The worst one-parameter subgroups for unstable cubic plane curves

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The worst one-parameter subgroups for unstable cubic plane curves

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Abstract

Algebraic geometry is a branch of mathematics, studying zero sets of multivariate polynomials. It is mainly based on the use of commutative algebra for solving geometrical problems about these sets of zeros. One of geometrical problems is a moduli problem which is the classification of geometric objects up to some notion of equivalence.

Geometric invariant theory is a method for constructing quotients by group actions in algebraic geometry, used to construct moduli spaces. It was advanced by David Mumford using ideas from David Hilbert in invariant theory.

I focus on the numerical criterion for stability in geometric invariant theory. For a linear action of a reductive group on a projective scheme, we get a numerical criterion called the Hilbert-Mumford criterion which can be used to determine stability of a point on that projective scheme.

In this thesis, I focus on the stability for a linear action of the special linear group of degree 3 over the complex field on the vector space of cubic plane curves. By the Hilbert-Mumford criterion, unstable cubic plane curves are plane algebraic curves whose zero set is a nonzero linear combination of $x^3$, $x^2y$, $xy^2$, $y^3$, and $y^2z$.

George Kempf solved the Mumford’s conjecture which is finding one-parameter subgroups such that our given unstable point is unstable with respect to them. Using this idea, I completely compute the worst one-parameter subgroups for unstable cubic plane curves like the following.

First, there are infinitely many unstable cubic plane curves, but I reduce them for the five cases up to projective equivalence. This is based on the classification of unstable cubic plane curves.

Second, I compute the five cases by the elementary theorem in the linear programming. The basic principle of it is based on the optimization theory and duality.

Finally, I check the worst one-parameter subgroups for unstable cubic plane curves by using the Hilbert-Mumford criterion. This procedure ensures
my work.

**Key words:** Geometric invariant theory, numerical criterion for stability, the worst one-parameter subgroups

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Chapter 1

Introduction and preliminaries

In order to solve our problem, we start with the concepts of moduli space. Especially, we focus on the moduli space of curves. Then we explain the basic concepts and properties of geometric invariant theory for applying our problem. After that, we revisit the concept of the worst one-parameter subgroup in Kempf’s theorem([3]), and introduce the crucial technique for solving our problem in Hyeon’s result([4]).

1.1 Moduli space

Mathematicians study the classification of mathematical objects under certain conditions. The Poincaré conjecture is the typical problem of the classification. Moduli spaces came from the classification problem.

Roughly speaking, a moduli space is an algebraic variety whose points are in a one-to-one correspondence with the set of isomorphism classes of algebra-geometric objects. The basic example is the moduli of elliptic curves. Before introducing a moduli space, we need an example of plane conics in algebraic geometry.

Example 1.1.1. How to find all possible plane conics? A plane conic is $Z(h) = \{(x, y) \in \mathbb{R}^2 | h(x, y) = 0\}$ where $h(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ for $a, b, c, d, e, f \in \mathbb{R}$. Since there are 6 scalars, one may think there are a lot of plane conics. However, note that a nonzero scalar multiple gives the
same plane conic. Hence, the space of plane conics \( M_C = \{(a, b, c, d, e, f) \in \mathbb{R}^6 | a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0 \}/\sim \) is called the moduli space of plane conics where \( \sim \) is an equivalence relation of a nonzero scalar multiple.

We must do the following two things. First, we make a category of objects, such as schemes, sheaves or morphisms, together with a notion of what it means to have a family of these objects over a scheme \( B \). Second, we take an equivalence relation \( \sim \) on the set \( S(B) \) of all such objects over each \( B \). See more details in [5], [12].

Hence, we start this idea by defining a functor \( F \) from the category of schemes to sets by the rule \( F(B) = S(B)/\sim \) and call \( F \) the moduli functor of our moduli problem.

**Definition 1.1.2.** ([12]) Let \( M : \text{Sch} \to \text{Set} \) be a moduli functor. Then a scheme \( M \) is a fine moduli space for \( M \) if it represents \( M \).

A fine moduli space is desirable, but it does not always exist and it is difficult to construct. See details in [5]. Hence, mathematicians develop a weaker notion of this such as following.

**Definition 1.1.3.** ([12]) A coarse moduli space for a moduli functor \( M \) is a scheme \( M \) and a natural transformation of functors \( \eta : M \to h_M \) such that \( \eta_{\text{Speck}} : M(\text{Speck}) \to h_M(\text{Speck}) \) is bijective and for any scheme \( N \) and natural transformation \( \nu : M \to h_N \), there exists a unique morphism of schemes \( f : M \to N \) such that \( \nu = h_f \circ \eta \) where \( h_f : h_M \to h_N \) is the corresponding natural transformation of presheaves.

In this paper, we focus on a moduli space of curves which is a geometric space whose points represent isomorphism classes of algebraic curves. This space \( M_g \) can be constructed by many techniques such as the Teichmüller approach, the Hodge theory approach, and the GIT approach. We concentrate on the GIT approach.

### 1.2 Geometric invariant theory

Invariant theory has been researched for a long time. After a lot of study, they knew that the invariant under choice of coordinates was really the invariance
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under an action of a group. Thus the general problem of invariant theory is
like the following: Given a "nice" action of a group $G$ as automorphisms of
the polynomial ring $A = k[x_1, \cdots, x_n]$, find the elements of $A$ that are left
invariant by $G$. See for details in [8]. Here is a famous example by Gauss.

**Example 1.2.1.** ([8]) Let $A = k[x_1, \cdots, x_n]$ be the polynomial ring. Let the
permutation group $S_n$ acts on $A$ like the following: If $\sigma \in S_n$ and $f \in A$, then
we define $\sigma(f)(x_1, \cdots, x_n) = f(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$.

Then the set of invariants $A^{S_n} := \{f \in A| \sigma(f) = f\}$ is called the ring
of symmetric functions which is a subring of $A$. It contains the elementary
symmetric functions

$$
\begin{align*}
  f_1(x_1, \cdots, x_n) &= x_1 + \cdots + x_n \\
  f_2(x_1, \cdots, x_n) &= \sum_{1 \leq i < j \leq n} x_i x_j \\
  \cdots \\
  f_n(x_1, \cdots, x_n) &= x_1 \cdots x_n
\end{align*}
$$

Then $A^{S_n}$ is generated as a $k$-algebra by $f_1, \cdots, f_n$.

In general, let $G$ be a group and $V$ be a vector space over a field $k$ which
is a representation space of $G$ and $k[V]$ be a ring of polynomial functions on
$V$. There exists an induced $G$-action on $k[V]$. We say $f \in k[V]$ is $G$-invariant
if $f(\sigma \cdot v) = f(v)$ for every $\sigma \in G$. Hilbert gave his 14th problem like the
following: Can we find finitely many generators $f_1, \cdots, f_n$ such that $k[V]^G =
 k[f_1, \cdots, f_n]$? See details in [8] and [10].

Actually, the answer is no by Nagata. However, Hilbert proved that if $G$
is a linearly reductive group, then $k[V]^G$ is finitely generated. See details in
[8].

From now on, we focus on a general group action. Let a group $G$ act on

a set $X$. Then the quotient space $X/G$ is called the set of orbits $G$ in $X$, and
we have a natural projection map $\pi : X \to X/G$ by sending $x \in X$ to its orbit
$Gx \in X/G$.

One might ask whether if $X$ has some property, then so is $X/G$. If $X$ is

a topological space, then so is $X/G$ by its quotient topology. Furthermore,
if $X$ is compact or connected, then so is $X/G$. Hence, we could expect this
property in our algebraic-geometric settings.
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Now, we change our situation a little bit. Assume that $X$ is an algebraic variety over $k$ and $G$ acts on $X$ by polynomial maps. One might expect that $X/G$ could be an algebraic variety and $\pi$ is a regular map of algebraic varieties. However, this does not hold always. Here is a typical example.

**Example 1.2.2.** Let $G = \mathbb{C}^*$, $X = \mathbb{C}$ and $G$ acts on $X$ by multiplication. Then $G/X$ has only two elements which consists of the orbit of 0 and the orbit of nonzero elements. Here, 0 is in the closure of the set of nonzero elements in $\mathbb{C}$. Note that any finite algebraic set has the discrete topology, where every point is closed. Therefore, $X/G$ must not be an algebraic variety.

Naturally, we need other notion of quotients for observing the construction. Let $G$ be an affine algebraic group acting on a scheme $X$ over $k$. By the above phenomena, we ask a universal quotient in the category of schemes.

**Definition 1.2.3.** ([10]) A categorical quotient for the action of $G$ on $X$ is a $G$-invariant morphism $\phi : X \to Y$ of schemes which is universal; that is, every other $G$-invariant morphism $f : X \to Z$ factors uniquely through $\phi$ so that there exists a unique morphism $h : Y \to Z$ such that $f = \phi \circ h$.

However, this categorical quotient cannot also admit an algebraic scheme structure. Hence, our problem is like the following: How can we construct $X/G$ by an algebraic scheme? In [5], David Mumford showed that for a reductive group $G$ acting on a quasi-projective scheme $X$, one can construct an open subvariety $U \subset X$ and a categorical quotient $U//G$ of the $G$-action on $U$ which is a quasi-projective scheme. We briefly explain this like the below.

From now on, let $G$ be a reductive algebraic group acting on an affine scheme $X = \text{Spec} A$ of finite type over algebraically closed field $k$. See [2] for definitions and properties of algebraic groups and [5], [7] for definitions and properties of schemes. Note that this group action induces an action of $G$ on the coordinate ring $O(X)$, which is finitely generated $k$-algebra.

**Definition 1.2.4.** ([10]) The affine GIT quotient is the morphism $\phi : X \to X//G := \text{Spec} O(X)^G$ of affine schemes associated to the inclusion $\phi^* : O(X)^G \hookrightarrow O(X)$.

Now, we get $X//G$ is an affine scheme. In particular, if $O(X)^G$ is finitely generated as a $k$-algebra, then the associated affine scheme $\text{Spec} O(X)^G$ is
also of finite type over $k$ and the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ induces a morphism $X \to X//G := \text{Spec} \mathcal{O}(X)^G$, which is a categorical quotient of the $G$-action on $X$. The affine GIT quotient $X \to X//G$ identifies any orbits whose closures meet, but restricts to an orbit space on an open subscheme of so-called stable points. See more details in [5]. Now, how about the projective case?

Let $k$ be an algebraically closed field, $G$ be a reductive group over $k$ acting on a projective scheme $X \subset \mathbb{P}^n$. Then the action of $G$ on $\mathbb{P}^n$ lifts to an action of $G$ on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$, i.e., $O(\mathbb{A}^{n+1}) = \bigoplus_{r \geq 0} k[x_0, \cdots, x_n]_r = \bigoplus_{r \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(r))$ and if $X \subset \mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X) \subset k[x_0, \cdots, x_n]$, then $\tilde{X} = \text{Spec} R(X) \subset \mathbb{A}^{n+1}$ where $R(X) = k[x_0, \cdots, x_n]/I(X)$.

Since the $G$-action on $\mathbb{A}^{n+1}$ is linear, it preserves the graded pieces. Hence, we get $O(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} k[x_0, \cdots, x_n]_r^G$. Similarly, $R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G$.

Define the semistable set $X^{ss} = X - N$ to be the open subset of $X$ given by the complement to the nullcone $N$ where $N$ is the closed subscheme of $X$ defined by the homogeneous ideal $R(X)^G := \bigoplus_{r \geq 0} R(X)_r^G$ in $R(X)$. Then we call the morphisms $X^{ss} \to X//G := \text{Proj} R(X)^G$ the GIT quotient of this action.

How can we define the GIT quotient for the general scheme $X$ of finite type over $k$? We can define it by using a linearization of the action of $G$ with respect to an ample line bundle $L$. See more details in [5] and [10]. Mumford proved the below theorem.

**Theorem 1.2.5.** ([10]) Let $G$ be a reductive group acting on a quasi-projective scheme $X$ and $L$ be a linearization of this action. Then there is a quasi-projective scheme $X//_L G$ and a good quotient $\phi : X^{ss}(L) \to X//_L G$ of the $G$-action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\phi^{-1}(Y^s) = X^s(L)$ and $\phi : X^s(L) \to Y^s$ is a geometric quotient for the $G$-action on $X^s(L)$.

Now, we focus on the numerical criterion of stability due to David Hilbert and David Mumford. See details in [5], [9], [10] and [13]. Let $k$ be an algebraically closed field, $G$ be a reductive group over $k$ acting on a projective scheme $X \subset \mathbb{P}^n$. We can study the closure of an orbit by using one-parameter subgroups of $G$. 
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**Definition 1.2.6.** A one-parameter subgroup of $G$ is a nontrivial homomorphism of algebraic groups $\lambda: G_m \to G$.

**Definition 1.2.7.** ([10]) Let $G$ be an affine algebraic group scheme over $k$. Then $G$ is a torus if $G \cong G_m^n$ for some $n > 0$. A maximal torus of $G$ is a torus $T \subset G$ which is not contained in any other torus.

**Remark 1.2.8.** For a torus $G$, we have commutative groups $X^*(T) := \text{Hom}(T, G_m)$ and $X_*(T) := \text{Hom}(G_m, T)$ which called the character group and cocharacter group respectively. Note that $\mathbb{Z} \cong X^*(G_m)$. Hence, we deduce that the (co)character groups are finite free $\mathbb{Z}$-modules of rank $\text{dim} T$ for a general torus $T$. Hence, there is a perfect pairing $\langle , \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ where $\langle \chi, \lambda \rangle := \chi \circ \lambda$. See more details in [5], [9] and [10].

We can pick a basis $e_0, \cdots, e_n$ of $k^{n+1}$ such that

$$\lambda(t) \cdot e_i = t^{r_i}e_i \text{ for } r_i \in \mathbb{Z}$$

since we use the blow up property and $\lambda$ is diagonalizable. See details in [10]. Now, we can define the Hilbert-Mumford weight of $\lambda$ which is crucial for the numerical criterion of stability.

**Definition 1.2.9.** ([10]) We define the Hilbert-Mumford weight of $x \in X$ at $\lambda$ to be

$$\mu(x, \lambda) := -\min\{r_i | x_i \neq 0\}.$$ 

Here is the Hilbert-Mumford criterion by using one-parameter subgroups of $G$.

**Theorem 1.2.10.** (Hilbert-Mumford criterion) ([5]) Let $G$ be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X(k)$, we have

- $x \in X^s \iff \mu(x, \lambda) > 0$ for all one-parameter subgroups $\lambda$ of $G$,
- $x \in X^{ss} \iff \mu(x, \lambda) \geq 0$ for all one-parameter subgroups $\lambda$ of $G$,
- $x \in X^{us} \iff \mu(x, \lambda) < 0$ for some one-parameter subgroups $\lambda$ of $G$. 


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1.3 Instability in invariant theory

In the previous part, we introduced the numerical criterion of stability in general settings by using GIT. From now on, let $k$ be an algebraically closed field, $G$ be a reductive group over $k$ and $V$ be an $n$-dimensional representation of $G$. The interesting examples are when $G = SL(n; \mathbb{C})$, $GL(n; \mathbb{C})$.

David Mumford had a natural question like the following: For given an unstable point $v \in V$, find a natural class $\Lambda_v$ of one-parameter subgroups $\lambda$ of $G$ such that $x$ is $\lambda$-unstable. It is called the Mumford’s conjecture. In [3], George Kempf found its solution. Before introducing it, we need the weight decomposition of $V$.

**Proposition 1.3.1.** ([10]) For a finite dimensional linear representation of a torus $\rho : T \to GL(V)$, there is a weight decomposition

$$V \cong \bigoplus_{\chi \in X^*(T)} V^\chi$$

where $V^\chi = \{ v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \in T \}$ are called the weight spaces and $\Xi := \{ \chi \mid V^\chi \neq 0 \}$ are called the weights of the action.

Note that $\text{Im}(\lambda)$ is a torus of $G$. He applied $\text{Im}(\lambda)$ by the weight decomposition for solving the Mumford’s conjecture. Then we define the notion of the worst one-parameter subgroup of $\lambda$ by the below theorem.

**Theorem 1.3.2.** ([3]) Let $v$ be a non-zero element of $V$. Let $B(v) = \sup -\frac{\mu(v, \lambda)}{\|\lambda\|}$ for all one-parameter subgroups $\lambda$ of $G$. Then there exists at least one $\lambda_0$ such that $B(v) = -\frac{\mu(v, \lambda_0)}{\|\lambda_0\|}$.

The idea of this theorem is like the following: Let $M$ be a finite dimensional real vector space with positive definite inner product $(\ , \ )$ and $\|m\| = (m, m)^{1/2}$ for $m \in M$. Let $S$ be the set of $m \in M$ satisfying $\|m\| = 1$ and $F$ be a finite set of real-valued linear functions on $M$.

Now, we define $h(m)$ by the minimum value $f(m)$ for all $f \in F$ for each point $m \in M$. In [3], Kempf proved that the following lemma.

**Lemma 1.3.3.** ([3]) Assume that the function $h$ actually has a positive value somewhere in $M$. Let $U = \{ m \in M \mid h(m) > 0 \}$. Then there is a unique point $p_h$ of $S \cap U$, where $h$ obtains a maximum value. In fact, $p_h$ is the only point of $S \cap U$, where $h$ has a relative (local) maximum value.
Kempf used this lemma and some elementary properties of $\lambda$, $\mu(v, \lambda)$ for proving his theorem. See more details in [3].

Kempf completely solved the Mumford’s conjecture, but he only showed the existence of the worst one-parameter subgroup, that is, it is not easy to find explicitly in general. In the next part, we find the worst one-parameter subgroups for our situation.

### 1.4 The Worst one-parameter subgroups for unstable cubic plane curves

Now, we focus on our problem. By the previous parts, we know the existence of the moduli of cubic plane curves by GIT. From now on, let $G = \text{SL}(3; \mathbb{C})$ and $V = \mathbb{P}(H^0(O_{P^2}(+3)))$ which is the vector space of cubic plane curves. Hence, we find the worst one-parameter subgroups for the case $\mathbb{P}(H^0(O_{P^2}(+3)))/\text{SL}(3; \mathbb{C})$.

We use the Hilbert-Mumford criterion for picking unstable cubic plane curves out. In [13], we get the idea which explains $f \in V$ is unstable if and only if in some coordinates, all non-zero coefficients of $f$ lie to one side of a line passing through $xyz$ like the below picture.

![Diagram of cubic plane curve coefficients]

After the change of coordinates, we may assume that $f$ is a linear combination of $x^3$, $x^2y$, $xy^2$, $y^3$ and $y^2z$. Now, we state our main theorem.

**Theorem 1.4.1.** Let $f = ax^3 + bx^2y + cxy^2 + dy^3 + ey^2z$ be an unstable cubic plane curve in $V$ for $a, b, c, d, e \in \mathbb{C}$ where the coefficients are not
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all 0. Then the weight vector for the worst one-parameter subgroup of \( f \) is \((1, 4, -5)\) if \( V(f) \) is an irreducible cuspidal, \((0, 1, -1)\) if \( V(f) \) is a conic and a tangent line, \((1, 1, -2)\) if \( V(f) \) is a three concurrent lines, \((1, 0, -1)\) if \( V(f) \) is a double line and a line, and \((2, -1, -1)\) if \( V(f) \) is a triple line.

Our plan attack is like the following:

First, we show that for a given unstable cubic plane curve \( f \), \( f \) is projectively equivalent to one and only one of \( x^3 + y^2z \) if \( V(f) \) is an irreducible cuspidal, \( x^2y + y^2z \) if \( V(f) \) is a conic and a tangent line, \( x^3 + (a + 1)x^2y + axy^2 \) for some \( a \neq 0 \) and \( a \neq 1 \) if \( V(f) \) is a three concurrent lines, \( x^2y \) if \( V(f) \) is a double line and a line, and \( x^3 \) if \( V(f) \) is a triple line.

This work is tedious a little bit. The idea is separating case by case and using projectively equivalence of cubic plane curves.

Second, we show that Newton polytopes of the each above polynomials is the farthest polynomial for the each cases. We use the idea of this in [4].

Let \( P \) denote the convex hull of \( \Xi = \{\chi_1, \cdots, \chi_s\} \) which is a finite subset of \( V \) and fix an isomorphism between \( V \) and \( V^* \). Then the following two problems are equivalent.

(A) Find \( \min \{||\chi||_|| \chi \in P || \} \);

(B) Find \( \max \{g(\lambda)|||\lambda|^* \leq 1\} \), where \( g(\lambda) = \min_{\chi \in \Xi} <\chi, \lambda> \).

In order to find the worst one-parameter subgroup of \( \lambda \), we use the below theorem via the observation of the equivalence between (A) and (B).

**Theorem 1.4.2.** Suppose that \( \{\lambda|g(\lambda) \geq 0\} \) is nonempty. If \( \chi_0 \) is the solution for the first problem and \( \chi_0 \neq 0 \), then the solution for the second is given by \( \chi_0 ||\chi_0|| \) (via the chosen isomorphism \( V \simeq V^* \)).

By the theorem, we can easily find the worst one-parameter subgroups for each cases by picking the farthest Newton polytope.

Lastly, we just compute the worst one-parameter subgroups for each cases. This can be done by the instability in invariant theory and the elementary mathematics.
Chapter 2

Moduli space

A moduli space is a geometric space whose points represent algebro-geometric objects of fixed kind, or isomorphism classes of such objects. The j-invariant classifying elliptic curves up to isomorphism is a typical example. In this section, we use some of the language of category theory.

2.1 Category theory

Recall that a morphism of categories \( C \) and \( D \) is given by a (covariant) functor \( F : C \to D \), which associates to every object \( C \in C \) an object \( F(C) \in D \) and to each morphism \( f : C \to C' \) in \( C \) a morphism \( F(f) : F(C) \to F(C') \) in \( D \) such that \( F \) preserves identity morphisms and composition. A contravariant functor \( F : C \to D \) reverses arrows such that \( F \) sends \( f : C \to C' \) in \( C \) a morphism \( F(f) : F(C') \to F(C) \) in \( D \).

A (covariant) functor \( F : C \to D \) induces a function \( F_{C,C'} : \text{Mor}_C(C, C') \to \text{Mor}_D(F(C), F(C')) \) for every pair of objects \( C, C' \in C \). Then \( F \) is said to be faithful if \( F_{C,C'} \) is injective, full if \( F_{C,C'} \) is surjective, and fully faithful if \( F_{C,C'} \) is bijective for each \( C, C' \in C \).

For (covariant) functors \( F, G : C \to D \), recall a morphism of \( F \) and \( G \) as a natural transformation \( \eta : F \to G \) which associates to every object \( C \in C \) a morphism \( \eta_C : F(C) \to G(C) \) in \( D \), we have a commutative diagram
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\[
\begin{array}{ccc}
F(C) & \xrightarrow{\eta_C} & G(C) \\
\downarrow F(f) & & \downarrow G(f) \\
F(C') & \xrightarrow{\eta_{C'}} & G(C').
\end{array}
\]

Lemma 2.1.1. (Yoneda’s Lemma) Let \( C \) be any category. Then for any \( C \in C \) and any presheaf \( F \in \text{Psh}(C) \), there exists a bijection

\[
\{ \text{natural transformations } \eta : h_C \to F \} \leftrightarrow F(C)
\]

which sends \( \eta \) to \( \eta_C(id_C) \).

Proof. Let us check surjectivity. For an object \( s \in F(C) \), define a natural transformation \( \eta(s) : h_C \to F \) as follows. For \( C' \in C \), let \( \eta_{C'} : h_{C'}(C') \to F(C') \) be the morphism of sets which sends \( f : C' \to C \) to \( F(f)(s) \). This is compatible with morphisms and \( \eta_C(id_C) = F(id_C)(s) = s \).

Now, let us check injectivity. Suppose that there are two natural transformations \( \eta, \eta' : h_C \to F \) such that \( \eta_C(id_C) = \eta'_C(id_C) \). Let \( g : C' \to C \).

Since \( \eta \) is a natural transformation, we have a commutative diagram

\[
\begin{array}{ccc}
h_C(C) & \xrightarrow{\eta_C} & F(C) \\
\downarrow h_C(g) & & \downarrow F(g) \\
h_{C'}(C) & \xrightarrow{\eta_{C'}} & F(C').
\end{array}
\]

Then we have \( (F(g) \circ \eta_C)(id_C) = (\eta_{C'} \circ h_{C'}(g))(id_C) = \eta_{C'}(g) \). Similarly, since \( \eta' \) is a natural transformation, we have \( (F(g) \circ \eta'_C)(id_C) = \eta'_C(g) \). Hence, \( \eta_{C'}(g) = (F(g) \circ \eta_C)(id_C) = F(g)(\eta_C(id_C)) = F(g)(\eta'_C(id_C)) = (F(g) \circ \eta'_C)(id_C) = \eta'_{C'}(g) \) as required.

Definition 2.1.2. Let \( C \) be a category and \( \text{Set} \) be the category of sets. A functor \( F : C \to \text{Set} \) is said to be representable if \( F \) is naturally isomorphic to \( \text{Hom}(A, -) \) for some object \( A \in C \).
2.2 Fine moduli spaces

As mentioned earlier, a moduli space is a geometric space whose points represent algebro-geometric objects of fixed kind, or isomorphism classes of such objects. Hence, a moduli problem is essentially a classification problem.

**Definition 2.2.1.** A (naive) moduli problem is a collection $A$ of objects and an equivalence relation $\sim$ on $A$.

**Example 2.2.2.** Let $A$ be the set of $k$-dimensional linear subspaces of an $n$-dimensional vector space and $\sim$ be equality. Then $(A, \sim)$ is a moduli problem.

A moduli problem defines a functor $M \in \mathbf{Psh}(\mathbf{Sch})$ given by $M(S) := \{\text{family over } S\}/\sim_S$ and $M(f : T \to S) = f^* : M(S) \to M(T)$. We will often refer to a moduli problem simply by its moduli functor.

**Definition 2.2.3.** Let $M : \mathbf{Sch} \to \mathbf{Set}$ be a moduli functor. Then a scheme $M$ is a fine moduli space for $M$ if it represents $M$.

Let’s see this definition. Since $M$ is representable by $M$, there exists a natural transformation $\eta : M \to h_M$. Therefore, for every scheme $S$, we have a bijection $\eta_S : M(S) := \{\text{family over } S\}/\sim_S \leftrightarrow h_M(S) := \{\text{morphisms } S \to M\}$.

**Remark 2.2.4.** Not every moduli functor has a fine moduli space.

**Definition 2.2.5.** Let $M$ be a fine moduli space for $M$. Then the family $U \in M(M)$ corresponding to the identity morphisms on $M$ is called the universal family.

2.3 Coarse moduli spaces

By the above remark, we need a weaker notion of fine moduli spaces. This is why coarse moduli spaces come out.

**Definition 2.3.1.** A coarse moduli space for a moduli functor $M$ is a scheme $M$ and a natural transformation of functors $\eta : M \to h_M$ such that $\eta_{\text{Spec}} : M(\text{Spec}) \to h_M(\text{Spec})$ is bijective and for any scheme $N$ and natural
transformation $\nu : M \to hN$, there exists a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$ where $h_f : hM \to hN$ is the corresponding natural transformation of presheaves.

In this paper, we focus on a moduli space of curves which is a geometric space whose points represent isomorphism classes of algebraic curves. This space $M_g$ can be constructed by many techniques such as the Teichmüller approach, the Hodge theory approach, and the GIT approach. We concentrate on the GIT approach.
Chapter 3

Geometric invariant theory

In this section, we start with algebraic groups and their quotient groups. Then we study affine geometric invariant theory and projective geometric invariant theory. Also, we use the numerical criterion for stability which is a crucial technique for geometric invariant theory.

3.1 Algebraic group actions and quotients

Definition 3.1.1. An algebraic group over \( k \) is a scheme \( G \) over \( k \) with morphisms \( e : \text{Spec}(k) \to G \) (identity element), \( m : G \times G \to G \) (group law) and \( i : G \to G \) (group inversion) such that we have commutative diagrams

\[
\begin{array}{ccc}
G \times G \times G & \xrightarrow{id \times m} & G \times G \\
\downarrow{m \times id} & & \downarrow{m} \\
G \times G & \xrightarrow{m} & G
\end{array}
\]

\[
\begin{array}{ccc}
\text{Spec}(k) \times G & \xrightarrow{e \times id} & G \times G \\
\downarrow{\cong} & & \downarrow{\cong} \\
G & \xleftarrow{m} & G \\
& \xleftarrow{id \times e} & G \times \text{Spec}(k)
\end{array}
\]
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We say that $G$ is an affine algebraic group if the underlying scheme $G$ is affine. We say $G$ is a group variety if the underlying scheme $G$ is a variety.

**Example 3.1.2.** The additive group $G_m = \text{Spec}(k[t])$ over $k$ is the algebraic group whose underlying variety is the affine line $\mathbb{A}_1$ over $k$ and whose group structure is given by addition. Also, the general linear group $GL_n$ over $k$ is the algebraic group over $k$ as the usual matrix multiplication and matrix inversion.

**Definition 3.1.3.** An algebraic group is linearly reductive if every finite dimensional representation is semisimple, i.e., a sum of simple representations.

**Example 3.1.4.** $SL(3; \mathbb{C})$ is a reductive algebraic group.

**Definition 3.1.5.** An (algebraic) action of an affine algebraic group $G$ on a scheme $X$ is a morphism of schemes $\sigma : G \times X \to X$ such that the following diagram commute

$$
\begin{array}{ccc}
\text{Spec}(k) \times X & \xrightarrow{\text{id}_X} & G \times X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & G \times X
\end{array}
$$

Suppose we have actions $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$ of an affine algebraic group $G$ on schemes $X$ and $Y$. Then a morphism $f : X \to Y$ is $G$-equivariant if the following diagram commutes

$$
\begin{array}{ccc}
G \times X & \xrightarrow{id_G \times f} & G \times Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma_X} & Y
\end{array}
$$

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If $Y$ is given the trivial action $\sigma_Y = \pi_Y : G \times Y \to Y$, then we refer to a $G$-equivariant morphism $f : X \to Y$ as a $G$-invariant morphism.

Now, we study quotients of algebraic groups.

**Definition 3.1.6.** A categorical quotient for the action of $G$ on $X$ is a $G$-invariant morphism $\phi : X \to Y$ of schemes which is universal; that is, every other $G$-invariant morphism $f : X \to Z$ factors uniquely through $\phi$ so that there exists a unique morphism $h : Y \to Z$ such that $f = \phi \circ h$.

**Remark 3.1.7.** Let $\phi : X \to Y$ be a categorical quotient of a $G$-action on $X$. Then $\phi$ preserves connectedness, irreducibility.

However, this categorical quotient cannot also admit an algebraic scheme structure. Hence, our problem is like the following: How can we construct $X/G$ by an algebraic scheme?

Now, let $G$ be an affine algebraic group acting on a scheme $X$ over $k$. The group $G$ acts on the $k$-algebra $O(X)$ of regular functions on $X$ by $g \cdot f(x) = f(g^{-1} \cdot x)$ and we denote the subalgebra of invariant functions by $O(X)^G := \{ f \in O(X) : g \cdot f = f \text{ for all } g \in G \}$.

**Definition 3.1.8.** A morphism $\phi : X \to Y$ is a good quotient for the action of $G$ on $X$ if
1) $\phi$ is $G$-invariant,
2) $\phi$ is surjective,
3) If $U \subset Y$ is an open subset, then the morphism $O_Y(U) \to O_X(\phi^{-1}(U))$ is an isomorphism onto the $G$-invariant functions $O_X(\phi^{-1}(U))^G$,
4) If $W \subset X$ is a $G$-invariant closed subset of $X$, then its image $\phi(W)$ is closed in $Y$,
5) If $W_1$ and $W_2$ are disjoint $G$-invariant closed subsets, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint,
6) $\phi$ is affine.
If moreover, the preimage of each point is a single orbit then we say $\phi$ is a geometric quotient.

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3.2 Affine geometric invariant theory

Now, we study affine geometric invariant theory based on algebraic group actions and quotients. This can be extended for projective geometric invariant theory which we want to use for our main problem.

**Definition 3.2.1.** The affine GIT quotient is the morphism \( \phi : X \to X//G := \text{Spec} \, O(X)^G \) of affine schemes associated to the inclusion \( \phi^* : O(X)^G \hookrightarrow O(X) \).

**Remark 3.2.2.** The double slash notation \( X//G \) used for the GIT quotient is a reminder that this quotient is not necessarily an orbit space and so it may identify some orbits.

However, is \( X//G \) an affine scheme? The following theorem says yes in the particular case.

**Theorem 3.2.3.** Let \( G \) be a reductive group acting on an affine scheme \( X \). Then the affine GIT quotient \( \phi : X \to X//G \) is a good quotient and, moreover, \( X//G \) is an affine scheme.

Now, we get \( X//G \) is an affine scheme. In particular, if \( O(X)^G \) is finitely generated as a \( k \)-algebra, then the associated affine scheme \( \text{Spec} \, O(X)^G \) is also of finite type over \( k \) and the inclusion \( O(X)^G \hookrightarrow O(X) \) induces a morphism \( X \to X//G := \text{Spec} \, O(X)^G \), which is a categorical quotient of the \( G \)-action on \( X \).

**Definition 3.2.4.** We say \( x \in X \) is stable if its orbit is closed in \( X \) and \( \dim G_x = 0 \). We let \( X^s \) denote the set of stable points.

**Proposition 3.2.5.** Suppose a reductive group \( G \) acts on an affine scheme \( X \) and let \( \phi : X \to Y := X//G \) be the affine GIT quotient. Then \( X^s \subset X \) is an open and \( G \)-invariant subset, \( Y^s := \phi(X^s) \) is an open subset of \( Y \) and \( X^s = \phi^{-1}(Y^s) \).

The affine GIT quotient \( X \to X//G \) identifies any orbits whose closures meet, but restricts to an orbit space on an open subscheme of so-called stable points. See more details in [5]. Now, how about the projective case?
3.3 Projective geometric invariant theory

The basic idea is that we want to construct our projective GIT quotient by gluing affine GIT quotients. In order to do this we wish to cover our scheme $X$ by affine open subsets which are invariant under the group action and glue the affine GIT quotients of these affine open subsets of $X$.

**Definition 3.3.1.** Let $X$ be a projective scheme with an action of an affine algebraic group $G$. A linear $G$-equivariant projective embedding of $X$ is a group homomorphism $G \to \text{GL}_{n+1}$ and a $G$-equivariant projective embedding $X \to \mathbb{P}^n$. We will simply say that the $G$-action on $X \to \mathbb{P}^n$ is linear to mean that we have a linear $G$-equivariant embedding of $X$ as above.

Suppose we have a linear action of a reductive group $G$ on a projective scheme $X \subset \mathbb{P}^n$. Then the action of $G$ on $\mathbb{P}^n$ lifts to an action of $G$ on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is $G$-equivariant, $O(\mathbb{A}^{n+1}) = \bigoplus_{r \geq 0} \mathbb{k}[x_0, \cdots, x_n]_r = \bigoplus_{r \geq 0} H^0(\mathbb{P}^n, O_X(r))$ and if $X \subset \mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X) \subset \mathbb{k}[x_0, \cdots, x_n]$, then $\tilde{X} = \text{Spec} R(X) \subset \mathbb{A}^{n+1}$ where $R(X) = \mathbb{k}[x_0, \cdots, x_n]/I(X)$.

Since the $G$-action on $\mathbb{A}^{n+1}$ is linear, it preserves the graded pieces. Hence, we get $O(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} \mathbb{k}[x_0, \cdots, x_n]_r^G$. Similarly, $R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G$. By Nagata’s theorem, $R(X)^G$ is finitely generated since $G$ is reductive. The inclusion of finitely generated graded $k$-algebras $R(X)^G \hookrightarrow R(X)$ determines a rational morphism of projective schemes $X \dashrightarrow \text{Proj} R(X)^G$ whose indeterminacy locus is the closed subscheme of $X$ defined by the homogeneous ideal $R(X)_+^G = \bigoplus_{r \geq 0} R(X)_r^G$.

**Definition 3.3.2.** For a linear action of a reductive group $G$ on a projective scheme $X \subset \mathbb{P}^n$, we define the nullcone $N$ to be the closed subscheme of $X$ defined by the homogeneous ideal $R(X)_+^G$ in $R(X)$. We define the semistable set $X^{ss} = X - N$ to be the open subset of $X$ given by the complement to the nullcone $N$. Then we call the morphisms $X^{ss} \to X//G := \text{Proj} R(X)^G$ the GIT quotient of this action. More precisely, $x \in X$ is semistable if there exits a $G$-invariant homogeneous function $f \in R(X)_r^G$ for $r > 0$ such that $f(x) \neq 0$. By construction, the semistable set is the open subset which is the domain of definition of the rational map $X \dashrightarrow \text{Proj} R(X)^G$. We call the morphisms $X^{ss} \dashrightarrow X//G := \text{Proj} R(X)^G$ the GIT quotient of this action.
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**Theorem 3.3.3.** For a linear action of a reductive group $G$ on a projective scheme $X \subset \mathbb{P}^n$, the GIT quotient $\phi : X^{ss} \to X//G$ is a good quotient of the $G$-action on the open subset $X^{ss}$ of semistable points in $X$. Furthermore, $X//G$ is a projective scheme.

How can we define the GIT quotient for the general scheme $X$ of finite type over $k$? We can define it by using a linearization of the action of $G$ with respect to an ample line bundle $L$. See more details in [5] and [10]. Mumford proved the below theorem.

**Theorem 3.3.4.** Let $G$ be a reductive group acting on a quasi-projective scheme $X$ and $L$ be a linearization of this action. Then there is a quasi-projective scheme $X//L G$ and a good quotient $\phi : X^{ss}(L) \to X//L G$ of the $G$-action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X//L G$ such that $\phi^{-1}(Y^s) = X^s(L)$ and $\phi : X^s(L) \to Y^s$ is a geometric quotient for the $G$-action on $X^s(L)$.

### 3.4 Numerical criterion for stability

We use numerical criterion for stability for some particular cases. Using it, we can easily determine stable points, semistable points, and unstable points.

**Definition 3.4.1.** A linear representation of an algebraic group $G$ on a vector space $V$ over $k$ is a homomorphism of group valued functors $\rho : G \to \text{GL}(V)$. If $V$ is a finite dimensional, this is equivalent to a homomorphism of algebraic groups $\rho : G \to \text{GL}(V)$, which we call a finite dimensional linear representation of $G$.

**Definition 3.4.2.** A one-parameter subgroup of $G$ is a nontrivial homomorphism of algebraic groups $\lambda : \mathbb{G}_m \to G$.

**Definition 3.4.3.** Let $G$ be an affine algebraic group scheme over $k$. Then $G$ is a torus if $G \cong \mathbb{G}_m^n$ for some $n>0$. A maximal torus of $G$ is a torus $T \subset G$ which is not contained in any other torus.

**Remark 3.4.4.** For a torus $G$, we have commutative groups $X^*(T) := \text{Hom}(T, \mathbb{G}_m)$ and $X_*(T) := \text{Hom}(\mathbb{G}_m, T)$ which called the character group and cocharacter group respectively. Note that $\mathbb{Z} \cong X^*(\mathbb{G}_m)$. Hence, we deduce that the
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(co)character groups are finite free \( \mathbb{Z} \)-modules of rank \( \dim T \) for a general torus \( T \). Hence, there is a perfect pairing \( \langle , \rangle : X^*(T) \times X_*(T) \to \mathbb{Z} \) where \( \langle \chi, \lambda \rangle := \chi \circ \lambda \). See more details in [5], [9] and [10].

We can pick a basis \( e_0, \ldots, e_n \) of \( k^{n+1} \) such that

\[
\lambda(t) \cdot e_i = t^{r_i}e_i \text{ for } r_i \in \mathbb{Z}
\]

since we use the blow up property and \( \lambda \) is diagonalizable. See details in [10]. Now, we can define the Hilbert-Mumford weight of \( \lambda \) which is crucial for the numerical criterion of stability.

**Definition 3.4.5.** We define the Hilbert-Mumford weight of \( x \in X \) at \( \lambda \) to be

\[
\mu(x, \lambda) := -\min\{r_i | x_i \neq 0\}.
\]

Here is the Hilbert-Mumford criterion by using one-parameter subgroups of \( G \).

**Theorem 3.4.6.** (Hilbert-Mumford criterion) Let \( G \) be a reductive group acting linearly on a projective scheme \( X \subset \mathbb{P}^n \). Then, for \( x \in X(k) \), we have

\[
x \in X^s \iff \mu(x, \lambda) > 0 \text{ for all one-parameter subgroups } \lambda \text{ of } G,
\]

\[
x \in X^{ss} \iff \mu(x, \lambda) \geq 0 \text{ for all one-parameter subgroups } \lambda \text{ of } G,
\]

\[
x \in X^{us} \iff \mu(x, \lambda) < 0 \text{ for some one-parameter subgroups } \lambda \text{ of } G.
\]
Chapter 4

Instability in invariant theory

In this section, we review the Kempf’s paper and the section 4 of the Hyeon’s paper. Throughout this section, let $G$ be a reductive group over an algebraically closed field $k$ of characteristic zero and $V$ be an $n$-dimensional representation space of $G$.

Note that a one-parameter subgroup $\lambda$ of $G$ is diagonalizable since $G$ is reductive. Hence, we may assume that $\lambda(t) = \begin{bmatrix} t^{r_1} & 0 \\ \vdots \\ 0 & t^{r_n} \end{bmatrix}$ be a one-parameter subgroup from $G_m$ to $G$.

Note that $\text{Im}(\lambda)$ is a torus of $G$. Let $\chi$ be a character of the image of $\lambda$. Then we have $\chi(\lambda(t)) = t^{\chi(\lambda)}$ for all $t$ in $G_m$ where $\chi(\lambda)$ is the integer defined by the formula. Hence, we can apply the weight space decomposition as in the introduction.

$$V \cong \bigoplus_{\chi \in X^*(\text{Im}(\lambda))} V^\chi$$

where $V^\chi = \{v \in V | \lambda(t) \cdot v = t^{\chi(\lambda)}v \text{ for all } t \in G_m\}$.

Thus, the characters of $\text{Im}(\lambda)$ and, hence, its eigenspaces in $V$ are linearly ordered by the integer $\chi(\lambda)$. Therefore, the Hilbert-Mumford weight of $v$ at $\lambda$ is $\mu(v, \lambda) = -\min(\langle \chi, \lambda \rangle | \chi \in \Xi)$. Note that $\mu(v, \lambda) = \mu(g \cdot v, g\lambda g^{-1})$.
for any element $g$ of $G$. By the Hilbert-Mumford criterion, $v \in V$ is unstable if and only if $\mu(v, \lambda) < 0$ for some one-parameter subgroup $\lambda$ of $G$.

Recall that the one-parameter subgroups of a torus $T$ are a free abelian group of rank equal the dimension of $T$. We can define a notion of length $\| \lambda \|$ to any one-parameter subgroup $\lambda$ of $G$ such that

1. $\| g\lambda g^{-1} \| = \| \lambda \|$ for all $g$ in $G$ and any one parameter subgroup $\lambda$ of $G$, and

2. for any maximal torus $T$ of $G$, there is a bilinear positive definite integral-valued bilinear form $(\ , \ )$ on the group of one-parameter subgroups $\lambda$ of $T$ such that $(\lambda, \lambda)^{1/2} = \| \lambda \|$.

Let $M$ be a finite dimensional real vector space with positive definite inner product $(\ , \ )$ and $\| m \| = (m, m)^{1/2}$ for $m \in M$. Let $S$ be the set of $m \in M$ satisfying $\| m \| = 1$ and $F$ be a finite set of real-valued linear functions on $M$.

Now, we define $h(m)$ by the minimum value $f(m)$ for all $f \in F$ for each point $m \in M$. For understanding the notion of the worst one-parameter subgroups, we need the following lemma.

**Lemma 4.0.1.** Assume that the function $h$ actually has a positive value somewhere in $M$. Let $U = \{ m \in M | h(m) > 0 \}$. Then there is a unique point $p_h$ of $S \cap U$, where $h$ obtains a maximum value. In fact, $p_h$ is the only point of $S \cap U$, where $h$ has a relative (local) maximum value.

Now, the below theorem gives the notion for the worst one-parameter subgroups.

**Theorem 4.0.2.** Let $v$ be a non-zero element of $V$. Let $B(v) = \sup -\frac{\mu(v, \lambda)}{\| \lambda \|}$ for all one-parameter subgroups $\lambda$ of $G$. Then there exists at least one $\lambda_0$ such that $B(v) = -\frac{\mu(v, \lambda_0)}{\| \lambda_0 \|}$.

**Proof.** By the conjugacy properties of $\mu$ and $\| \cdot \|$, it is enough to show that there exists a one-parameter subgroup $\lambda_0$ of a fixed maximal torus $T$ such that

$$-\frac{\mu(gv, \lambda_0)}{\| \lambda_0 \|} \geq -\frac{\mu(gv, \lambda)}{\| \lambda \|}$$

for all $g \in G$ and $\lambda \in X_*(T)$. 

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Let \( R \) be the state of a vector \( v' \in V \) with respect to \( T \). For any \( \lambda \in X_*(T) \), we have \( -\mu(v',\lambda) = \min \langle \chi, \lambda \rangle \) for \( \chi \in R \). Since \( \langle \chi, \lambda \rangle \) are integral-valued linear functions of \( \lambda \in X_*(T) \), we may apply the above lemma to find a one-parameter subgroup \( \lambda_R \in X_*(T) \) such that

\[
-\frac{\mu(v',\lambda_R)}{\|\lambda_R\|} \geq -\frac{\mu(v',\lambda)}{\|\lambda\|}
\]

for any \( \lambda \in X_*(T) \).

Note that this theorem just gives the existence of the worst one-parameter subgroup. For finding explicitly it, we need some technique.

From the remaining part of this section, we review the section 4 of the Hyeon’s paper. Let \( V \) be a finite dimensional vector space and \( (, ) \) be a positive definite real valued symmetric bilinear form on \( V \). Let \( \Xi = \{\chi_1, \cdots, \chi_s\} \) be a finite subset of \( V \) and \( P \) denote its convex hull. Let \( |\chi| = (\chi, \chi)^{\frac{1}{2}} \) denote the associated norm and \( |\cdot|^* \), the dual norm on \( V^* \) : \( |\lambda|^* = \max \{\langle \chi, \lambda \rangle | |\chi| \leq 1 \} \). Choose a suitable basis (and its dual) with respect to which the pairing \( \langle \chi, \lambda \rangle \) is the matrix multiplication \( \chi^\top \lambda \). We also use the basis to fix an isomorphism between \( V \) and \( V^* \). The duality is between the following two problems :

(A) Find \( \min \{ |\chi| | \chi \in P \} \);

(B) Find \( \max \{g(\lambda) | |\lambda|^* \leq 1 \} \), where \( g(\lambda) = \min_{\chi \in \Xi} \langle \chi, \lambda \rangle \).

Hence, we get the following theorem.

**Theorem 4.0.3.** Suppose that \( \{ \lambda | g(\lambda) \geq 0 \} \) is nonempty. If \( \chi_0 \) is the solution for the first problem and \( \chi_0 \neq 0 \), then the solution for the second is given by \( \frac{\chi_0}{|\chi_0|} \) (via the chosen isomorphism \( V \simeq V^* \)).

Hence, this theorem is the core idea for finding explicitly the worst one-parameter subgroups.
Chapter 5

The Worst one-parameter subgroups for unstable cubic plane curves

In this section, we calculate the worst one-parameter subgroups for unstable cubic plane curves with the previous sections. Throughout this section, we concentrate on $G = \text{SL}(3; \mathbb{C})$, $V = \mathbb{P}(\mathcal{H}^0(\mathcal{O}_{P^2}(+3)))$ which is the vector space of cubic plane curves.

Indeed, we must check whether $G$ is a reductive algebraic group or not and $V$ is a finite dimensional representation of $G$ or not.

Now, let us define a group action. Let $f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x^{i_1} y^{i_2} z^{i_3} \in V$ for $a_{i_1i_2i_3} \in \mathbb{C}$ and

$$
\lambda(t) = \begin{bmatrix}
  t^{r_1} & 0 & 0 \\
  0 & t^{r_2} & 0 \\
  0 & 0 & t^{r_3}
\end{bmatrix}
$$

be the one-parameter subgroup from $G_m$ to $G$ where $r_1 + r_2 + r_3 = 0$ and $r_1, r_2, r_3 \in \mathbb{Z}$.

Then, our group action is like the following:

$$
\lambda(t)f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{r_1i_1+r_2i_2+r_3i_3} x^{i_1} y^{i_2} z^{i_3}.
$$

Now, we must check whether our action is a group action.
Proposition 5.0.1. The above group action is well-defined.

Proof. For all \( f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x_i^y y_i^z z_i^3 \in V \) where \( a_{i_1i_2i_3} \in \mathbb{C} \),
\[
I_3 f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{0i_1+0i_2+0i_3} x_i^y y_i^z z_i^3 = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x_i^y y_i^z z_i^3 = f.
\]

For all
\[
\lambda_1(t) = \begin{bmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{r_2} & 0 \\ 0 & 0 & t^{r_3} \end{bmatrix} \quad \text{and} \quad \lambda_2(t) = \begin{bmatrix} t^{s_1} & 0 & 0 \\ 0 & t^{s_2} & 0 \\ 0 & 0 & t^{s_3} \end{bmatrix} \in G
\]
where \( r_1 + r_2 + r_3 = 0 \), \( s_1 + s_2 + s_3 = 0 \) and \( r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z} \), all \( f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x_i^y y_i^z z_i^3 \in V \) where \( a_{i_1i_2i_3} \in \mathbb{C} \),
\[
(\lambda_1 \lambda_2(t)) f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{(r_1+s_1)i_1+r_2i_2+r_3i_3} x_i^y y_i^z z_i^3 = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{r_1i_1+r_2i_2+r_3i_3} x_i^y y_i^z z_i^3 = \lambda_1 (\lambda_2(t) f).
\]
Hence, our group action is well-defined. \( \square \)

Therefore, we verified that \( V \) is a finite dimensional representation of \( G \). Note that \( G \) is a reductive algebraic group since it is linearly reductive. Hence, we can use the Hilbert-Mumford criterion.

In order to apply the Kempf’s theorem, we must calculate \( \mu(f, \lambda) \).
Let \( f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x_i^y y_i^z z_i^3 \in V \) for \( a_{i_1i_2i_3} \in \mathbb{C} \) and
\[
\lambda(t) = \begin{bmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{r_2} & 0 \\ 0 & 0 & t^{r_3} \end{bmatrix}
\]
be the one-parameter subgroup from \( G_m \) to \( G \) where \( r_1 + r_2 + r_3 = 0 \) and \( r_1, r_2, r_3 \in \mathbb{Z} \).
Let \( G \) act on \( V \) like the above. Then \( V^x = \{ f \in V : \lambda(t) \cdot f = t^{<\chi, \lambda>} f \ \text{for all} \ t \in \mathbb{C}^* \} \).
Let's observe that \( \lambda(t) \cdot f = t^{<\chi, \lambda>} f \) for all \( t \in \mathbb{C}^* \). Hence, we get
\[
\sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{r_1i_1+r_2i_2+r_3i_3} x_i^y y_i^z z_i^3 = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} t^{<\chi, \lambda>} x_i^y y_i^z z_i^3
\]
for all \( t \in \mathbb{C}^* \).
Therefore, if \( a_{i_1i_2i_3} \neq 0 \), then \( r_1i_1 + r_2i_2 + r_3i_3 = \langle \chi, \lambda \rangle \).

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This means that $V^\chi$ is generated by $x^{r_1}y^{r_2}z^{r_3}$ where $r_1i_1 + r_2i_2 + r_3i_3 = < \chi, \lambda >$. By convention, we denote this $\chi$ by $\chi(i_1,i_2,i_3)$.

Hence, $\mu(f, \lambda) = -\min(\chi(i_1,i_2,i_3), \lambda)$ where $f = \sum_{i_1+i_2+i_3=3} a_{i_1i_2i_3} x^{i_1}y^{i_2}z^{i_3} \in V$ for $a_{i_1i_2i_3} \in \mathbb{C}$ and $a_{i_1i_2i_3} \neq 0$.

**Example 5.0.2.** Let $f = x^3 + y^2z$ and

$$\lambda(t) = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-1} \end{bmatrix}$$

Then the state $\Xi(f)$ of $f$ with respect to $\text{Im}(\lambda)$ is the set of $\chi(3,0,0)$ and $\chi(0,2,1)$. Hence, $\mu(f, \lambda) = -\min((3,0,0) \cdot (2,-1,-1), (0,2,1) \cdot (2,-1,-1)) = 3$.

Now we want to apply the theorem in the Hyeon’s paper. Before doing this, we need a definition.

**Definition 5.0.3.** Let $f \in \mathbb{C}[x_1, \cdots, x_n]$, and write

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha.$$ 

The Newton polytope of $f$, denoted by $\text{NP}(f)$, is the lattice polytope

$$\text{NP}(f) = \text{the convex hull of } \{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}.$$ 

**Example 5.0.4.** Any polynomial of the form $f = axy + bx^2 + cy^5 + d$ with $a, b, c, d \neq 0$ has Newton polytope equal to the triangle $Q = \text{Conv}((1,1),(2,0),(0,5),(0,0))$.

In the section 2, $f$ is unstable if and only if $\mu(f, \lambda) < 0$ for some one-parameter subgroup $\lambda$. This means that all points of $\text{NP}(f)$ and $\lambda$ must be positively intersected. Therefore, we get the below proposition.

**Proposition 5.0.5.** A polynomial $f \in V$ is unstable if and only if in some coordinates, $\text{NP}(f)$ lies to one side of some line $L$ passing through $(1,1,1)$.

By the above proposition, for any unstable cubic plane curve $f$, we may assume that $f$ is a linear combination of $x^3, x^2y, xy^2, y^3, y^2z$ because of the below picture.
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\[
\begin{align*}
&\bullet x^3 & \bullet x^2y \\
\frac{x^2y}{x^2z} & \bullet xy^2 & \bullet xz^2 \\
\frac{xy}{xz} & \bullet y^3 & \bullet y^2z \\
\frac{y^2}{z^3} & \bullet \quad \bullet & \bullet \quad \bullet \\
\end{align*}
\]

Now, we can apply the duality theorem of the section 2 and compute the worst one-parameter subgroups by the nearest point of the Newton polytope. Since \(G = SL(3; \mathbb{C})\), we only need to find the direction for \(\lambda\), i.e., we don’t care about the length of \(\lambda\).

Now, here is the main theorem.

**Theorem 5.0.6.** Let \(f = ax^3 + bx^2y + cxy^2 + dy^3 + ey^2z\) be an unstable cubic plane curve in \(V\) for \(a, b, c, d, e \in \mathbb{C}\) where the coefficients are not all 0. Then the weight vector for the worst one-parameter subgroup of \(f\) is (1, 4, -5) if \(V(f)\) is an irreducible cuspidal, (0, 1, -1) if \(V(f)\) is a conic and a tangent line, (1, 1, -2) if \(V(f)\) is a three concurrent lines, (1, 0, -1) if \(V(f)\) is a double line and a line, and (2, -1, -1) if \(V(f)\) is a triple line.

**Proof.** Note that since \(f\) has degree 3, \(V(f)\) is one of an irreducible cuspidal, a conic and a tangent line, a three concurrent lines, a double line and a line, and a triple line.

1. First, I show that \(f\) is projectively equivalent to one and only one of \(x^3 + y^2z\) if \(V(f)\) is an irreducible cuspidal, \(x^2y + y^2z\) if \(V(f)\) is a conic and a tangent line, \(x^3 + (a + 1)x^2y + axy^2\) for some \(a \neq 0\) and \(a \neq 1\) if \(V(f)\) is a three concurrent lines, \(x^2y\) if \(V(f)\) is a double line and a line, and \(x^3\) if \(V(f)\) is a triple line.

If \(e = 0\), then the curve is reducible. It must be the union of three lines meeting \((0, 0, 1)\). Hence \(V(f)\) must be isomorphic to one of a three concurrent lines, a double line and a line, and a triple line.

Assume that \(V(f)\) is a three concurrent lines. Then we may assume that \(f = (ax + by)(cx + dy)(ex + gy)\) for some \(a, b, c, d, e, g \in \mathbb{C}\) which \(a : b\)
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\( \neq c : d, a : b \neq e : g, c : d \neq e : g \). In this case, we may assume that \( f = y(ax + by)(cx + dy) \) for some \( a, b, c, d \in \mathbb{C} \) which \( a : b \neq c : d \), and \( a, b, c, d \neq 0 \) by scaling. If \( e \neq 0 \), then by replacing \( x \) by \( x - \frac{a}{e}y \), we get \( f = e(ax - \frac{a}{e}ay + by)(cx - \frac{a}{e}cy + dy)x \). Since \( a : b \neq c : d \), we get \( a : -\frac{a}{e}a + b \neq c : -\frac{a}{e}c + d \). Hence, we may assume that \( f = x(ax + by)(cx + dy) \) for some \( a, b, c, d \in \mathbb{C} \) which \( a : b \neq c : d \), and \( a, b, c, d \neq 0 \) by scaling.

The above both cases are the same by replacing \( y \) by \( x \). Therefore, we may assume that \( f = x(ax + by)(cx + dy) \) for some \( a, b, c, d \in \mathbb{C} \) which \( a : b \neq c : d \), and \( a, b, c, d \neq 0 \). Since \( a, b, c, d \neq 0 \), we may assume that \( f = x(x + ay)(x + by) \) for some \( a, b \in \mathbb{C} \) which \( a \neq b \), and \( a, b \neq 0 \) by scaling again.

Then, by scaling again, we may assume that \( f = x(x + y)(x + ay) = x^3 + (a + 1)x^2y + axy^2 \) for some \( a \in \mathbb{C} \) which \( a \neq 0 \) and \( a \neq 1 \).

Assume that \( V(f) \) is a double line and a line. Then we may assume that \( f = (ax + by)^2(cx + dy) \) for some \( a, b, c, d \in \mathbb{C} \) which \( a : b \neq c : d \), \( a, b \) are not both 0 and \( c, d \) are not both 0. Replacing \( x \) by \( \frac{1}{ad - bc}(dx - by) \) and \( y \) by \( \frac{1}{ad - bc}(-cx + ay) \), we get \( f = x^2y \).

Assume that \( V(f) \) is a triple line. Then we may assume that \( f = (ax + by)^3 \) for some \( a, b \in \mathbb{C} \) which they are not both 0. If \( a = 0 \), then \( b \neq 0 \). Replacing \( y \) by \( \frac{x}{b} \), we get \( f = x^3 \). If \( a \neq 0 \), then by replacing \( x \) by \( \frac{1}{a}(x - by) \), we get \( f = x^3 \). By the above arguments, we may assume that \( f = x^3 \).

Now, assume that \( e \neq 0 \). Replacing \( z \) by \( \frac{1}{e}(z - cx - dy) \), we get \( f = ax^3 + bx^2y + y^2z \).

Assume that \( a = 0 \). Then \( f = bx^2y + y^2z = y(bx^2 + yz) \). If \( b = 0 \), then \( f = y^2z \), so \( f \) is projectively equivalent to \( x^2y \). In this case, \( V(f) \) is a double line and a line. If \( b \neq 0 \), then by replacing \( x \) by \( \frac{1}{b}x \), we get \( f = x^2y + y^2z \). In this case, \( V(f) \) is a conic and a tangent line.

Assume that \( a \neq 0 \). Then by scaling, we may assume that \( f = x^3 + bx^2y + y^2z \). Replacing \( x \) by \( x - \frac{1}{b}by \), we get \( f = x^3 - \frac{1}{b}bxy^2 + \frac{2}{b}b^2y^3 + y^2z \). Again, replacing \( z \) by \( \frac{1}{b}bx - \frac{3}{2}b^2y + z \), we get \( f = x^3 + y^2z \). In this case, \( V(f) \) is an irreducible cuspidal.

By the above arguments, we classified that \( f \) is projectively equivalent to one and only one of \( x^3 + y^2z \) if \( V(f) \) is an irreducible cuspidal, \( x^2y + y^2z \) if
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\( V(f) \) is a conic and a tangent line, \( x^3 + (a + 1)x^2y + axy^2 \) for some \( a \neq 0 \) and \( a \neq 1 \) if \( V(f) \) is a three concurrent lines, \( x^2y \) if \( V(f) \) is a double line and a line, and \( x^3 \) if \( V(f) \) is a triple line.

2. Second, I show that Newton polytopes of the each above polynomials is the farthest polynomial for the each cases, i.e., each above polynomials are the maximization of the case (B) of the duality theorem in the section 2.

Note that there are 5 different distances for all worst one-parameter subgroups of unstable cubic plane curves from \( xyz \) in order of distance like the below pictures.
Let \( f = ax^3 + bx^2y + cxy^2 + dy^3 + ey^2z \) be an unstable cubic plane curve in \( V \) for \( a, b, c, d, e \in \mathbb{C} \) where the coefficients are not all 0.

(a) Assume that \( V(f) \) is an irreducible cuspidal. By the result of 1, \( a \neq 0 \) and \( e \neq 0 \). Hence, \( \text{NP}(f) \) must contain terms \( x^3 \) and \( y^2z \). Therefore, the distance from \( xyz \) to \( \text{NP}(f) \) cannot be farther than \( \text{NP}(x^3 + y^2z) \).

(b) Assume that \( V(f) \) is a conic and a tangent line. By the result of 1, \( a = 0 \), \( b \neq 0 \), and \( e \neq 0 \). Hence, \( \text{NP}(f) \) must not contain the term \( x^3 \), and
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contain terms $x^2y$ and $y^2z$. Therefore, the distance from $xyz$ to $\text{NP}(f)$ cannot be farther than $\text{NP}(x^2y + y^2z)$.

(c) Assume that $V(f)$ is a three concurrent lines. Note that the fourth and fifth picture contain at least double lines. Therefore, the distance from $xyz$ to $\text{NP}(f)$ cannot be farther than $\text{NP}(x^2y + y^2z)$.

(d) Assume that $V(f)$ is a double line and a line. Note that the fifth picture contain a triple line. Therefore, the distance from $xyz$ to $\text{NP}(f)$ cannot be farther than $\text{NP}(x^3 + xy^2)$.

(e) Assume that $V(f)$ is a triple line. Clearly, the fifth picture is the farthest one. Therefore, the distance from $xyz$ to $\text{NP}(f)$ cannot be farther than $\text{NP}(x^3)$.

3. Lastly, I compute the weight-vector of the worst one-parameter subgroup for each five cases.

(a) The weight-vector of the worst one-parameter subgroup of $f$ is $(1, 4, -5)$ if $V(f)$ is an irreducible cuspidal.

We must find the worst one-parameter subgroup of $f = x^3 + y^2z$. Note that there are two distinct nonzero characters $\chi_{(3,0,0)}$ and $\chi_{(0,2,1)}$. Let

$$\lambda(t) = \begin{bmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{r_2} & 0 \\ 0 & 0 & t^{r_3} \end{bmatrix}$$

be the one-parameter subgroup from $G_m$ to $G$ where $r_1 + r_2 + r_3 = 0$ and $r_1, r_2, r_3 \in \mathbb{Z}$.

Then we get $\mu(f, \lambda) = -\min(3r_1, 2r_2 + r_3)$.

Since $(r_1+1, r_2+1, r_3+1)$ lies on $\text{NP}(f)$, this means that $(r_1+1, r_2+1, r_3+1)$ satisfies the equation $\frac{x-3}{3} = \frac{y}{2} = \frac{z}{1}$. Let $\frac{r_1-2}{3} = \frac{r_2+1}{2} = \frac{r_3+1}{1} = t$ for some $-1 \leq t \leq 0$. Then $r_1 = 3t + 2$, $r_2 = -2t - 1$, and $r_3 = -t - 1$. Hence, we must find supremum of $\min(9t+6, -5t-3)$ for $-1 \leq t \leq 0$. By checking this for drawing graphs, we get $t = -\frac{2}{3}$, so $(r_1, r_2, r_3) = (1, \frac{4}{3}, -\frac{5}{3})$. Therefore, $(r_1, r_2, r_3) = (1, 4, -5)$ by scaling. Hence, the weight-vector of the worst one-parameter subgroup of $f$ is $(1, 4, -5)$ and $\mu(f, \lambda) = -3$.

(b) The weight-vector of the worst one-parameter subgroup of $f$ is $(0, 1, -1)$ if $V(f)$ is a conic and a tangent line.

We must find the worst one-parameter subgroup of $f = x^2y + y^2z$. Note
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that there are two distinct nonzero characters $\chi_{(2,1,0)}$ and $\chi_{(0,2,1)}$. Let

$$
\lambda(t) = \begin{bmatrix} t^r & 0 & 0 \\
0 & t^s & 0 \\
0 & 0 & t^t \end{bmatrix}
$$

be the one-parameter subgroup from $G_r$ to $G$ where $r_1 + r_2 + r_3 = 0$ and
$r_1, r_2, r_3 \in \mathbb{Z}$.

Then we get $\mu(f, \lambda) = -\min(2r_1 + r_2, 2r_2 + r_3)$.

Since $(r_1 + 1, r_2 + 1, r_3 + 1)$ lies on NP$(f)$, this means that $(r_1 + 1, r_2 + 1, r_3 + 1)$ satisfies the equation $\frac{x^2}{2} = \frac{y^2}{2} = t$. Let $\frac{r_1}{2} = \frac{r_2}{2} = \frac{r_3}{2} = t$ for some $-1 \leq t \leq 0$. Then $r_1 = 2t + 1$, $r_2 = -t$, and $r_3 = -t - 1$. Hence, we must find supremum of $\min(3t + 2, -3t - 1)$ where $-1 \leq t \leq 0$. By checking this for drawing graphs, we get $t = -\frac{1}{2}$, so $(r_1, r_2, r_3) = (0, \frac{1}{2}, -\frac{1}{2})$. Therefore, $(r_1, r_2, r_3) = (0, 1, -1)$ by scaling. Hence, the weight-vector of the worst one-parameter subgroup of $f$ is $(0, 1, -1)$ and $\mu(f, \lambda) = -1$.

(c) The weight-vector of the worst one-parameter subgroup of $f$ is $(1, 1, -2)$ if $V(f)$ is a three concurrent lines.

We must find the worst one-parameter subgroup of $f = x^3 + xy^2$. Note that there are two distinct nonzero characters $\chi_{(3,0,0)}$ and $\chi_{(1,2,0)}$. Let

$$
\lambda(t) = \begin{bmatrix} t^r & 0 & 0 \\
0 & t^s & 0 \\
0 & 0 & t^t \end{bmatrix}
$$

be the one-parameter subgroup from $G_r$ to $G$ where $r_1 + r_2 + r_3 = 0$ and
$r_1, r_2, r_3 \in \mathbb{Z}$.

Then we get $\mu(f, \lambda) = -\min(3r_1, r_1 + 2r_2)$.

Since $(r_1 + 1, r_2 + 1, r_3 + 1)$ lies on NP$(f)$, this means that $(r_1 + 1, r_2 + 1, r_3 + 1)$ satisfies the equation $\frac{x^2}{2} = \frac{y^2}{2} = t$. Let $\frac{r_1}{2} = \frac{r_2 + 1}{2} = t$ for some $-1 \leq t \leq 0$ and $r_3 = -1$. Then $r_1 = 2t + 2$, $r_2 = -2t - 1$, and $r_3 = -1$. Hence, we must find supremum of $\min(6t + 6, -2t)$ where $-1 \leq t \leq 0$. By checking this for drawing graphs, we get $t = -\frac{3}{4}$, so $(r_1, r_2, r_3) = (1, \frac{1}{2}, -\frac{3}{2})$. Therefore, $(r_1, r_2, r_3) = (1, 1, -2)$ by scaling. Hence, the weight-vector of the worst one-parameter subgroup of $f$ is $(1, 1, -2)$ and $\mu(f, \lambda) = -3$.

(d) The weight-vector of the worst one-parameter subgroup of $f$ is $(1, 0, -1)$ if $V(f)$ is a double line and a line.

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We must find the worst one-parameter subgroup of $f = x^2y$. Note that there is only one nonzero character $\chi_{(2,1,0)}$. Hence, the weight-vector of the worst one-parameter subgroup of $f$ is $(1, 0, -1)$ and $\mu(f, \lambda) = -2$.

(e) The weight-vector of the worst one-parameter subgroup of $f$ is $(2, -1, -1)$ if $V(f)$ is a triple line.

We must find the worst one-parameter subgroup of $f = x^3$. Note that there is only one nonzero character $\chi_{(3,0,0)}$. Hence, the weight-vector of the worst one-parameter subgroup of $f$ is $(2, -1, -1)$ and $\mu(f, \lambda) = -6$. 

\[ \square \]
Bibliography


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국문초록

대수기학은 수학의 한 분야로서, 다변수 다항식들의 균들을 연구하는 학문이다. 이러한 균들에 대한 기하적 문제들을 풀기 위해 주로 가환대수학이 사용된다. 기하적 문제들 중 하나인 모듈리 문제는 기하적 대상들을 동형으로 식으로 분류하는 것이다.

기하적 불변 이론은 대수기학에서 균 작용에 의한 똑같이 만드는 방법으로써 모듈리 공간을 만드는데 사용된다. 이는 불변 이론에 있는 대이비드 헨버트의 아이디어를 사용하여 대이비드 멤포드에 의해 발견되었다.

나는 기하적 불변 이론 안에 있는 안정성에 대한 수치적 방법에 집중한다. 사영 스크립 위에 가까운 선형 작용에서 우리는 그 사영 스크립 위에 있는 점의 안정성을 결정하는데 사용되는 헨버트-멤포드 방법이라고 불리우는 수치적 방법을 얻게 된다.

이 학위 논문에서 나는 3차 평면 곡선의 벡터 공간 위에 3차원 특수 선형군의 선형 작용에 있는 안정성에 집중한다. 헨버트-멤포드 방법으로써, 불안정 3차 평면 곡선들은 $x^3, x^2y, xy^2, y^3, y^2z$의 선형 결합의 근인 평면 대수 곡선이 된다.

조지 켐프는 주어진 불안정 점을 불안정하게 만드는 1모수 부분군들을 찾는 멤포드 가설을 풀었다. 이를 이용하여, 나는 다음과 같이 불안정 3차 평면 곡선들에 대한 가장 안 좋은 1모수 부분군들을 완벽하게 분류하였다.

첫째, 무한히 많은 불안정 3차 평면 곡선들이 있지만, 나는 사영 동형으로써 이를 5가지 경우로 줄인다. 이는 불안정 3차 평면 곡선의 분류에 바탕을 둔다.

둘째, 나는 선형 프로그래밍에 있는 기초적인 정리를 사용하여 이 5가지 경우를 계산한다. 이것의 기초적 원리는 최적화 이론과 삼차원에 바탕을 둔다.

마지막으로, 나는 헨버트-멤포드 방법을 사용하여 불안정 3차 평면 곡선들에 대한 가장 안 좋은 1모수 부분군들을 확인한다. 이러한 과정이 나의 작업을 보장한다.

주요이휘: 기하적 불변 이론, 안정성에 대한 수치적 방법, 가장 안 좋은 1모수 부분군
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