공학 수학 I

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Lesson 1: Introduction to matrix

- terminologies
- addition and scalar multiplication
- product of matrices
- transpose of a matrix

Matrix (행렬) & Vector (벡터)

행렬(벡터)의 addition & scalar multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

합과 스칼라 곱의 연산법칙

For $A, B, C \in \mathbb{R}^{m \times n}$ and $c, k \in \mathbb{R}$,

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C) \\ A + 0 &= A \\ A + (-A) &= 0 \end{aligned}$$

and

$$\begin{aligned} c(A + B) &= cA + cB \\ (c + k)A &= cA + kA \\ c(kA) &= (ck)A \\ 1A &= A \end{aligned}$$

행렬의 곱

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

행렬 곱의 연산법칙

For $A, B, C \in \mathbb{R}^{m\times n}$ and $k \in \mathbb{R}$,

$$\begin{aligned}(kA)B &= k(AB) = A(kB)\\ A(BC) &= (AB)C\\ (A+B)C &= AC + BC\\ C(A+B) &= CA + CB\end{aligned}$$

Transposition

$$\begin{aligned}(A^\top)^\top &= A\\ (A+B)^\top &= A^\top + B^\top\\ (cA)^\top &= cA^\top\\ (AB)^\top &= B^\top A^\top\end{aligned}$$

예 : 토지의 용도 변경

예 : 회전 변환

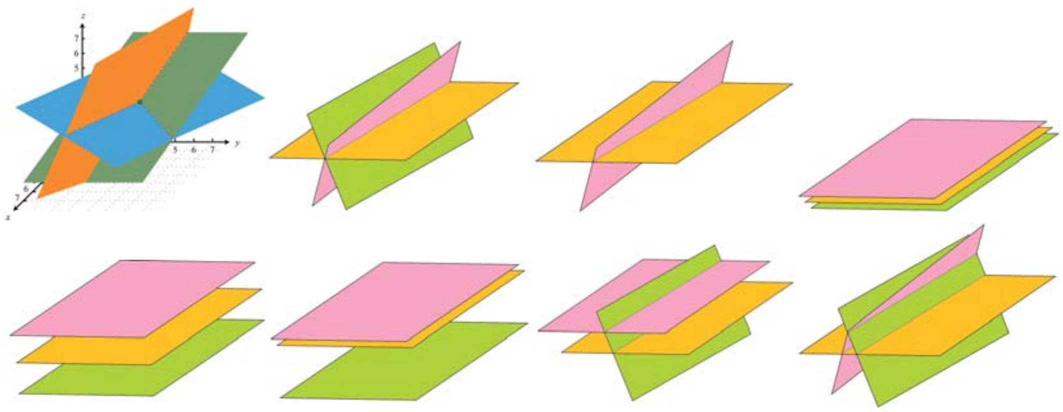
Lesson 2: System of linear equations, Gauss elimination

- existence and uniqueness of solution
- elementary row operation
- Gauss elimination, pivoting
- echelon form

선형연립방정식 (system of linear equations) & 해 (solution)

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Existence and uniqueness of solution (해의 존재성과 유일성)



해를 구하는 법

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \end{aligned}$$

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30 \end{aligned}$$

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$$

1. 두 식의 위치 교환
2. 한 식을 다른 식에 더하기
3. 한 식에 0 아닌 상수 곱하기
4. 한 식을 상수배하여 다른 식에 더하기

1. 두 행의 위치 교환
2. 한 행을 다른 행에 더하기
3. 한 행에 0 아닌 상수 곱하기
4. 한 행을 상수배하여 다른 행에 더하기

Gauss elimination

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Gauss elimination (partial pivoting)

$$
\begin{array}{rrrl}
x_1 - & x_2 + & x_3 & = 0 \\
2x_1 - & 2x_2 + & 2x_3 & = 0 \\
 & 10x_2 + & 25x_3 & = 90 \\
20x_1 + & 10x_2 & & = 80
\end{array}
$$

Gauss elimination (the case of infinitely many solutions)

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss elimination (the case of no solution)

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Echelon form (계단 형태)

Gauss elimination:

$$[A \quad b] \quad \Rightarrow \quad [R \quad f]$$

$$[R, f] = \begin{bmatrix} r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\ & & \ddots & & & \vdots & \vdots \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ & & & & & & f_{r+1} \\ & & & & & & \vdots \\ & & & & & & f_m \end{bmatrix}$$

Lesson 3: Rank of a matrix, Linear independence of vectors

- linear combination (of vectors)
- linear independence (of vectors)
- rank (of a matrix)
- practice using MATLAB

Linear combination (of vectors) & linear independence (of a set of vectors)

Example

$$\underline{a}_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$$
$$\underline{a}_2 = \begin{bmatrix} -6 & 42 & 24 & 54 \end{bmatrix}$$
$$\underline{a}_3 = \begin{bmatrix} 21 & -21 & 0 & -15 \end{bmatrix}$$

Rank of a matrix

DEF: $\text{rank}\ A =$ 행렬 A에서 선형독립인 row vector의 최대 수

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Properties of 'rank'

THM: elementary row operation을 해서 얻는 모든 행렬들은 같은 rank를 가진다.
(Rank는 elementary row operation에 대하여 invariant 하다.)

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Properties of ‘rank’

THM: $\text{rank } A$는 A의 선형독립인 column vector의 최대 수와도 같다.
(따라서 $\text{rank } A = \text{rank } A^\top$.)

Properties of ‘rank’

- For $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank} A \le \min\{m, n\}$.
- For $v_1, \cdots, v_p \in \mathbb{R}^n$, if $n < p$, then they are linearly dependent.
- Let $A = [v_1, v_2, \ldots, v_p]$ where $v_i \in \mathbb{R}^n$.
  If $\operatorname{rank} A = p$, then they are linearly independent.
  If $\operatorname{rank} A < p$, then they are linearly dependent.

Ex:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

MATLAB을 사용한 실습

`http://www.mathworks.com`

Lesson 4: Vector space

- vector space (in $\mathbb{R}^n$), subspace
- basis, dimension
- column space, null space of a matrix
- existence and uniqueness of solutions
- vector space (in general)

Vector space

선형연립방정식의 해: 존재성과 유일성

$$Ax = b \qquad \text{with } A \in \mathbb{R}^{m\times n} \text{ and } b \in \mathbb{R}^m$$

1. existence: a solution x exists iff
   - $b \in$ column space of A
   - $\operatorname{rank} A = \operatorname{rank}\,[A\ b]$
2. uniqueness: when a solution x exists, it is the unique solution iff
   - $\dim(\text{null space of } A) = 0$
   - $\operatorname{rank} A = n$
3. existence & uniqueness: the solution x uniquely exists iff
   - $\operatorname{rank} A = \operatorname{rank}\,[A\ b] = n$
4. existence for any $b \in \mathbb{R}^m$: a solution x exists for *any* $b \in \mathbb{R}^m$ iff
   - $\operatorname{rank} A = m$
5. unique existence for any $b \in \mathbb{R}^m$: the unique solution x exists for *any* $b \in \mathbb{R}^m$ iff
   - $\operatorname{rank} A = m$ and $\operatorname{rank} A = n$ (i.e., $A \in \mathbb{R}^{n\times n}$ has 'full rank')

Ex: $\operatorname{rank} A = r < n \quad \Rightarrow$

Homogeneous case

$$Ax = 0 \qquad A \in \mathbb{R}^{m\times n}$$

- non-trivial solution exists iff $\operatorname{rank} A = r < n$
- 방정식의 수가 미지수의 수보다 적은 경우 항상 non-trivial solution을 가진다.

Q: Dimension of the 'solution space' =

Nonhomogenous case

$$Ax = b \neq 0 \qquad A \in \mathbb{R}^{m \times n}$$

- Any solution x can be written as

$$x = x_0 + x_h$$

where x_0 is *a solution* to $Ax = b$ and x_h is *a solution* to $Ax = 0$.

Vector space

: set of vectors with "addition" and "scalar multiplication"

For $A, B, C \in V$ and $c, k \in \mathbb{R}$,

$$A + B = B + A$$
$$(A + B) + C = A + (B + C)$$
$$A + 0 = A$$
$$A + (-A) = 0$$

and

$$c(A + B) = cA + cB$$
$$(c + k)A = cA + kA$$
$$c(kA) = (ck)A$$
$$1A = A$$

Examples of vector space

Normed space

: vector space with "norm"

ex: for $v \in \mathbb{R}^n$, the norm is $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$

Inner product space

: vector space with “inner product”

1. $(c_1 A + c_2 B, C) = c_1(A, C) + c_2(B, C)$
2. $(A, B) = (B, A)$
3. $(A, A) \geq 0$ and $(A, A) = 0$ iff $A = 0$

Lesson 5: Determinant of a matrix

- determinant (of a matrix)
- Cramer's rule

Determinant (of a matrix)

For $A \in \mathbb{R}^{n\times n}$,

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ & & & a_{mn} \end{vmatrix} =$$

Elementary row operation & determinant

1. 두 행을 바꾸면 determinant의 부호가 반대가 됨
2. 똑같은 행이 존재하는 행렬의 determinant는 0
3. 한 행의 상수 배를 다른 행에 더해도 determinant 불변
4. 한 행에 0 아닌 c를 곱하면 determinant는 c배가 됨
   ($c = 0$인 경우도 성립하지만 쓸모는 없음)

Properties of ‘determinant’

- 앞 페이지의 1번–4번은 행 대신 열에 대해서도 똑같이 성립한다.
- $\det A = \det A^\top$
- zero row나 zero column이 있으면 determinant는 0
- 두 행이나 두 열이 비례관계이면 determinant는 0

Properties of ‘determinant’

THM: A matrix $A \in \mathbb{R}^{m\times n}$ has rank $r(\geq 1)$ iff

- A has a $r \times r$ submatrix whose determinant is non-zero, and
- determinants of submatrices of A, whose size is larger than $r \times r$, are zero (if exists).

Cramer's rule

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} x = b, \qquad A \in \mathbb{R}^{n\times n}, \quad \det A =: D \neq 0$$

Cramer's rule:

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots \quad x_n = \frac{D_n}{D}$$

where

$$D_k = \begin{bmatrix} a_1 & \cdots & a_{k-1} & b & a_{k+1} & \cdots & a_n \end{bmatrix}$$

Ex:

$$\begin{aligned} 2x - y &= 1 \\ 3x + y &= 2 \end{aligned}$$

Lesson 6: Inverse of a matrix

- inverse (of a matrix)
- Gauss-Jordan elimination (computing inverse)
- formula for the inverse
- properties of inverse and nonsingular matrices

Inverse of a matrix

- For $A \in \mathbb{R}^{n \times n}$, the inverse of A is a matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

  and we denote B by A^{-1}.
- A^{-1} exists iff $\operatorname{rank} A = n$ iff $\det A \neq 0$ iff A is 'non-singular'

Computing the inverse: Gauss-Jordan elimination

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Downarrow$$

$$[A|I] = \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

A formula for the inverse

For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$,

Properties about nonsingular matrix, inverse, and determinant

- Inverse of 'diagonal matrix' is easy.
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- For $A, B, C \in \mathbb{R}^{n \times n}$, if A is nonsingular (i.e., $\operatorname{rank} A = n$),
  - $AB = AC$ implies $B = C$.
  - $AB = 0$ implies $B = 0$.
- For $A, B \in \mathbb{R}^{n \times n}$, if A is singular, then AB and BA are singular.
- $\det(AB) = \det(BA) = \det A \det B$

Lesson 7: Eigenvalues and eigenvectors

- eigenvalues and eigenvectors
- symmetric, skew-symmetric, and orthogonal matrices

Eigenvalue and eigenvector of a matrix

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = -3$$

$$A - 5I = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Lesson 8: Similarity transformation, diagonalization, and quadratic form

- similarity transformation
- diagonalization
- quadratic form

Similarity transformation

행렬 $A \in \mathbb{R}^{n\times n}$가 n개의 선형독립인 e.vectors를 가질 때...

언제 행렬 A가 n개의 선형독립인 e.vectors를 갖나? (1)

언제 행렬 A가 n개의 선형독립인 e.vectors를 갖나? (2)

$$A_1 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -3 \end{array}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = -2$$

$$A_3 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = -2$$

Diagonalization

Diagonalization이 안되는 경우

Quadratic form

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

못 다룬 것들

교재의 연습 문제:

trace,
positive definite matrix, positive semi-definite matrix

out of the scope:

(induced) norm of a matrix,
(generalized eigenvectors,)
Jordan form

further study:

`http://snuon.snu.ac.kr` [최신제어기법]
`http://snui.snu.ac.kr` [최신제어기법]
`http://lecture.cdsl.kr` [선형대수 및 선형시스템 기초]

Lesson 9: Introduction to differential equation

- function, limit, and differentiation
- differential equation, general and particular solutions
- direction field, solving DE by computer

Function, limit, and differentiation

Basic concepts and ideas

$$
\begin{aligned}
&y'(x) + 2y(x) - 3 = 0 \\
&y'(x) = -27x + x^2 \\
&y'(t) = 2t \\
&y''(x) + y'(x) + y(x) = 0 \\
&y''(x)y'(x) + \sin(y(x)) + 2 = 0 \\
&\begin{cases} y_1'(x) + 2y_2(x) + 3 = 0 \\ y_2'(x) + 2y_1'(x) + y_2(x) = 2 \end{cases} \\
&2\frac{\partial y}{\partial x}(x,z) + 3\frac{\partial y}{\partial z}(x,z) - 2x = 0
\end{aligned}
$$

* ODE (ordinary differential equation) / PDE (partial differential equation)
* Solving DE:

* Explicit/implicit solution

Why do we have to study DE?

General solution and particular solution

Direction fields (a geometric interpretation of $y' = f(x, y)$)

An idea of solving DE by computer

Lesson 10: Solving first order differential equations

- separable differential equations
- exact differential equations

Separable DE

f, g: continuous functions

$$g(y)y' = f(x) \qquad \Rightarrow \qquad g(y)dy = f(x)dx$$

$y' = g\left(\frac{y}{x}\right)$

replacing $ay + bx + k$ with v

$$(2x - 4y + 5)y' + (x - 2y + 3) = 0$$

Exact differential equation: introduction

(observation:) For $u(x, y)$,

$$du = \frac{\partial u}{\partial x}(x, y)dx + \frac{\partial u}{\partial y}(x, y)dy \quad : \qquad \text{differential of } u.$$

So, if $u(x, y) = c$ (constant), then $du = \qquad$.

Exact differential equation

Given DE: $M(x, y) + N(x, y)\frac{dy}{dx} = 0$

If $\exists$ a function $u(x, y)$ s.t.

$$\frac{\partial u}{\partial x}(x, y) = M(x, y) \quad \& \quad \frac{\partial u}{\partial y}(x, y) = N(x, y)$$

then

$$u(x, y) = c$$

is a general sol. to the DE.

The DE is called “exact DE”.

How to check if the given DE is exact?

How to solve the exact DE?

Lesson 11: More on first order differential equations

- integrating factor
- linear differential equation
- Bernoulli equation
- obtaining orthogonal trajectories of curves
- existence and uniqueness of solutions to initial value problem

Integrating factor

$$P(x, y)dx + Q(x, y)dy = 0$$

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

Linear DE

$$y' + p(x)y = r(x)$$

Bernoulli DE

$$y' + p(x)y = g(x)y^a, \qquad a \neq 0 \text{ or } 1$$

Verhulst logistic model (population model):

$$y' = Ay - By^2, \qquad A, B > 0$$

Orthogonal trajectories of curves

Existence of solutions to initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

THM 1: IF $f(x, y)$ is continuous, and bounded such that $|f(x, y)| \leq K$, in the region

$$R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$$

THEN the IVP has at least one sol. $y(x)$ on the interval $|x - x_0| < \alpha$ where $\alpha = \min(a, b/K)$.

Uniqueness of solutions to initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

THM 2: IF $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous, and bounded such that $|f(x, y)| \leq K$ and $|\frac{\partial f}{\partial y}(x, y)| \leq M$ in R, THEN the IVP has a *unique* sol. $y(x)$ on the interval $|x - x_0| < \alpha$ where $\alpha = \min(a, b/K)$.

Lesson 12: Solving the second order linear DE

- overview
- homogeneous linear DE
- reduction of order
- homogeneous linear DE with constant coefficients

Overview: Linear ODEs of second order

$$y'' + p(x)y' + g(x)y = r(x), \qquad y(x_0) = K_0, \quad y'(x_0) = K_1$$

1. The homogeneous linear ODE:
$$y'' + p(x)y' + g(x)y = 0 \tag{1}$$
has two "linearly independent" solutions $y_1(x)$ and $y_2(x)$.
2. Let $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ with two constant coefficients c_1 and c_2, which is again a solution to (1).
3. Solve
$$y'' + p(x)y' + g(x)y = r(x) \tag{2}$$
without considering the initial condition. Let the solution be $y_p(x)$.
4. The general solution is
$$y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$
Determine c_1 and c_2 with the initial condition.

Homogeneous linear ODEs of second order

$$y'' + p(x)y' + g(x)y = 0$$

Claim: Linear homogeneous ODE of the second order has two linearly independent solutions.

How to obtain a basis if one sol. is known? (Reduction of order)

Obtaining another $y_2(x)$ with a known $y_1(x)$

Homogeneous linear ODEs with constant coefficients

$$y'' + ay' + by = 0$$

Lesson 13: The second order linear DE

- case study: free oscillation
- Euler-Cauchy equation
- existence and uniqueness of a solution to IVP
- Wronskian and linear independence of solutions

Modeling: Free oscillation

Euler-Cauchy equation

$$x^2y'' + axy' + by = 0$$

Existence and uniqueness of a solution to IVP

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = K_0, \quad y'(x_0) = K_1$$

THM: IF $p(x)$ and $q(x)$ are continuous (on an open interval $I \ni x_0$),
THEN $\exists$ a unique sol. $y(x)$ (on the interval I).

Wronskian and linear independence of solutions

With $y_1(x)$ and $y_2(x)$ being the solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

Wronski determinant (Wronskian) of y_1 and y_2 is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

THM:

1. two sol. y_1, y_2 are linearly dep. on $I \Leftrightarrow$ $W(y_1(x), y_2(x)) = 0$ at some $x^* \in I$
2. If $W(y_1(x), y_2(x)) = 0$ at some $x^* \in I$, then $W(y_1(x), y_2(x)) \equiv 0$ on I.
3. If $W(y_1(x), y_2(x)) \neq 0$ at some $x^* \in I$, then y_1 and y_2 are linearly indep. on I.

$y'' + p(x)y' + q(x)y = 0$ has two indep. sol. y_1 and y_2

so, it has a general sol. $y(x) = c_1y_1(x) + c_2y_2(x)$

Any sol. to $y'' + p(x)y' + q(x)y = 0$ has the form of $c_1y_1(x) + c_2y_2(x)$

Lesson 14: Second order nonhomogeneous linear DE

- nonhomogeneous linear DE
- solution by undetermined coefficient method
- solution by variation-of-parameter formula

Nonhomogeneous linear DE

$$y'' + p(x)y' + q(x)y = r(x)$$

Candidate for $y_p(x)$ in $y'' + p(x)y' + q(x)y = r(x)$

Term in $r(x)$	Candidate for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n, n \geq 0$ integer	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

The above rules are applied for each term $r(x)$.

If the candidate for $y_p(x)$ happens to be a sol. of the homogeneous equation, then multiply $y_p(x)$ by x (or by x^2 if this sol. corresponds to a double root of the characteristic eq. of the homogeneous equation).

$$y'' + 4y = 8x^2$$

$$y'' - 3y' + 2y = e^x$$

$$y'' + 2y' + y = e^{-x}$$

$$y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$$

$$y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 2x$$

$$y'' + 2y' + 5y = 1.25e^{0.5x} + 40e^{-x}\cos 2x$$

Solution by variation of parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

Lesson 15: Higher order linear DE

- higher order homogeneous linear DE
- higher order homogeneous linear DE with constant coefficients
- higher order nonhomogeneous linear DE

Higher order homogeneous linear DE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad \text{(H)}$$

General sol.: $y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$
where $y_i(x)$'s are linearly indep. sol. to (H).

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \qquad y^{(i)}(x_0) = K_i$$

THM: If all p_i's are conti. (on I), then IVP has a unique sol. (on I).

THM: With all p_i's being conti.,

sol. $\{y_1, \cdots, y_n\}$ are lin. dep. on I

$$\Leftrightarrow \quad W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0 \quad \text{at some } x_0 \in I$$

$$\Leftrightarrow \quad W(y_1, \cdots, y_n) \equiv 0 \quad \text{on } I$$

$$y'''' - 5y'' + 4y = 0$$

THM: With all p_i's being conti., the (H) has n lin. indep. sol. (i.e., there is a general solution).

THM: With all p_i's being conti., the general sol. includes all solutions.

Higher order homogeneous linear DE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

* distinct roots

* multiple roots

Higher order nonhomogeneous linear DE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$$

* undetermined coefficient method:

* variation-of-parameter formula:

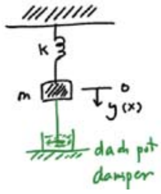
$$y_p(x) = y_1 \int \frac{W_1 r}{W} dx + y_2 \int \frac{W_2 r}{W} dx + \cdots + y_n \int \frac{W_n r}{W} dx$$

where $W = W(y_1, \cdots, y_n)$ and W_j: j-th column in W replaced by $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Lesson 16: Case studies

- mass-spring-damper system: forced oscillation
- RLC circuit
- elastic beam

Case study: forced oscillation ($my'' + cy' + ky = r$)



$$y_p(t) = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \cos \omega t + F_0 \frac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \sin \omega t, \quad y(t) = y_h(t) + y_p(t)$$

$$y(t) = y_h(t) + F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \cos \omega t + F_0 \frac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \sin \omega t$$

Modeling: RLC circuit

RLC circuit: forced response

Elastic beam

Lesson 17: Systems of ODEs

- introduction
- existence and uniqueness of solutions to IVP
- linear homogeneous case
- linear homogeneous constant coefficient case

Systems of ODE

Existence and uniqueness of solutions to IVP

$$y' = f(t, y), \qquad y(t_0) = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$$

THM: If all $f_i(t, y)$ and $\frac{\partial f_i}{\partial y_j}(t, y)$ are conti. on some region of $(t, y_1, y_2, \cdots, y_n)$-space containing $(t_0, k_1, \cdots, k_n)$, then a sol. $y(t)$ exists and is unique in some local interval of t around t_0.

$$y' = A(t)y + g(t), \qquad y(t_0) = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$$

THM: If $A(t)$ and $g(t)$ are conti. on an interval I, then a sol. $y(t)$ exists and is unique on the interval I.

Linear homogeneous case

$$y' = A(t)y$$

General sol.: $y(t) = c_1 y^{(1)}(t) + c_2 y^{(2)}(t) + \cdots + c_n y^{(n)}(t)$
where $y^{(i)}(t)$'s are lin. indep. sol.

Linear homogeneous constant coefficient case

$$y' = Ay$$

Handling complex e.v/e.vectors

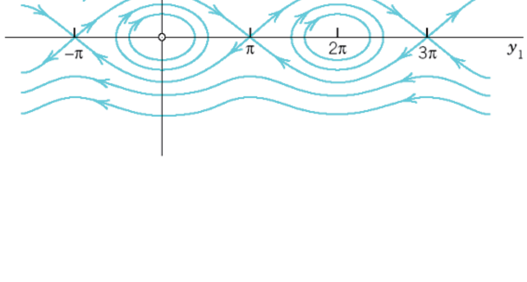
Lesson 18: Qualitative properties of systems of ODE

- phase plane and phase portrait
- critical points
- types and stability of critical points

Phase plane and phase portrait

Critical point (= equilibrium)

Example: undamped pendulum



Types of critical points: node

Types of critical points: saddle / center

Types of critical points: spiral / degenerate node

Stability

DEF: stability of a critical point $P_0(= y^*)$:

- all trajectories of $y' = f(y)$ whose initial condition $y(t_0)$ is sufficiently close to P_0 remain close to P_0 for all future time
- for each $\epsilon > 0$, there is $\delta > 0$ such that,

$$|y(t_0) - y^*| < \delta \qquad \Rightarrow \qquad |y(t) - y^*| < \epsilon, \quad \forall t \geq t_0$$

DEF: asymptotic stability of P_0 = stability + attractivity ($\lim_{t\to\infty} y(t) = y^*$)

Example: second order system

Lesson 19: Linearization and nonhomogeneous linear systems of ODE

- linearization
- nonhomogeneous case

Linearization

$$y' = f(y)$$

Let $y = 0$ be a critical point (without loss of generality; WLOG), and be isolated.

$$y_1' = f_1(y_1, y_2) = f_1(0,0) + \frac{\partial f_1}{\partial y_1}(0,0)y_1 + \frac{\partial f_1}{\partial y_2}(0,0)y_2 + h_1(y_1, y_2)$$
$$y_2' = f_2(y_1, y_2) = f_2(0,0) + \frac{\partial f_2}{\partial y_1}(0,0)y_1 + \frac{\partial f_2}{\partial y_2}(0,0)y_2 + h_2(y_1, y_2)$$

$$y' = f(y) \qquad \Rightarrow \qquad y' = Ay = \frac{\partial f}{\partial y}\Big|_{y=0} y$$

- If no e.v. of A lies in the imaginary axis, then stability of the critical point of the nonlinear system is determined by A.
  - If $\mathrm{Re}(\lambda) < 0$ for all λ, it is asymptotically stable.
  - If $\mathrm{Re}(\lambda) > 0$ for at least one λ, it is unstable.
- If all e.v.'s are distinct and no e.v. of A lies in the imaginary axis, then the type of the critical point of the nonlinear system is determined by A.
  - The node, saddle, and spiral are preserved, but center may not be preserved.

Nonhomogeneous linear case

Method of undetermined coefficients (for time-invariant case)

Method of variation of parameters (for time-varying case)

Method of diagonalization (for time-invariant case)

Lesson 20: Series solutions of ODE

- power series method
- Legendre equation

Power series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$$
$$+ a_{n+1}(x-x_0)^{n+1} + \cdots$$

For a given x_1,

if $\lim_{n\to\infty} S_n(x_1)$ exists (or, $\lim_{n\to\infty} R_n(x_1) = 0$,
or for any $\epsilon > 0$, $\exists N(\epsilon)$ s.t. $|R_n(x_1)| < \epsilon$ for all $n > N(\epsilon)$),

then the series is called "convergent at $x = x_1$" and we write $S(x_1) = \lim_{n\to\infty} S_n(x_1)$.

Radius of convergence

If

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}}, \qquad \text{or} \qquad R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

is well-defined, then the series is convergent for x s.t. $|x - x_0| < R$.

Power series method

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

If p, q, and r are analytic at $x = x_0$,

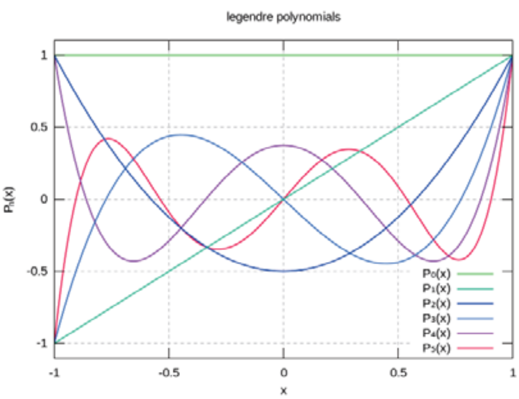
then there exists a power series solution around x_0 (i.e., $R > 0$):

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \qquad n : \text{real number}$$

Legendre polynomial (of degree n)



"Legendrepolynomials6" by Geek3

Lesson 21: Frobenius method

- Frobenius method
- Euler-Cauchy equation revisited

Frobenius method

The DE

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

where b and c are analytic at $x = 0$, has at least one sol. around $x = 0$ of the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \cdots).$$

- Case 1: distinct roots, not differing by an integer
- Case 2: double roots
- Case 3: distinct roots differing by an integer

General sol.: $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where

- Case 1:

$$y_1(x) = x^{r_1}(a_0 + a_1 x + \cdots)$$
$$y_2(x) = x^{r_2}(A_0 + A_1 x + \cdots)$$

- Case 2: $r = (1 - b_0)/2$

$$y_1(x) = x^{r}(a_0 + a_1 x + \cdots)$$
$$y_2(x) = y_1(x)\ln x + x^{r}(A_1 x + A_2 x^2 + \cdots)$$

- Case 3: $r_1 > r_2$

$$y_1(x) = x^{r_1}(a_0 + a_1 x + \cdots)$$
$$y_2(x) = k y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + \cdots)$$

Example: Euler-Cauchy equation revisited

Lesson 22: Bessel DE and Bessel functions

- example for Frobenius method
- Bessel DE and its solutions

Example: a simple hypergeometric equation

$$x(x-1)y'' + (3x-1)y' + y = 0$$

Example: another simple hypergeometric equation

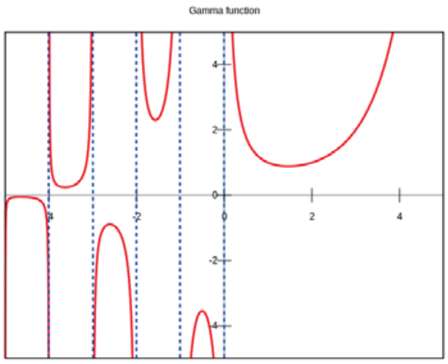
$$x(x-1)y'' - xy' + y = 0$$

Gamma function

$$\Gamma(\nu) := \int_0^\infty e^{-t} t^{\nu-1} dt$$

has the properties:

1. $\Gamma(\nu + 1) = \nu\Gamma(\nu)$
2. $\Gamma(1) = 1$
3. $\Gamma(n + 1) = n!$



"Gamma plot" by Alessio Damato

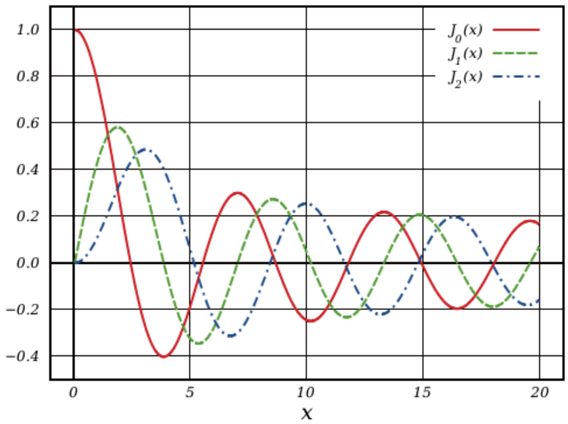
Bessel's DE

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \qquad \nu \geq 0$$

Computing $y_1(x)$

Bessel function of the first kind of order n

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

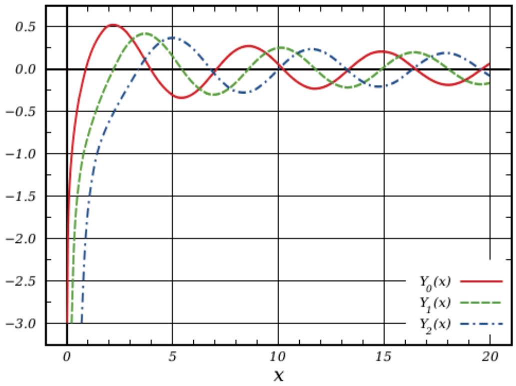


"Bessel Functions" by Inductiveload

Finding $y_2(x)$

Bessel function of the second kind of order ν

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$
$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) = \cdots$$



"Bessel Functions" by Inductiveload

Lesson 23: Laplace transform I

- introduction to Laplace transform
- linearity, shifting property
- existence and uniqueness of Laplace transform
- computing inverse Laplace transform
- partial fraction expansion & Heaviside formula

Laplace transform

$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

(Property) Linearity: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

(Property) s-shifting property: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Transform table: $f(t) \leftrightarrow F(s)$

$$1 \leftrightarrow \frac{1}{s}$$

$$t \leftrightarrow \frac{1}{s^2}$$

$$t^2 \leftrightarrow \frac{2!}{s^3}$$

$$t^n \leftrightarrow \frac{n!}{s^{n+1}}, \quad n = \text{integer}$$

$$t^a \leftrightarrow \frac{\Gamma(a+1)}{s^{a+1}}, \quad a > 0$$

$$e^{at} \leftrightarrow \frac{1}{s-a}$$

$$\cos\omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

$$\sin\omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$\cosh at \leftrightarrow \frac{s}{s^2-a^2}$$

$$\sinh at \leftrightarrow \frac{a}{s^2-a^2}$$

$$e^{at}\cos\omega t \leftrightarrow \frac{s-a}{(s-a)^2+\omega^2}$$

$$e^{at}\sin\omega t \leftrightarrow \frac{\omega}{(s-a)^2+\omega^2}$$

Existence and uniqueness of Laplace transform

IF $f(t)$ is piecewise continuous on every finite interval in $\{t : t \geq 0\}$, and

$$|f(t)| \leq Me^{kt}, \qquad t \geq 0$$

with some M and k,

THEN $\mathcal{L}\{f(t)\}$ exists for all $\mathrm{Re}(s) > k$.

Computing inverse Laplace transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = ?$$

* Partial fraction expansion:

Finding coefficients in partial fraction expansion: Heaviside formula

$Y(s) = \frac{s+1}{s^3+s^2-6s} = \frac{A_1}{s} + \frac{A_2}{s+3} + \frac{A_3}{s-2}$

$Y(s) = \frac{s^3-4s^2+4}{s^2(s-2)(s-1)} = \frac{A_2}{s^2} + \frac{A_1}{s} + \frac{B}{s-2} + \frac{C}{s-1}$

$$Y(s) = \cdots = \frac{A_3}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_1}{s-1} + \frac{B_2}{(s-2)^2} + \frac{B_1}{s-2}$$

$$Y(s) = \frac{20}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}$$

Lesson 24: Laplace transform II

- transform of derivative and integral
- solving linear ODE
- unit step function and t-shifting property
- Dirac's delta function (impulse)

(Property) Transform of differentiation: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

(Property) Transform of integration: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s)$

Solving IVP of linear ODEs with constant coefficients

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \quad y'(0) = K_1$$

Unit step function (Heaviside function)

(Property) t-shifting property: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

(Dirac's) delta function

$\delta(t)$ is a (generalized) function such that

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \qquad \text{and} \qquad \int_{-a}^{a} \delta(t)dt = 1 \quad \text{for any } a > 0$$

sifting property: $\displaystyle\int_0^\infty g(t)\delta(t-a)dt = g(a),$ $\quad g$: conti., $a > 0$

Lesson 25: Laplace transform III

- convolution
- impulse response
- differentiation and integration of transforms
- solving system of ODEs

(Property) Convolution: $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$

Properties of convolution:

$$
\begin{aligned}
f * g &= g * f \\
f * (g_1 + g_2) &= f * g_1 + f * g_2 \\
(f * g) * v &= f * (g * v) \\
f * 0 &= 0 * f = 0, \qquad f * 1 \neq f
\end{aligned}
$$

Impulse response

(Property) Differentiation of transform: $\mathcal{L}\{tf(t)\} = -F'(s)$

(Property) Integration of transform: $\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(\tilde{s})d\tilde{s}$

Solving system of ODEs