Linkedness of Allocations and the Rate of Convergence of the Core

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This paper is based on a theorem of Robert Anderson (1987) which proves that in a general sequence of finite economies with smooth preferences, the rate of convergence of the competitive gap with respect to the gap-minimizing prices is the inverse of the square of the number of agents. We argue that the assumptions of the theorem are too restrictive and try to relax some of them. We actually prove that the same rate of convergence can be obtained with a much weaker assumption called the uniform linkedness of allocations. (*JEL* C71, D50)

I. Introduction

In a recent paper, Anderson (1987) showed that the rate of convergence of the *average* competitive gap to zero with respect to suitably chosen prices is the inverse of the square of the number of agents, in a general sequence of economies with smooth preferences. This improvement upon the previous results which had established the rate $O(1/n)$ could be obtained essentially by combining the flattening effect of increasing the number of agents with smooth preferences at the tangent points, with the fundamental result of Anderson (1978).

Like others in the literature (see Debreu 1975; Grodal 1975), Anderson makes two restrictive assumptions: (i) all agents have strictly positive endowments of all commodities, and (ii) the closure of each indifference curve is contained in $\mathbb{R}^k_{++}$ (boundary condition). With these conditions, the consumption set can be confined to a subset of $\mathbb{R}^k_{++}$ without encountering any unpleasant boundary problems. For example, the possibility that core allocations lie on the boundary of the consumption set is excluded.

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[Seoul Journal of Economics 1992, Vol. 5, No. 3]

Cheng (1981) was able to relax one of the two assumptions, the boundary condition, to obtain the rate $O(1/n)$ of convergence of the *maximum* competitive gap to zero with respect to the supporting prices at the core allocations in a sequence of type economies, by introducing a rather mild assumption which he called the Indecomposability condition. The condition essentially says that we cannot partition the set of agents into two groups such that each group consumes disjoint set of commodities.

Anderson conjectured that Cheng's condition or some variant of it could replace the two assumptions in his theorems as well. Indeed, this turns out to be true in a general sequence of economies converging to a well-defined continuum economy. The purpose of this paper is to show how Cheng's condition which we will call the linkedness condition following Mas-Colell (1985), can be used to replace the two assumptions in Anderson (1987)'s theorem. As in Cheng (1981) and Mas Colell (1985), we assume that all the Walrasian allocations of the limit economy are linked. But they apply linkedness at the core allocations with respect to the supporting prices. Since any core allocation is, in particular, a feasible allocation, it follows that any convergent subsequence of core allocations converges to a Walrasian allocation of the limit economy. Therefore, uniform linkedness of Walrasian allocations in the limit economy implies uniform linkedness of the core allocations. In our case, linkedness is applied at the expenditure-minimizing points (denoted $h_n(a)$) with respect to the gap-minimizing prices, which typically do not form a feasible allocation. The limit of a convergent subsequence of the $\{h_n\}$ will be a selection from the demand correspondence of the limit economy at an equilibrium price, but it need not be a feasible allocation, hence it need not be Walrasian, and hence need not be linked. Strong convexity of preferences of the limit economy is sufficient to ensure that the limit of a convergent subsequence of the $\{h_n\}$ is Walrasian, and so it is assumed in our main result (Theorem 1). We also give two other results: In the first (Theorem 2), we show that the convexity assumption can be dropped, provided we are willing to accept an arbitrarily small slowing of the rate of convergence. In the second (Theorem 3), we show that the quadratic convergence rate holds for an open and dense set of limit economies, using a result of Mas-Colell and Neuefeind (1977).

The basic references are Anderson (1987) and Mas-Colell (1985). We follow their convention in definitions and notations and readers are referred to them for the details. Their theorems and lemmas will be freely cited whenever needed, sometimes without proof.

II. Definitions and Notations

We begin with the concept of preference preorder $\geq$ on $\mathbb{R}^k_+$, which is defined as a reflexive and transitive binary relation on $\mathbb{R}^k_+$. Throughout the paper, we assume that $\geq$ is complete, continuous, strongly monotone, and semismooth. The definition of semismoothness of $\geq$ given by Anderson (1987) is adjusted to cover the boundary points of $\mathbb{R}^k_+$ as follows: for all $x \in \mathbb{R}^k_+$, $\exists\, y \neq x \in \mathbb{R}^k_+$ such that if $\| z - y \| < \| x - y \|$ for $z \in \mathbb{R}^k_+$, then $z > x$. Let $\mathcal{P}$ denote the set of all such preferences. $\mathcal{P}$ is endowed with the topology of closed convergence. The set of all strongly convex preferences in $\mathcal{P}$ is denoted by $\mathcal{P}_{sc}$.

We will need a uniform version of semismoothness. $P \subset \mathcal{P}$ is said to be equisemismooth, if for every compact $Q \subset \mathbb{R}^k_+$ there exists $\rho > 0$ such that for all $\geq \in \mathcal{P}$, for all $x \in Q$, $\exists\, y \in \mathbb{R}^k_+$, such that $\| x - y \| = \rho$, and if $\| z - y \| < \rho$ for $z \in \mathbb{R}^k_+$ then $z \geq x$.

An exchange economy is a map ε: $\mathcal{A} \to \mathcal{P} \times \mathbb{R}^k$, where $\mathcal{A}$ is a finite set or [0, 1]. $\mathcal{E}(a) = (\geq_a, e(a))$ has a standard interpretation: an agent $a \in \mathcal{A}$ in an exchange economy $\mathcal{E}$ is characterized by a preference preorder $\geq_a$ and an endowment vector $e(a) \in \mathbb{R}^k_+$. We assume that total endowments are strictly positive, i.e., $\Sigma_{a \in \mathcal{A}}\, e(a) \gg 0$ (for a continuum economy, $\int_{\mathcal{A}} e\, d\mu \gg 0$ where μ is Lebesgue measure on [0, 1]). For a continuum economy with $\mathcal{A} = [0, 1]$, we require that $\mathcal{E}$ be Borel measurable and supp$\mathcal{E}$ be compact in $\mathcal{P} \times \mathbb{R}^k_+$ with the product topology.

An allocation f is an integrable function f: $\mathcal{A} \to \mathbb{R}^k_+$. A *feasible* allocation is an allocation satisfying $\int f\, d\mu = \int e\, d\mu$. The set of core allocations $C(\mathcal{E})$, and the set of Walrasian allocations $W(\mathcal{E})$ are defined as usual. The set of distributions of Walrasian allocations of $\mathcal{E}$ is denoted by $\mathcal{D}W(\mathcal{E})$. $\mathcal{D}f$ denotes the distribution of an allocation f. A price p is an element of the set $\Delta = \{p \in \mathbb{R}^k_+ \mid \| p \| = 1\}$, where $\| p \|$ denotes the Euclidean length of p. Define $\Delta^0 = \{p \in \Delta \mid p \gg 0\}$. The set of equilibrium prices of $\mathcal{E}$ is denoted by $\Pi(\mathcal{E})$. The demand set is defined as $\varphi(\mathcal{E}(a), p) = \{x \in \mathbb{R}^k_+ \mid p \cdot x \leq p \cdot e(a)$, and $y >_a x \Rightarrow p \cdot y > p \cdot x\}$.

We consider a sequence of finite exchange economies $\{\mathcal{E}_n\}$ which converges to a continuum economy $\mathcal{E}$. For simplicity we assume that the number of agents $\#\mathcal{A}_n$ of the economy $\mathcal{E}_n$ is n. Let ν denote the distribution of characteristics induced by $\mathcal{E}$. Then we say $\mathcal{E}_n \to \mathcal{E}$, if $\nu_n \to \nu$ weakly, and $e_n(a)$ is uniformly bounded. Because of the latter condition, we assume that there exists a compact $Q \in \mathbb{R}^k_+$ such that $\mathcal{E}_n$: $\mathcal{A}_n \to \mathcal{P} \times Q$ and $\mathcal{E}$: $[0, 1] \to \mathcal{P} \times Q$. The sequence $\mathcal{E}_n \to \mathcal{E}$ is purely competitive (Hildenbrand 1974, Definition 4, p. 138). Note that we do not assume supp$\mathcal{E}_n \to$ supp$\mathcal{E}$.

It is well-known that for two economies $\mathcal{E}_1$ and $\mathcal{E}_2$ with the same distribution v of characteristics, the sets $W(\mathcal{E}_1)$ and $W(\mathcal{E}_2)$ may be different. But if $\mathcal{E}_1$ and $\mathcal{E}_2$ are continuum economies (i.e., $\mathcal{A} = [0, 1]$) with the same v, then $\mathcal{D}W(\mathcal{E}_1)$ and $\mathcal{D}W(\mathcal{E}_2)$ have the same closure with respect to the weak convergence, and there is an $\mathcal{E}$ with the same v such that $D\mathcal{D}W(\mathcal{E})$ is closed (Hildenbrand 1974, Proposition 5 and 6, pp. 155-6).

An allocation f is *linked*, if there are no partitions of agents ($\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$) and commodities ($L = L_1 \cup L_2$) such that $f^j(a) = 0$ whenever $(a, j) \in \mathcal{A}_1 \times L_2$ or $\mathcal{A}_2 \times L_1$. This linkedness condition is also called the Indecomposability condition in Cheng (1981) or the No Isolated Community condition in Smale (1974). In their models, this guarantees that the supporting price at a core allocation is *unique*.

A $(k-1)$ collection of pairs of commodities $\mathcal{J} = \{J_1, \dots, J_{k-1}\}$ is *linked* if $\bigcup_{h=1}^{k-1} J_h = L$ and for any partition $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{J}$, $\{i \in J_h \mid J_h \in \mathcal{J}_1\} \cap \{i \in J_h \mid J_h \in \mathcal{J}_2\} \neq \phi$. It is not hard to show that if f is linked, then there is $\delta > 0$ and a linked collection $\mathcal{J}$ such that $\#\{a \in \mathcal{A}_n \mid f^j{}_n(a) > \delta$ for each $j \in J\}/n > \delta$ for all $J \in \mathcal{J}$. In this case we say f is δ-linked. For a continuum economy, the fractions are replaced by Lebesgue measure μ on $[0, 1]$.

We now define a uniform version of the linkedness condition. The collection of pairs of a coalition $S_\alpha (\subset \mathcal{A}_\alpha)$ and an allocation f_α, $\{(S_\alpha, f_\alpha)\}_{\alpha \in \mathcal{A}}$, is *$\delta$-uniformly linked*, if there exists $\delta > 0$ such that for any $\alpha \in \mathcal{A}$, $\#S_\alpha \geq \delta\#\mathcal{A}_\alpha$ and there exists a linked collection $\mathcal{J}$, such that $\#\{a \in S_\alpha \mid f^j{}_\alpha(a) > \delta$ for each $j \in J\}/\#\mathcal{A}_\alpha > \delta$, for all $J \in \mathcal{J}$. When $S_\alpha = \mathcal{A}_\alpha$ for all $\mathcal{A}$, we use the notation $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ for abbreviation. The idea of δ-uniform linkedness is adapted from δ-balancedness in Mas-Colell (1985). The following fact is very useful and interesting in its own right.

***Proposition* 1**

If every Walrasian allocation of $\mathcal{E}$: $[0, 1] \to \mathcal{P} \times \mathbb{R}^k_+$ is linked, then there exists $\delta > 0$ such that $W(\mathcal{E})$ is δ-uniformly linked.

Proof: Define the extended economy $\tilde{\mathcal{E}}$: $[0, 1] \times [0, 1] \to \mathcal{P} \times \mathbb{R}^k_+$ such that $\tilde{\mathcal{E}}(a, t) = \mathcal{E}(a)$. Then obviously, the distributions of characteristics of $\tilde{\mathcal{E}}$ and $\mathcal{E}$ are the same. By Theorem 4 of Hildenbrand (1974, p. 140). $\overline{\mathcal{D}W(\mathcal{E})} = \overline{\mathcal{D}W(\tilde{\mathcal{E}})}$, where $\bar{X}$ denotes the closure of X. By adapting the proof of Proposition 6 of Hildenbrand (1974, p. 156), we can show that

$\mathcal{DW}(\mathcal{E})$ is closed. In fact, it is compact, because supp$\tilde{\mathcal{E}}$ = supp$\mathcal{E}$ is compact. Therefore, $\overline{\mathcal{DW}(\mathcal{E})} = \mathcal{DW}(\tilde{\mathcal{E}})$.

We now show that $\tilde{f} \in W(\tilde{\mathcal{E}})$ is linked. Let $\psi(a) = \cup_{t \in [0,1]} \tilde{f}(a, t)$ for $a \in [0, 1]$ and $g(a) = \int_0^1 \tilde{f}(a, t)\, dt$. Then $g(a) \in \mathrm{co}\psi(a)$, where $\mathrm{co}\psi(a)$ denotes the convex hull of $\psi(a)$. But $\int g = \int\int \tilde{f} = \int e$. Therefore, $\int e \in \int \mathrm{co}\psi = \mathrm{co} \int \psi$. This implies that there is $f \in W(\mathcal{E})$ such that $f(a) \in \psi(a)$ and so supp$\mathcal{D}f$ $\subset$ supp$\mathcal{D}\tilde{f}$. Since $f \in W(\mathcal{E})$ is linked by assumption, f is linked.

Then, compactness of $\mathcal{DW}(\tilde{\mathcal{E}})$ implies that $W(\tilde{\mathcal{E}})$ is δ-uniformly linked for some $\delta > 0$. But, $\overline{\mathcal{DW}(\mathcal{E})} = \mathcal{DW}(\tilde{\mathcal{E}})$. There $W(\mathcal{E})$ is δ-uniformly linked.

Q.E.D.

Given an economy $\mathcal{E}$, an allocation f, and a price $p \in \Delta$, the *competitive gap* ϕ for an agent $a \in \mathcal{A}$ is defined as: $\phi(p, f, a) = |\, p \cdot (f(a) - e(a))\,| + |\, \inf\{p \cdot (y - e(a)) \mid y \geq_a f(a)\}\,|$. The average competitive gap is denoted by

$$\phi(p, f) = (1/\#\mathcal{A})\, \Sigma_{a \in \mathcal{A}}\, \phi(p, f, a).$$

III. Results

Now we can state and prove the main theorem.

Theorem 1

Let $\mathcal{E}_n$: $\mathcal{A}_n \rightarrow P \times Q$ be a sequence of finite exchange economies converging to a continuum economy $\mathcal{E}: [0, 1] \rightarrow \mathcal{P}_{sc} \times Q$, where P is an equisemismooth subset of $\mathcal{P}$ and Q is a compact subset of $\mathbb{R}^k_+$. If all the Walrasian allocations of $\mathcal{E}$ are linked, then there is a constant $M > 0$ such that for all n and core allocations $f_n \in C(\mathcal{E}_n)$, there exist price vectors $p_n \in \Delta$ such that $\phi(p_n, f_n) \leq M/n^2$.

Theorem 1 says that under the almost same hypothesis as in Proposition 7.4.12 in Mas-Colell (1985), the *average* competitive gap with respect to the "gap-minimizing" prices, is the order of $1/n^2$. Note that we do not assume supp$\mathcal{E}_n \rightarrow$ supp$\mathcal{E}$, nor strict positivity of endowment vectors.

The proof will closely follow those of Anderson (1987) and Mas-Colell (1985). We begin with the definitions: for an exchange economy $\mathcal{E}$ and a core allocation f, define $\gamma(a) = \{x - e(a) \mid x \geq_a f(a)\}$ and $V = \Sigma_{a \in \mathcal{A}}\, \gamma(a) \cup \{0\}$. $\bar{p}$ which maximizes $\inf p \cdot V$ is called the "gap-minimizing" price. Define $\alpha = p \cdot V$. For $\xi \in [0, 1)$, let $\Gamma = \cup\ \#S \geq \xi\#\mathcal{A}\Sigma_{a \in S}\, \gamma(a)$. $\bar{q}$ maximizes inf $q \cdot \Gamma$. Define β= inf $\bar{q}$ and $g(a) \in \operatorname{argmin}\, \bar{q} \cdot \gamma(a)$. Define $g(S) = \Sigma_{a \in S}\, g(a)$

and $\beta(S) = q \cdot g(S)$.

The next two lemmas have been proved by Anderson (1987).

Lemma 1

For any finite economy $\mathcal{E}_n$: $\mathcal{P} \to \mathbb{R}^k_+$ with $\| e_n(a) \| \leq K$ and a core allocation $f_n \in C(\mathcal{E}_n)$, and for $\xi \in [0, 1)$, we have:

$$\sum_{a \in \mathcal{A}_n} |\bar{q}_n \cdot [f_n(a) - e_n(a)]| \leq 2\sqrt{k}(k+1)K / (1 - \xi) \tag{1}$$

$$\sum_{a \in \mathcal{A}_n} |\bar{q}_n \cdot g_n(a)| \leq 2\sqrt{k}(k+1)K / (1 - \xi). \tag{2}$$

Proof: This follows immediately from Theorem 3.2 and Lemma 3.3 of Anderson (1987), since $e_n(a)$ is uniformly bounded.

Q.E.D.

Lemma 2

For any finite economy $\mathcal{E}_n$: $\mathcal{P} \times \mathbb{R}^k_+$ with $\| e_n(a) \| \leq K$ and a core allocation $f_n \in C(\mathcal{E}_n)$, and for $\xi \in [0, 1)$, there exists a coalition S_n with $\# S_n \geq \xi \# \mathcal{A}_n - (k + 1)$ such that $\beta(S_n) \leq (1 - \xi)\alpha$, and

$$\| g_n(S_n) \| \leq (k+1)K \left[\frac{2\sqrt{k}(1-\xi)^{-1} + 1}{\min\{q_1, \ldots, q_n\}} + 1 + \sqrt{k} \right].$$

Proof: See the proof of Theorem 3.4 of Anderson (1987).

The following is the central lemma in this paper.

Lemma 3

Let all the assumptions of Theorem 1 hold. Then there exist $N > 0$, $\delta > 0$, and a constant $M_1 > 0$ such that if f_n is a core allocation of $\mathcal{E}_n$, $n > N$, then $\beta(S_n) \geq -M_1/n$ for $\xi = 1 - \delta$ and S_n defined as in Lemma 2.

Proof: *Step 1.* Define an allocation h_n such that for all $a \in \mathcal{A}_n$, $h_n(a) = g_n(a) + e_n(a)$, which, by definition of $g_n(a)$, minimizes $\bar{q}_n \cdot x$ over $\{x \mid x \geq_a f(a)\}$. Then it follows from the proof of Theorem 1 of Hildenbrand (1974 p. 179) that $\{h_n\}$ has a subsequence converging in distribution to an allocation h of the limit economy $\mathcal{E}$ and the corresponding subsequence of $\{\bar{q}_n\}$ converges to an equilibrium price $q \in \Pi(\mathcal{E})$. Therefore, $h(a) \in \varphi(\mathcal{E}(a), q)$ for a.e. $a \in \mathcal{A}$. In Hildenbrand (1974), it is proved that any sequence of core allocations $\{f_n\}$ has the same convergence property as above. But $\{h_n\}$ has all the properties of $\{f_n\}$ used for the proof, except that f_n's are feasible allocations: Lemma 1 above provides the refined version of Lemma 1 of Hildenbrand (1974, p. 180) concerning the property of h_n,

h_n is uniformly bounded as will be shown later in Step 2(2). And the feasibility of f_n is not needed to prove that the limit of $\{f_n\}$ has the above properties.

Since strong convexity implies the unique demand, the allocation h, the elements of which is in the demand set at an equilibrium price, is indeed a Walrasian allocation of $\mathcal{E}$. By Proposition 1, $W(\mathcal{E})$ is 4δ-uniformly linked for some δ. Consequently, there exists $N > 0$ such that $\{h_n\}_{n>N}$ is 2δ-uniformly linked. Since $S_n \geq \xi n - (k + 1)$, for $\xi = 1 - \delta$, $\{(S_n, h_n)\}_{n>N}$ is δ-uniformly linked.

Step 2. We now show that for $n > N$ and $\xi = 1 - \delta$, we can find a constant $M_1 > 0$ such that $g_n(S_n) \cdot \bar{q}_n \geq - M_1/n$. Let $n > N$ and $\xi = 1 - \delta$ for the rest of the proof.

(1) First we show that $\bar{q}_n$ is uniformly bounded away from zero, i.e., $\inf_n \min_k \{\bar{q}_n^1, \ldots, \bar{q}_n^k\} = d > 0$. To do this, we adapt the proof of Lemma 4 of Anderson (1981). Since $\mathcal{P}$ is compact in the topology of closed convergence, strong monotonicity implies equimonotonicity, which could replace the equiconvexity assumption. See Anderson (1987) for the definition of equimonotionicity. And $\mathcal{E}_n \to \mathcal{E}$ and $\int_{\mathcal{A}} e \, d\mu \gg 0$ imply that there is $\varepsilon > 0$ such that $\{a \in \mathcal{A}_n | e_n^i(a) > \varepsilon\} / n > \varepsilon$. Moreover, as long as $(1 - \xi)n \to 1$ as $n \to \infty$, by Lemma 1, $\bar{q}_n$ has the property required for the proof of Lemma 4 of Anderson (1981). Thus, all the hypotheses are satistfied and we get the desired result.

(2) Since $e_n(a) \in Q$, there is $K > 0$, such that $\| e_n(a) \| \leq K$. Lemma 1 (2) implies that for all $a \in \mathcal{A}_n$ and all n, $| \bar{q}_n(h_n(a) - e_n(a)) | \leq \sum_{a \in \mathcal{A}_n} |\bar{q}_n \cdot g_n(a) | \leq 2\sqrt{k}(k+1)K / (1 - \xi)$. Since $\bar{q}_n$ is uniformly bounded away from zero and $\| e_n(a) \| \leq K$, there is compact $H \subset \mathbb{R}_+^k$ such that $h_n(a) \in H$ for all n and $a \in \mathcal{A}_n$. Therefore by equisemismoothness of P, there exists $\rho > 0$ such that for all $h_n(a) \in \mathcal{P}$, there is $y_n(a) \in \mathbb{R}_+^k$ such that $\| h_n(a) - y_n(a) \| = \rho$ and if $\| z - y_n(a) \| < \rho$ for $z \in \mathbb{R}_+^k$, then $z >_a h_n(a)$. We can choose ρ such that $\rho \leq \delta$. Strong monotonicity, together with $y_n(a) \in \mathbb{R}_+^k$, implies $\nabla_n(a) = y_n(a) - h_n(a) > 0$. Since $\bar{q}_n$ is uniformly bounded away from zero, there is $\kappa > 0$ such that for all n and $a \in \mathcal{A}_n$, $\nabla_n(a) \cdot \bar{q}_n \geq \kappa$.

(3) We now prove the following claim.

Claim

Let $\theta = \delta\rho\kappa/\kappa^2$. Then, $\sum_{a \in S_n} \{x \mid x >_a h_n(a)\}$ contains the open ball B of radius $n\theta$ and center $\sum_{a \in S_n} h_n(a) + n\theta\bar{q}_n$.

Remark

This Claim is the generalized version of the simple geometric fact that the sum of the n sets each of which contains the ball of radius θ includes the ball of radius $n\theta$. Cheng (1981) and Mas-Colell (1985) did not prove this, but they could prove directly that the zero point is not in the ball B.

Proof of the Claim: Since $\{(S_n, h_n)\}$ is δ-uniformly linked, for a given (S_n, h_n), there is a linked collection $\mathcal{J} = \{J_1,\ldots, J_{k-1}\}$. Let unit vectors $v_i \in \mathbb{R}^k_+$, $i = 1,\ldots, k-1$ be such that for all i, $v_i \cdot \bar{q}_n = 0$ and $v_i^j \neq 0$ if and only if $j \in J_i$. Then $\{v_1,\ldots, v_{k-1}, \bar{q}_n\}$ constitutes a basis of $\mathbb{R}^k_+$.

Now we can partition S_n into k disjoint coalitions S_n^i, $i = 1,\ldots, k$, such that (i) for $i = 1,\ldots, k-1$, $h_n^j(a) > \delta$ for each $j \in J_i$ and $a \in S_n^i$, and (ii) for all i, $\#S_n^i \geq \delta n/k$. (i) implies that if $a \in S_n^i$, then $v_i \cdot \nabla_n(a) = 0$. Thus if $x = h_n(a) + \alpha v_i + \beta\nabla_n(a)$ and $\alpha^2 + (\rho - \beta)^2 < \rho^2$, then $x >_a h_n(a)$ for $a \in S_n^i$. From a simple geometric fact, we can infer that for $a \in S_n^i$, if $x = h_n(a) + \alpha v_i + \beta\bar{q}_n$ and $\alpha^2 + (\rho\kappa - \beta)^2 < (\rho\kappa)^2$, where κ is a lower bound of $\nabla_n(a) \cdot \bar{q}_n$, then $x >_a h_n(a)$.

Pick any z in B. Then there are α_i, $i = 1,\ldots, k-1$ such that, $z = \sum_{a\in S_n} h_n(a) + \Sigma\alpha_i v_i + (z \cdot \bar{q}_n)\bar{q}_n$. Since v_i could be chosen such that for all i, $\alpha_i \geq 0$, we can assume $|\alpha_i| \leq \|\Sigma\alpha_i v_i\|$ for all i. Define $\bar{h}_n(a) = h_n(a) + (1/\#S_n^i)\{\alpha_i v_i + [(z \cdot \bar{q}_n)/(k-1)]\bar{q}_n\}$, if $a \in S_n^i$, $i = 1,\ldots, k-1$. And $\bar{h}_n(a) = h_n(a)$, if $a \in S_n^k$. Then $z = \sum_{a\in S_n} \bar{h}_n(a)$. Now consider

$$t = \frac{\alpha^2}{(\#S_n^i)^2} + \left[\rho\kappa - \frac{z\cdot\bar{q}_n}{(k-1)\#S_n^i}\right]^2 - \rho^2\kappa^2.$$

Since $z \in B$, $\|\Sigma\alpha_i v_i\|^2 + (n\theta - z \cdot \bar{q}_n)^2 < n^2\theta^2$, or

$$\|\Sigma\,\alpha_i v_i\|^2 = \left[\frac{\delta\rho\kappa n}{k^2} - z\cdot\bar{q}_n\right]^2 < \left[\frac{\delta\rho\kappa n}{k^2}\right]^2.$$

Thus, $\alpha_i^2 \leq \|\Sigma\alpha_i v_i\|^2 < -(z \cdot \bar{q}_n)^2 + 2\delta\rho\kappa n(z \cdot \bar{q}_n)/k^2$.

Therefore,

$$t < \frac{1}{(\#S_n^i)^2}\cdot\left[-(z\cdot\bar{q}_n)^2 + \frac{2\delta\rho\kappa n}{k^2}(z\cdot\bar{q}_n)\right] + \left[\frac{z\cdot\bar{q}_n}{(k-1)\#S_n^i}\right]^2 - \frac{2\rho\kappa(z\cdot\bar{q}_n)}{(k-1)\#S_n^i}$$

$$\leq \frac{2z\cdot\bar{q}_n}{\#S_n^i}\cdot\left[\frac{\delta\rho\kappa n}{k^2\#S_n^i} - \frac{\rho\kappa}{k-1}\right] + \left[\frac{z\cdot\bar{q}_n}{\#S_n^i}\right]^2\left[\frac{1}{(k-1)^2} - 1\right] \leq 0,$$

since $z\cdot\bar{q}_n \geq 0$ and $\#S_n^i \geq \delta n/k$.

Therefore, $\bar{h}_n(a) >_a h_n(a)$ for $a \in S_n^i$, $i = 1, \ldots, k-1$. For $a \in S_n^k$, $\bar{h}_n(a) = h_n(a)$. This implies that the closure of $\sum_{a \in S_n} \{x \mid x >_a h_n(a)\}$ contains B, since preferences are continuous. Therefore, $\sum_{a \in S_n} \{x \mid x >_a h_n(a)\}$ contains B, and this proves the claim.

(4) $\sum_{a \in S_n} e_n(a) \notin \sum_{a \in S_n} \{x \mid x >_a h_n(a)\}$. If not, S_n would be a blocking coalition for f_n. And this is a contradiction to the fact that f_n is a core allocation. Hence,

$$\left\|\sum_{a \in S_n} h_n(a) + n\theta\bar{q}_n - \sum_{a \in S_n} e_n(a)\right\|^2 = \left\|g_n(S_n) + n\theta\bar{q}_n\right\|^2 \geq n^2\theta^2,$$

where $\theta = \delta\rho\kappa/k^2$.

Thus, $\| g_n(S_n) \|^2 + 2n\theta g_n(S_n)\bar{q}_n \geq 0$ or,

$$\beta(S_n) = g_n(S_n) \cdot \bar{q}_n \geq -\frac{\|g_n(S_n)\|^2}{2\theta_n}.$$

Let $M_1 = \| g_n(S_n) \|^2/2\theta > 0$. Then by Lemma 2, $M_1 = \frac{(k+1)^2K^2}{2\theta} \cdot [\frac{2\sqrt{k}\delta^{-1}}{d} + 1 + \sqrt{k}]^2 \cdot \frac{1}{n}$, since $\xi = 1 - \delta$ and $\min_k\{\bar{q}_n^1, \ldots, \bar{q}_n^k\} \geq d$. This concludes the proof of Lamma 3.

Q.E.D

***Proof of Theorem* 1:** Lemma 2 and 3 establish that for $n > N$ and $\xi = 1 - \delta$, $\alpha \geq -M_1/(1-\xi)n$. But it immediately follows from the proof of Theorem 1 of Anderson (1978) that with the gap minimizing price $\bar{p}$, $\sum_{a \in \mathcal{A}_n} \phi(\bar{p}, f, a) \leq -4\alpha$. The proof is completed by letting $M = 4M_1/(1-\xi)$.

Q.E.D.

If we assume smooth preferences, equisemismoothness can be dropped. Let $\mathcal{P}^2_{sc}$ denote the space of C^2, strongly convex preferences endowed with the topology of uniform C^2 convergence on compacta (Mas-Colell 1985, Definition 2.4.1).

Corollary 1

Let $\mathcal{E}_n$: $\mathcal{A}_n \to \mathcal{P}^2_{sc} \times Q$ and $\mathcal{E}_n \to \mathcal{E}$, where Q is a compact subset of $\mathbb{R}^k_+$. If every Walrasian allocation of $\mathcal{E}$ is linked, then for any $f_n \in C(\mathcal{E}_n)$, there exists $p_n \in \Delta$ such that $\phi(p_n, f_n) = O(1/n^2)$.

Proof: $\mathcal{P}^2_{sc} \times \mathbb{R}^k_+$ is separable and metrible (Mas-Colell 1985, p. 70). By definition of convergence $\mathcal{E}_n \to \mathcal{E}$, $\nu_n \to \nu$ weakly and $\{\nu_n, \nu\}$ for each n is tight because suppε is compact. Therefore, by a fact about tight measures (32) on page 49 in Hildenbrand (1974), $\{\nu, \nu_1, \nu_2, \ldots\}$ is tight. So there exists a compact set $P' \times Q \subset \mathcal{P}^2_{sc} \times \mathbb{R}^k_+$ such that $\nu_n(P' \times Q) > 1 - \delta/2$ for all n. Since preferences are smooth, the compact set P' is equisemismooth (see Anderson 1987, p. 6). And P' contains the preferences of the sufficient fraction of the agents: For all n, $\#\{a \in \mathcal{A}_n \mid \succeq_a \in P'\} > 1 - \delta/2$. Therefore, we can work with P' instead of P to get the same result, in (2) and (3) of step 2 in the proof of Lemma 3, The rest of the proof of the Theorem 1 applies without any modification.

Q.E.D.

Convexity of preferences of the limit economy was used in Theorem 1 to ensure that for large n, h_n be close to a Walrasian allocation of the limit economy. But, even without convexity, h_n could be converging to a Walrasian allocation, if we choose ξ increasing to 1, as n goes to infinity. This enables us to have an almost quadratic convergence rate when we drop the convexity assumption.

Theorem 2

Let $\mathcal{E}_n$: $\mathcal{A}_n \rightarrow P \times Q$ converge to a continuum economy $\mathcal{E}$: $[0, 1] \rightarrow P \times Q$.Assume $P \subset \mathcal{P}$ is equisemismooth and $Q \subset \mathbb{R}^k_+$ is compact. If every Walrasian allocation is linked, then for any t which is an increasing function of n such that $t(n) \rightarrow \infty$, and for any $f_n \in C(\mathcal{E}_n)$, there is $p_n \in \Delta$ such that $\phi(p_n, f_n) = O(\frac{t(n)}{n^2})$.

Remark

An example of such $t(n)$ is log n. It is easy to see that for any $\varepsilon > 0$, log $n/n^\varepsilon \rightarrow 0$ as $n \rightarrow 0$. Therefore, $\phi(p_n, f_n) = O(1/n^{2-\varepsilon})$ for any small $\varepsilon > 0$. Hence, the convergence rate is almost quadratic.

***Proof*:** Define an increasing function $r(n)$ such that $r(n) \geq 1$ and $r(n) \rightarrow \infty$, but $r(n)/n \rightarrow 0$ as $n \rightarrow \infty$. Let $\xi = 1 - \delta/4r(n)$. Define $h'_n(a) = h_n(a)$ if $a \in S_n$, and 0 otherwise. Then $\#\{a \in \mathcal{A}_n \mid h'_n(a) \neq h_n(a)\}/n \leq 1 - \xi + (k + 1)/n = \delta/4r(n) + (k + 1)/n \rightarrow 0$ as $n \rightarrow \infty$. The proof of the Theorem 1 can be used with the following modifications:

(1) Even if ξ is increasing, Lemma 1 still implies that $(1/n) \sum_{a \in \mathcal{A}_n} | \bar{q}_n \cdot [f_n(a) - e_n(a)] | \rightarrow 0$, and $(1/n) \sum_{a \in \mathcal{A}_n} | \bar{q}_n \cdot g_n(a) | \rightarrow 0$ as $n \rightarrow \infty$. Thus the conclusion of Lemma 4 of Anderson (1981) still holds: $\bar{q}_n$ is uniformly bounded away from zero.

(2) $h_n(a)$ may not be uniformly bounded. But it follows from $(1/n) \sum_{a \in \mathcal{A}_n} | \bar{q}_n \cdot g_n(a) | \rightarrow 0$ as $n \rightarrow \infty$ that $h_n(a)$ is uniformly integrable (see Hildenbrand 1974, p. 182 for the definition) and so for large enough n, $\#\{a \in \mathcal{A}_n \mid h_n(a) \notin H\}/n \leq \delta/8k$ for some compact $H \subset \mathbb{R}^k_+$. Obiously,

the same is true for $h'_n(a)$.

(3) Therefore, we can use the Theorem 1 of Hildenbrand (1974) here also for h'_n. In addition, the proof of Theorem 3.4 of Anderson (1987) shows that there exist $0 \leq \lambda_a \leq 1$ for $a \in \mathcal{A}_n \setminus S_n$ such that $\sum_{a \in S_n} g_n(a) + \sum_{a \in \mathcal{A}_n \setminus S_n} \lambda_a g_n(a) \ll 0$. But $\lambda_a g_n(a) \geq -e_n(a)$. Therefore, $\sum_{a \in \mathcal{A}_n} [h'_n(a) - e_n(a)] = \sum_{a \in \mathcal{A}_n \setminus S_n} e_n(a) \ll 0$. Let h' be the limit of a subsequence of $\{h'_n\}$. Then $\int h'\, d\mu \leq \int e\, d\mu$. But $\int h'\, d\mu < \int e\, d\mu$, since the limit of the corresponding subsequence of $\{\bar{q}_n\}$ is an equilibrium price and preferences are strongly monotonic. Thus, we can conclude that h' is a Walrasian allocation of $\mathcal{E}$. By the same argument as in the proof of Lemma 3, for large enough n, $\{(S_n, h'_n)\}_{n>N}$ is $\delta/4$-uniformly linked.

(4) In Step 2 of the proof of Lemma 3, the only modification needed is that each S_n^i now contains only $\delta n/8k$ agents with $h_n(a) \in H$. H denotes a compact subset of $\mathbb{R}^k_+$ obtained in (2) above. Then $\theta = \delta\rho\kappa/8k^2$. Therefore, considering $1 - \xi = \delta/4r(n)$, we get:

$$\phi(p_n, f_n) \leq 8\delta^{-1}\theta^{-1}(k+1)^2 K^2 [1 + d^{-1} + \sqrt{k} + 8\sqrt{k}\delta^{-1}d^{-1}r(n)]^2 r(n)/n^2.$$

Thus $\phi(p_n, f_n) = O\left(\frac{r(n)^3}{n^2}\right)$. By the definition of $r(n)$, we get the desired result.

Q.E.D.

We may take another direction to relax the convexity assumption. Let T be a compact subset of $\mathcal{P} \times \mathbb{R}^k_+$ and E denote the set of all continuum economies $\mathcal{E}$: $[0, 1] \to T$. Since a continuum economy $\mathcal{E}$ can be characterized by the associated distribution of characteristics, which is a measure on T, E can be viewed as the set of all such measures. Endowed with the topology of weak convergence of measures (since T is compact, $v_n \to v$ weakly implies $\int e_n d\mu \to \int e\, d\mu$), E is compact (Hildenbrand 1974, D(30), p. 49).

Now define $\Delta_n = \{p \in \Delta \mid \text{ for all } i, p^i \geq 1/n\}$ and $\Delta^\circ_n = \{p \in \Delta \mid \text{ for all } i, p^i > 1/n\}$ for $n = 1, 2, \ldots$. Let $E_n = \{\mathcal{E} \in E \mid \Pi(E) \subset \Delta^\circ_n\}$. Then E_n is open in E and $E = \cup_n E_n$, since $\Pi(\mathcal{E})$ is a compact subset of Δ° and $\Pi(\cdot)$ is u.h.c. (Hildenbrand 1974, Proposition 4, p. 152). Now consider the set $D_\varepsilon = \{\mathcal{E} \in E \mid \text{diam} \int \varphi(\mathcal{E}(a), p) d\mu < \varepsilon \text{ for all } p \in (\mathcal{E})\}$ for $\varepsilon > 0$. It can be rewritten as the union of the sets, $\{\mathcal{E} \in E_n \mid \text{diam} \int \varphi(\mathcal{E}(a), p) d\mu < \varepsilon$ for all $p \in \Pi(\mathcal{E})\}$, each of which is an open and dense subset of E_n, because of the following observation by Mas-Colell and Neuefeind (1977, Remark 3, p. 597): For any compact $K \subset \Delta^\circ$ and $\varepsilon > 0$, the set $\{\mathcal{E} \in E \mid \text{diam} \int \varphi(E(a), p) d\mu < \varepsilon$ for all $p \in K\}$ is open and dense in E. Therefore

D_ε is an open and dense subset of E. Using this fact, we now can prove that the rate of convergence is "generically" quadratic in non-convex economies. First, we prove two lemmas.

***Lemma* 4**

For any $\delta > 0$, there is an open and dense set D'_δ such that if $\mathcal{E} \in D'_\delta$, then $\mu\{a \in [0, 1] \mid \text{diam}\varphi(\mathcal{E}(a), p) < \delta \text{ for all } p \in \Pi(\mathcal{E})\} > 1 - \delta$.

Proof: For simplicity, let $\psi(a)$ denote $\varphi(\mathcal{E}(a), p)$. Obviously, for any $q \in \mathbb{R}^k_+$, $\sup\{q \cdot z \mid z \in \int \psi\} + \sup\{-q \cdot z \mid z \in \int \psi\} \le \sup_{q \in \mathbb{R}^k}[\sup\{q \cdot z \mid z \in \int \psi\} + \sup\{-q \cdot z \mid z \in \int \psi\} = \text{diam} \int \psi$. But, for any $q \in \mathbb{R}^k$, $\sup\{q \cdot z \mid z \in \int \psi\} = \int \sup\{q \cdot x \mid x \in \int \psi(\cdot)\}$ (Hildenbrand 1974, Proposition 6, p. 63). So, for some $q \in \mathbb{R}^k$, $\sup\{q \cdot z \mid z \in \int \psi\} + \sup\{-q \cdot z \mid z \in \int \psi\} = \int \sup_q[\sup\{q \cdot x \mid x \in \int \psi(\cdot)\} + \sup\{-q \cdot x \mid x \in \psi(\cdot)\}] = \int \text{diam}\psi$. Therefore, $\int \text{diam}\psi(\mathcal{E}(a), p)d\mu \le \text{diam} \int \psi$. If $\mathcal{E} \in D_\varepsilon$ and $p \in \Pi(\mathcal{E})$, then, $\int \text{diam}\varphi(\mathcal{E}(a), p)d\mu \le \text{diam} \int \varphi(\mathcal{E}(a), p)d\mu < \varepsilon$. Therefore for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $\mathcal{E} \in D_\varepsilon$, $\mu\{a \in [0, 1] \mid \text{diam}\varphi(\mathcal{E}(a), p) < \delta \text{ for all } p \in \Pi(\mathcal{E})\} > 1 - \delta$. The proof is done by letting $D'_\delta, \in D_\varepsilon$, because D_ε is open and dense.

Q.E.D.

***Lemma* 5**

Let E^n denote the set $\{\mathcal{E} \in E \mid W(\mathcal{E}) \text{ is } 1/n\text{-uniformly linked}\}$ for $n = 1, 2, \ldots$. Then E^n is open in E.

Proof: Openness of E^n follows from the following facts. Firstly, from the definition of δ-linkedness, it is obvious that the set $\{\mathcal{D}f \in M \mid f \text{ is a } 1/n\text{-linked allocation}\}$ is open in M, where M denotes the set of all measures on $\mathbb{R}^k_+$ with the topology of weak convergence. Secondly, the proof of Proposition 1 implies that E^n is equal to the set of the economies $\mathcal{E}$ for which the set of allocations $\{f \mid \mathcal{D}f \in \overline{\mathcal{D}W(\mathcal{E})}\}$ is $1/n$-uniformly linked. Finally, $\overline{\mathcal{D}W(\cdot)}$ is u.h.c. (Hildenbrand 1974, Theorem 3, p. 159).

Q.E.D.

***Theorem* 3**

Let $\mathcal{E}_n$: $\mathcal{A}_n \to P \times Q$, where $P \subset \mathcal{P}$ is equisemismooth and $Q \subset \mathbb{R}^k_+$ is compact. Then, there is an open and dense set $D \subset E$ such that if the sequence $\{\mathcal{E}_n\}$ converges to $\mathcal{E} \in D$, and every Walrasian allocation of E is linked, then for any $f_n \in C(\mathcal{E})$ there is $p_n \in \Delta$ such that $\phi(p_n, f_n) = O(1/n^2)$.

Proof: Let E^n be as defined in Lemma 5 and E^∞ denote the interior of $E\backslash$

$(\cup_n E_n)$. Then $E = (\cup_n E^n) \cup \bar{E}^\circ$. Define $D = (\cup_n (E^n \cap D'_{1/2n})) \cup E^\circ$. Since E^n $(n = 0, 1, 2, \ldots)$ is open and $D'_{1/2n} (n = 1, 2, \ldots)$ is open and dense in E, D is open and dense in E. Now assume that $\mathcal{E} \in D$ and every Walrasian allocation is linked. Then, $W(\mathcal{E})$ is $1/n$-uniformly linked for some n. To avoid confusion, we let $1/n = 8\delta$. Since $\mathcal{E} \in D$, $\mathcal{E} \in D'_{4\delta}$. Therefore, for any allocation h such that $h(a) \in \varphi(\mathcal{E}(a), p)$ for some $p \in \Pi(\mathcal{E})$ and for a.e. $a \in [0, 1]$, there exists a Walrasian allocation f such that $\mu\{a \in [0, 1] \mid \| h(a) - f(a) \| < 4\delta\} > 1 - 4\delta$. Thus there is a linked collection $\mathcal{J}$ such that $\mu\{a \in [0, 1] \mid h^j_n(a) > 4\delta \text{ for each } j \in J\} > 4\delta$ for all $J \in \mathcal{J}$. But it was proved in step 1 of the proof of Lemma 3, that $\{h_n\}$ defined there has a subsequence converging to such an allocation h. Therefore, if $\mathcal{E}_n \to \mathcal{E} \in D$, then we can find $N > 0$ such that $\{h_n\}_{n>N}$ is 2δ-uniformly linked. And this is the only modification needed of the proof of Theorem 1.

Q.E.D.

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