proposition 3.2, we see that Q^* is an appropriate set of equilibrium asset prices and that Q and Q^* appear as cones with vertex zero under Assumption (A6). We observe that Q^* may not be convex. Therefore we consider $\hat{Q}$ which is the convex hull of Q^*. Then $\hat{Q}$ is a non-empty convex cone.

We now impose a portfolio survival condition, which states that there is no constraint on trading for sufficiently small amount of portfolios.

(A7) $0 \in int(\Theta_i(q))$ for every $q \in cl(\hat{Q}) \backslash \{0\}$.

To analyze the effects of redundant assets on risk-sharing in constrained asset markets, we need to examine feasible zero-income portfolios. We call portfolios in $C_i(q) \cap V^{\perp}$ *scale-free feasible zero-income portfolios* for agent i in that, if $v_i \in C_i(q) \cap V^{\perp}$, we have $\lambda v_i \in \Theta_i(q)$ and $R \cdot (\lambda v_i) = 0$ for every $\lambda \geq 0$. Particularly, if $q \in cl(\hat{Q}) \backslash \{0\}$, some agent i may have a portfolio $v_i \in C_i(q) \cap V^{\perp}$ satisfying $q \cdot v_i \leq 0$. Therefore, in the presence of scale-free feasible zero-income portfolios, some agent's portfolio choices may be unbounded with his budget constraint satisfied. To prevent such negative effect of scale-free feasible zero-income portfolios at the aggregate level, we need the following assumption:

(A8) For every $q \in cl(\hat{Q}) \backslash \{0\}$, if $v_i \in C_i(q) \cap V^{\perp}$, $\forall i \in \boldsymbol{I}$ and $\sum_{i \in \boldsymbol{I}} v_i = 0$, then $v_i = 0$, $\forall i \in \boldsymbol{I}$.

If Assumption (A8) does not hold, there is an asset price $q \in cl(\hat{Q}) \backslash \{0\}$, such that some agent i has a scale-free feasible zero-income portfolio

$v_i \in C_i(q) \cap V^{\perp}$, which is supported by the other agents because $-v_i \in \sum_{j \neq i} C_j(q) \cap V^{\perp}$. Therefore, agent i can hold an indefinite amount of portfolios in the direction of v_i such that the budget constraints of all agents and the market clearing condition are not violated. This possibility is eliminated by Assumption (A8).

IV. Examples of Endogenous Portfolio Constraints

Financial intermediaries prohibit short-selling above specific limits, which can depend on asset prices. Financial regulation prohibits the purchase of some securities above given limits, which may also depend on asset prices. Let continuous functions a_i: $\mathbb{R}^J \to \mathbb{R}^J$ and b_i: $\mathbb{R}^J \to \mathbb{R}^J$ take the values of the short-selling limits or buying limits of agent i on securities, respectively. The portfolio constraints of agent i can therefore be described by

$$\Theta_i(q) = \{\theta_i \in \mathbb{R}^J : \theta_i \geq a_i(q)\} \quad \text{or} \quad \Theta_i(q) = \{\theta_i \in \mathbb{R}^J : \theta_i \leq b_i(q)\}, \tag{3}$$

where $0 \in (a_i(q), b_i(q))$, $\forall q \in \mathbb{R}^J$, a_i: $\mathbb{R}^J \to \mathbb{R}^J$, and b_i: $\mathbb{R}^J \to \mathbb{R}^J$ are continuous functions and homogeneous of degree zero in q.

Financial intermediaries can provide credit to agents with limits that depend on asset prices. In this case, the trading strategies of agent i are restricted such that

$$\Theta_i(q) = \{\theta_i \in \mathbb{R}^J : -q \cdot \theta_i \leq a_i(q),\ R \cdot \theta_i \geq -b_i(q)\}, \tag{4}$$

where a_i: $\mathbb{R}^J \to \mathbb{R}_{++}$ is a continuous function and homogeneous of degree one in q and b_i: $\mathbb{R}^J \to \mathbb{R}^S_{++}$ is a continuous function and homogenous of degree zero in q.

Financial assets such as collateralized debt obligation (CDO) are used as debt instruments and should be backed by a pool of other financial assets. Supposing that security 1 is a risk-less bond, we can express portfolio constraints in the following form:[11]

$$\Theta_i(q) = \{\theta_i \in \mathbb{R}^J :\ q \cdot \theta_i^- \leq a_i q \cdot \theta_i^+ + b_i \mid q_1 \mid\}, \tag{5}$$

[11] This example is adapted from Elsinger and Summer (2001).

where $\theta_i^- = (-\min\{0, \theta_{ij}\})_{j=1}^J$, $\theta_i^+ = (\max\{0, \theta_{ij}\})_{j=1}^J$, $a_i \in \mathbb{R}_+$, and $b_i \in \mathbb{R}_{++}$. It is obvious that the portfolio correspondences of the above examples satisfy Assumptions (A6) and (A7).

As in Heath and Jarrow (1987), portfolio constraints that involve margin requirements can be described as

$$\Theta_i(q) = \{\theta_i \in \mathbb{R}^J : \max_{j \in J}\{|\, q_j\theta_{ij} \,|\} \le a_i(q \cdot \theta_i + b_i \,|\, q_1 \,|)\}, \tag{6}$$

where security 1 is risk-less bond, $a_i \ge 2$, and $b_i \in \mathbb{R}_{++}$. For example, assume that $J=2$ and $b_i=0$.[12] Suppose that security 1 is risk-less bond and security 2 is a stock. Now suppose that agent i shorts one stock and maintains a margin account with m_i proportion of the stock price in the bond. The portfolio constraint is therefore reduced to

$$\max\{|\, m_i q_2 \,|, \;\; |\, -q_2 \,|\} \le a_i(m_i q_2 - q_2) = a_i(m_i - 1)q_2, \tag{7}$$

which implies that $m_i \ge a_i/(a_i - 1)$. In the case where $a_i = 3$, we have $m_i \ge 3/2$, that is, agent i should put the money from shorting the stock and an additional fifty percent of the stock price in his margin account.

Won (2003) provides a more generalized form of the example in Heath and Jarrow (1987). Assuming that security 1 is a risk-less bond with $q_i = 1$, we modify his example to present the portfolio constraint set of agent i at $q \in cl(Q)\backslash\{0\}$ by

$$\Theta_i(q) = \left\{\theta_i \in \mathbb{R}^J : \max_{j \in J}\left\{c_{ij} \,|\, q_j(\theta_{ij} + \delta_{ij}) \,|\right\} \le a_i\left(q \cdot (\theta_i + \delta_i) + \sum_{j \in J} b_{ij} q_j(\theta_{ij} + \delta_{ij})\right) + d_i \,|\, q_1 \,|\right\}, \tag{8}$$

where $a_i \ge 1$, $b_{ij} \ge 0$, $c_{ij} \ge 0$, $d_i > 0$, and $\delta_{ij} > 0$ are constants for every i and j and $\delta_i = (\delta_{ij})$. It is obvious that Θ_i has a closed graph and satisfies the homogeneity of degree zero. If we assume that

$$\max_{j \in J}\left\{c_{ij} \,|\, q_j \delta_{ij} \,|\right\} < a_i\left(q \cdot \delta_i + \sum_{j \in J} b_{ij} q_j \delta_{ij}\right) + d_i \,|\, q_1 \,|, \; \forall q \in c\ell(Q)\backslash\{0\}, \tag{9}$$

[12] To be precise, b_i should be sufficiently close to zero.

we have $0 \in Int(\Theta_i(q))$, $\forall q \in cl(Q) \setminus \{0\}$. To see that Θ_i is convex-valued, we define continuous function f_i: $\mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$ by

$$f_i(q, \theta_i) = a_i \left(q \cdot (\theta_i + \delta_i) + \sum_{j \in J} b_{ij} q_j (\theta_{ij} + \delta_{ij}) \right) + d_i \mid q_1 \mid - \max_{j \in J} \left\{ c_{ij} \mid q_j (\theta_{ij} + \delta_{ij}) \mid \right\}. \tag{10}$$

The portfolio constraint correspondence is then given by

$$\Theta_i(q) = \left\{ \theta_i \in \mathbb{R}^J : f_i(q, \theta_i) \geq 0 \right\}. \tag{11}$$

We can observe that $\max_j\{\cdot\}$ is a convex function on $\mathbb{R}^J$ and $|\cdot|$ is a convex function on $\mathbb{R}$, which implies that $-\max_{j \in J}\{c_{ij} | q_j(\theta_{ij} + \delta_{ij}) |\}$ is a concave function of θ_i. Hence, we see that f_i is a concave function function of θ_i and therefore Θ_i is convex-valued.

To show that Θ_i is lower hemicontinuous, we define correspondence Θ_i°: $\mathbb{R}^J \to 2^{\mathbb{R}^J}$ by

$$\Theta_i^\circ(q) = \left\{ \theta_i \in \mathbb{R}^J : f_i(q, \theta_i) > 0 \right\}. \tag{12}$$

Suppose $\theta_i^* \in \Theta_i^\circ(q^*)$, that is, $f_i(q^*, \theta_i^*) > 0$. Take a sequence $\{(q^n, \theta_i^n)\}$ converging to (q^*, θ_i^*). Since f_i is continuous, for sufficiently large n,

$$f_i(q^n, \theta_i^n) > 0, \tag{13}$$

which implies that $\theta_i^n \in \Theta_i^\circ(q^n)$. Thus we see that Θ_i° is lower hemicontinuous. Noting that $\Theta_i(q) = cl(\Theta_i^\circ(q))$, we see that Θ_i is lower hemicontinuous. Hence, Θ_i satisfies Assumptions (A6) and (A7). □

V. Existence of a Competitive Equilibrium

In this section, we will show that there exists a competitive equilibrium of economy ***E***. We define the sets of normalized prices by $\Delta = \Delta_0 \times \Delta_1$, where

$$\Delta_0 = \left\{ (p(0),\ q) \in \mathbb{R}^L \times c\ell(\hat{Q}) :\ \| p(0) \| + \| q \| \ \le\ 1 \right\},$$
$$\Delta_1 = \prod_{s \in \boldsymbol{S}} \Delta_s \qquad \text{with} \qquad \Delta_s = \left\{ p(s) \in \mathbb{R}^L :\ \| p(s) \| \ \le\ 1 \right\}.^{13} \tag{14}$$

We observe that Δ is compact and convex.

Let $X := \Pi_{i \in \boldsymbol{I}} X_i$ and $A_X := \{(x_1, \ldots, x_I) \in X \colon \sum_{i \in \boldsymbol{I}} (x_i - e_i) = 0\}$. We denote by $\hat{X}_i$ the projection of X_i onto A_X and let $\hat{X}\colon \Pi_{i \in \boldsymbol{I}} \hat{X}_i$. To consider a sequence of truncated economies, we take an increasing sequence $\{(K_n, M_n)\}$ of compact convex cube pairs with center 0 such that $K_n \subset \mathbb{R}^\ell$ with $\hat{X}_i \subset int(K_1)$, and $M_n \subset \mathbb{R}^J$ with $0 \in int(M_1)$ which satisfy $\cup_n K_n = \mathbb{R}^\ell$ and $\cup_n M_n = \mathbb{R}^J$. For each n and $i \in \boldsymbol{I}$, we define $X_i^n := X_i \cap K_n$, $\Theta_i^n(q) := \Theta_i(q) \cap M_n$, $X^n := \Pi_{i \in \boldsymbol{I}} X_i^n$, and $\Theta^n(q) := \Pi_{i \in \boldsymbol{I}} \Theta_i^n(q)$. Moreover, the preference correspondence P_i^n: $X_i^n \to 2^{X_i^n}$ is defined by $P_i^n(x_i) := P_i(x_i) \cap X_i^n$.

We denote by $\boldsymbol{E}^n$ the truncated economy $\langle (X_i^n,\ P_i^n,\ e_i,\ \Theta_i^n)_{i \in \boldsymbol{I}} \rangle$. We observe that each X_i^n is compact and each Θ_i^n is lower hemicontinuous with non-empty compact convex values and has a closed graph. Moreover, each P_i^n inherits the properties of P_i. We define function $\gamma : \Delta \to \mathbb{R}^{S+1}$ by $\gamma(p,\ q) = (\gamma_s(p,\ q))_{s \in \boldsymbol{S}_0}$ with $\boldsymbol{S}_0 = \boldsymbol{S} \cup \{0\}$, where

$$\gamma_s(p, q) = \begin{cases} 1 - (\| (p(0) \| + \| q \|), & \text{if } s = 0, \\ 1 - \| p(s) \|, & \text{if } s \in \boldsymbol{S}\ . \end{cases} \tag{15}$$

Let $\Psi_i^n = M_n$, $\forall i \in \boldsymbol{I}$, and $\Psi^n = \Pi_{i \in \boldsymbol{I}} \Psi_i$. For every $i \in \boldsymbol{I}$ and every n, we define correspondences $B_i^n\colon \Delta \to 2^{X_i^n \times \Psi_i^n}$ and $\bar{B}_i^n\colon \Delta \to 2^{X_i^n \times \Psi_i^n}$ as follows:

$$B_i^n(p, q) = \left\{ (x_i, \theta_i) \in X_i^n \times \Theta_i^n(q) : p \ \square\ (x_i - e_i) \ll W(q) \cdot \theta_i + \gamma(p, q) \right\},$$
$$\bar{B}_i^n(p, q) = \left\{ (x_i, \theta_i) \in X_i^n \times \Theta_i^n(q) : p \ \square\ (x_i - e_i) \le W(q) \cdot \theta_i + \gamma(p, q) \right\}. \tag{16}$$

***Proposition* 5.1:** Under Assumptions (A1)–(A7), for each n, there is a profile $(p^n,\ q^n,\ x^n,\ \theta^n) \in \Delta \times X^n \times \Theta^n(q^n)$ such that

(a) $(x_i^n, \theta_i^n) \in \bar{B}_i^n(p^n, q^n)$, $\forall i \in \boldsymbol{I}$,

(b) $[P_i^n(x_i^n) \times \Theta_i^n(q^n)] \cap B_i^n(p^n, q^n) = \varnothing$, $\forall i \in \boldsymbol{I}$,

(c) $\sum_{s \in S_0} p^n(s) \cdot z^n(s) + q^n \cdot \sum_{i \in I} \theta_i^n \ge \sum_{s \in S_0} p(s) \cdot z^n(s) + q \cdot \sum_{i \in I} \theta_i^n$, $\forall (p, q) \in \Delta$,

13 $\| \cdot \|$ is the Euclidean norm.

(d) $z^n = 0$ and $\sum_{i \in \boldsymbol{I}} \theta_i^n = 0$,

(e) $\gamma(p^n, q^n) = 0$,

where $z^n(s) := \sum_{i \in \boldsymbol{I}} (x_i^n(s) - e_i(s))$ for every $s \in \boldsymbol{S}_0$.

Proof: See Appendix. ■

***Lemma* 5.1:** Suppose that Assumption (A6) holds. Let $\{(q^n, \theta_i^n)\}$ be a sequence in $\mathbb{R}^J \times \mathbb{R}^J$ with $q^n \to q^*$ and $\theta_i^n \in \Theta_i(q^n)$. Suppose that $\{a_n\}$ be a sequence in $\mathbb{R}_+$, such that $a_n \to 0$. If sequence $\{a_n \theta_i^n\}$ converges to v_i, then $v_i \in C_i(q^*)$.

Proof: Apply 3.2 Lemma on p. 396 of Page (1987) to Θ_i. ■

From Proposition 5.1, we obtain an equilibrium existence theorem for economy ***E***.

Theorem* 5.1:** Under Assumptions (A1)–(A8), economy ***E has a competitive equilibrium.

Proof: Take a sequence $\{(p^n, q^n, x^n, \theta^n)\}$ of profiles obtained in Propostion 5.1. Since each X_i is closed and bounded from below, $\hat{X}_i$ is compact and so is $\hat{X}$. Noting that $\{(p^n, q^n, x^n)\} \in \Delta \times \hat{X}$, without loss of generality, we may assume that $\{(p^n, q^n, x^n)\}$ converges to $(p^*, q^*, x^*)\} \in \Delta \times \hat{X}$.

***Claim* 1:** $\sum_{i \in \boldsymbol{I}} (x_i^* - e_i) = 0$ and $\sum_{i \in \boldsymbol{I}} \theta_i^* = 0$, where $(x_i^*, \theta_i^*) \in X_i \times \Theta_i(q^*)$ for each $i \in \boldsymbol{I}$.

Proof: From (d) of Proposition 5.1, it is immediate that $\sum_{i \in \boldsymbol{I}} (x_i^* - e_i) = 0$. To show $\sum_{i \in \boldsymbol{I}} \theta_i^* = 0$, we claim that sequences $\{\theta_i^n\}$ for each $i \in \boldsymbol{I}$ are bounded. Suppose otherwise. For each n, we set $a_n = (1 + \sum_{i \in \boldsymbol{I}} \| \theta_i^n \|)^{-1}$, which converges to 0. We see that $a_n \theta_i^n \in \Theta_i(q^n)$ and sequence $\{a_n \theta_i^n\}$ for each $i \in \boldsymbol{I}$ are bounded. Thus, without loss of generality, it converges to v_i for each $i \in \boldsymbol{I}$. Since $\sum_{i \in \boldsymbol{I}} a_n \theta_i^n = 0$ for all n, it holds that $\sum_{i \in \boldsymbol{I}} v_i = 0$ and $\sum_{i \in \boldsymbol{I}} \| v_i \| = 1$, which implies that $v_i \neq 0$ for some $i \in \boldsymbol{I}$.

Using Lemma 5.1, we see that $v_i \in C_i(q^*)$. On the other hand, $p^n(s) \cdot (x_i^n(s) - e_i(s)) \leq r(s) \cdot \theta_i^n + \gamma_s(p^n, q^n)$ for all n and $s \in \mathbf{S}$. By multiplying both sides of the inequalities by a_n and passing to the limit, we obtain $R \cdot v_i \geq 0$. In view of $\sum_{i \in \boldsymbol{I}} v_i = 0$, we obtain $R \cdot v_i = 0$, that is, $v_i \in V^{\perp}$. This implies that $v_i \in C_i(q^*) \cap V^{\perp}$. Since $\sum_{i \in \boldsymbol{I}} v_i = 0$, by Assumption (A8), we obtain $v_i = 0$ for all $i \in \boldsymbol{I}$, which leads to a contradiction.

Therefore, $\{\theta_i^n\}$ is bounded for each $i \in \boldsymbol{I}$. Without loss of generality, we may assume that $\{\theta_i^n\}$ converges to θ_i^*. From Assumption (A6) and (d) of Proposition 5.1, it follows that $\theta_i^* \in \Theta_i(q^*)$ and $\sum_{i\in \mathbf{I}} \theta_i^* = 0$. □

***Claim* 2:** $\gamma(p^*, q^*) = 0$.

Proof: This immediately follows from (e) of Proposition 5.1. □

***Claim* 3:** $(x_i^*, \theta_i^*) \in \bar{B}_i(p^*, q^*)$.

Proof: This directly follows from (a) of Proposition 5.1 and Claims 1 and 2. □

***Claim* 4:** $p^*(0) \neq 0$.

Proof: If $p^*(0)=0$, agent i has $x_i \in X_i$ such that $x_i \succ_i x_i^*$ and $(x_i, \theta_i^*) \in \bar{B}_i (p^*, q^*)$ in view of Assumption (A4) and Claim 3. Since $p^*(s) \neq 0$, $\forall s \in \mathbf{S}$ due to Claim 2, by Assumption (A5), there is $x_i^\circ \in int(X_i)$ such that $p^*(s) \cdot x_i^\circ(s) \ll p^*(s) \cdot e_i(s)$, $\forall s \in \mathbf{S}$. Since $\| q^* \| = 1$ by Claim 2, Assumption (A7) ensures that there exists $\theta_i^\circ \in int(\Theta_i(q^*))$ such that $q^* \cdot \theta_i^\circ < 0$. Then, for $t \in (0, 1)$ sufficiently close to 1, we see that $tx_i + (1-t)x_i^\circ \succ_i x_i^*$ and $p^* \square [tx_i + (1-t)x_i^\circ - e_i] \ll W(q^*) \cdot [t\theta_i^* + (1-t)\theta_i^\circ]$ with $t\theta_i^* + (1-t)\theta_i^\circ \in \Theta_i(q^*)$. Since Θ_i is lower hemicontinuous, there exists a sequence $\{\hat{\theta}_i^n\}$ converging to $t\theta_i^* + (1-t)\theta_i^\circ$ such that $\hat{\theta}_i^n \in \Theta_i(q^n)$, $\forall n$. Therefore, for sufficiently large n, we have $tx_i + (1-t)x_i^\circ \succ_i x_i^n$ and $p^n \square [tx_i + (1-t)x_i^\circ - e_i] \ll W(q^n) \cdot \hat{\theta}_i^n$ with $tx_i + (1-t)x_i^\circ \in X_i^n$ and $\hat{\theta}_i^n \in \Theta_i^n(q^n)$. This is a contradiction in view of of Proposition (b) and (e) of Proposition 5.1. Hence, it follows that $p^*(0) \neq 0$. □

***Claim* 5:** $q^* \in Q^*$.

Proof: First, we show that $q^* \in Q$. Suppose otherwise. Then there is some agent i who has a portfolio $\theta_i \in C_i(q^*)$ satisfying $W(q^*) \cdot \theta_i > 0$. Assumption (A4) ensures that there exists $\delta \in \mathbb{R}^\ell$ such that $x_i^* + \delta \succ_i x_i^*$ and $p^* \square \delta < W(q^*) \cdot \theta_i$. Claim 3 implies that $p^* \square (x_i^* + \delta - e_i) < W(q^*) \cdot (\theta_i^* + \theta_i)$ with $\theta_i^* + \theta_i \in \Theta_i(q^*)$. Note that Claims 2 and 4 imply that $p^*(s) \neq 0$, $\forall s \in \mathbf{S}_0$. Assumption (A5) allows us to take $x_i^\circ \in int(X_i)$, such that $p^* \square x_i^\circ \ll p^* \square e_i$. Therefore, for $t \in (0, 1)$ sufficiently close to 1, we obtain $t(x_i^* + \delta) + (1-t)x_i^\circ \succ_i x_i^*$ and $p^* \square [t(x_i^* + \delta) + (1-t)x_i^\circ - e_i] \ll W(q^*) \cdot [t(\theta_i^* + \theta_i)]$. Since $t(\theta_i^* + \theta_i) \in \Theta_i(q^*)$ and Θ_i is lower hemicontinuous, there exists a sequence $\{\hat{\theta}_i^n\}$ converging to $t(\theta_i^* + \theta_i)$ with $\hat{\theta}_i^n \in \Theta_i(q^n)$. For sufficiently large n,

$$t(x_i^n+\delta)+(1-t)x_i^\circ \succ_i x_i^n \text{ and} \tag{17}$$
$$p^n \Box [t(x_i^n+\delta)+(1-t)x_i^\circ - e_i] \ll W(q^n)\cdot \hat{\theta}_i^n$$

with $t(x_i^n+\delta)+(1-t)x_i^\circ \in X_i^n$ and $\hat{\theta}_i^n \in \Theta_i^n(q^n)$. This is a contradiction in view of (b) and (e) of Proposition 5.1. Hence, $q^* \in Q$.

We now show that $q \in N_0(q^*)^\perp$, that is, $q^* \cdot v=0$ for all $v \in N_0(q^*)$. Suppose otherwise. Then we have some $v \in N_0(q^*)$ such that $q^* \cdot v<0$ without loss of generality. Since there exists $v_i \in N_i(q^*) \cap V^\perp$, $\forall i \in \boldsymbol{I}$ such that $v \in \sum_{i \in \boldsymbol{I}} v_i$, it follow that $q^* \cdot v_i<0$ for some i. Noting that $v_i \in C_i(q^*)$ and $R \cdot v_i=0$, we know that v_i is a constrained arbitrage opportunity at q^*. Applying the same arguments presented in the previous paragraph, we arrive at a contradiction. Hence, $q^* \in N_0(q^*)^\perp$ and therefore $q^* \in Q \cap N_0(q^*)^\perp$, that is, $q^* \in Q^*$. □

Let us now define the open budget set of agent i by

$$B_i(p,\ q) := \{(x_i,\ \theta_i) \in X_i \times \Theta_i(q) : p \Box (x_i - e_i) \ll W(q)\cdot \theta_i\}. \tag{18}$$

***Claim* 6:** $[P_i(x_i^*)\times\Theta_i(q^*)] \cap B_i(p^*, q^*) = \varnothing$.

Proof: Suppose that the claim does not hold. Then there is some $i \in \boldsymbol{I}$ with $(\hat{x}_i,\ \hat{\theta}_i) \in [P_i(x_i^*)\times\Theta_i(q^*)] \cap B_i(p^*,\ q^*)$. Noting that P_i^{-1} is open-valued by Assumption (A3), we see that P_i^{-1} is lower hemicontinuous. Since P_i and Θ_i are lower hemicontinuous and B_i has an open graph, the correspondence $(p, q, x_i) \mapsto [P_i(x_i)\times\Theta_i(q)] \cap B_i(p, q)$ is lower hemicontinuous. Therefore there is a sequence $\{(\hat{x}_i^n,\ \hat{\theta}_i^n)\}$ converging to $(\hat{x}_i,\ \hat{\theta}_i)$ such that $(\hat{x}_i^n,\ \hat{\theta}_i^n) \in [P_i(x_i^n)\times\Theta_i(q^n)] \cap B_i(p^n,\ q^n)$. For each $t \in (0,\ 1)$ and each $n \in \mathbb{N}$, we set $y_i^n(t) := (t\hat{x}_i^n+(1-t)x_i^n,\ t\hat{\theta}_i^n+(1-t)\theta_i^n)$. Observe that, for sufficiently large n, we obtain $y_i^n(t) \in x_i^n \times \Theta_i^n(q^n)$ and thus $y_i^n(t) \in B_i^n(p^n,\ q^n)$ by Claim 2. Therefore, for sufficiently large n, we have $y_i^n(t) \in [P_i^n(x_i^n)\times\Theta_i^n(q^n)] \cap B_i^n(p^n,\ q^n)$, which contradicts (b) of Proposition 5.1. Hence, $[P_i(x_i^*)\times\Theta_i(q^*)] \cap B_i(p^*,\ q^*) = \varnothing$. □

***Claim* 7:** For every $i \in \boldsymbol{I}$, $[P_i(x_i^*)\times\Theta_i(q^*)] \cap \bar{B}_i(p^*, q^*) = \varnothing$.

Proof: Suppose that the claim does not hold. Then there is some $i \in \boldsymbol{I}$ with $(x_i,\ \theta_i) \in [P_i(x_i^*)\times\Theta_i(q^*)] \cap \bar{B}_i(p^*,\ q^*)$. Since Claims 2 and 4 imply that $p^*(s) \neq 0$, $\forall s \in \mathbf{S}_0$, by Assumption (A5), we can take $(x_i',\ \theta_i') \in B_i(p^*,\ q^*) \neq \varnothing$. Assumption (A3) implies that for $t \in (0,\ 1)$ sufficiently close to 1,

$t(x_i,\ \theta_i)+(1-t)(x_i',\ \theta_i')\in[P_i(x_i^*)\times\Theta_i(q^*)]\cap B_i(p^*,\ q^*)$, which contradicts Claim 6. Hence, $[P_i(x_i^*)\times\Theta_i(q^*)]\cap\bar{B}_i(p^*,\ q^*)=\varnothing$. □

By Claims 1, 3, and 7, we prove that $(p^*,\ q^*,\ x^*,\ \theta^*)$ is a competitive equilibrium for economy $\boldsymbol{E}$. ■

***Example* 5.1:** We consider an exchange economy with $I=2$, $L=1$, $J=3$, and $S=3$. The utility functions and initial endowments of agents are provided as follows:

$$\begin{aligned} u_i(x) &= \sum_{s=0}^{3} \ln x_i(s),\ \forall i=1,\ 2; \\ e_1 &= (26/9,\ 11/6,\ 35/6,\ 2), \\ e_2 &= (64/9,\ 25/6,\ 1/6,\ 2). \end{aligned} \tag{19}$$

Let $X_t=\mathbb{R}^4_+$, $\forall i=1,\ 2$ and consider the commodity as a numéraire. Payoff matrix is given by

$$R=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

This allows us to restrict no-arbitrage asset prices to $\mathbb{R}^3_{++}$. Note that $V^\perp=\{v\in\mathbb{R}^3:\ v=\lambda(1,1,-1),\lambda\in\mathbb{R}\}$. Portfolio constraints for agents are described by:

$$\begin{aligned} \Theta_1(q) = \{(a,b,c)\in\mathbb{R}^3 &: q_1a+q_2b+q_3c \\ &\geq -(q_1/2+q_2/2+q_3/3), b\geq -1, c\geq -1\}, \\ \Theta_2(q) = \{(a,b,c)\in\mathbb{R}^3 &: q_1a+q_2b+q_3c \\ &\geq -(q_1/2+q_2/2+q_3/3), a\geq -1, c\geq -1\}. \end{aligned} \tag{21}$$

The recession cones of these constraints are:

$$\begin{aligned} C_1(q) &= \{(a,b,c)\in\mathbb{R}^2 : q_1a+q_2b+q_3c\geq 0, b\geq 0, c\geq 0\}, \\ C_2(q) &= \{(a,b,c)\in\mathbb{R}^2 : q_1a+q_2b+q_3c\geq 0, a\geq 0, c\geq 0\}. \end{aligned} \tag{22}$$

Define $\mathcal{R}_+ := \{\theta \in \mathbb{R}^3 : R \cdot \theta > 0\}$. Since $C_i(q) \cap \mathcal{R}_+ = \mathbb{R}^3_+$ for all i and $q \in \mathbb{R}^3_{++}$, we find that $Q = \mathbb{R}^3_{++}$, which is a nonempty open convex cone. We denote a competitive equilibrium of the economy by $(q^*, (x_1^*, x_2^*), (\theta_1^*, \theta_3^*)) \in \mathbb{R}^2_+ \times (\mathbb{R}^4)^2 \times (\mathbb{R}^3)^2$. Then it follows that

$$\begin{aligned} &q^* = (1,1,10/3);\ x_1^* = (5,3,3,2),\ \ x_2^* = (5,3,3,2); \\ &\theta_1^* = (3/2,-5/2,-1/3),\ \theta_2^* = (-3/2,5/2,1/3). \end{aligned} \tag{23}$$

Since $C_i(q^*) \cap V^{\perp} = \{0\}$ for all $i = 1,\ 2$, we see that Assumption (A8) is trivially holds. Note that the law of one price does not hold and that the first inequality constraint of agent 1 is binding at the equilibrium. □

VI. Concluding Remarks

It is shown that there exists a competitive equilibrium in a two-period exchange economy with incomplete markets where redundant assets are present and portfolio constraints are represented by a lower hemicontinuous correspondence of asset prices. Most of general equilibrium models, which study incomplete markets with endogenous portfolio constraints, either express portfolio constraints in terms of differentiable restriction functions that describe the boundary of constraints, or *de facto* exclude redundant assets. The present paper not only models endogenous portfolio constraints via correspondences of asset prices, but also considers the risk-sharing role of redundant assets in incomplete markets. Assumption (A8) plays a key role of excluding the unboundedness of scale-free zero-income portfolios, which arises due to redundant assets. Future possible directions of research include weakening Assumption (A8) for more general results and extending the results of this paper to economies with multiperiod incomplete markets.

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Appendix

***Proof of Proposition* 5.1:** We observe that, for each n, economy $\boldsymbol{E}^n$ satisfies Assumptions (A1)-(A7).

***Claim* 1:** For every $i \in \boldsymbol{I}$ and for every $n \in \mathbb{N}$, the following hold:

(i) B_i^n is lower hemicontinuous with nonempty convex values on Δ.
(ii) $\bar{B}_i^n$ is lower hemicontinuous with nonempty convex values on Δ.

Proof: (i) Since Θ_i is convex-valued, B_i^n is clearly convex-valued. To show that B_i^n is nonempty-valued, note that by Assumption (A5), there exists $x_i^\circ \in X_i^n$ such that $p \,\Box\, (x_i^\circ - e_i) \ll \gamma(p, q)$ if $p(0) \neq 0$ or $q = 0$. Then $(x_i^\circ, 0) \in B_i^n(p, q)$. If $p(0) = 0$ and $q \neq 0$, by Assumption (A7), there exists $\xi_i \in \Theta_i(q)$ such that $q \cdot \xi_i < 0$. Since, by Assumption (A5), there exists $x_i^\circ \in X_i^n$ such that $p(s) \cdot (x_i^\circ(s) - e_i(s)) < \gamma_s(p, q)$ for all $s \in \boldsymbol{S}$, for sufficiently small $\alpha > 0$, we obtain $p \,\Box\, (x_i^\circ - e_i) \ll W(q) \cdot (\alpha \xi_i) + \gamma(p, q)$. That is, $(x_i^\circ, \alpha \xi_i) \in B_i^n(p, q)$. Hence, B_i^n is nonempty-valued.

To prove that B_i^n is lower hemicontinuous, we define correspondence $B_i' : \Delta \to 2^{\mathbb{R}^\ell \times \mathbb{R}^J}$ by

$$B_i'(p, q) := \left\{(x_i, \theta_i) \in \mathbb{R}^\ell \times \mathbb{R}^J : p \,\Box\, (x_i - e_i) \ll W(q) \cdot \theta_i + \gamma(p, q)\right\}. \quad (24)$$

Obviously, B_i' has an open graph. Furthermore, correspondence $X_i^n \times \Theta_i^n(\cdot) : \Delta \to 2^{X_i^n \times \Psi_i^n}$ is lower hemicontinuous. Since $B_i^n(p, q) = B_i'(p, q) \cap [X_i^n \times \Theta_i^n(q)]$, it follows that B_i^n is lower hemicontinuous. □

(ii) Since B_i^n is nonempty-valued on Δ, it is the case that $\bar{B}_i^n(p, q) = cl\,(B_i^n(p, q))$. Hence, (i) implies that $\bar{B}_i^n$ is lower hemicontinuous with nonempty convex values on Δ. □

Let us construct the following correspondences $\varphi_0^n : \Delta \times X^n \times \Psi^n \to 2^\Delta$ and $\varphi_i^n : \Delta \times X^n \times \Psi^n \to 2^{X_i^n \times \Psi_i^n}$ for every $i \in \boldsymbol{I}$:

$$\varphi_0^n(p, q, x, \theta) = \{(p', q') \in \Delta : \sum_{s \in \boldsymbol{S}_0} [p'(s) - p(s)] \cdot z(s) + (q' - q) \cdot \sum_{i \in \boldsymbol{I}} \theta_i > 0\}, \quad (25)$$

$$\varphi_i^n(p, q, x, \theta) = \begin{cases} \bar{B}_i^n(p, q), & \text{if } (x_i, \theta_i) \notin \bar{B}_i^n(p, q), \\ [P_i(x_i) \times \Theta_i^n(q)] \cap B_i^n(p, q), & \text{if } (x_i, \theta_i) \in \bar{B}_i^n(p, q), \end{cases}$$

where $z(s) = \sum_{i\in \boldsymbol{I}} [x_i(s) - e_i(s)]$ for each $s \in \mathbf{S}_0$.

Claim 2: Correspondence φ_i^n is lower hemicontinuous with convex values for every $i \in \boldsymbol{I}_0 := \boldsymbol{I} \cup \{0\}$ and for every $n \in \mathbb{N}$.

Proof: It is obvious that φ_i^n is convex-valued for every $i \in \boldsymbol{I}_0$ and that φ_0^n is lower hemicontinuous. To show that φ_i^n is lower hemicontinuous for every $i \in \boldsymbol{I}$, take any open set V in $X_i^n \times \Psi_i^n$ and let

$$U_i^n = \{(p, q, x, \theta) \in \Delta \times X^n \times \Psi^n : \varphi_i^n(p, q, x, \theta) \cap V \neq 0\}. \tag{26}$$

Now define

$$\begin{aligned} U_i^n(1) &= \{(p, q, x, \theta) \in \Delta \times X^n \times \Psi^n : \bar{B}_i^n(p, q) \cap V \neq 0\}, \\ U_i^n(2) &= \{(p, q, x, \theta) \in \Delta \times X^n \times \Psi^n : ([P_i^n(x_i) \times \Theta_i^n(q)] \cap B_i^n(p, q)) \cap V \neq 0\}. \end{aligned} \tag{27}$$

Then we see that $U_i^n = U_i^n(1) \cup U_i^n(2)$. Since $\bar{B}_i^n$ is lower hemicontinuous on Δ by (ii) of Claim 1, the set $U_i^n(1)$ is open in $\Delta \times X^n \times \Psi^n$. We observe that the correspondence $[P_i^n \times \Theta_i^n] \cap B_i^n : \Delta \times X^n \times \Psi^n \rightarrow 2^{X_i^n \times \Psi_i^n}$ defined by

$$([P_i^n \times \Theta_i^n] \cap B_i^n)(p, q, x, \theta) = [P_i^n(x_i) \times \Theta_i^n(q)] \cap B_i^n(p, q) \tag{28}$$

is lower hemicontinuous because P_i^n is lower hemicontinuous on X_i^n, Θ_i^n is lower hemicontinuous on $cl(Q)$, and B_i^n has an open graph on Δ. Therefore, $U_i^n(2)$ is open in $\Delta \times X^n \times \Psi^n$. As a result, $U_i^n = U_i^n(1) \cup U_i^n(2)$ is open in $\Delta \times X^n \times \Psi^n$, which implies that φ_i^n is lower hemicontinuous. □

From Claim 2, we know that φ_i^n is lower hemicontinuous and convex-valued for each $i \in \boldsymbol{I}_0$ and each $n \in \mathbb{N}$. By applying the fixed point theorem of Gale and Mas-Colell (1975, 1979) to φ_i^n's, we obtain $(p^n, q^n, x^n, \theta^n) \in \Delta \times X^n \times \Theta^n(q^n)$ that satisfies (a), (b), and (c).

To prove (d), suppose that $z^n(0) \neq 0$ or $\sum_{i\in \boldsymbol{I}} \theta_i^n \neq 0$. Then (c) implies that $\| p^n(0) \| + \| q^n \| = 1$ and $p^n(0) \cdot z^n(0) + q^n \cdot \sum_{i\in \boldsymbol{I}} \theta_i^n > 0$. However, (a) implies that $p^n(0) \cdot z^n(0) + q^n \cdot \sum_{i\in \boldsymbol{I}} \theta_i^n \leq I \cdot (1 - \| p^n(0) \| - \| q^n \|) = 0$, which is a contradiction. If $z^n(s) \neq 0$ for some $s \in \mathbf{S}$, then (c) implies that $\| p^n(s) \| = 1$ and $p^n(s) \cdot z^n(s) > 0$. However, (a) implies that $p^n(s) \cdot z^n(s) \leq r(s) \cdot \sum_{i\in \boldsymbol{I}} \theta_i^n = 0$, which is a contradiction.

To show (e), we observe that Assumption (A4) and (a) imply that

$$p^n \square (x_i^n - e_i) = W(q^n) \cdot \theta_i^n + \gamma(p^n,\ q^n),\ \forall i \in \boldsymbol{I}. \tag{29}$$

Summing this up over $i \in \boldsymbol{I}$, we obtain $I \cdot \gamma(p^n, q^n) = 0$ so that $\gamma(p^n, q^n) = 0$, which implies the claim. ■

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